ON ZEEMAN'S FILTRATION IN HOMOLOGY

1994 Russ. Acad. Sci. Sb. Math. 77 477

(http://iopscience.iop.org/1468-4802/77/2/A13)

Росс. Акад. Наук Матем. Сборник
Том 183 (1992), № 12

Russian Acad. Sci. Sb. Math.
Vol. 77 (1994), No. 2

ON ZEEMAN'S FILTRATION IN HOMOLOGY

UDC 515.142.21

E. G. SKLYARENKO

ABSTRACT. For a finite complex K, Zeeman constructed a spectral sequence, converging to the homology of the complex, of the form $E_2^{pq} = H^q(K; \mathscr{H}_p) \Rightarrow H_{p-q}(K)$. Special attention was given to the corresponding filtration in the homology of K, essentially dependent on the cohomology:

$$H_r(K) = F^0 H_r(K) \supset F^1 H_r(K) \supset \cdots \supset F^q H_r(K) \supset \cdots,$$
$$E_\infty^{pq} = F^q H_r(K)/F^{q+1} H_r(K), \qquad r = p - q,$$

where $\mathscr{H}_p$ is the coefficient system determined by the local homology groups $H_p^x = H_p(K, K \setminus x)$.

The object of the present paper is to show that the Zeeman filtration, although defined in terms of the simplicial structure of the complex, is, in the end, of a general-categorical nature. Due to this fact, a more complete description of its connection with the topology of the space and with the product is obtained.

Bibliography: 19 titles.

INTRODUCTION

Starting with the structure of the double complex generated by pairs of simplices $\tau \subset \sigma$ of a given simplicial complex K, with the total differential defined by the boundary and coboundary operators of the simplices τ and σ, respectively, Zeeman [18], [19] constructed for a finite complex K a spectral sequence, converging to the homology of the complex, of the form $E_2^{pq} = H^q(K; \mathscr{H}_p) \Rightarrow H_{p-q}(K)$. Special attention was given to the filtration corresponding to this spectral sequence, in the homology of K, dependent essentially on the cohomology:

$$H_r(K) = F^0 H_r(K) \supset F^1 H_r(K) \supset \cdots \supset F^q H_r(K) \supset \cdots,$$
$$E_\infty^{pq} = F^q H_r(K)/F^{q+1} H_r(K), \qquad r = p - q.$$

Here $\mathscr{H}_r$ is the coefficient system determined by the local homology groups $H_p^x = H_p(K, K \setminus x)$.

Zeeman observes that this filtration, being connected with the dimension of the part of the space K that contains the cycles determining a given homology class, reflects to a certain extent the "degree of freedom" of the distribution of these cycles in K. This is corroborated by his proof of the relation $H_p(K) \frown H^q(K) \subset F^q H_r(K)$ (homology and cohomology with integer coefficients or with coefficients in any principal ideal ring R). When K is a manifold, all terms of the filtration coincide with $H_r(K)$.

To establish the topological invariance of the spectral sequence and the corresponding filtration, Zeeman extends his construction to more general topological spaces, using in place of simplicial a combination of singular chains and Čech cohomology.

1991 *Mathematics Subject Classification*. Primary 55N25, 55N30; Secondary 18G35, 18G40, 55M05.

In this more general situation the coefficient systems $\mathscr{H}_p$ become local-homology sheaves generated by the presheaves $U \to H_p(X, X \setminus U)$ (where the U are open sets in the space X), and the spectral sequence itself in fact coincides, as he observes, with the sequence used (in its degenerate form) by Cartan [9] to prove Poincaré duality between the singular homology and Čech cohomology of a topological manifold. He observes also that in terms of local Čech homology (which coincides in categories of polyhedra with the singular or ordinary homology) a similar spectral sequence was independently studied by István Fáry in 1955. Since then, this sequence has found a number of applications. In a very general form it is described in [8], Chapter 5, §8 (see also [6], Chapter 5, §4.5).

A considerable simplification of Zeeman's original construction of the spectral sequence and his proof of its topological invariance was obtained in [14]. Taken into account there were not only finite, but also infinite locally finite simplicial complexes of finite dimension. Also, a relative version of this spectral sequence was obtained:

$$E_2^{pq} = H_c^q(A; \mathscr{H}_p(K)) \Rightarrow H_{p-q}^c(K, K \setminus A)$$

(Čech cohomology of a closed subset $A \subset K$, with compact supports, and ordinary homology, i.e., singular homology with compact supports). This made it possible to obtain, for the first time, the following geometric interpretation of the Zeeman filtration: the filtration of an element $h \in H_{p-q}^c(K)$ coincides with the largest integer q such that h is contained in the images of the homomorphisms $H_{p-q}^c(K \setminus A) \to H_{p-q}^c(K)$ for any closed subsets $A \subset K$ for which $\dim A \leq q - 1$. From this it is obvious why the filtration (as well as the spectral sequence) is not homotopy invariant.

The proof of this result, however, is based on still another interpretation of Zeeman's spectral sequence, using, as observed in §8 of [14], a new definition of the filtration itself. Consider the filtration, in chains, generated by the "co-skeletons" of a simplicial complex K, i.e., the unions K^q of the barycentric stars of the simplices $\Delta^q \in K$ of dimension q (the simplices of the barycentric subdivision of K that are transverse to Δ^q). The spectral sequence corresponding to this filtration is a special case of the spectral sequence $^*E$ of [17]. It can be shown that, up to reindexing, this "coskeleton" spectral sequence coincides with the spectral sequence of Zeeman; there results, in particular, its topological invariance.

The basic constructions in [14] depend, obviously, on the simplicial structure of the complex K. Essential use is made of the cap product, and so, as with Zeeman, homology with coefficients in some ring R is considered. Also important is the condition that K be finite-dimensional, which guarantees finiteness of the decreasing filtration in chains and, as a consequence, convergence of the spectral sequence.

The object of the present paper is to show that the Zeeman filtration is in the end dictated by neither the simplicial structure of the space in question, nor the spectral sequences of Cartan type (generalizing Poincaré duality) or the double complexes generating them. As in the case of Poincaré duality (cf. [5]), the Zeeman filtration in homology is in essence of a general-categorical nature. The basic results described above are in fact valid for the homology of any (metrizable) locally compact spaces, and in full measure for any locally finite cellular polyhedra (not necessarily finite-dimensional). They are valid not only for the ordinary homology H_*^c, but also for the homology H_*^φ with supports in a paracompactifying family φ—in particular, for the homology H_* of the second kind (defined by arbitrary "infinite" cycles). As an incidental development we obtain a filtration in the homology of $K \setminus A$, determined by the cohomology of the pair (K, A) (the second of two relative versions).

§1. Description of the filtration

Let T be a left-exact additive covariant functor on an abelian category $\mathscr{K}$, amplified to a cohomological functor $\{T^q\}$ the sense of Grothendieck (i.e., an exact ∂-functor; see §2.1 of [12]) such that $T^0 = T$ and $T^q = 0$ for $q < 0$. The classical example of a cohomological functor is the set $\{T^q\}$ of right derived functors of a functor T (which exist whenever the category $\mathscr{K}$ has enough injectives). Let $\mathscr{C}_*$ be a complex consisting of objects in $\mathscr{K}$, with boundary operator of degree -1:

$$\mathscr{C}_*: \cdots \leftarrow \mathscr{C}_{n-1} \overset{\partial}{\leftarrow} \mathscr{C}_n \overset{\partial}{\leftarrow} \mathscr{C}_{n+1} \leftarrow \cdots .$$

As usual, we write $\mathscr{Z}_n = \operatorname{Ker}\partial$, $\mathscr{B}_n = \operatorname{Im}\partial$, $\mathscr{B}_n \subset \mathscr{Z}_n \subset \mathscr{C}_n$.

We shall be concerned with the homology groups $H_n(T(\mathscr{C}_*))$ of the chain complex $T(\mathscr{C}_*)$ in the case that all the objects $\mathscr{C}_n$ are T-acyclic (i.e., that $T^q(\mathscr{C}_n) = 0$ for $q > 0$). If $\mathscr{C}_*$ is a flabby sheaf of chains on a locally compact space X (see §3 below), and $T = \Gamma_\varphi$ is the functor for sections with supports in some family φ, then $H_n(T(\mathscr{C}_*)) = H_n^\varphi(X; \mathscr{G})$ is the homology of X with supports in φ (where $\mathscr{G}$ is a locally constant coefficient sheaf). The category $\mathscr{K}$ is in this case a category of sheaves of abelian groups or modules, and for $\mathscr{A} \in \mathscr{K}$ the values $T^q(\mathscr{A})$ are the cohomology groups $H_\varphi^q(X; \mathscr{A})$ of X with coefficients in $\mathscr{A}$ and supports in φ.

In view of the T-acyclicity of the objects $\mathscr{C}_p$, the exact sequences $0 \to \mathscr{Z}_p \to \mathscr{C}_p \to \mathscr{B}_{p-1} \to 0$ determine for $q > 1$ isomorphisms $\delta: T^{q-1}(\mathscr{B}_{p-1}) \to T^q(\mathscr{B}_p)$; hence the inclusions $\mathscr{B}_p \subset \mathscr{Z}_p$ imply the existence of homomorphisms $T^q(\mathscr{Z}_p) \to T^{q-1}(\mathscr{Z}_{p-1})$. This gives rise for each n to a sequence of homomorphisms

$$\cdots \to T^q(\mathscr{Z}_{n+q}) \to T^{q-1}(\mathscr{Z}_{n+q-1}) \to \cdots \to T^1(\mathscr{Z}_{n+1}) \subset H_n(T(\mathscr{C}_*)).$$

The final inclusion here follows from the exact sequence

$$0 \to T(\mathscr{Z}_{n+1}) \to T(\mathscr{C}_{n+1}) \to T(\mathscr{B}_n) \to T^1(\mathscr{Z}_{n+1}) \to 0,$$

corresponding to the short exact sequence $0 \to \mathscr{Z}_{n+1} \to \mathscr{C}_{n+1} \to \mathscr{B}_n \to 0$. Indeed, from the left exactness of T we have the inclusions $T(\mathscr{B}_n) \subset T(\mathscr{Z}_n) \subset T(\mathscr{C}_n)$, and so the $T(\mathscr{Z}_q)$ are cycles in the complex $T(\mathscr{C}_*)$, while the subgroups of boundaries in $T(\mathscr{C}_*)$ are the images of the homomorphisms $T(\mathscr{C}_{p+1}) \to T(\mathscr{B}_p)$. Consequently, the quotient group of $T(\mathscr{B}_n)$ module the image of $T(\mathscr{C}_{n+1})$, which coincides with $T^1(\mathscr{Z}_{n+1})$, is a subgroup of $H_n(T(\mathscr{C}_*))$.

Let $F^q H_n(T(\mathscr{C}_*))$ be the image of $T^q(\mathscr{Z}_{n+q})$ in $H_n(T(\mathscr{C}_*))$. There is determined a decreasing filtration

$$H_n(T(\mathscr{C}_*)) = F^0 H_n(T(\mathscr{C}_*)) \subset F^q H_n(T(\mathscr{C}_*)) \supset \cdots \supset F^q H_n(T(\mathscr{C}_*)) \supset \cdots .$$

By the filtration of an element $h \in H_n(T(\mathscr{C}_*))$ we mean the largest integer q such that $h \in F^q H_n(T(\mathscr{C}_*))$.

§2. Comparison with the Zeeman filtration

Let $\mathscr{J}^*(\mathscr{C}_*)$ be a resolution, in the sense of Cartan-Eilenberg ([10], Chapter 17), of the complex $\mathscr{C}_*$ in the category $\mathscr{K}$, consisting of T-acyclic objects (it contains in a natural fashion resolutions $\mathscr{J}^*(\mathscr{Z}_p)$, $\mathscr{J}^*(\mathscr{B}_p)$ of the objects $\mathscr{Z}_p$ and $\mathscr{B}_p$). If every object in $\mathscr{K}$ is contained in an injective object, then such a resolution can be constructed out of injective objects in $\mathscr{K}$. In the situation we are interested in, where $\mathscr{C}_*$ is a differential sheaf of chains of a topological space, for $\mathscr{J}^*(\mathscr{C}_*)$ we can take $\mathscr{C}^*(\mathscr{C}_*)$, where $\mathscr{C}^*$ is the canonical flabby resolution of Godement [11].

As is known ([12], Chapter 2, §2.4), any such resolution determines a spectral sequence (of Cartan-Zeeman type) with second term $E_2^{pq} = T^q(\mathscr{H}_p)$, where $\mathscr{H}_p = \mathscr{Z}_p/\mathscr{B}_p$. Under certain conditions (e.g., if $\mathscr{C}_p = 0$ for $p > p_0$; see also §3 below), the spectral sequence converges to the homology groups $H_{p-q}(T(\mathscr{C}_*))$. In this section we show that the filtration in the homology $H_n(T(\mathscr{C}_*))$ corresponding to this spectral sequence coincides with the filtration described in §1. It follows, in particular, that the filtration in $H_n(T(\mathscr{C}_*))$ defined by means of $\mathscr{J}^*(\mathscr{C}_*)$ is independent of the choice of the resolution $\mathscr{J}^*(\mathscr{C}_*)$. The results of this section will be used below in §5.

In the double complex $\mathscr{J}^*(\mathscr{C}_*) = \{\mathscr{J}^q(\mathscr{C}_p)\}$ we define the total differential $d = d' + (-1)^p d''$, where d' and d'' are the homomorphisms induced by the boundary homomorphism in $\mathscr{C}_*$ and the differential in $\mathscr{J}^*$. As usual, we have $d^2 = 0$. Using the left exactness of the functor T and the T-acyclicity of the objects $\mathscr{C}_p$ and $\mathscr{J}^q(\mathscr{C}_p)$, we find by a standard diagram chase (cf. the argument of Lemma B.32 and Theorem B.32 in [16]) that the imbedding of $\mathscr{C}_*$ into $\mathscr{J}^0(\mathscr{C}_*) \subset \mathscr{J}^*(\mathscr{C}_*)$ induces an isomorphism of the groups $H_n(T(\mathscr{C}_*))$ to the homology of the chain complex $T(\mathscr{J}^*(\mathscr{C}_*))$, taken with the total grading $n = p - q$ (relative to which the n-dimensional chains are $\bigoplus_{p-q=n} T(\mathscr{J}^q(\mathscr{C}_p)))$. This follows also from the spectral sequence of the double complex $T(\mathscr{J}^*(\mathscr{C}_*))$ corresponding to its filtration by the subcomplexes $K_p = \bigoplus_{i\leq p} T(\mathscr{J}^*(\mathscr{C}_i))$. Application of the spectral sequence is possible in view of the fact that this filtration is obviously regular in the sense of [11] (cf. Theorem 4.8.1 in Chapter 1 of the latter).

The filtration we are interested in (of Zeeman type) is determined in the homology groups $H_n(T(\mathscr{C}_*)) = H_n(T(\mathscr{J}^*(\mathscr{C}_*)))$ by the subcomplexes $K^q = \bigoplus_{i\geq q} T(\mathscr{J}^i(\mathscr{C}_*))$. Let $\mathscr{L}^q(\mathscr{C}_*)$ be the kernel of $d''\colon \mathscr{J}^q(\mathscr{C}_*) \to \mathscr{J}^{q+1}(\mathscr{C}_*)$. As above, we find that the imbedding $\mathscr{L}^q(\mathscr{C}_*) \subset K^q$ induces an isomorphism of the homology of these complexes; therefore, any element $h \in H_n(T(\mathscr{C}_*))$ falling into the image of $H_n(K^q)$ under the imbedding $K^q \subset T(\mathscr{J}^*(\mathscr{C}_*))$ is defined by a p-cycle z_p^q of the complex $T(\mathscr{L}^q(\mathscr{C}_*))$ (belonging to the group $T(\mathscr{L}^q(\mathscr{C}_p)))$, where $p = n+q$, or, in view of the left exactness of the functor T, by a q-cycle z_p^q of the complex $T(\mathscr{J}^*(\mathscr{Z}_p))$ (since $\mathscr{J}^*(\mathscr{Z}_p) \subset \mathscr{J}^*(\mathscr{C}_p))$. The imbeddings $T(\mathscr{C}_i) \subset K_p$ for $i \leq p$ induce an isomorphism $H_n(T(\mathscr{C}_*)) = H_n(K_p)$ (again the same argument); therefore, an isomorphism of n-dimensional homology is induced also by the imbedding $K_p \subset T(\mathscr{J}^*(\mathscr{C}_*))$. Thus, h belongs to the image in $H_n(K_p) = H_n(T(\mathscr{C}_*))$ of the group $H^q(T(\mathscr{J}^*(\mathscr{Z}_p))) = T^q(\mathscr{Z}_p)$ under the imbedding $T(\mathscr{J}^*(\mathscr{Z}_p)) \subset K_p$ $(p = n + q)$.

A cycle in $T(\mathscr{C}_*)$ defining h is obtained from z_p^q by a diagram chase. Namely, from the acyclicity of the complex $T(\mathscr{J}^*(\mathscr{C}_p))$ we find that $z_p^q = (-1)^p d''c$, where c is some element of $T(\mathscr{J}^{q-1}(\mathscr{C}_p))$; therefore, $z_p^q = dc - d'c$, i.e., z_p^q is homologous to a cycle $z_{p-1}^{q-1} = d'c$. Continuing this operation, we obtain the required cycle in $T(\mathscr{C}_*)$.

A somewhat different procedure for defining the filtration was used in §1. With the short exact sequence $0 \to \mathscr{Z}_p \to \mathscr{C}_p \to \mathscr{B}_{p-1} \to 0$ we associate the exact triple of T-acyclic resolutions

$$0 \to \mathscr{J}^*(\mathscr{Z}_p) \to \mathscr{J}^*(\mathscr{C}_p) \to \mathscr{J}^*(\mathscr{B}_{p-1}) \to 0$$

and the connecting homomorphism $\delta\colon T^{q-1}(\mathscr{B}_{p-1}) \to T^q(\mathscr{Z}_p)$. In accordance with the definition of δ, the cycles z_{p-1}^{q-1} and z_p^q must be connected by the relations $z_{p-1}^{q-1} = d'c$ and $z_p^q = d''c$. Clearly, the cycles $z_{p-1}^{q-1} \in T(\mathscr{J}^*(\mathscr{B}_{p-1})) \subset T(\mathscr{J}^*(\mathscr{Z}_{p-1}))$ that correspond to z_p^q in these two cases can differ only in sign, which has no effect on the definitions of the filtrations.

§3. Homology of locally compact spaces

Any choice of chains defining the homology of a topological space X gives rise to differential sheaves of chains C_*. These are determined by the differential presheaves that associate to open sets $U \subset X$ the chain complexes of the pairs $C_*(X, X\setminus U; \mathscr{G})$ (where $\mathscr{G}$ is a locally constant coefficient system). The criterion for the value of one or the other approach to a description of homology is, it would appear, not only (and not so much) the generality achieved, but also (above all) the effectiveness of the constructions involved as regards applications.

The most suitable approach has for a long time been taken to be the singular theory. It was precisely for this reason that the topological invariance of the Zeeman spectral sequence (and filtration) was proved by means of its mapping (shown to be an isomorphism) into the spectral sequence of the type of §2 above corresponding to the differential sheaf of singular chains and its resolution composed of Čech cochains (see [19], [14]). Convergence and isomorphism of the spectral sequences was guaranteed by the requirement of finite dimensionality of the space under examination (in this connection see [8], Chapter 4, §2.9). In general, without this requirement, convergence to the homology of the space breaks down (see [6], Chapter 5, §4.5).

In the present problem, however, the singular approach is far from being either the most general or the simplest. Indeed, the sections of the sheaves $\mathscr{C}_*$ coincide with the chains defining them (and can be used to define homology) only over the whole space X ([8], Chapter 1, Exercise 12, or [16], Chapter 2, §2), while the sheaves themselves are only homotopically fine ([8], Chapter 2, Exercise 32(b), or [16], Chapter 6, Proposition 7 and its corollary). Furthermore, an examination of singular theory from the standpoint of sheaf theory shows (see, e.g., [6], Chapter 3, §5.2, or [7], Chapter 3, §1) that outside the bounds of the category of (weakly locally contractible spaces the singular groups in general cease to be homology and cohomology groups (see [6], Chapter 2, §1.1).

Thus, the results of [19] and [14] are limited (in accordance with their authors' intentions) to the category of (finite-dimensional) locally finite simplicial polyhedra. The coincidence of the Zeeman filtration and the filtration of the present paper follows from the content of §4 below.

It is well known that the Borel-Moore homology for locally compact spaces likewise suffers from many essential deficiencies. Called up with the intent of replacing the singular homology where the latter "does not work", it proved to be too artificial—actually, a version of "co-cohomology" with respect to sheaf cohomology (see the beginning of Chapter 5 in [8]). For locally constant coefficients with finitely generated stalks, the theory is isomorphic to that of Steenrod-Sitnikov homology, which finally established itself in the literature a little later. Without this restriction the Borel-Moore theory is not well defined (see in this regard [7], Chapter 2, §2, and Chapter 3, §6). The sheaves of chains in Borel-Moore theory are fine (and therefore soft), and for homology with coefficients in the ground ring even flabby. They are defined formally in such a fashion that the complexes of sections over open sets $U \subset X$ determine the homology of the pairs $(X, X \setminus U)$.

For Steenrod-Sitnikov homology, the chains that stack up into sheaves were defined in [3] (see also [7], Chapter 2, §2). The differential sheaves of chains $\mathscr{C}_*$ they give rise to are flabby, and the chain complexes $C_*(X, X \setminus U; \mathscr{G})$ that determine these coincide with the complexes of sections of these sheaves over open sets $U \subset X$. We recall that flabby sheaves are always φ-acyclic for any family of supports φ (not just a paracompactifying; see Theorem 4.4.3 in [11], Chapter 2). The derived sheaves $\mathscr{H}_n$ are determined by the presheaves $U \to H_n(X, X \setminus U; \mathscr{G})$; their stalks are the local homology groups $H_n^x = (X, X \setminus x; \mathcal{G})$. Similar sheaves of chains arise in using the homology construction proposed by Massey [13] ([7], Chapter 2, §2). That Massey homology is equivalent to Steenrod-Sitnikov is proved in §6 of [4].

Since the constructions of the chains that define the Steenrod-Sitnikov homology can differ, the question arises whether the filtration in homology is independent of the choice of chains. In accordance with [6], Chapter 5, §2.2, and [4], §6, for any two approaches to a description of chains there is a third, comparable with the first two in the sense that the corresponding sheaf of chains is included with either of the first in a transformation of the form $\mathcal{C}_*' \to \mathcal{C}_*''$ giving an isomorphism of the homology of the space X, as well as of all pairs of the form $(X, X \setminus U)$. Consequently, the transformations also induce the identity isomorphism of the derived sheaves $\mathcal{H}_p$. Thus, the fact that the filtration does not depend on the choice of chains is a consequence of the following general assertion.

3.1. **Proposition.** *Let $\alpha: \mathcal{C}_* \to \mathcal{C}_*'$ be a transformation of chain complexes in a category $\mathcal{K}$ (consisting of T-acyclic objects) that induces an isomorphism of homology $H_n(T(\mathcal{C}_*)) \to H_n(T(\mathcal{C}_*'))$ and of the derived objects $\mathcal{H}_n \to \mathcal{H}_n'$ then α also induces an isomorphism of the filtrations in the homology of these complexes.*

Proof. In view of the argument at the end of §1, the pair $T^1(\mathcal{Z}_{n+1}) \subset H_n(T(\mathcal{C}_*))$ is obtained from the pair $T(\mathcal{B}_n) \subset T(\mathcal{Z}_n)$ by factoring modulo the image of the group $T(\mathcal{C}_{n+1})$. Therefore, the short exact sequence $0 \to \mathcal{B}_n \to \mathcal{Z}_n \to \mathcal{H}_n \to 0$ determines an exact sequence

$$\begin{aligned} 0 \to T^1(\mathcal{Z}_{n+1}) \to H_n(T(\mathcal{C}_*)) \to T(\mathcal{H}_n) \to T^1(\mathcal{B}_n) \\ \to T^1(\mathcal{Z}_n) \to T^1(\mathcal{H}_n) \to T^2(\mathcal{B}_n) \to \cdots , \end{aligned}$$

which transforms into the corresponding sequence for the complex $\mathcal{C}_*'$. Using the isomorphisms $T^q(\mathcal{B}_{p-1}) = T^{q+1}(\mathcal{Z}_p)$ for $q \geq 1$, the Five Lemmas, and an elementary induction on q, we obtain the existence of isomorphisms $T^q(\mathcal{Z}_p) \to T^q(\mathcal{Z}_p')$, that guarantee the coincidence of the filtrations.

Thus, let X be a locally compact space, $\mathcal{C}_* = \mathcal{C}_*(X; \mathcal{G})$ a flabby differential sheaf of chains, $\mathcal{G}$ a locally constant coefficient sheaf, and φ a family of supports. In accordance with §1, there is determined for each $n \geq 0$ a sequence of cohomology groups and homomorphisms, giving rise to a filtration in the group $H_n^\varphi(X; \mathcal{G})$:

$$\begin{aligned} \cdots \to H_\varphi^s(X; \mathcal{Z}_{n+s}) \to H_\varphi^{s-1}(X; \mathcal{Z}_{n+s-1}) \to \cdots \\ \to H_\varphi^2(X; \mathcal{Z}_{n+2}) \to H_\varphi^1(X; \mathcal{Z}_{n+1}) \subset H_n^\varphi(X; \mathcal{G}). \end{aligned} \tag{1}$$

In particular, ordinary homology ($\varphi = c =$ compact sets) is filtered by cohomology of the second kind, and homology of the second kind ($\varphi =$ all closed sets) by ordinary (sheaf) cohomology.

Let A be a subspace of X, and $B = X \setminus A$. As is known, the sections of the sheaves of chains $\mathcal{C}_*$ with supports in φ that are contained in B (i.e., with supports in the family $\varphi|B$) determine the subspace homology $H_n^\varphi(B; \mathcal{G})$, while the homology $H_n^\varphi(X, B; \mathcal{G})$ is determined by the quotient complex of the complex of all chains $\mathcal{C}_*^\varphi(X; \mathcal{G})$ modulo this part (see, e.g., [7], Chapter 2, §2.4, or [8], Chapter 5, §5). We note that in the case of flabby sheaves of cochains, the cohomology of the pair (X, A) and of the subspace A can be determined similarly in any of the following cases: a) A open; b) A closed, X paracompact, and φ a paracompactifying family; c) A an arbitrary set, X a hereditarily paracompact set, and φ a paracompactifying family (see [8], or [7], Chapter 1, §5.3). Indeed, in each of these cases the restriction to A

of the resolution defining the cohomology is φ-acyclic, and the sections of sheaves of the resolution over A with supports in $\varphi \cap A$ extend to sections over A with supports in φ. The necessity of the paracompactifying requirement for φ is shown by the examples in [7], Chapter 1, §5.3 and [6], Chapter 8, §5.4. Thus, we have the following result.

3.2. **Theorem.** *Under the above restrictions on A, X, and φ, there exist the following sequences of homomorphisms, determining filtrations in their final terms*:

$$(2)\qquad \begin{aligned} &\cdots \to H^s_{\varphi\cap A}(A; \mathscr{Z}_{n+s}) \to H^{s-1}_{\varphi\cap A}(A; \mathscr{Z}_{n+s-1}) \to \\ &\quad\cdots \to H^2_{\varphi\cap A}(A; \mathscr{Z}_{n+2}) \to H^1_{\varphi\cap A}(A; \mathscr{Z}_{n+1}) \subset H^\varphi_n(X, X \setminus A; \mathscr{G}), \end{aligned}$$

$$(3)\qquad \begin{aligned} &\cdots \to H^s_\varphi(X, A; \mathscr{Z}_{n+s}) \to H^{s-1}_\varphi(X, A; \mathscr{Z}_{n+s-1}) \to \\ &\quad\cdots \to H^2_\varphi(X, A; \mathscr{Z}_{n+2}) \to H^1_\varphi(X, A; \mathscr{Z}_{n+1}) \subset H^{\varphi|X\setminus A}_n(X \setminus A; \mathscr{G}). \end{aligned}$$

The inclusion $A \subset X$ induces a natural transformation of the sequence (1) into (2), and similarly of the sequence (3) into (1).

3.3. **Corollary.** *If the filtration of an element $h \in H^\varphi_n(X; \mathscr{G})$ is at least s, then h belongs to the image of the homomorphism $H^{\varphi|X\setminus A}_n(X \setminus A; \mathscr{G}) \to H^\varphi_n(X; \mathscr{G})$ for any subspace $A \subset X$ for which $\dim A < s$, in the following cases*:

a) *for open A*;
b) *for closed A, when X is paracompact and φ is paracompactifying family*;
c) *for arbitrary $A \subset X$, when X is hereditarily paracompact and φ paracompactifying.*

Indeed, h is the image of some $h' \in H^s_\varphi(X; \mathscr{Z}_{n+s})$. But $H^s_{\varphi\cap A}(A; \mathscr{Z}_{n+s}) = 0$; therefore (in view of the exactness of the cohomology sequence of the pair (X, A)), h' is the image of some $k' \in H^s_\varphi(X, A; \mathscr{Z}_{n+s})$. Hence h is the image of the element k determined by k' in the group $H^{\varphi|X\setminus A}_n(X \setminus A; \mathscr{G})$.

A similar result is obtained in [14] for the case $\varphi = c$, $\mathscr{G} = R$ (the ground ring), $\dim A < \infty$, and the set A closed.

§4. The case of a locally finite polyhedron

Let $X = K$ be a locally finite cell complex, and φ either the family c of all compact subsets or the family of *all* closed sets. As usual, we denote by K^q the q-skeleton of K.

4.1. **Theorem.** *The filtering subgroups $F^s H^\varphi_n(K; \mathscr{G}) \subset H^\varphi_n(K; \mathscr{G})$ coincide with the image of the natural homomorphisms*

$$H^{\varphi|K\setminus K^{s-1}}_n(K \setminus K^{s-1}; \mathscr{G}) \to H^\varphi_n(K; \mathscr{G}).$$

Thus (cf. 3.3 above), the group $F^s H^\varphi_n(K; \mathscr{G})$ coincides with the intersection of the images in $H^\varphi_n(K; \mathscr{G})$ of the groups $H^{\varphi|B}_n(B; \mathscr{G})$ taken over all sets $B \subset K$ (not just the open or closed) for which $\dim(K \setminus B) < s$. Similar results were obtained in [14] for finite-dimensional (locally finite) simplicial complexes in the case that $\varphi = c$, $\mathscr{G} = R$ (that ground ring), and the sets B are open.

4.2. **Lemma.** $H^r_\varphi(K, K^s; \mathscr{H}_j) = 0$ *for* $r \le s$.

We note that this would not be true for any arbitrary coefficient sheaf $\mathscr{H}$. For example, if a sheaf $\mathscr{H}$ on the space K is concentrated on a closed subspace $L \subset K \setminus K^s$ and is constant on it, then in accordance with [11], Chapter 2, §4.10, the group $H^r(K, K^s; \mathscr{H})$ coincides with $H^r(L; \mathscr{H}|L)$.

To prove the lemma we start with the pair (K^{s+1}, K^s). Its cohomology is the cohomology of $K^{s+1}\setminus K^s$ with coefficients $\mathscr{H}_j|K^{s+1}\setminus K^s$ and supports in $\varphi|K^{s+1}\setminus K^s$. But since $K^{s+1}\setminus K^s$ is a discrete union of open $(s+1)$-cells, the group $H^r_\varphi(K^{s+1}, K^s; \mathscr{H}_j)$ is, depending on φ, either the direct sum or the direct product of the cohomology groups of such cells. Since the $\mathscr{H}_j$ are local-homology sheaves (with locally constant coefficients!), they are constant on these cells. Hence for $r \neq s+1$ the cohomology groups of the cells (which coincide with the classical cohomology groups of balls modulo their boundaries) are zero.

Using the exact cohomology sequence of the triple (K^{s+k}, K^{s+1}, K^s) it is easily shown by induction on k that $H^r_\varphi(K^N, K^s, \mathscr{H}_j) = 0$ for all $N < s$. If $\varphi = c$, then any cocycle of the pair (K, K^s), realized as a section, with compact support, of a certain differential sheaf of cochains, is for sufficiently large N concentrated on K^N, hence is cohomologous to zero; i.e., $H^r_c(K, K^s; \mathscr{H}_j) = 0$. In the case of ordinary cohomology ($\varphi =$ all closed sets), we can suppose the mappings $C^*(K^{N+1}, K^s; \mathscr{H}_j) \to C^*(K^N, K^s; \mathscr{H}_j)$, of the cochain complexes of the indicated pairs, to be epimorphic. Indeed, the cochains of the pair (K^{N+1}, K^s), realized via the Godement resolution $\mathscr{C}^* = \mathscr{C}^*(\mathscr{H}_j)$, for example, are sections of the restriction of $\mathscr{C}^*$ to K^{N+1} that vanish on K^s; and since the space K is paracompact, the sheaves $\mathscr{C}^*|K^{N+1}$ are soft, so that sections of these sheaves extend from K^N to K^{N+1}. Thus, the projective system of the cochain complexes of the pairs (K^N, K^s) is $\varprojlim$-acyclic. Furthermore, since the complex K is locally finite, we have $C^*(K, K^s; \mathscr{H}_j) = \varprojlim_N C^*(K^N, K^s; \mathscr{H}_j)$. As a result, we are in the situation described, for example, at the beginning of §1.5 in [6], Chapter 8; and since the projective system is countable, there is determined an exact sequence

$$0 \to \varprojlim_N{}^1 H^{r-1}(K^N, K^s; \mathscr{H}_j) \to H^r(K, K^s; \mathscr{H}_j) \to \varprojlim_N H^r(K^N, K^s; \mathscr{H}_j) \to 0.$$

From all the preceding it follows that the middle term here is zero.

To prove the theorem, let $h \in H^\varphi_n(K; \mathscr{G})$ be an element of filtration s (see the end of §1). In accordance with 3.3 above, h is contained in the image of the n-dimensional homology group of the space $K \setminus K^{s-1}$, and it suffices to show that it is not contained in the image of the homology of the subspace $K \setminus K^s$. Consider the commutative diagram of mappings corresponding to the short exact sequence $0 \to \mathscr{B}_{n+r} \to \mathscr{Z}_{n+r} \to \mathscr{H}_{n+r} \to 0$ and the inclusion $K^s \subset K$:

$$\begin{array}{ccccccccc}
\cdots \to & H^{r-1}_\varphi(K; \mathscr{H}_{n+r}) & \xrightarrow[\delta]{} & H^r_\varphi(K; \mathscr{B}_{n+r}) & \xrightarrow[\gamma]{} & H^r_\varphi(K; \mathscr{Z}_{n+r}) & \to & H^r_\varphi(K; \mathscr{H}_{n+r}) & \to \cdots \\
& \Vert & & \downarrow{\scriptstyle\nu} & & {\scriptstyle\mu}\downarrow & & \cap & \\
\cdots \to & H^{r-1}_\varphi(K^s; \mathscr{H}_{n+r}) & \xrightarrow[\delta']{} & H^r_\varphi(K^s; \mathscr{B}_{n+r}) & \xrightarrow[\rho]{} & H^r_\varphi(K^s; \mathscr{Z}_{n+r}) & \to & H^r_\varphi(K; \mathscr{H}_{n+r}) & \to \cdots
\end{array}$$

In the case we are are interested in, $r \leq s$; the isomorphism and monomorphism in the extreme columns are consequences of the lemma.

Suppose first that $r = s$, and let $h_s \in H^s_\varphi(K; \mathscr{Z}_{n+s})$ be any element representing h. Since h has filtrations, it follows that h_s is not in the image of γ, and $\mu(h_s) \neq 0$.

For the next step, let $r = s-1$. Because of the isomorphism $H^s_\varphi(K; \mathscr{Z}_{n+s}) = H^{s-1}_\varphi(K; \mathscr{B}_{n+s-1})$ and the corresponding isomorphism for the subspace K^s, we have $h_s \in H^{s-1}_\varphi(K; \mathscr{B}_{n+s-1})$ and $\nu(h_s) \neq 0$. Then $\rho\nu(h_s) \neq 0$. Otherwise we should have $\nu(h_s) = \delta'(k)$, and then $\gamma(h_s - \delta(k)) = \gamma(h_s)$, i.e., $h'_s = h_s - \delta(k)$ likewise represents h, but $\nu(h'_s) = 0$, contradicting the observation just made (which holds for all elements of the group in question that represent h). Thus, for all elements $h_{s-1} \in$

$H_\varphi^{s-1}(K; \mathscr{Z}_{n+s-1})$ representing h and belonging to the image of γ, $\mu(h_{s-1}) \neq 0$. For the same reasons as in the first step of the argument, this conclusion holds also for those elements of $H_\varphi^{s-1}(K; \mathscr{Z}_{n+s-1})$ representing h that do not belong to the image of γ.

Thus, $\mu(h_{s-1}) \neq 0$ for any element $h_{s-1} \in H_\varphi^{s-1}(K; \mathscr{Z}_{n+s-1})$ representing h. Repeating the argument (i.e., using the isomorphisms $H_\varphi^{s-1}(K; \mathscr{Z}_{n+s-1}) = H_\varphi^{s-2}(K; \mathscr{B}_{n+s-2})$, $H_\varphi^{s-1}(K^s; \mathscr{Z}_{n+s-1}) = H_\varphi^{s-2}(K^s; \mathscr{B}_{n+s-2})$, etc.) for $r = s-2, s-3, \ldots$, we arrive at the conclusion that for an element $h_1 \in H_\varphi^1(K; \mathscr{Z}_{n+1})$ representing h the restriction to K^s is nonzero. Since $H_\varphi^1(K^s; \mathscr{Z}_{n+1}) \subset H_n^\varphi(K, K \setminus K^s; \mathscr{G})$ (see (2) in §3), this means that the image of h in $H_n^\varphi(K, K \setminus K^s, \mathscr{G})$ is nonzero. From the exactness of the homology sequence of the pair $(K, K \setminus K^s)$ it follows that h does not belong to the image of the group $H_n^{\varphi|K\setminus K^s}(K \setminus K^s; \mathscr{G})$. This proves the theorem.

§5. Connection with the cap-product

Let $\mathscr{G}$ and $\mathscr{A}$ be locally constant sheaves whose stalks G and A are flat R-modules (torsion-free in the case that R is a principal ideal ring; see [11], Chapter 1, §5.3), and let φ and ψ be two families of supports (the space X is locally compact).

5.1. **Theorem.** *Under the condition that the family* $\varphi \cap \psi$ *is paracompactifying,*(1)

$$H_p^\varphi(X; \mathscr{G}) \frown H_\psi^q(X; \mathscr{A}) \subset F^q H_{p-q}^{\varphi\cap\psi}(X; \mathscr{G} \otimes_R \mathscr{A}).$$

In the special case that $\mathscr{G} = \mathscr{A} = R$, the ground ring, this result was obtained for finite simplicial complexes by Zeeman (Theorem 3 of [19]), and with the additional condition $\varphi = \psi = c$ for finite-dimensional locally finite simplicial complexes in [14]. As noted above, the result was used in studying the filtration itself.

5.2. *Remark.* The construction of the cap product used below was outlined in [6], Chapter 8, §5.3 and does not outwardly differ from the construction of the cup product of cohomology classes described in [11], Chapter 2, §6.6, (see also [6], Chapter 6, §1).

Proof of the theorem. It is well known that the stalks of a tensor product of sheaves are tensor products of the stalks of the sheaves (see [11], Chapter 2, §2.8). From the fact that the stalks of $\mathscr{A}$ are flat and the functor $\otimes$ is right-exact, it follows easily that the kernels and images of the boundary operator of the differential sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$ are the sheaves $\mathscr{Z}_p(\mathscr{G}) \otimes_R \mathscr{A}$ and $\mathscr{B}_p(\mathscr{G}) \otimes_R \mathscr{A}$, respectively, and the corresponding derived sheaves are $\mathscr{H}_p(\mathscr{G}) \otimes_R \mathscr{A}$. Since the sections of Godement sheaves identify with elements of direct products of the stalks, and the stalks are direct limits (over neighborhoods) of the groups of these sections, we conclude from the stalkwise homotopic triviality of the Godement resolution (i.e., its splitting at stalks; see Remark 4.3.1 in Chapter 2 of [11]) that the stalks of the sheaves $\mathscr{C}^*(\mathscr{A})$ are also flat R-modules (cf. [10], Chapter 6, Exercises 3, 4, and 17). Hence by the same argument as above we conclude that $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ naturally contains resolutions $\mathscr{Z}_p(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ and $\mathscr{B}_p(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ of the sheaves $\mathscr{Z}_p(\mathscr{G}) \otimes \mathscr{A}$ and $\mathscr{B}_p(\mathscr{G}) \otimes_R \mathscr{A}$. By the same token, $\mathscr{H}_p(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ is a resolution of the sheaf $\mathscr{H}_p(\mathscr{G}) \otimes_R \mathscr{A}$ (exactness at the index q is a consequence of the stalkwise splitting of $\mathscr{C}^*(\mathscr{A})$). Thus, the sheaves $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ constitute a resolution of the sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$ in the sense of Chapter 17 of [10], applicable in studying hyperhomology (in the present case, for the section functor).

(1)The author has been able to show that the result holds for arbitrary $\mathscr{G}$, $\mathscr{A}$, φ, and ψ.

For the ring of integers Z, the sheaf $\mathscr{C}^0(Z)$ coincides, obviously, with the sheaf of germs of zero-dimensional Alexander-Spanier cochains, and is, therefore, φ-fine for any paracompactifying family φ (see Example 3.7.1 in Chapter 2 of [11]). Also it is clear that for any sheaf $\mathscr{A}$ the Godement sheaf $\mathscr{C}^0(\mathscr{A})$ is a $\mathscr{C}^0(Z)$-module and therefore also φ-fine ([11], Chapter 2, §3.7). Consequently, the whole resolution $\mathscr{C}^*(\mathscr{A})$ consists of φ-fine sheaves. By Theorem 3.7.3 in Chapter 2 of [11], any sheaves of the form $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ are then φ-fine. Since the family φ here is arbitrary, under the paracompactifying condition on $\varphi \cap \psi$ the sheaves $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ constitute a $(\varphi \cap \psi)$-acyclic resolution of the sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$.

Let us observe that in accordance with Exercise 18 in Chapter 2 of [8], the sheaves $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$ are c-soft (for any sheaf $\mathscr{A}$) and therefore (since the space X is locally compact) soft (cf. Theorem 3.4.1 in Chapter 2 of [11]) and $(\varphi \cap \psi)$-acyclic. Thus, by the argument already used at the beginning of §2 above, the sections with supports in the paracompactifying family $\varphi \cap \psi$ of the differential sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(\mathscr{A})$ taken with the total grading $n = p - q$, determine the same homology as the corresponding sections of the differential sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$. Consequently, there is defined via tensor multiplication the cap product of homology $H_p^{\varphi}(X; \mathscr{G})$ and cohomology $H_{\psi}^q(X; \mathscr{A})$, the result being in the $(p-q)$-dimensional homology group of the chain complex $\Gamma_{\varphi\cap\psi}(\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A})$. In accordance with the content of §2, if for the complex $\mathscr{C}_*$ there we take the differential sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$, and for $\mathscr{J}^*(\mathscr{C}_*)$ the bigraded differential sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{C}^*(A)$ we obtain for the section functor $T = \Gamma_{\varphi\cap\psi}$ the inclusion

$$H_p^{\varphi}(X; \mathscr{G}) \frown H_{\psi}^q(X; \mathscr{A}) \subset F^q H_{p-q}(\Gamma_{\varphi\cap\psi}(\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A})).$$

Here F^q is the filtration considered in §2. The cycle z_p^q mentioned there is obtained by tensor multiplication of a p-cycle (with support in φ) by a q-cocycle (with support in ψ). To complete the proof of the theorem it remains to compare the sheaf $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A}$ with the sheaf $\mathscr{C}_*(\mathscr{G} \otimes_R \mathscr{A})$ that determines the required homology of X.

Since the filtration defined above in §2 is functorial, obviously, with respect to chain complex morphisms, it suffices to exhibit a homomorphism $\mathscr{C}_*(\mathscr{G}) \otimes_R \mathscr{A} \to \mathscr{C}_*(\mathscr{G} \otimes_R \mathscr{A})$. In accordance with the description of sheaves of chains given in [3] (see also Chapter 5 in [6]; similar constructions are applicable to the Massey chains of [13]), the groups of sections of sheaves of the type of $\mathscr{C}_p(\mathscr{G})$ have the structure of direct products $\prod G$, and the homomorphism in question is obtained as a consequence of the natural transformation $\kappa\colon (\prod G) \otimes_R A \to \prod (G \otimes_R A)$ (a description of this transformation in a special case is given in Lemma 7 of [3]). This proves the theorem.

We note that in view of the argument used for Lemma 7 of [3], the transformation κ is an isomorphism precisely under the condition that the module A be finitely presentable. Hence in the special case that the stalk $\mathscr{G} = \mathscr{R}$ is the ground ring R, the sheaf $\mathscr{C}_*(\mathscr{R}) \otimes_R \mathscr{A}$ is equal to the sheaf $\mathscr{C}_*(R) \otimes_R (\mathscr{R} \otimes_R \mathscr{A})$ and determines the Borel-Moore homology (cf. §3 above) of the space X with coefficients in $\mathscr{R} \otimes_R \mathscr{A}$ (the fact that the original description of Borel-Moore theory uses instead of $\mathscr{C}_*(R)$ the "dual sheaf to the differential cosheaf" plays no role; see the editor's remarks to §3, Chapter 5, of the Russian translation of [8], or [7], Chapter 3, §6.1). However, the transformation of differential sheaves described at the end of the above proof can in this situation induce an isomorphism of homology not just for finitely presentable stalks A. Thus, for homology H_*^c with compact supports we have an isomorphism for any $\mathscr{A}$ in the case that X is metrizable and homologically locally connected over R (see [2]). When X is a cellular polyhedron and the sheaves $\mathscr{R}$ and $\mathscr{A}$ are constant, the transformation induces an isomorphism for homology with arbitrary closed supports (homology "of the second kind"). Indeed, in this case the Borel-Moore homology is $\prod$-additive ([7], Chapter 3, §6.2), and since in the category of locally finite polyhedra and proper mappings it satisfies all the usual Eilenberg-Steenrod axioms, it coincides with the ordinary (i.e., classical) homology "of the second kind" (see the uniqueness theorem in [1]).

§6. Remarks apropos the Zeeman filtration in cohomology

Using the same means as for homology, Zeeman defined a filtration also in the cohomology of a finite complex [19]. A simplified version was then extended in [14] to finite-dimensional simplicial polyhedra. In the tensor product $C^*(K) \otimes C_*(K)$ of integral cochains and infinite chains there was considered the (double) subcomplex generated by the pairs $\Delta > \Delta'$ (Δ' a face of the simplex Δ). Its spectral sequence corresponding to the filtration in cochain dimension is degenerate (in computing its initial terms, using the boundary operator in chains for fixed Δ and all $\Delta' < \Delta$ one already discovers the acyclicity of Δ), and the homology of the double complex, the latter being provided with a suitable total grading, coincides with the cohomology of K.

In computing the second spectral sequence (corresponding to the filtration in chain dimension), the use at the first step of the coboundary operator for fixed Δ' and all $\Delta > \Delta'$ reveals an obstruction to degeneration in the cohomology of the stars of the Δ' in K modulo the complements to the Δ', i.e., in the systems of local cohomology groups $h^p(K)$—as a result of which we have, for the second term of the spectral sequence, $E^2_{pq} = H_q(K; h^p(K)) \Rightarrow H^{p-q}(K)$. Corresponding to this spectral sequence there is determined a filtration in cohomology $H^s(K) \supset \cdots \supset F_qH^s(K) \supset F_{q-1}H^s(K) \supset \cdots \supset F_0H^s(K)$, for which

$$E^\infty_{pq} = F_qH^{p-q}(K)/F_{q-1}H^{p-q}(K).$$

Zeeman showed that if for an element $\alpha \in H^s(K)$ of filtration q the restriction to $K\backslash A$ is equal to zero for some closed set $A \subset K$, then dim $A \geq q$; and he conjectured that this is a defining property for the filtration. The conjecture was confirmed in [14] (in the more general situation considered there).

One feels that these results are of the same general nature as those presented here for homology. Realization of the dual constructions has, however, been stymied by the fact that homology theory with nonconstant coefficients (i.e., homology theory with coefficients, say, in cosheaves in which homology groups would be derived functors on zero-dimensional groups), despite the many attempts undertaken, has not yet come into being. Certainly the well-known Borel-Moore theory does not fill the bill (see §3 above), although the homology is defined there formally with coefficients in arbitrary sheaves. A well-defined homology theory exists for the present only for arbitrary locally constant coefficients.

Bibliography

1. S. V. Petkova, *The axioms of homology theory*, Mat. Sb. **90 (132)** (1973), 607–624; English transl. in Math. USSR-Sb. **19** (1973).

2. ———, *Coincidence of homologies on homologically locally connected spaces*, C. R. Acad. Bulgare Sci. **35** (1982), 427–430. (Russian)

3. E. G. Sklyarenko, *Homology theory and the exactness axiom*, Uspekhi Mat. Nauk **24** (1969), no. 5 (149), 87–140; English transl. in Russian Math. Surveys **24** (1969).

4. ———, *On the homology theory associated with Aleksandrov-Čech cohomology*, Uspekhi Mat. Nauk **34** (1979), no. 6 (210), 90–118; English transl. in Russian Math. Surveys **34** (1979).

5. ———, *Poincaré duality and correlations between the functors* Ext *and* Tor, Mat. Zametki **28** (1980), 769–776; English transl. in Math. Notes **28** (1980).

6. ———, *The homology and cohomology of general spaces*, Itogi Nauki i Tekhniki: Sovremennye Problemy Mat.: Fundamental' nye Napravleniya, vol. 50, VINITI, Moscow, 1989, pp. 129–266; English transl. in Encyclopaedia of Math. Sci., vol. 50 [General Topology, II], Springer-Verlag, Berlin (to appear).

7. ———, *General homology and cohomology theory. The current state and typical applications*, Itogi Nauki i Tekhniki: Algebra, Topologiya, Geometriya vol. 27, VINITI, Moscow, 1989, pp. 125–228; English transl. in J. Soviet Math. **57** (1991), no. 2.

8. Glen E. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967; Russian transl., "Nauka", Moscow, 1988.

9. Séminaire Henri Cartan de L' École Normale Supérieure, 1950/51, *Cohomologie des groupes, suite spectrale, faisceaux*, 2nd ed., Secrétariat Math., Paris, 1955.

10. Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1956.

11. Roger Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Indust., no. 1252, Hermann, Paris, 1957.

12. Alexandre Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–211.

13. William S. Massey, *Homology and cohomology theory*, Marcel Dekker, New York, 1978.

14. Clint McCrory, *Zeeman's filtration of homology*, Trans. Amer. Math. Soc. **250** (1979), 147–166.

15. Pierre Schapira, *Théorie des hyperfonctions*, Lecture Notes in Math., vol. 126, Springer-Verlag, Berlin, 1970.

16. Richard G. Swan, *The theory of sheaves*, Lecture notes, Math. Inst., Oxford 1958; 2nd rev. aug. ed., Univ. of Chicago Press, Chicago, IL, 1964.

17. George W. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283.

18. E. C. Zeeman, *Dihomology*, Proc. Internat. Congr. Math. (Amsterdam, 1954), Vol. II: Short lectures (Abstracts), Noordhoff, Groningen, and North-Holland, Amsterdam, 1954, pp. 269–270.

19. ———, *Dihomology*. III: *A generalization of the Poincaré duality for manifolds*, Proc. London Math. Soc. (3) **13** (1963), 155–183.

Moscow

Received 31/JAN/92

Translated by J. A. ZILBER