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Let X and Y be compact metric spaces and let a map $f: X \to Y$ be onto. The Vietoris Mapping Theorem as proved by Vietoris [**8**] states that if for all $0 \leqq r \leqq n-1$ and all $y \in Y$, $H_r(f^{-1}(y))=0$ (augmented Vietoris homology mod two) then the induced homomorphism $f_*: H_r(X) \to H_r(Y)$ is an isomorphism onto for $r \leqq n-1$ and onto for $r=n$. Begle [**1**; **2**] has generalized this theorem to nonmetric spaces and more general coefficient groups. Simple examples show that an analogous theorem does not hold directly for homotopy. However by imposing strong local connectedness conditions, results can be obtained. That is the idea of this paper. We prove:

Main Theorem. *Let $f: X \to Y$ be proper and onto where X and Y are 0-connected, locally compact, separable metric spaces, X is LC^n, and for each $y \in Y$, $f^{-1}(y)$ is LC^{n-1} and $(n-1)$-connected. Then*

(A) *Y is LC^n and*

(B) *the induced homomorphism $f_\#: \pi_r(X) \to \pi_r(Y)$ is an isomorphism onto for all $0 \leqq r \leqq n-1$ and onto for $r=n$.*

We recall that a map is called *proper* if the inverse image of a compact set is compact. Clearly any map between compact Hausdorff spaces is proper. A space X is said to be n-connected if $\pi_r(X)=0$ for $0 \leqq r \leqq n$. As above we often suppress the base point of a homotopy group.

Part (A) of the Main Theorem is a homotopy analogue of a theorem of Wilder [**9**, p. 31]. The proof of the Main Theorem can be pieced together from Theorems 8 and 9 of §2 and Theorem 11 of §3. These theorems taken together in fact say a little more than the Main Theorem. It should be mentioned that the Vietoris Mapping Theorem has been generalized using proper maps of noncompact spaces; for example see [**10**].

1. It will be assumed that all spaces are locally compact, separable, and metric. A proof of the following may be found in [**7**].

Lemma 1. *Let $f: X \to Y$ be proper and onto. Suppose $y_0 \in Y$ and U is an open set of X containing $f^{-1}(y_0)$. Then there exists a neighborhood V of y_0 such that $f^{-1}(V) \subset U$.*

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The following theorem may be found in [5, p. 82] where the terms are defined.

THEOREM 1. *If X is compact and LC^n, then given any $\epsilon>0$, there exists $\eta=\eta^n(X, \epsilon)>0$ such that every dense partial realization of mesh $<\eta$ of a finite complex of dimension $\leqq n+1$ can be extended to a full realization of mesh $<\epsilon$.*

If f and g are two maps of a compact space X into a space Y, by $d(f, g)$ we will mean max $\{d(f(x), g(x)) \mid x\in X\}$. The next theorem is a special case of one which may be found in [6, p. 48].

THEOREM 2. *Given a compact set F in an LC^n space X and $\epsilon>0$, there exists an $\eta=\eta^n(\epsilon, F)$ with the following property: If K is a polyhedron of dimension $\leqq n$, and if f_0, f_1 map K into F satisfying $d(f_0,f_1)<\eta$, there exists a homotopy $f_t\colon K\rightarrow X$ between f_0 and f_1 such that for each $x\in K$ the curve $f_t(x)$ has diameter $<\epsilon$.*

We will use the same symbol to denote a polyhedron and one of its underlying complexes. If K is a complex, K^r will, as usual, mean the rth skeleton of K. If $X\subset Y$, the symbol $\pi_r(X/Y)$ denotes the image of $\pi_r(X)$ in $\pi_r(Y)$ under the homomorphism induced by inclusion. We say that X is semi-$r=LC$ if for every $x\in X$ there exists a neighborhood V of x such that $\pi_r(V/X)=0$. Obviously if X is $r=LC$ it is semi-$r=LC$.

THEOREM 3. *Given a compact set F in a semi-$n=LC$, LC^{n-1} space X, there exists an $\eta=\eta^n(F)$ with the following property: If K is a polyhedron of dimension $\leqq n$, and if f_0, f_1 map K into F satisfying $d(f_0, f_1)<\eta$, there exists a homotopy $f_t\colon K\rightarrow X$ between f_0 and f_1.*

PROOF. By the local compactness of X, choose $\alpha>0$ so that Cl $(U(F, \alpha))=F'$ is compact. Since X is semi-$n=LC$ we can find ϵ with $0<\epsilon<\alpha$ so that if V is a neighborhood in F' of diameter $<\epsilon$ then $\pi_n(V/X)=0$. It will be shown that $\eta_0=\eta^{n-1}(\epsilon/3, F')$ of Theorem 2 may be taken as the $\eta^n(F)$ demanded by our theorem.

Let f_0, f_1 and K be as given with $d(f_0, f_1)<\eta_0$. Take a subdivision Sd of K so fine that if $\sigma\in Sd$ then max diameter $(f_0(\sigma), f_1(\sigma))<\epsilon/3$. The choice of η_0 yields a homotopy $h_t\colon Sd^{n-1}\rightarrow X$ between f_0 and f_1 restricted to Sd^{n-1} with $h_t(x)$ having diameter $<\epsilon/3$ for each $x\in Sd^{n-1}$. Since $\epsilon<\alpha$, $h_t(Sd^{n-1})\subset F'$ for each $t\in I$.

If $\sigma^n\in Sd$, the maps f_0, f_1, and h_t define in an obvious fashion a map H of $A=\sigma^n\times 0\cup\sigma^n\times 1\cup\dot{\sigma}^n\times I$ into F'. From the choice of η_0 and Sd it follows that the diameter of $H(A)<\epsilon$. Then by the choice of ϵ,

H may be extended to $\sigma^n \times I$. Thus is obtained our desired homotopy f_t. q.e.d.

By S^n is meant the n-sphere.

THEOREM 4. *Let X be LC^n and contain a compact LC^{n-1} subset M and an open set $P \supset M$. Then there exists an open set $Q = Q^n(P, M)$ such that $P \supset Q \supset M$, with the following property: If $g: S^n \to Q$ is given, then there is a homotopy $g_t: S^n \to P$ of g with $g_1(S^n) \subset M$.*

PROOF. Choose $\alpha > 0$ so that Cl $(U(M, \alpha)) = F'$ is compact and contained in P. Let $\eta_0 = \eta^n(\alpha, F')$ be given by Theorem 2. Let $\eta_1 = \eta^{n-1}(M, \eta_0/3)$ be given by Theorem 1. It will be shown that $Q = U(M, \eta_1/3)$ may be taken as the Q of our theorem.

Let $g: S^n \to Q$ be given. Take a subdivision Sd of S^n so fine that for $\sigma \in Sd$, the diameter of $g(\sigma)$ is less than $\eta_1/3$. A map $\bar{g}: Sd \to M$ is constructed as follows. If v is a vertex of Sd let $\bar{g}(v)$ be a point of M at a distance less than $\eta_1/3$ from $g(v)$. This defines a dense partial realization of S^n in M which is easily shown to have mesh less than η_1. The choice of η_1 yields a full realization $\bar{g}$ of S^n into M with mesh less than $\eta_0/3$. It is easily checked that for every $x \in S^n$, $d(g(x), \bar{g}(x)) < \eta_0$. Then by the choice of η_0 we obtain our desired homotopy between g and $\bar{g}$. q.e.d.

THEOREM 5. *Let X be LC^{n-1} and semi-n-LC and let M be a compact LC^{n-1} subset of X. Then there exists a $Q = Q^n(M)$ containing M with this property: For every map $g: S^n \to Q$ there is a homotopy $g_t: S^n \to X$ of g with $g_1(S^n) \subset M$.*

Theorem 5 is proved in the same way as the proceeding one only this time using Theorem 3 instead of Theorem 2.

2. THEOREM 6. *Let $f: X \to Y$ be proper and onto. Suppose X is LC^{n-1}, and for each $y \in Y$, $f^{-1}(y)$ is LC^{n-2} and $(n-1)$-connected. Let be given*

(1) *$\eta > 0$,*
(2) *a subcomplex L of an n-dimensional (or less) complex K,*
(3) *a map $g: K \to Y$,*
(4) *a map $\bar{g}: L \to X$ such that $f\bar{g} = g_{|L}$.*

Then there exists an extension G of $\bar{g}$ to K such that $d(fG, g) < \eta$.

PROOF. We use induction on n. The theorem is trivially true for $n = 0$ (we interpret LC^{-1} and (-1)-connected to mean no condition is implied).

Now suppose the theorem is true for $n = k - 1$. We will show that then it is true for $n=k$. Let η, L, K, g, and $\bar{g}$ be given as in (1), (2), (3), and (4) (with $n=k$). Choose β, $0<\beta<\eta$, so that Cl $(U(g(K), \beta)) = B$ is compact.

For each $y \in B$, a system $(y, U_y, P_y, F_y, Q_y, V_y)$ is defined as follows: U_y is a neighborhood of y of diameter less than β, $P_y = f^{-1}(U_y)$, and $F_y = f^{-1}(y)$. As defined in Theorem 4, Q_y is $Q^{k-1}(P_y, F_y)$. Finally V_y is a neighborhood of y with $f^{-1}(V_y) \subset Q_y$ as given by Lemma 1.

Let δ be the Lebesgue number of the covering $\{V_y | y \in B\}$ of B. Take a subdivision Sd of K so fine that for every simplex σ of K, $g(\sigma)$ has diameter less than $\delta/3$.

The induction hypothesis can be applied to yield an extension (still denoted by $\bar{g}$) of $\bar{g}$ to $L \cup Sd^{k-1}$ such that $d(g', f\bar{g}) < \delta/3$ where g' denotes g restricted to $L \cup Sd^{k-1}$.

If σ^k is a k-simplex of Sd which is not in L, then from the last two sentences it follows that the diameter of $f\bar{g}(\dot{\sigma}^k)$ is less than δ. Then some V_0 of $\{V_y | y \in B\}$ contains $f\bar{g}(\dot{\sigma}^k)$. Let the corresponding system as defined above be denoted by $(y_0, U_0, P_0, F_0, Q_0, V_0)$.

By the choice of V_0, $\bar{g}(\dot{\sigma}^k) \subset Q_0$. Then by the choice of Q_0, there is a homotopy $\bar{g}_t: \dot{\sigma}^k \to P_0$ of $\bar{g}_k$ ($\bar{g}_k$ denotes $\bar{g}$ restricted to $\dot{\sigma}^k$) such that $\bar{g}_1(\dot{\sigma}^k) \subset F_0$. Since $\pi_{k-1}(F_0)=0$, $\bar{g}_1$ can be extended to σ^k. Then $\bar{g}_k$ can be extended to a map $\bar{g}_k' : \sigma^k \to P_0$. From the choice of P_0 it follows easily that $d(f\bar{g}_k', g_k) < \eta$ (g_k denotes g restricted to σ^k). The desired extension is obtained by repeating the above process on each k-simplex of Sd.

THEOREM 7. *Let $f: X \to Y$ be proper and onto where X is LC^{n-2} and semi-$(n-1)$-LC. Suppose for each $y \in Y$, $f^{-1}(y)$ is LC^{n-2}, $(n-2)$-connected and $\pi_{n-1}(f^{-1}(y)/X)=0$. Let be given*

(1) *a subcomplex L of an n-dimensional complex K,*

(2) *a map $g: K \to Y$, and*

(3) *a map $\bar{g}: L \to X$ such that $f\bar{g} = g_{|L}$.*

Then there exists an extension of $\bar{g}$ to all of K.

The proof of Theorem 7 is very similar to that of the preceding theorem except that Theorem 5 is used in place of Theorem 4. It will not be given.

THEOREM 8. *Let $f: (X, x_0) \to (Y, y_0)$ be proper and onto where X is LC^{n-2} and semi-$(n-1)=LC$. Suppose for each $y \in Y$, $f^{-1}(y)$ is LC^{n-2}, $(n-2)$-connected, and $\pi_{n-1}(f^{-1}(y)/X)=0$. Then the induced homomorphism $f_\#: \pi_{n-1}(X, x_0) \to \pi_{n-1}(Y, y_0)$ is one-to-one.*

PROOF. Let $g: (\dot{I}^n, p) \to (X, x_0)$ represent an element of $\pi_{n-1}(X, x_0)$ such that $fg: (\dot{I}^n, p) \to (Y, y_0)$ can be extended to I^n. It is sufficient to show that g can be extended to I^n. Theorem 7 says that this indeed can be done. q.e.d.

THEOREM 9. *Let $f: (X, x_0)\to(Y, y_0)$ be proper and onto where X and Y are LC^{n-1} and Y is also semi-n-LC. Suppose for all $y\in Y$, $f^{-1}(y)$ is LC^{n-1} and $(n-1)$-connected. Then the induced homomorphism $f_\#: \pi_n(X, x_0)\to\pi_n(Y, y_0)$ is onto.*

PROOF. Let $g: (S^n, p)\to(Y, y_0)$ represent an element of $\pi_n(Y, y_0)$. Choose $\alpha>0$ so that Cl $(U(g(S^n), \alpha))=F$ is compact. Choose by Theorem 3 $\eta_0=\eta^n(F)$ with $\eta_0<\alpha$. Theorem 6 yields a map $\bar{g}: (S^n, p)\to(X, x_0)$ with $d(g, f\bar{g})<\eta_0$. By the choice of η_0, g and $f\bar{g}$ are homotopic in Y. This proves Theorem 9.

3. THEOREM 10. *Let $f: X\to Y$ be proper and onto and suppose for each $y\in Y$, $f^{-1}(y)$ is 0-connected and 0-LC. Let X be LC^1. Then Y is LC^1.*

PROOF. That Y is 0-LC is well-known. Let $p\in Y$ and W, a neighborhood of p be given. Let $P=f^{-1}(W)$ and $F=f^{-1}(p)$. Choose by Theorem 4, $Q=Q^1(P, F)$. From the construction of Q and the fact that F is 0-connected it follows that we may assume Q is 0-connected. By Lemma 1 let V be a neighborhood of p with $f^{-1}(V)\subset Q$. To prove the theorem it is sufficient to show $\pi_1(V/W)=1$. Let $g: S^1\to V$ be given.

For each $t\in S^1$ and $\epsilon>0$ we define a system $(W(t, \epsilon), P(t, \epsilon), F(t, \epsilon), Q(t, \epsilon), V(t, \epsilon))$ similar to the one used in the proof of Theorem 6 and in the preceding paragraph. Let $W(t, \epsilon)=U(g(t), \epsilon/2)$, $P(t, \epsilon)=f^{-1}(W(t, \epsilon))$, and $F(t, \epsilon)=f^{-1}(g(t))$. Then $Q(t, \epsilon)$ is chosen by Theorem 4 equal to $Q^1(P(t, \epsilon), F(t, \epsilon))$. As in the previous paragraph we will assume $Q(t, \epsilon)$ to be 0-connected. Choose $V(t, \epsilon)$, a neighborhood of $g(t)$, by Lemma 1 so that $f^{-1}(V(t, \epsilon))\subset Q(t, \epsilon)$.

Take $\epsilon_1>0$ so that $U(g(S^1), \epsilon_1)\subset V$. Define $\mathfrak{V}_1$ to be the collection $\{V(t, \epsilon_1) \mid t\in S^1\}$. Take a subdivision Sd_1 of S^1 so fine that for each $\sigma\in Sd_1$, $g(\sigma)$ is contained in an element of $\mathfrak{V}_1$ say V_σ. Denote the corresponding system as defined above by $(W_\sigma, P_\sigma, F_\sigma, Q_\sigma, V_\sigma)$. Note that by the choice of ϵ_1, W_σ and V_σ are contained in V and $Q_\sigma\subset Q$ for each $\sigma\in Sd_1$.

We now define a map $\bar{g}_1: Sd_1\to Q$ with the property $\bar{g}_1(\sigma)\subset Q_\sigma$ for each $\sigma\in Sd_1$. If v is a vertex of Sd_1 let $\bar{g}_1(v)$ be an arbitrary point of $f^{-1}(g(v))$. Then if σ is a 1-simplex of Sd_1, by the choice of V_σ, $\bar{g}_1(\dot{\sigma})\subset Q_\sigma$. Extend $\bar{g}_1$ to all of σ by the 0-connectedness of Q_σ. This defines $\bar{g}_1$. Let $g_1=f\bar{g}_1$. It is seen readily that $d(g, g_1)<\epsilon_1$.

By the choice of Q, $\bar{g}_1$ is homotopic in P to a map of S^1 into F. This implies that g_1 is homotopic in W to p.

Choose ϵ_2 such that $0<\epsilon_2<\min\{d(CV_\sigma, g(\sigma))\mid \sigma\in Sd_1\}$ where CV_σ is the complement of V_σ in Y. Let Sd_2 be a subdivision of Sd_1 so fine that for every $\sigma\in Sd_2$, $g(\sigma)$ is contained in some element, say V_σ, of $\mathcal{V}_2=\{V(t, \epsilon_2)\mid t\in S^1\}$. Denote the corresponding system by $(W_\sigma, P_\sigma, F_\sigma, Q_\sigma, V_\sigma)$ as before.

A map $\bar{g}_2\colon Sd_2\rightarrow Q$ is defined as follows. If v is a vertex of Sd_1, let $\bar{g}_2(v)=\bar{g}_1(v)$. The rest of the definition of $\bar{g}_2$ is analogous to that of $\bar{g}_1$ using Sd_2 and elements of $\mathcal{V}_2$ instead of Sd_1 and $\mathcal{V}_1$. Then for each $\sigma\in Sd_2$, $\bar{g}_2(\sigma)\subset Q_\sigma$. Let $g_2=f\bar{g}_2$. Then $d(g_2, g)<\epsilon_2$.

We will now construct a homotopy $h_1\colon S^1\times I\rightarrow V$ between g_1 and g_2. Let $\sigma_1\in Sd_1$, $\sigma_2\in Sd_2$ and $\sigma_2\subset\sigma_1$. From the choice of ϵ_2 it follows that $W_{\sigma_2}\subset V_{\sigma_1}$. Then $Q_{\sigma_2}\subset Q_{\sigma_1}$ since $Q_{\sigma_2}\subset f^{-1}(W_{\sigma_2})\subset f^{-1}(V_{\sigma_1})\subset Q_{\sigma_1}$. This implies $\bar{g}_2(\sigma_1)\subset Q_{\sigma_1}$. Let $A=\sigma_1\times 0\cup\sigma_1\times 1\cup\dot{\sigma}_1\times I\subset\sigma_1\times I$ and define $\bar{h}_1\colon A\rightarrow Q_{\sigma_1}$ by $\bar{h}_1(t, 0)=\bar{g}_1(t)$, $\bar{h}_1(t, 1)=\bar{g}_2(t)$ and for $t\in\dot{\sigma}_1$, $\bar{h}_1(t, v)=\bar{g}_1(t)=\bar{g}_2(t)$. By the choice of Q_{σ_1}, $\bar{h}_1$ is homotopic in P_{σ_1} to a map of A into F_{σ_1}. This implies that $h_1=f\bar{h}_1$ can be extended to $\sigma_1\times I$ in W_{σ_1}. This yields the homotopy h_1 between g_1 and g_2 with the property that for each $(t, v)\in S^1\times I$, $d(h_1(t, v), g(t))<\epsilon_1$.

Continuing as above one obtains for each natural number i a map $g_i\colon S^1\rightarrow V$ and a homotopy $h_i\colon S^1\times I\rightarrow V$, with $h_i(t, 0)=g_i(t)$, $h_i(t, 1)=g_{i+1}(t)$, and for all $(t, v)\in S^1\times I$, $d(h_i(t, v), g(t))<\epsilon_i$ where we may assume that the ϵ_i converge to zero.

A homotopy $H\colon S^1\times I\rightarrow V$ between $g_1(t)$ and $g(t)$ is defined as follows:

$$H(t, v) = h_1(t, 2v) \qquad 0\leqq v\leqq 1/2,$$

$$H(t, v) = h_k(t, 2^k v - 2^k - 2) \quad \frac{2^{k-1}-1}{2^{k-1}}\leqq v\leqq\frac{2^k-1}{2^k},\ k=2, 3, \cdots,$$

$$H(t, 1) = g(t).$$

From the facts in the previous paragraph it is easily checked that H is well-defined and continuous. As we have already shown that g_1 is homotopic to p in W, this proves Theorem 10.

For homology in the rest of the paper we will use augmented Čech theory with compact carriers over the integers. The following theorem is the goal of this section. It generalizes Theorem 10.

THEOREM 11. *Let $f\colon X\rightarrow Y$ be proper and onto. Suppose for each $y\in Y$, $f^{-1}(y)$ is $(n-1)$-connected and LC^{n-1}. Let X be LC^n. Then Y is LC^n.*

PROOF. First, an argument that Y is lc^n will be roughly sketched. For the case of field coefficients this would follow from a theorem of Wilder [9].

By a local theorem of Hurewicz [4], since X is LC^n it is lc^n. For each $y \in Y$, $f^{-1}(y)$ is $(n-1)$-connected. Then by the Hurewicz Theorem, the augmented singular homology groups of $f^{-1}(y)$ vanish up through dimension $n-1$. By a theorem in [5], since $f^{-1}(y)$ is LC^{n-1} this implies that the Čech homology groups $H_r(f^{-1}(y))$ vanish for $0 \leqq r \leqq n-1$. Thus f is $(n-1)$-monotone over the integers in the sense of Wilder [9].

Let $p \in Y$ and U, a neighborhood of p, be given. Let $F = f^{-1}(p)$ and $P = f^{-1}(U)$. By an easily proved homology analogue of Theorem 4 one chooses $Q \supset F$ so that an r-cycle (r fixed less than $n+1$) on Q is homologous in P to one in F. Choose a neighborhood V of p so that $f^{-1}(V) \subset Q$ by Lemma 1.

Let z_r be an r-cycle of V. By the Begle-Vietoris theory [1; 2; 3] using the fact that X is lc^n, one can find an r-cycle w_r of Q so that $f(w_r)$ is homologous to z_r. By the choice of Q this implies that z_r is homologous to zero in U. Thus Y is lc^n.

By Theorem 10 Y is LC^1. Then by the previously mentioned theorem of Hurewicz in [4] it follows that Y is LC^n. q.e.d.

BIBLIOGRAPHY

1. E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. of Math. vol. 51 (1950) pp. 534–543.

2. ———, *The Vietoris mapping theorem for bicompact spaces*, II, Michigan Mathematical Journal vol. 3 (1956).

3. ———, *A fixed point theorem*, Ann. of Math. vol. 51 (1950) pp. 544–550.

4. W. Hurewicz, *Homotopie, Homologie, und lokaler Zusammenhang*, Fund. Math. vol. 25 (1935) pp. 467–485.

5. S. Lefschetz, *Topics in topology*, Annals of Mathematics Studies, Princeton, 1942.

6. M. H. A. Newman, *Local connection in locally compact spaces*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 44–53.

7. S. Smale, *A note on open maps*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 391–393.

8. L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. vol. 97 (1927) pp. 454–472.

9. R. L. Wilder, *Mappings of manifolds*, Summer Institute on Set Theoretic Topology, Madison, 1955.

10. ———, *Some mapping theorems with applications to non-locally connected spaces*, Algebraic Geometry and Topology, Princeton, 1956.

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