Transfer and ramified coverings

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Abstract

In this note we introduce a general class of finite ramified coverings $\pi: \tilde{X} \downarrow X$. Examples of ramified covers in our sense include: finite covering spaces, branched covering spaces and the orbit map $Y \downarrow Y/G$ where G is a finite group and Y an arbitrary G-space. For any d-fold ramified covering $\pi: \tilde{X} \downarrow X$ we construct a transfer homomorphism

$$\pi_{\mathfrak{h}}: H_{\ast}(X; \mathbb{Z}) \to H_{\ast}(X; \mathbb{Z}),$$

with the expected property that

 $\pi_*.\pi_{\natural}:H_*(X;\mathbb{Z})$

is multiplication by d. As a consequence we obtain a simple proof of the Conner conjecture; viz. the orbit space of an arbitrary finite group action on a Q-acyclic space is again Q acyclic.

1. Ramified coverings

Fix a natural number $d \in \mathbb{N}$ and a topological space Y. Introduce the following notations:

$$P^{d}(Y) := \overleftarrow{Y \times \ldots \times Y},$$

$$UP^{d}(Y) := P^{d}(Y) \times_{\Sigma_{d}} \{1, \ldots, d\},$$

$$SP^{d}(Y) := P^{d}(Y) / \Sigma_{d},$$

where Σ_d is the symmetric group on *d* elements acting on $[d] := \{1, ..., d\}$ via permutations and on $P^d(Y)$ via permutation of the coordinates. The space $SP^d(Y)$ is called the *d*-fold symmetric product of Y (2). There is the projection map

$$p: UP^{d}(Y) \to SP^{d}(Y) = P^{d}(Y) \times_{\Sigma d} [1]$$

which serves as the generic example of a d-fold ramified covering. Specifically we introduce:

Definition. A surjective finite to one map $\pi: \tilde{X} \downarrow X$ is called a *d*-fold ramified covering iff there is a map

$$\mu: X \to \mathbb{N}$$

called the multiplicity map (N.B. μ is part of the structure) such that

(1) $\forall x \in X \sum_{\widetilde{x} \in \pi^{-1}(x)} \mu(\widetilde{x}) = d,$

(2) the map

$$f_{\pi}: X \to SP^{d}(\tilde{X})$$

given by sending x into $\pi^{-1}(x)$, where each $\tilde{x} \in \pi^{-1}(x)$ occurs $\mu(\tilde{x})$ times, is continuous.

PROPOSITION 1.1. For any space Y and $d \in \mathbb{N}$ the map $p: UP^d(Y) \downarrow SP^d(Y)$ is a d-fold ramified covering.

Before taking up the proof we need to make explicit the multiplicity map

$$\mu: UP^d(Y) \to \mathbb{N}.$$

To this end we regard $P^d(Y)$ as the space of functions

$$Y^{[d]} := \{ y \colon [d] \to Y \}.$$

The symmetric group Σ_d acts on $P^d(Y)$ via

$$\sigma \cdot y(j) = y(\sigma^{-1}(j)) \colon \forall y \in P^d(Y), \quad \sigma \in \Sigma_d.$$

 $UP^{d}(Y)$ is the orbit space of $P^{d}(Y) \times [d]$ under the diagonal action. To each point $(y, j) \in P^{d}(Y) \times [d]$ we associate the subset

$$A(y,j) := y^{-1}(y(j)) \subseteq [d]$$

Note that $A(\sigma \cdot (y,j)) = \sigma A(y,j)$ so that |A(y,j)| depends only on the orbit of (y,j) in $UP^{d}(Y) = P^{d}(Y) \times_{\Sigma_{d}}[d]$ and thus we may define the multiplicity function μ by

$$\mu([y,j]) := |A(y,j)|,$$

where | | denotes cardinality. This defines

$$\mu: UP^d(Y) \to \mathbb{N},$$

completing the requisite structure for a ramified cover.

Proof of (1.1). We begin by analysing the map $p: UP^d(Y) \downarrow SP^d(Y)$. If $[y] \in SP^d(Y) = P^d(Y)/\Sigma_d$ is represented by $y \in P^d(Y)$, then we claim

 $p^{-1}([y]) = \{[y, 1], ..., [y, d]\} \subset UP^d(Y).$

To see this suppose (y', j'), $(y'', j'') \in P^d(Y) \times [d]$ represent the same point in $UP^d(Y)$. Then there exists $\sigma \in \Sigma_d$ such that

$$j' = \sigma(j'')$$

 $y'(i) = y''(\sigma(i)) : \forall i \in [d].$

In particular the elements of $p^{-1}([y])$ are representable by some $(y, j) \in P^d(Y) \times [d]$. Therefore unravelling the definitions we see that the sets

$$A(y, 1), \dots, A(y, d) \subseteq [d],$$

define a partition $\pi[y]$ of [d] and hence

$$\sum_{[y',j'] \in p^{-1}(y,j)} \mu([y',j']) = \sum_{A \in \pi[y]} |A| = d,$$

as required of the multiplicity function.

The preceding analysis shows that the map

$$f_p: SP^d(Y) \to SP^d(UP^d(Y)),$$

arises as follows. Let

$$l: P^d(Y) \to P^d(UP^d(Y)),$$

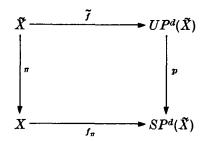
be defined by

$$l(y)(j) = [y,j]: j \in [d].$$

Then l is Σ_d equivariant and so induces a continuous map on quotient spaces, which upon unravelling the definitions is seen to be f_p , so f_p is continuous.

486

PROPOSITION 1.2. If $(\tilde{X} \downarrow^{\pi} X, \mu)$ is a d-fold ramified covering then there is a cartesian square



Proof. To define \tilde{f} let $\tilde{x} \in \tilde{X}$ and set $x = \pi(\tilde{x})$. Choose an arrangement of the points of $\pi^{-1}(x)$ counted according to multiplicity so that all $\mu(\tilde{x})$ copies of \tilde{x} occur at the beginning; say for example

$$(\underbrace{\widetilde{x},\ldots,\widetilde{x}}_{\mu(\widetilde{x})},\widetilde{x}',\ldots)\in P^d(\widetilde{X}).$$

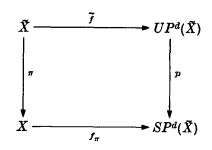
Define

(*)
$$\tilde{f}(\tilde{x}) := [(\tilde{x}, ..., \tilde{x}, \tilde{x}', ...), 1] \in UP^d(\tilde{X}),$$

where the square brackets denote equivalence class. To see that this is well defined let $\Sigma_{d-1} \hookrightarrow \Sigma_d$ as the isotropy group of $1 \in [d]$. Then for any $\sigma \in \Sigma_{d-1}$ and $(\tilde{x}_1, \ldots, \tilde{x}_d)$ belonging to $P^d(\tilde{X})$ we have

in
$$UP^d(X)$$
. Thus
$$\begin{split} &[(\tilde{x}_1,\ldots,\tilde{x}_d),1]=[(\tilde{x}_1,\tilde{x}_{\sigma(2)},\ldots,\tilde{x}_{\sigma(d)}),1],\\ &[(\tilde{x},\ldots,\tilde{x},\tilde{x}',\ldots),1]\in UP^d(\tilde{X}), \end{split}$$

does not depend on the choice of the ordering of $(\tilde{x}', ...)$ and so $\tilde{f}(\tilde{x}) \in UP^d(\tilde{X})$ is well defined, and makes the diagram



commute.

To establish the continuity of \tilde{f} regard $P^{d}(\tilde{X})$ as the function space $\tilde{X}^{[d]}$. The evaluation map

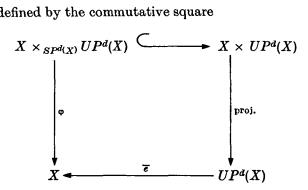
$$e: P^d(\tilde{X}) \times [d] \to \tilde{X}: e(\tilde{x}, j) = \tilde{x}(j),$$

is Σ_d equivariant and so defines

$$\bar{e}: UP^d(\tilde{X}) \to \tilde{X}.$$

LARRY SMITH

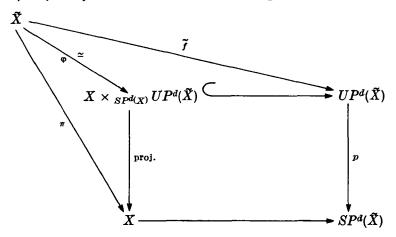
The composite φ defined by the commutative square



is closed since f is continuous. Moreover the map

$$\psi \colon \tilde{X} \to X \times_{SP^{d}(\tilde{X})} UP^{d}(\tilde{X}) \colon \psi(\tilde{x}) = (\pi(\tilde{x}), \tilde{f}(\tilde{x})),$$

is inverse to φ so φ is bijective and hence a homeomorphism. Thus



establishes the continuity of \tilde{f} and the cartesian nature of the square.

PROPOSITION 1.3. Suppose $(\tilde{Y} \downarrow^{\pi} Y, \mu)$ is a d-fold ramified covering and $\varphi: X \to Y$ is a continuous map. Then the pullback $\pi_{\varphi}: \widetilde{X} \downarrow X$ is a ramified covering.

Proof. To begin note

$$\pi_{\varphi} \colon \tilde{X} := X \times_{Y} \tilde{Y} \downarrow X \colon \pi_{\varphi}(x, \tilde{y}) \coloneqq x$$

We define

$$\mu \colon \tilde{X} \to \mathbb{N} \colon \mu(x, \tilde{y}) = \mu(\tilde{y}).$$

Then for any $x \in X$ we have

$$\sum_{\tilde{x}\in\pi_{\varphi}^{-1}(x)}\mu(\tilde{x})=\sum_{\tilde{y}\in\pi_{\varphi}^{-1}(x)}\mu(\tilde{y})=d,$$

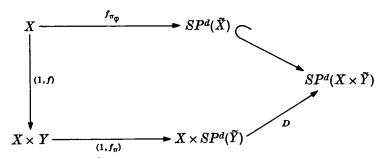
as required. To obtain the continuity of

$$f_{\pi_{\mathfrak{P}}}: X \to SP^{d}(\tilde{X}),$$

consider the inclusion $\tilde{X} \hookrightarrow X \times \tilde{Y}$. This induces an inclusion

$$SP^{d}(\tilde{X}) \hookrightarrow SP^{d}(X \times \tilde{Y}).$$

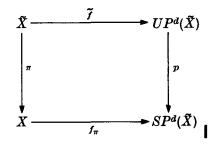
There is the commutative diagram



where D is induced by the diagonal $X \hookrightarrow P^d(X)$. Since the composition $D \cdot (1 \times f_{\pi}) \cdot (1, f)$ is continuous and $SP^d(\tilde{X}) \hookrightarrow SP^d(X \times \tilde{Y})$ has the subspace topology it follows that $f_{\pi_{\varphi}}$ is continuous.

By combining the preceding results we obtain

COROLLARY 1.4. $\pi: \tilde{X} \downarrow^{\pi} X$ is a d-fold ramified covering iff there is a cartesian square



PROPOSITION 1.5. Let G be a finite group and Y a G-space. Then the orbit map

 $\pi\colon Y\to Y/G,$

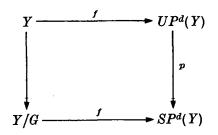
admits the structure of a d := |G| fold ramified covering.

Proof. Choose an ordering g_1, \ldots, g_d of the elements of G. There are the continuous maps

$$f: Y \to UP^{d}(Y) | f(y) = [g_{1}y, ..., g_{d}y, 1]$$

$$f: Y/G \to SP^{d}(Y) | f[y] = [g_{1}y, ..., g_{d}y],$$

and the square



commutes. To see it is cartesian note that we can reinterpret Y as follows. A point y of Y is the pointed G-orbit $\pi^{-1}\pi(y)$ with basepoint y. But by definition this is

$$(Y/G) \times_{SP^{d}(Y)} UP^{d}(Y).$$

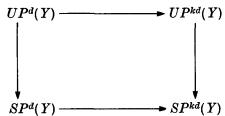
LARRY SMITH

N.B. Since the group G is finite the orbit map $\pi: Y \to Y/G$ is clopen. For if A is open then $\pi^{-1}\pi(A) = \bigcup_{g \in G} gA$ is a union of open sets in Y, so open, whereas if A is closed the finiteness of the union implies $\pi^{-1}\pi(A)$ is closed. This assures the continuity of f and \tilde{f} .

Remark 1. A 2-fold ramified covering $\pi: \tilde{X} \downarrow X$ is nothing but an involution $T: \tilde{X} \to \tilde{X}$.

To see this note that by (1.5) an involution defines a ramified double covering. On the other hand switching the points in each fibre of a ramified double covering $\tilde{X} \downarrow^{\pi} X$ defines an involution on \tilde{X} .

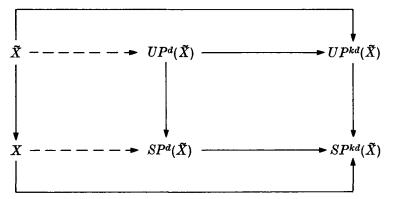
Remark 2. By (1.1) the quotient map $p: UP^d(Y) \downarrow SP^d(Y)$ is a *d*-fold ramified covering. Note that for any $k \in \mathbb{N}$ the *k*-fold diagonal map $\Delta_k: Y \to P^k(Y)$ induces a cartesian square:



so that $UP^d(Y) \downarrow SP^d(Y)$ may be regarded as a kd-fold ramified covering. Since $p: UP^d(Y) \downarrow SP^d(Y)$ is the 'universal example' of a d-fold ramified covering, this says that by 'multiplying' all multiplicities by k we can regard any d-fold covering as a kd-fold ramified covering.

One can also examine the converse, namely when can one suitably redefine the multiplicity function of a kd-fold ramified cover so as to obtain a d-fold ramified covering. In this connection we have the useful:

Observation. Suppose given the diagram of solid arrows



defining $\tilde{X} \downarrow X$ as a kd-fold ramified covering. If the dotted arrows exist as functions, then they are continuous.

Proof. $SP^{d}(\tilde{X}) \hookrightarrow SP^{kd}(\tilde{X})$ and $UP^{d}(\tilde{X}) \hookrightarrow UP^{kd}(\tilde{X})$ have the subspace topology. This implies:

COROLLARY 1.6. Let $G \times Y \to Y$ be an action of a finite group G. If $d' \in \mathbb{N}$ is such that $d' \mid [G; Gy]$ for all $y \in Y$, where Gy is the isotropy group of y, then it is possible to define multiplicities so as to make the orbit map $Y \to Y/G$ into a d'-fold ramified covering.

For completeness we note:

PROPOSITION 1.7. If $\tilde{X} \downarrow^{\pi} X$ is a finite covering then assigning the multiplicity 1 to every point of X makes it a ramified covering.

Proof. Only the continuity of f_{π} is at stake. But this is a local question, so it suffices to look at the situation

$$V \times [d] \downarrow V : (v, j) \downarrow v$$

where d is the number of sheets. However in this case continuity is clear.

In the usual definition of a *branched* covering (4) one can compute the multiplicity of points from 'local data'. This suggests that for a *d*-fold ramified cover $\tilde{X} \downarrow X$ one introduce the set

$$X_d := \{x \in X \mid |\pi^{-1}(x)| = d\}.$$

N.B. It can well happen that $X_d = \emptyset$. A *d*-fold ramified cover is called *nice* iff $X_d \subset X$ is a dense subset. In this case we see that

$$f_{\pi}: X \to SP^d(\tilde{X}),$$

is determined by its restriction to X_d . So to compute the multiplicity of a point $\tilde{x} \in \tilde{X}$ we choose a net of points $x_{\lambda} \in X$ converging to $\pi(\tilde{x})$. Then $f_{\pi}(x_{\lambda}) \in SP^d(\tilde{X})$ converges to $f_{\pi}(\pi(\tilde{x}))$ which is a *d*-tuple of points of \tilde{X} containing \tilde{x} exactly $\mu(\tilde{x})$ times.

The following theorem of Chernowski (1) shows that the above discussion applies in the manifold situation.

THEOREM. If a finite group G acts effectively on a smooth manifold M^n such that M^n/G is again a manifold, then the branching set $B \subset M^n/G$ has codimension $\leq n-2$. In particular the set $M^n/G - B = (M^n/G)_d$; d := |G| is open and dense.

2. Transfer for ramified coverings: applications

Let $\pi: \tilde{X} \downarrow X$ be a *d*-fold ramified covering. Then there is the map

$$f_{\pi}: X \to SP^d(\tilde{X}).$$

Consider the composite

$$\pi_{\mathfrak{h}}: X \xrightarrow{f_{\pi}} SP^{d}(X) \xrightarrow{c} SP^{\infty}(X)$$

where *i* is the standard inclusion induced by a choice of basepoint $\tilde{x} \in \tilde{X}$. By the theorem of Dold and Thom (2) there is a weak homotopy equivalence

$$SP^{\infty}(\tilde{X}) \to K(\tilde{H}_{*}(\tilde{X};\mathbb{Z})),$$

where $K(A_*)$ denotes the Eilenberg-Mac Lane space for the graded abelian group A_* . Thus the based homotopy class of π_{\sharp} defines an element

$$[\pi_{\mathfrak{h}}] \in \dot{H}^{\ast}(X; \dot{H}_{\ast}(\ddot{X}; \mathbb{Z})),$$

and composing with the universal coefficient map

gives a map

$$\pi_{\natural}: \tilde{H}_{\ast}(X; \mathbb{Z}) \to \tilde{H}_{\ast}(\tilde{X}; \mathbb{Z}),$$

which we call the *transfer homomorphism* for the ramified covering $\pi: \tilde{X} \downarrow X$.

LARRY SMITH

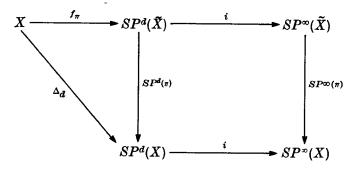
PROPOSITION 2.1. The transfer construction is natural with respect to pull backs and compositions of ramified coverings. |

PROPOSITION 2.2. If $\tilde{X} \downarrow^{\pi} X$ is a d-fold ramified covering then

$$\pi_*.\pi_{\natural}:\tilde{H}_*(X;\mathbb{Z})$$

is multiplication by d.

Proof. One simply notes that



commutes, where Δ_d is the *d*-fold diagonal map. The composition

$$SP^{\infty}(\pi) \cdot i \cdot f_{\pi} \colon X \to SP^{\infty}(X)$$

represents, by definition,

whereas

 $i\Delta_d \in \tilde{H}^*(X; \tilde{H}_*(X; \mathbb{Z}))$

 $\pi_*([\pi_{\mathtt{b}}]) \in \tilde{H}^*(X; \tilde{H}_*(X; \mathbb{Z}))$

under the universal coefficient map goes to multiplication by $d: \tilde{H}_*(X; \mathbb{Z}) \to \tilde{H}_*(X; \mathbb{Z})$, which completes the proof.

Remark 1. The above construction defines an integral homology transfer, and hence by tensoring a rational homology transfer. To define a transfer for homology with finite coefficients \mathbb{Z}/n we replace the map

$$i: SP^{d}(\widetilde{X}) \hookrightarrow SP^{\infty}(\widetilde{X}),$$

with the map

$$i_n: SP^d(\tilde{X}) \longrightarrow \lim_{\rightarrow} \{SP^d(\tilde{X}) \xrightarrow{\Delta_n} SP^{nd}(\tilde{X}) \xrightarrow{\Delta_n} \ldots \}.$$

A moment's reflection shows that

$$\lim_{\longrightarrow} \{SP^d(\tilde{X}) \xrightarrow{\Delta_n} SP^{nd}(\tilde{X}) \longrightarrow \ldots\} \sim K(\tilde{H}_*(\tilde{X}; \mathbb{Z}/n)),$$

which leads to a transfer for \mathbb{Z}/n homology.

Remark 2. If G is a group acting on \tilde{X} and X such that the ramified covering map $\pi: \tilde{X} \downarrow X$ is a G-map, then

$$\pi_{\natural} \colon \tilde{H}_{\ast}(X; \mathbb{Z}) \to \tilde{H}_{\ast}(\tilde{X}; \mathbb{Z}),$$

is a $\mathbb{Z}(G)$ -module homomorphism. This follows by simply noting that the map

$$f_{\pi}: X \to SP^{d}(\tilde{X}),$$

is G-equivariant.

COROLLARY 2.3. If $\tilde{X} \downarrow^{\pi} X$ is a d-fold ramified cover and \tilde{X} is \mathbb{Q} , resp. \mathbb{Z}/n acyclic, then X is \mathbb{Q} acyclic, resp. \mathbb{Z}/n acyclic provided $d \in \mathbb{Z}/n^*$.

Proof. One has

where k is either Q or \mathbb{Z}/n . But $\tilde{H}_*(\tilde{X}; k) = 0$, and the result follows.

COROLLARY 2.4. (Conner Conjecture for finite groups): If a finite group G acts on a rationally acyclic manifold M, then M/G is rationally acyclic.

In a similar vein we have:

PROPOSITION 2.4. Let G be a finite group acting on the space Y with orbit map π : $Y \rightarrow Y/G$ structured as a d := |G| fold ramified covering. Then

is given by

$$\pi_{\natural} \cdot \pi_{\ast} \colon \tilde{H}_{\ast}(Y) \mathfrak{I}$$
$$\pi_{\natural} \cdot \pi_{\ast}(u) = \sum_{q \in G} g_{\ast}(u).$$

$$Y \xrightarrow{\pi} Y/G \xrightarrow{f_{\pi}} SP^d(Y)$$

is given by

$$y \mapsto \{gy \mid g \in G\},\$$

regarded as an unordered d-tuple with repeats. Since

$$Y \to Y/G \to SP^d(Y) \to SP^{\infty}(Y)$$

represents π_{\flat} . π_{\star} the result follows.

COROLLARY 2.5. Let G be a finite group acting on a space Y with orbit map $\pi: Y \to Y/G$ structured as a d := |G| fold ramified covering. Let $k = \mathbb{Q}$ or \mathbb{Z}/n where (n, d) = 1. Then

$$\pi_* \colon \hat{H}_*(Y;k)^G \to \hat{H}_*(Y/G;k),$$

is an isomorphism.

Proof. By (2.2) $\pi_*\pi_{\natural}: \tilde{H}_*(Y/G; k) \supset$ is multiplication by d while by (2.4) $\pi_{\natural}\pi_{\ast}: H(Y; k)^G \supset$ is also multiplication by d. Since $d \in k^*$ the result follows.

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