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Invariants for immersions with applications to covering spaces

By

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Let M^m be a closed smooth m-dimensional manifold $M^m \overset{\varphi}{\subseteqq} \mathbb{R}^{m+k}$ an immersion, and $\tilde{M}^m \downarrow^{\pi} M^m$ a finite covering. The composite $\tilde{\varphi}\colon \tilde{M}^m \downarrow^{\pi} M \subseteqq \mathbb{R}^{m+k}$ is then also an immersion. What relations are there between $\varphi\colon M^m \subseteqq \mathbb{R}^{m+k}$ and $\tilde{\varphi}\colon \tilde{M}^m \subseteqq \mathbb{R}^{m+k}$? For example, if φ is an embedding, can $\tilde{\varphi}$ be regularly homotopic to an embedding? (For a non-trivial covering π, $\tilde{\varphi}$ can never be an embedding on the nose.) The case $m = k$ seems particularly interesting in view of Whitney's exhaustive study of immersions in double dimension. Whitney showed that there is always an embedding $M^m \overset{\varphi}{\subset} \mathbb{R}^{2m}$, that is, an immersion without double points. For m even, the number of double points of a generic representative, counted with appropriate orientations, is in fact the only invariant of the regular homotopy class of an immersion $\varphi\colon M^m \subseteqq \mathbb{R}^{2m}$. Furthermore, this is just half the Euler class of the normal bundle $\nu \downarrow M$ of the immersion. For m odd the only invariant of the regular homotopy class of φ is the congruence class modulo 2 of the number of double points of a generic representative. For this invariant there is no simple characteristic class interpretation in terms of the normal bundle $\nu \downarrow M^m$ of φ.

The requirement that φ be generic (in order to count the number of double points) is incompatible with questions involving covering spaces. For m even however, the Euler class interpretation allows questions involving covering spaces to be dealt with. The situation for m odd is not so straight forward as the theorem of Brown [3] shows.

Our first step is to introduce an invariant for immersions $\varphi\colon M^m \subseteqq \mathbb{R}^{2m+1-k}$ that is well adapted to the study of immersions of the form $\varphi\colon \tilde{M}^m \downarrow^{\pi} M^m \overset{\varphi}{\subseteqq} \mathbb{R}^{2m+1-k}$ where $\tilde{M}^m \downarrow M^m$ is a covering. The invariant is a homotopy class $\varDelta(\varphi) \in \pi_{2m+1}(\Sigma^k K(\mathbf{Z}/2, m+1-k))$, whose introduction is motivated by the work of Dupont [5]. For $k = 1$ the invariant is complete, that is:

Proposition. *An immersion $\varphi\colon M^m \subseteqq \mathbb{R}^{2m}$ is regularly homotopic to an embedding iff $\varDelta(\varphi) = 0 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)) \cong \mathbf{Z}/2$.*

Remark. *The group $\pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)) \cong \pi_{2m}(\Omega\Sigma K(\mathbf{Z}/2, m))$ is detected by the Hurewicz map*

$$h\colon \pi_{2m}(\Omega\Sigma K(\mathbf{Z}/2, m)) \to H_{2m}(\Omega\Sigma K(\mathbf{Z}/2, m); \mathbf{Z}/2).$$

This leads to a cohomological interpretation of $\Delta(\varphi)$ which will be explained elsewhere.

We apply the invariant $\Delta(\varphi)$ to the study of immersions of the form $\tilde{M} \downarrow^{\pi} M \overset{\varphi}{\subseteqq} \mathbb{R}^{2m}$, where π is a finite covering, and we obtain

Proposition. *Let $\tilde{M} \downarrow^{\pi} M$ be an odd order covering of $Z/2$ homologyspheres $\varphi\colon M^m \subseteqq \mathbb{R}^{2m}$ an immersion $\tilde{\varphi} = \varphi \circ \pi\colon \tilde{M} \subseteqq \mathbb{R}^{2m}$. Then $\Delta(\tilde{\varphi}) = \Delta(\varphi)$.*

Naturally one can ask for an evaluation of $\Delta(\tilde{\varphi})$ in some familiar situations. For immersions $\tilde{\varphi}\colon S^m \downarrow^{\pi} \mathbb{R}P(m) \overset{\varphi}{\subseteqq} \mathbb{R}^{2m}$ it is a consequence of [3] that $\Delta(\tilde{\varphi}) = 0$ iff $m \neq 2^q - 1$. In contrast to Brown's theorem the preceeding proposition shows that in the odd order lens space situation

$$\tilde{\varphi}\colon S^{2k+1} \downarrow^{\pi} L(p; 2k+1) \overset{\varphi}{\subseteqq} \mathbb{R}^{4k+2} \quad \text{that} \quad \Delta(\tilde{\varphi}) = \Delta(\varphi).$$

For an odd dimensional π-manifold M^m there is always an immersion $\varphi\colon M^m \subseteqq \mathbb{R}^{2m}$ with normal bundle $\nu \downarrow M^m$ isomorphic to the tangent bundle. Using [5] we show that for this immersion class φ, $\Delta(\varphi) = \chi_{1/2}(M) \bmod 2$, $m \neq 1, 3, 7$, wheie $\chi_{1/2}(M)$ is the Kervaire semi-characteristic of M.

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1. The immersion invariant $\Delta(\varphi) \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$. Let M^m be a closed smooth m-dimensional manifold and $\varphi\colon M^m \subseteqq \mathbb{R}^{2m}$ an immersion. We denote by $T(\nu \downarrow M)$ the Thom space of the normal bundle $\nu \downarrow M^m$ of the immersion φ. Let

$$t\colon T(\nu \downarrow M^m) \to K(\mathbf{Z}/2, m)$$

represent the $\mathbf{Z}/2$ cohomology Thom class of $T(\nu \downarrow M)$. Fix an embedding $\psi\colon M^m \subset \mathbb{R}^{2m+1}$, and denote by $\hat{\nu} \downarrow M^m$ the normal bundle of ψ. According to Whitney [10] there is a unique regular homotopy class of immersions $M^m \subseteqq \mathbb{R}^{2m+1}$. Therefore the stabilized immersion $M^m \overset{\varphi}{\subseteqq} \mathbb{R}^{2m} \subset \mathbb{R}^{2m+1}$ is regularly homotopic to ψ and we obtain a well defined homotopy class of bundle isomorphism [5] $\alpha\colon \hat{\nu} \xrightarrow{\sim} \nu \oplus \mathbb{R} \downarrow M$ and hence a well defined homotopy equivalence

$$T(\alpha)\colon T(\hat{\nu} \downarrow M) \to T(\nu \oplus \mathbb{R} \downarrow M).$$

We define $\Delta(\varphi) \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$ to be the homotopy class of the composition

$$S^{2m+1} \xrightarrow{c} T(\nu \downarrow M) \xrightarrow{T(\alpha)} T(\nu \oplus \mathbb{R} \downarrow M) \overset{\theta}{\underset{\cong}{\to}} \Sigma T(\nu \downarrow M) \xrightarrow{\Sigma t} \Sigma K(\mathbf{Z}/2, m)$$

where c is the Pontrjagin-Thom map associated to the embedding ψ, and θ is the standard identification. According to Wu [11] there is a unique isotopy class of embeddings $M^m \subset \mathbb{R}^{2m+1}$, and therefore $\Delta(\varphi)$ does not depend on the choice of the embedding ψ.

Lemma 1.1. *With the notations preceeding:*

1) *The homotopy class of $\Delta(\varphi)$ depends only on the regular homotopy class of φ.*
2) *If φ is regularly homotopic to an embedding then*

$$\Delta(\varphi) = 0 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)).$$

Proof. Suppose that $\varphi', \varphi'': M^m \subseteq \mathbb{R}^{2m}$ are immersions with normal bundles $\nu', \nu'' \downarrow M^m$. A regular homotopy from φ' to φ'' induces a bundle isomorphism $f: \nu' \cong \nu'' \downarrow M$. By Hirsch's theorem [5] the homotopy class of f does not depend on the choice of regular homotopy. A simple diagram chase then yields (1). To prove (2), note from (1), we may suppose that $\varphi: M^m \subseteq \mathbb{R}^{2m}$ is already an embedding, in which case $\Delta(\varphi)$ is represented by the suspension of the composite

$$S^{2m} \xrightarrow{c} T(\nu \downarrow M) \xrightarrow{t} K(\mathbf{Z}/2, m)$$

which is null homotopic since $\pi_{2m}(K(\mathbf{Z}/2, m)) = 0$. ■

Remark. The homotopy of $\Sigma K(\mathbf{Z}/2, m)$ has been studied in [2] where it is shown that $\pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)) \cong \mathbf{Z}/2$.

Proposition 1.2. *Let m be odd, $m \neq 1, 3, 7$, and $\omega: S^m \subseteq \mathbb{R}^{2m}$ the Whitney immersion, that is ω has exactly one double point. Then $\Delta(\omega) \neq 0 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$.*

Proof. This will follow from (2.1) as soon as one knows that the normal bundle of the Whitney immersion is isomorphic to the tangent bundle of S^m. To see this we note the normal bundle $\nu \downarrow S^m$ of the Whitney immersion is stably trivial, as one sees from the bundle equation

$$\nu \oplus \mathbb{R}^{m+1} \cong \nu \oplus \tau \oplus \mathbb{R} \cong \mathbb{R}^{2m} \oplus \mathbb{R} \downarrow S^m.$$

From [1] it follows that (recall m odd, $m \neq 1, 3, 7$)

$$\operatorname{Ker}\{\pi_m(BSO(m)) \to \pi_m(BSO)\} \cong \mathbb{Z}/2$$

with generator $\tau \downarrow S^m$. Thus $\nu \cong \mathbb{R}^m \downarrow S^m$ or $\nu \cong \tau \downarrow S^m$. Moreover, since

$$\tau \oplus \mathbb{R}^m \cong \mathbb{R}^{2m} \downarrow S^m \quad \text{and} \quad \tau \oplus \tau \cong \mathbb{R}^{2m} \downarrow S^m$$

both τ and $\mathbb{R}^m$ occur as normal bundles of immersions $S^m \subseteq \mathbb{R}^{2m}$ [6]. Since $\mathbb{R}^m$ occurs as the normal bundle of the canonical embedding, and ω is not regularly homotopic to an embedding, it follows that $\nu \cong \tau \downarrow S^m$. ■

Suppose given immersions $\varphi_i: M_i^m \subseteq \mathbb{R}^{2m}$, for $i = 1, 2$. By forming the connected sum of M_1 and M_2 along a disk of regular points we obtain an immersion

$$\varphi_1 \# \varphi_2: M_1 \# M_2 \subseteq \mathbb{R}^{2m}$$

called the connected sum of φ_1 and φ_2.

Proposition 1.3. *In the notations preceeding*

$$\Delta(\varphi_1 \# \varphi_2) = \Delta(\varphi_1) + \Delta(\varphi_2) \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)).$$

Proof. Let $p: M_1 \# M_2 \to M_1 \vee M_2$ be the map induced by pinching the boundary of the disk used to join M_1 to M_2 in forming the connected sum to a point.

Then the normal bundle of the immersion $\varphi_1 \# \varphi_2$ is given by

$$\nu \cong p^*(\nu_1 \vee \nu_2) \downarrow M_1 \# M_2 .$$

The remainder of the proof is a chase around the obvious diagram. ∎

Recall that according to Smale [8] the regular homotopy classes of immersions $S^m \overset{\varphi}{\subseteqq} \mathbb{R}^{m+k}$ are classified by their Smale invariant $\sigma_\varphi \in \pi_m(V_{m+1}(\mathbb{R}^{m+1+k}))$, provided $m \neq 1 \neq k$. Thus (1.2) provides an isomorphism

$$\varDelta: \pi_m(V_{m+1}(\mathbb{R}^{2m+1})) \cong \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m)) .$$

It would be interesting to know if this isomorphism is induced by a map of some kind.

Theorem 1.4. *Let $m \neq 1, 3, 7$ be odd and $\varphi: M^m \subseteqq \mathbb{R}^{2m}$ an immersion. Then*

$$\varDelta(\varphi) = 0 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$$

iff φ is regularly homotopic to an embedding.

Proof. It remains to prove for a non-embedding $\varphi: M^m \subseteqq \mathbb{R}^{2m}$ that $\varDelta(\varphi) \neq 0$. According to the analysis of Whitney [10], there are exactly two regular homotopy classes of immersions in double dimension, one of which contains an embedding $\varphi_e: M^m \subset \mathbb{R}^{2m}$, and one of which contains an immersion $\varphi_i: M^m \subseteqq \mathbb{R}^{2m}$ with a single isolated double point. By change of viewpoint we may decompose φ_i as $\varphi_e \# \omega$ where $\omega: S^m \subseteqq \mathbb{R}^{2m}$ is the Whitney immersion. Then by (1.3)

$$\varDelta(\varphi_i) = \varDelta(\varphi_e) + \varDelta(\omega) \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$$

and the result follows from (1.1) and (1.2). ∎

Remark. It is clear how to define an analogous invariant

$$\varDelta(\varphi) \in \pi_{2m+1}(\Sigma^k K(\mathbf{Z}/2\ m+1-k))$$

for immersions $M^m \overset{\varphi}{\subseteqq} \mathbb{R}^{2m+1-k}$. Those invariants are studied in the Dissertation of R. Wiegmann.

2. π-Manifolds. Let M^m be an odd dimensional π-manifold with tangent bundle $\tau \downarrow M^m$. Being odd dimensional τ has a cross-section so $\tau \cong \mathbb{R} \oplus \hat{\tau} \downarrow M^m$, and being a π-manifold $\mathbb{R} \oplus \tau \cong \mathbb{R}^{m+1} \downarrow M$. Thus

$$\tau \oplus \tau \cong \hat{\tau} \oplus \mathbb{R} \oplus \tau \cong \hat{\tau} \oplus \mathbb{R}^{m+1} \cong \tau \oplus \mathbb{R}^m \cong \mathbb{R}^{2m} \downarrow M^m .$$

Hence by Hirsch's theorem [6] there is an immersion $\varphi: M^m \subseteqq \mathbb{R}^{2m}$ whose normal bundle ν satisfies $\nu \cong \tau \downarrow M^m$.

Proposition 2.1. *Let M^m be an odd dimensional π-manifold and $\varphi\colon M^m \subsetneqq \mathbb{R}^{2m}$ an immersion with normal bundle $\nu \cong \tau \downarrow M$. Assume $m \neq 1, 3, 7$. Then $\Delta(\varphi) = \chi_{1/2}(M) \in \mathbb{Z}/2$ where* [7]

$$\chi_{1/2}(M) := \sum_{n \equiv 0(2)} \dim H_n(M; \mathbb{Z}/2) : \mathrm{mod}\, 2$$

is the mod 2 *Kervaire semi-characteristic.*

The proof (2.1) requires a pair of preliminary lemmas and can then be completed as in [5; Thm. 3.4].

Lemma 2.2. *Let $j\colon S^{m+1} \subset \Sigma K(\mathbb{Z}/2, m)$ be the inclusion of the bottom cell. For $m \neq 1, 3, 7$*

$$j_* = 0\colon \pi_{2m+1}(S^{m+1}) \to \pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)).$$

Proof. We make use of the fact that the non-zero element

$$[h] \in \pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2$$

is detected by Sq^{m+1} in the mapping cone of h. (This fact is either well known or mysterious depending on who you ask about it, so a proof is appended.) A routine diagram chase then shows: If $j_*[f] \neq 0$ for $[f] \in \pi_{2m+1}(S^{m+1})$ then

$$Sq^{m+1} \neq 0 : H^{m+1}(Kf; \mathbb{Z}/2) \to H^{2m+2}(Kf; \mathbb{Z}/2)$$

where $Kf = S^{m+1} \cup_f e^{2m+2}$, hence by [1] $m + 1 = 2, 4$ or 8. ■

Fact. $[h] \neq 0 \in \pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2$ is detected by Sq^{m+1} in the mapping cone of h.

Proof. First of all we show that $\pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2$. To this end let

$$\sigma\colon \Sigma K(\mathbb{Z}/2, m) \to K(\mathbb{Z}/2\; m + 1)$$

classify the class $\Sigma i \neq 0 \in H^{m+1}(\Sigma K(\mathbb{Z}/2, m); \mathbb{Z}/2)$. Let

$$f := \Omega\sigma : \Omega\Sigma K(\mathbb{Z}/2, m) \to \Omega K(\mathbb{Z}/2, m + 1) = K(\mathbb{Z}/2, m)$$

and denote by $L(\mathbb{Z}/2, m)$ the fibre of f. Since f is a loop map the space $L(\mathbb{Z}/2, m)$ is a loop space and the fibration f principal. The canonical map

$$\chi\colon K(\mathbb{Z}/2, m) \to \Omega\Sigma K(\mathbb{Z}/2, m)$$

satisfies

$$\chi^* f^*(i_m) = i_m$$

and hence χ is homotopic to a cross-section. Being principal, it follows that the fibration f is a product fibration, and in particular there is a homotopy equivalence

$$(*) \qquad \Omega\Sigma K(\mathbb{Z}/2, m) \cong L(\mathbb{Z}/2, m) \times K(\mathbb{Z}/2, m).$$

Therefore

$$(**) \qquad \pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)) \cong \pi_{2m}(\Omega\Sigma K(\mathbb{Z}/2, m)) \cong \pi_{2m}(L(\mathbb{Z}/2, m)).$$

The cohomology of $K(\mathbb{Z}/2, m)$ and $\Omega\Sigma K(\mathbb{Z}/2, m)$ are well known, and by contemplating $(*)$ one sees that $L(\mathbb{Z}/2, m)$ is $2m-1$ connected and $H_{2m}(L(\mathbb{Z}/2, m); \mathbb{Z}) \cong \mathbb{Z}/2$. Whence by the Hurewicz theorem it follows that $\pi_{2m}(L(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2$. From $(**)$ it then follows that

$$\pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2 .$$

It remains to show that the non-zero element $[h] \in \pi_{2m+1}(\Sigma K(\mathbb{Z}/2, m))$ is detected by Sq^{m+1} in the mapping cone Kh of h. To this end we give an explicit description of h. Let $i\colon S^m \subset K(\mathbb{Z}/2, m)$ be the canonical inclusion representing the non-zero element of $\pi_m(K(\mathbb{Z}/2, m)) \cong \mathbb{Z}/2$. Let

$$j\colon S^m \times S^m \xrightarrow{i \times i} K(\mathbb{Z}/2, m) \times K(\mathbb{Z}/2, m) \xrightarrow{\mu} K(\mathbb{Z}/2, m)$$

where μ is the H-space structure of $K(\mathbb{Z}/2, m)$. Let

$$h := h(j)\colon S^{2m+1} = S^m * S^m \to \Sigma K(\mathbb{Z}/2, m)$$

be the Hopf construction on j. There is then the following diagram of cofibrations

$$\begin{array}{ccccc} S^{2m+1} = S^m * S^m & \xrightarrow{h} & \Sigma K(\mathbb{Z}/2, m) & \to & Kh \\ \downarrow{\scriptstyle i*i} & & \| & & \downarrow{\scriptstyle g} \\ K(\mathbb{Z}/2, m) * K(\mathbb{Z}/2, m) & \xrightarrow{h\mu} & \Sigma K(\mathbb{Z}/2, m) & \to & Kh\mu \end{array}$$

where $h\mu$ is the Hopf construction of μ. The mapping cone $K(h\mu)$ of $h\mu$ is the projective plane of $K(\mathbb{Z}/2, m)$. Thus $K(h\mu)$ has the homotopy type of the classifying space $K(\mathbb{Z}/2, m+1)$ of $K(\mathbb{Z}/2, m)$ through dimension $3m-1$. In particular if $i_{m+1} \in H^{m+1}(K(h\mu); \mathbb{Z}/2)$ is the class pulling back to $\Sigma i_m \in H^m(\Sigma K(\mathbb{Z}/2, m); \mathbb{Z}/2)$, where $i_m \in H^m(K(\mathbb{Z}/2, m); \mathbb{Z}/2)$ is the fundamental class, then $Sq^{m+1} i_{m+1} = i_{m+1}^2 \neq 0$. By naturality applied to the above diagram we see that

$$Sq^{m+1} \neq 0\colon H^{m+1}(Kh; \mathbb{Z}/2) \to H^{2m+2}(Kh; \mathbb{Z}/2)$$

as desired. ∎

Lemma 2.3. *Suppose M^m is a π-manifold, $\varphi\colon M^m \subseteq \mathbb{R}^{2m}$ an immersion with normal bundle $\nu \downarrow M$, is odd, and $m \neq 1, 3, 7$. Then for any degree one map*

$$d\colon S^{2m+1} \to \Sigma T(\nu \downarrow M)$$

the composition

$$S^{2m+1} \xrightarrow{d} \Sigma T(\nu \downarrow M) \xrightarrow{\Sigma t} \Sigma K(Z/2, m)$$

represents $\Delta(\varphi) \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$. (Here $t\colon T(\nu \downarrow M) \to K(\mathbf{Z}/2, m)$ represents the Thom class.)

Proof. First of all note that $\nu \oplus \mathbb{R} \cong \mathbb{R}^{m+1} \downarrow M$, so that ν is classified by a map $f\colon M^m \to V_1(\mathbb{R}^{m+1}) = S^m$. That is, there is a map $f\colon M^m \to S^m$ so that $f^* \tau_{S^m} \cong \nu$. Therefore the Thom class t can be factored as follows:

$$t\colon T(\nu \downarrow M) \xrightarrow{Tf} T(\tau \downarrow S^m) \xrightarrow{u} K(\mathbf{Z}/2, m),$$

where u represents the Thom class of τ. Let $c\colon S^{2m+1} \to \Sigma T(\nu \downarrow M)$ be the Pontrjagin-Thom collapse map for an embedding regularly homotopic to the stabilized immersion $M^m \overset{\varphi}{\subseteqq} \mathbb{R}^{2m} \subset \mathbb{R}^{2m+1}$. Then

$$e := d - c\colon S^{2m+1} \to \Sigma T(\nu \downarrow S^m)$$

falls off the top cell, so there is a factorization

$$\begin{tikzcd}
S^{2m+1} \arrow[r, "e"] \arrow[dr, "\hat{e}"'] & \Sigma T(\nu \downarrow M) \arrow[r, "Tf"] & \Sigma T(\tau \downarrow S^m) \arrow[r, "u"] & \Sigma K(\mathbf{Z}/2, m) \\
 & \Sigma T(\nu \downarrow M)^{2m} \arrow[u, hook] \arrow[r, "Tf"] & S^{m+1} \arrow[u, hook] \arrow[ur, "j"'] &
\end{tikzcd}$$

where $\Sigma T(\nu \downarrow M)^{2m}$ is the $2m$ skeleton of $\Sigma T(\nu \downarrow M)$. Thus $(\Sigma t) \cdot e$ is null homotopic by (2.2), whence $[\Sigma t \cdot d] = [\Sigma t \cdot c] = \Delta(\varphi)$. ■

Proof of 2.1. In view of (2.2) it is sufficient to show that for a conviently constructed degree one map

$$d\colon S^{2m+1} \to \Sigma T(\tau \downarrow M)$$

that the composite

$$S^{2m+1} \xrightarrow{d} \Sigma T(\tau \downarrow M) \xrightarrow{\Sigma t} \Sigma K(\mathbf{Z}/2, m)$$

represents $\chi_{1/2}(M) \in Z/2 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$. This however follows as in [5; Theorem 3.4] and/or [9; pp. 106—107]. ■

3. Odd order coverings and immersions.

Definition. *M^m is called a $\mathbb{Z}/2$ homology sphere iff $H_i(M; \mathbf{Z}/2) = 0$ for $0 < i < m$, and $H_i(M; \mathbb{Z}/2) = \mathbb{Z}/2$ for $i = 0, m$.*

Example. Lens spaces with odd order fundamental group.

Theorem 3.1. *Let M^m be an $\mathbb{Z}/2$ homology sphere, $\tilde{M}^m \downarrow^{\pi} M^m$ an odd order covering and $\varphi\colon M \subseteqq \mathbb{R}^{2m}$ an immersion. Set $\tilde{\varphi} =: \varphi \cdot \pi\colon \tilde{M} \subseteqq \mathbb{R}^{2m}$. Then for $m \neq 1, 3, 7$ $\Delta(\tilde{\varphi}) = \Delta(\varphi)$. In particular $\tilde{\varphi}$ is regularly homotopic to an immersion iff φ is also.*

The following Lemma will be useful.

Lemma 3.2. *Let X be a CW complex such that $H_*(X; \mathbf{Z}/2) \cong H_*(S^{m+1}; \mathbf{Z}/2)$ $m + 1 \neq 2, 4, 8$. Let $i \neq 0 \in H^{m+1}(\Sigma K(\mathbf{Z}/2, m); \mathbb{Z}/2)$. Suppose given maps*

$$S^{2m+1} \xrightarrow{f} X \xrightarrow{g} \Sigma K(\mathbf{Z}/2, m)$$

such that $g^*(i) \neq 0$. Then $[g \cdot f] = 0 \in \pi_{2m+1}(\Sigma K(\mathbf{Z}/2, m))$.

Proof. Consider the diagram of cofibrings

$$\begin{tikzcd} S^{2m+1} \arrow[r, "f"] \arrow[d, equal] & X \arrow[r] \arrow[d, "g"] & Kf \arrow[d, "k"] \\ S^{2m+1} \arrow[r, "h"] & \Sigma K(\mathbf{Z}/2, m) \arrow[r] & Kh \end{tikzcd}$$

where $h := g \cdot f$ and Kf, Kh are the mapping cones of f and h respectively. Assume $[h] \neq 0$. Using naturality and the fact that h is detected by Sq^{m+1} in $H^*(Kh; \mathbf{Z}/2)$ we see that

$$Sq^{m+1}\colon H^{m+1}(Kf; \mathbf{Z}/2) \to H^{2m+2}(Kf; \mathbf{Z}/2)$$

is non-zero. By [1] this is only possible for the excluded values $m + 1 = 2, 4, 8$ whence $[h] = 0$. ■

Proof of Theorem 3.1. Let $\nu \downarrow M$, $\tilde{\nu} = \pi^* \nu \downarrow \tilde{M}$ be the normal bundles of the immersions $\varphi\colon M \subseteqq \mathbb{R}^{2m}$ and $\tilde{\varphi}\colon \tilde{M} \subseteqq \mathbb{R}^{2m}$ respectively. Choose an embedding $\psi\colon M \subset \mathbb{R}^{2m+1}$ whose normal bundle is identified with $\nu \oplus \mathbb{R} \downarrow M$. By general position the composite

$$\tilde{M} \xrightarrow{\pi} M \overset{S_0}{\subset} D(\nu \oplus \mathbb{R} \downarrow M),$$

where S_0 is the 0-section, is regularly homotopic to an embedding

$$\bar{\psi}\colon \tilde{M} \subset D(\nu \oplus \mathbb{R} \downarrow M).$$

The composite embedding

$$\tilde{M} \subset D(\nu \oplus \mathbb{R} \downarrow M) \subset \mathbb{R}^{2m+1}$$

is regularly homotopic to the stabilization $\tilde{M} \overset{\tilde{\varphi}}{\subseteqq} \mathbb{R}^{2m} \subset \mathbb{R}^{2m+1}$. So we obtain inclusions

$$D(\tilde{\nu} \oplus \mathbb{R} \downarrow \tilde{M}) \subset D(\nu \oplus \mathbb{R} \downarrow M) \subset \mathbb{R}^{2m+1}.$$

Applying the Pontrjagin-Thom construction gives a commutative diagram

$$\begin{tikzcd} & T(\tilde{\nu} \oplus \mathbb{R} \downarrow \tilde{M}) \\ S^{2m+1} \arrow[ur, "\tilde{c}"] \arrow[dr, "c"'] & \\ & T(\nu \oplus \mathbb{R} \downarrow M). \arrow[uu, "\pi!"'] \end{tikzcd}$$

There is also the commutative diagram

$$\begin{tikzcd} T(\tilde{\nu} \downarrow \tilde{M}) \arrow[dr, "\tilde{t}"] \arrow[dd, "T\pi"'] & \\ & K(\mathbf{Z}/2, m) \\ T(\nu \downarrow M) \arrow[ur, "t"'] & \end{tikzcd}$$

and combining these two diagrams gives the diagram

$$\begin{tikzcd} & \Sigma T(\tilde{\nu} \downarrow \tilde{M}) \arrow[dr, "\Sigma \tilde{t}"] \arrow[dd, shift left, "\Sigma T\pi"] & \\ S^{2m+1} \arrow[ur, "\tilde{c}"] \arrow[dr, "c"'] & & \Sigma K(\mathbf{Z}/2, m). \\ & \Sigma T(\nu \downarrow M) \arrow[uu, shift left, "\pi!"] \arrow[ur, "\Sigma t"'] & \end{tikzcd}$$

Thus $\Delta(\tilde{\varphi}) = \Sigma \tilde{t} \cdot \tilde{c} = \Sigma t \cdot \Sigma T\pi \cdot \pi ! \cdot c$. Let $\lambda := \Sigma T\pi \cdot \pi ! \cdot c - dc$ where d is the order of the covering, the difference being taken in the suspension structure. Let X be the $2m$ skeleton of $\Sigma T(\nu \downarrow M)$. Since the covering $\tilde{M} \downarrow M$ is of odd order d the composite

$$S^{2m+1} \underset{\lambda}{\rightarrow} T(\nu \downarrow M) \underset{q}{\rightarrow} S^{2m+1}$$

where q is collapse onto the top cell, is null homotopic, so there is a factorization by the Blakers-Massey theorem (see the formulation [4; p. 17] and below)

$$\lambda\colon\ S^{2m+1} \xrightarrow{f} X \subset \Sigma T(\nu \downarrow M).$$

Since M is a $\mathbb{Z}/2$ homology sphere, we have $H_*(X; \mathbf{Z}/2) \cong H_*(S^{m+1}; \mathbf{Z}/2)$. Consider

$$\begin{tikzcd}
 & X \arrow[dr, "g"] \arrow[d, hook] & \\
S^{2m+1} \arrow[ur, "f"] \arrow[r, "\lambda"'] & \Sigma T(\nu \downarrow M) \arrow[r, "\Sigma t"'] & \Sigma K(\mathbf{Z}/2, m)
\end{tikzcd}$$

By the lemma $[g \cdot f] = 0$, so by commutativity $[\Sigma t \cdot \lambda] = 0$. Thus

$$\begin{aligned}
\Delta(\varphi) - \Delta(\varphi) &= \Sigma \tilde{t} \cdot \tilde{c} - \Sigma t \cdot c \\
&= \Sigma t \cdot (\Sigma T\pi \cdot \pi ! \cdot c - c) \\
&= \Sigma t \cdot (\Sigma T\pi \cdot \pi ! \cdot c - dc) \\
&= \Sigma t \cdot \lambda = 0
\end{aligned}$$

because the group is a $\mathbf{Z}/2$. ∎

Footnote. The composite

$$S^{2m+1} \underset{\lambda}{\rightarrow} \Sigma T(\nu \downarrow M) \underset{q}{\rightarrow} S^{2m+1}$$

is null homotopic. Let X be the $2m$ skeleton of $\Sigma T(\nu \downarrow M)$.

Claim. λ factors through X. First of all there is a homotopy equivalence

$$h\colon\ X \vee S^{2m+1} \rightarrow \Sigma T(\nu \downarrow M).$$

To see this note

$$S^{2m+1} \xrightarrow{c} \Sigma T(\nu \downarrow M) \xrightarrow{q} S^{2m+1}$$

has degree 1. Let $i\colon X \subset \Sigma T(\nu \downarrow M)$ be the inclusion. There is the exact sequence

$$\cdots \rightarrow H_i(X) \xrightarrow{i^*} H_i(\Sigma T(\nu \downarrow M) \underset{\curvearrowleft c^*}{\xrightarrow{q^*}} H_i(S^{2m+1}) \rightarrow \cdots$$

that is split by c_*. Then defining

$$h := i \vee c : X \vee S^{2m+1} \rightarrow \Sigma T(\nu \downarrow M)$$

we see h induces a homology isomorphism, so is a homotopy equivalence since everything in sight is simply connected. Thus by Blakers-Massey

$$\pi_i(\Sigma T(\nu \downarrow M)) \cong \pi_i(X) \oplus \pi_i(S^{2m+1})$$

for

$$i < 3m.$$

(X is m connected, S^{2m+1} is $2m$ connected).

Hence $q_*[\lambda] = 0 \Rightarrow \lambda$ factors through X. ■

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