BOUNDARY LINKS

by

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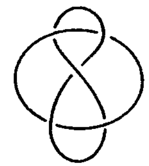
I. In 1936 Eilenberg [1] introduced the concepts of weak linking and n-linking. In summary

a) A polyhedral link L of two components ℓ_1, ℓ_2 in S^3 is said to be 1-linked if for every pair of polyhedra P_1, P_2 such that $\ell_i \sim 0$ in P_i, we have $P_1 \cap P_2 \neq \emptyset$. (0-linked means the components have non-zero linking number. In general, ℓ_1 is n-linked with ℓ_2 if every polyhedron P such that $\ell_1 \sim 0$ in P, contains a cycle which is (n-1)-linked with ℓ_2.) The link is 1-linked but not 0-linked.



b) L is weakly linked (faiblement enlace) if every pair of 1-cycles γ_1 in V_1, γ_2 in V_2, where V_i is a regular neighborhood of ℓ_i, such that $\gamma_i \not\sim 0$ in V_i, are 1-linked.

Using this theory of multicoherence, Eilenberg was able to characterize weakly linked links as: L is weakly linked if and only if there exists no homomorphism of $\pi_1(S^3 - L)$ onto a free group of rank two.

The question was raised, does there exist a link which is 1-linked but not weakly linked?

II. Some time ago, in an unpublished manuscript, R. H. Fox considered a generalization of 1-linking to links of more than two components.

A boundary link $L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_n$ in S^3 is one whose components bound disjoint orientable surfaces.

Clearly, a link of 2 components is a boundary link if and only if it is not 1-linked.

These links have various nice properties; e.g., the elementary ideals $\mathcal{E}_0(t_1 \cdots t_n), \ldots \mathcal{E}_{n-1}(t_1 \cdots t_n)$ are all 0, and $\mathcal{E}_n(t, t, \ldots, t)$ is principal, generated by a knot polynomial. Also the longitudes of L lie in the second commutator subgroup of $\pi_1(S^3 - L)$.

III. Weak linking can also be generalized to links of many components, although the concept becomes somewhat unrecognizable. [3] $L = \ell_1 \cup \ell_2 \cup \dots \cup \ell_n$ in S^3 (with solid torus regular neighborhoods $V_1, V_2, \dots, V_n$) will be called an <u>homology boundary link</u> if

1) There exist disjoint orientable surfaces $S_1 \cdots S_n$ in $S^3 - \cup \operatorname{Int} V_i$ such that $S_i \cap V_j = \dot{S}_i \cap \dot{V}_j$ consists of a number of simple closed curves $\lambda_{ij1}, \lambda_{ij2}, \dots, \lambda_{ijm}$ (with orientation induced from S_i) which are longitudes of V_j.

2) $\dot{S}_i = \cup_{j,k} \lambda_{ijk}$

3) $\Sigma_k \lambda_{ijk} \sim 0$ in V_j if $j \neq i$

4) $\Sigma_k \lambda_{ijk} \sim \ell_i$ in V_i.

(Shrinking V_i down to its core ℓ_i, we see that L is a boundary link except that the surfaces are allowed to have singularities along the components of L.)

Then a link of 2 components is an homology boundary link if and only if it is not weakly linked. This follows from

THEOREM: L is a homology boundary link if and only if there exists a homomorphism $f\colon \pi_1(S^3 - L) \to F(n)$ onto a free group of rank n. Furthermore, L is a boundary link if and only if there exists meridians $\alpha_1, \dots, \alpha_n$ of $\ell_1, \dots, \ell_n$ such that $f(\alpha_1), \dots, f(\alpha_n)$ freely generate $F(n)$.

To construct such a homomorphism given a homology boundary link, first thicken the surfaces S_i to open sets $N_i = \mathring{S}_i \times (0, 1)$; then $S^3 - \cup V_i$ can be mapped onto a wedge of n circles by mapping points outside $\cup N_i$ to the vertex, and a point of $S_i \times \{t\}$ to the point t of the i-th circle. Since $\cup S_i$ cannot separate $S^3 - \cup V_i$, the induced mapping of fundamental groups is the required homomorphism.

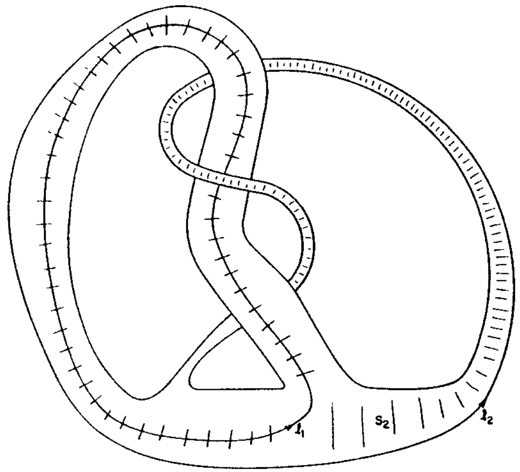
The idea of the proof in the other direction is to construct a retraction of $S^3 - \cup V_i$ onto a wedge of n circles. The surfaces will then appear as the inverse images of non-critical points (non-vertices). In the case of a boundary link, the n-leafed rose must be embedded in $S^3 - L$ as a set of meridians.

This algebraic characterization leads immediately to the fact that a homology boundary link has an Alexander matrix $A(t_1, \dots, t_n)$ with n columns of zeros (if the link is actually a boundary link, these columns correspond to meridians of $\pi_1(S^3 - L)$). Thus again $\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_{n-1} = 0$.

Also, the homomorphism f induces isomorphisms $f_\alpha\colon G/G_\alpha \approx F/F_\alpha$ of the quotient groups of the lower central series, where $G = \pi_1(S^3 - L)$. Since $F_\omega = 1$, we have $f_\omega\colon G/G_\omega \approx F$, and $G_\omega = \ker f$. The longitudes of L lie in G_ω, since they lie on the surfaces. The Milnor isotopy invariants $\bar{\mu}(i_1 \cdots i_k)$ of L are therefore all 0. (In fact much more can be said: it turns out that any link which is isotopic to a homology boundary link under a "nice" (e.g., polyhedral) isotopy, is an homology boundary link.) See [3], [4].

Eilenberg's question becomes: Does there exist a homology boundary link which is not a boundary link?

The answer is yes. The link of the figure is not a boundary link (it is easy to show that one of its longitudes does not lie in G"). But



it is a homology boundary link (either by constructing the surfaces, or constructing a map of G onto a free group of rank 2—but of course there is no such map under which a pair of meridians are free generators).

The surface S_2 is shown in the diagram, to be thought of with two boundaries lying parallel to ℓ_1 on a regular neighborhood V_1 of ℓ_1. The surface S_1 is to have 3 boundaries, all longitudes of V_1, one having opposite orientation to the others (two lie "above" S_2, and one "below").

<u>Questions</u>:

(1) (Fox) Suppose the longitudes of L lie in G". Is L a boundary link?
(2) Similarly, suppose $A(t_1 \cdots t_n)$ has n columns of zeros. Is L a homology boundary link (or a boundary link, if the columns correspond to meridians)?
(3) Is every link isotopic to a boundary link also a boundary link (under a "nice" isotopy)?
(4) Is there a corresponding theory for links in higher dimensions?
(5) Generalize n-linking to arbitrary numbers of components and characterize algebraically.
(6) There are a number of questions related to n-linking—see Fox [2].
(7) (Milnor) What does G_ω look like? Can invariants (say isotopy invariants) of G_ω be found to distinguish between homology boundary links?

REFERENCES

[1] S. Eilenberg, "Multicoherence I, II," Fundamenta Math., 27 (1936) p. 153; 29 (1937) p. 107.

[2] R. H. Fox, Some Problems in Knot Theory, Topology of 3-Manifolds, Prentice-Hall, 1962.

[3] N. Smythe, Isotopy Invariants of Links, Ch. IV, Princeton Ph.D. Thesis (1965).

[4] J. Milnor, "Isotopy of Links, Algebraic Geometry and Topology" (Lefschetz Symposium) Princeton University Press, Princeton, 1957.

SURFACES IN E^3*

by

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We present a brief summary, with references, of developments during the last fifteen years that are related to tame embeddings of 2-manifolds in 3-manifolds. This is the basis of a twenty minute talk presented in the Special Session on Recent Developments in Topology at the meeting of the American Mathematical Society at Iowa City on November 27, 1965. For simplicity in the presentation, we restrict our discussion to 2-spheres in E^3, but most of the theorems can be extended to 2-manifolds in a 3-manifold.

We have included an extensive list of references on this topic, but we presume that there are some serious omissions and some lack of current information in this list. We find it necessary to place some restriction on the number of papers listed here, and we suggest that other work essential to this development can be found in the references cited in the papers that are included in this list. For example, see Harrold's expository paper [37].

1. Fundamental work related to tame imbeddings of 2-spheres in E^3. A 2-sphere S is defined to be tame in E^3 if there is a homeomorphism h of E^3 onto itself that carries S onto the surface of a tetrahedron. Alexander [1] showed in 1924 that every polyhedral 2-sphere in E^3 is tame under this definition. Graeub [30] and Moise [41, II], working independently showed in 1950 that every polyhedral sphere in E^3 can be carried onto the surface of a tetrahedron with a piecewise linear homeomorphism of E^3 onto itself.

A fundamental characterization of tame spheres in E^3 was given by Bing in 1959 with the following theorem. (In the various theorems on characterizations of tame surfaces in E^3, we do not include the obvious statement that every tame sphere satisfies the requirements of the characterization.)

1.1 A 2-sphere is tame if it can be homeomorphically approximated in each of its complementary domains. (Bing [10].)

The proof of this theorem depended upon Bing's Approximation Theorem for Spheres [8], which was later generalized to the following Side Approximation Theorem.

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* This paper was not presented at the seminar, but it is included by invitation since it treats recent developments and questions discussed at the seminar. This work was supported by NSF Grant GP-3882.