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Contents

Preface

Were I to take an iron gun,
And fire it off towards the sun;
I grant 'twould reach its mark at last,
But not till many years had passed.

But should that bullet change its force,
And to the planets take its course,
'Twould never *reach the* nearest *star,*
Because it is so very *far.*

from FACTS by Lewis Carroll [55]

Let me begin by describing the two purposes which prompted me to write this monograph. This is a book about algebraic topology and more especially about homotopy theory. Since the inception of algebraic topology [217] the study of homotopy classes of continuous maps between spheres has enjoyed a very exceptional, central role. As is well known, for homotopy classes of maps $f : S^n \longrightarrow S^n$ with $n \geq 1$ the sole homotopy invariant is the degree, which characterises the homotopy class completely. The search for a continuous map between spheres of different dimensions and not homotopic to the constant map had to wait for its resolution until the remarkable paper of Heinz Hopf [111]. In retrospect, finding an example was rather easy because there is a canonical quotient map from S^3 to the orbit space of the free circle action $S^3/S^1 = \mathbb{CP}^1 = S^2$. On the other hand, the problem of showing that this map is not homotopic to the constant map requires either ingenuity (in this case Hopf's observation that the inverse images of any two distinct points on S^2 are linked circles) or, more influentially, an invariant which does the job (in this case the Hopf invariant). The Hopf invariant is an integer which is associated to any continuous map of the form $f : S^{2n-1} \longrightarrow S^n$ for $n \geq 1$. Hopf showed that when n is even, there exists a continuous map whose Hopf invariant is equal to any even integer. On the other hand the homotopy classes of continuous maps $g : S^m \longrightarrow S^n$ in almost all cases with $m > n \geq 1$ form a finite abelian group. For the study of the 2-Sylow subgroup of these groups the appropriate invariant is the Hopf invariant modulo 2. With the construction of mod p cohomology operations by Norman Steenrod it became possible to define the mod 2 Hopf invariant for any g but the only possibilities for non-zero mod 2 Hopf invariants occur when $m - n + 1$ is a power of two ([259] p. 12).

As described in Chapter 1, § 1, when $n \gg 0$ the homotopy classes of g's form the stable homotopy group $\pi_{m-n}(\Sigma^\infty S^0)$, which is a finite group when $m > n$. The p-Sylow subgroups of stable homotopy groups were first organised systematically by the mod p Adams spectral sequence, constructed by Frank Adams in [1]. Historically, the case when $p = 2$ predominates. As one sees from Chapter 1, Theorem 1.1.2, on the line $s = 1$, four elements exist denoted by h_0, h_1, h_2, h_3 in the 2-Sylow subgroups of $\pi_j(\Sigma^\infty S^0)$ when $j = 0, 1, 3$ and 7, respectively. In positive dimensions the homotopy classes with non-zero mod 2 Hopf invariant would all be represented on the $s = 1$ line in dimensions of the form $j = 2^k - 1$, by Steenrod's result ([259] p. 12). However, a famous result due originally to Frank Adams ([2]; see also [11], [248] and Chapter 6, Theorem 6.3.2) shows that only h_1, h_2, h_3 actually correspond to homotopy classes with non-zero mod 2 Hopf invariant.

From this historical account one sees that the resolution of the behaviour on the $s = 2$ line of the mod 2 Adams spectral sequence qualifies as a contender to be considered the most important unsolved problem in 2-adic stable homotopy theory. This basic unsolved problem has a history extending back over fifty years. Inspection of the segment of the spectral sequence which is given in Chapter 1, Theorem 1.1.2 correctly gives the impression that this problem concerns whether or not the classes labelled h_i^2 represent elements of $\pi_{2^{i+1}-2}(\Sigma^\infty S^0)$. The invariant which is capable of detecting homotopy classes represented on the $s = 2$ line is due to Michel Kervaire [138] as generalised by Ed Brown Jr. [50]. Bill Browder discovered the fundamental result [47], the analogue of Steenrod's result about the Hopf invariant, that the Arf-Kervaire invariant could only detect stable homotopy classes in dimensions of the form $2^{i+1} - 2$.

Stable homotopy classes with Hopf invariant one (mod 2) only exist for dimensions $1, 3, 7$ and currently (see Chapter 1, Sections One and Eight) stable homotopy classes with Arf-Kervaire invariant one (mod 2) have only been constructed in dimensions $2, 6, 14, 30$ and 62 (see [247] and [145] – I believe that [183] has a gap in its construction). Accordingly the following conjecture seems reasonable:

Conjecture. Stable homotopy classes with Arf-Kervaire invariant one (mod 2) exist only in dimensions 2, 6, 14, 30 and 62.

This brings me to my first purpose. As ideas for progress on a particular mathematics problem atrophy and mathematicians at the trend-setting institutions cease to direct their students to study the problem, then it can disappear. Accordingly I wrote this book in order to stem the tide of oblivion in the case of the problem of the existence of framed manifolds of Arf-Kervaire invariant one. During the 1970's I had heard of the problem – during conversations with Ib Madsen, John Jones and Elmer Rees in Oxford pubs and lectures at an American Mathematical Society Symposium on Algebraic Topology at Stanford University in 1976. However, the problem really came alive for me during February of 1980. On sabbatical from the University of Western Ontario, I was visiting Princeton University and sharing a house with Ib Madsen and Marcel Böksted. I went to the airport with Wu Chung Hsiang one Sunday to collect Ib from his plane. During dinner somewhere in New York's Chinatown Ib explained to Wu Chung his current work with Marcel, which consisted of an attempt to construct new framed manifolds of Arf-Kervaire invariant one. I did not understand the sketch given that evening but it was very inspirational. Fortunately the house we were sharing had a small blackboard in its kitchen and, by virtue of being an inquisitive pest day after day, I received a fascinating crash course on the Arf-Kervaire invariant one problem. A couple of months later, in April 1980, in that same sabbatical my family and I were enjoying a month's visit to Aarhus Universitet, during the first two days of which I learnt a lot more about the problem from Jorgen Tornehave and [247] was written. For a brief period overnight during the writing of [247] we were convinced that we had the method to make all the sought-after framed manifolds – a feeling which must have been shared by lots of topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem, despite having moved away from algebraic topology as a research area.

In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This goes some way to explain why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll (aka Charles Lutwidge Dodgson [55]).

My second purpose is to introduce a new technique, which I have christened upper triangular technology, into 2-adic classical homotopy theory. The method derives its name from the material of Chapter 3 and Chapter 5, which gives a precise meaning to the Adams operation ψ^3 as an upper triangular matrix. Briefly this is a new and easy to use method to calculate the effect of the unit maps

$$\pi_*(bo \wedge X) \longrightarrow \pi_*(bu \wedge bo \wedge X)$$

and

$$\pi_*(bu \wedge X) \longrightarrow \pi_*(bu \wedge bu \wedge X)$$

induced by the unit $\eta : S^0 \longrightarrow bu$ from the map on $bo_*(X)$ (respectively, $bu_*(X)$) given by the Adams operation ψ^3. Here bu and bo denote the 2-adic, connective complex and real K-theory spectra (see Chapter 1, § 1.3.2(v)).

There is a second point of view concerning upper triangular technology. Upper triangular technology is the successor to the famous paper of Michael Atiyah concerning operations in periodic unitary K-theory [25]. The main results of [25] are (i) results about the behaviour of operations in KU-theory with respect to the filtration which comes from the Atiyah-Hirzebruch spectral sequence and (ii) results which relate Adams operations in KU-theory to the Steenrod operations of

[259] in mod p singular cohomology in the case of spaces whose integral cohomology is torsion free. As explained in the introduction to Chapter 8, Atiyah's result had a number of important applications. Generalising Atiyah's results to spaces with torsion in their integral cohomology has remained unsolved since the appearance of [25]. At least when X is the mapping cone of $\Theta_{2n} : \Sigma^\infty S^{2n} \longrightarrow \Sigma^\infty \mathbb{RP}^{2n}$, which is a spectrum with lots of 2-ary torsion in its K-theory, I shall apply the upper triangular technology of Chapters 3 and 5 to offer a solution to this problem, although the solution will look at first sight very different from [25]. As explained in Chapter 1, § 8 the existence of framed manifolds of Arf-Kervaire invariant one may be equivalently rephrased in terms of the behaviour of the mod 2 Steenrod operations on the mapping cone of Θ_{2n}. By way of application, in Chapter 8, the upper triangular technology is used to give a new, very simple (here I use the word "simple" in the academic sense of meaning "rather complicated" and I use the phrase "rather complicated" in the non-academic sense of meaning "this being mathematics, it could have been a lot worse"!) proof of a conjecture of Barratt-Jones-Mahowald, which rephrases K-theoretically the existence of framed manifolds of Arf-Kervaire invariant one.

I imagine that the upper triangular technology will be developed for p-adic connective K-theory when p is an odd prime. This will open the way for the applications to algebraic K-theory, which I describe in Chapter 3, and to motivic cohomology, which I mention in Chapter 9. The connection here is the result of Andrei Suslin ([264], [265]) which identifies the p-local algebraic K-theory spectrum of an algebraically closed field containing $1/p$ as that of p-local bu-theory.

The contents of the book are arranged in the following manner.

Chapter 1 contains a sketch of the algebraic topology background necessary for reading the rest of the book. Most likely many readers will choose to skip this, with the exception perhaps of § 8. § 1 covers some history of computations of the stable homotopy groups of spheres and § 2 describes how the Pontrjagin-Thom construction rephrases these stable homotopy groups in terms of framed manifolds. § 3 describes the classical stable homotopy category and gives several examples of spectra which will be needed later. § 4 introduces the classical mod 2 Adams spectral sequence which ever since its construction in [1] has remained (along with its derivatives) the central computational tool of stable homotopy theory. § 5 introduces the Snaith splitting and the Kahn-Priddy Theorem. The latter is needed to rephrase the Arf-Kervaire invariant one problem in terms of $\Theta_{2n} : \Sigma^\infty S^{2n} \longrightarrow \Sigma^\infty \mathbb{RP}^{2n}$ and the former is needed both to prove the latter and to establish in Chapter 3 the first episode of the upper triangular saga. This section introduces the space $QX = \Omega^\infty \Sigma^\infty X$ as well as finitely iterated loopspaces $\Omega^r \Sigma^r X$. § 6 recapitulates the properties of the mod 2 Steenrod algebra, which is needed in connection with the Adams spectral sequences we shall use and to prove the results of Chapter 2. § 7 introduces the Dyer-Lashof algebra, which is also needed to prove the results of Chapter 2. § 7 describes the Arf-Kervaire invariant one problem and its equivalent reformulations which are either used or established in the course of this book – in particular, the upper triangular technology is used in Chapter 8 to prove the reformulation which is given in Theorem 1.8.10.

Chapter 2 considers the adjoint $\mathrm{adj}(\Theta_{2n}) : S^{2n} \longrightarrow Q\mathbb{RP}^{2n}$ and derives a number of formulae for the Arf-Kervaire invariant in this formulation. In particular, the equivalent formulation in terms of a framed manifold M^{2n} together with an orthogonal vector bundle gives rise to a formula for the Arf-Kervaire invariant (Theorems 2.2.2 and 2.2.3) which leads to a very easy construction in § 3 of a framed manifold of Arf-Kervaire invariant one in dimension 30 (and, of course, in dimensions $2, 6$ and 14). The material of Chapter 2 is taken from my joint paper with Jorgen Tornehave [247] which I mentioned above.

Chapter 3 introduces the upper triangular technology. § 1 describes left bu-module splittings of the 2-local spectra $bu \wedge bu$ and $bu \wedge bo$ which were originally discovered by Mark Mahowald – although I learnt about their proof from Don Anderson, via his approach, just prior to a lecture at the Vancouver ICM in 1974 in which Jim Milgram talked about their proof and properties via his approach! My account tends to follow the account given by Frank Adams in [9] except that I make use of the multiplicative properties of the Snaith splitting of $\Omega^2 S^3$. In § 2, Mahowald's splittings are used to show that the group of homotopy classes of left-bu-module automorphisms of the 2-local spectrum $bu \wedge bo$ which induce the identity on mod 2 homology is isomorphic to the group of upper triangular matrices with entries in the 2-adic integers. In § 3 I explain how this result may be applied to the construction of operations in algebraic K-theory and Chow groups (or even Spencer Bloch's higher Chow groups – see Chapter 9). The material of Chapter 3 is taken from [252].

Chapter 4 is included in order to amplify the comments in Chapter 3, § 3 concerning the connections between bu and algebraic K-theory. It sketches the ingenious proof of [265]. Suslin's proof is very sophisticated in its prerequisites and this chapter arose originally as a 10 lecture course which presented the result to an audience with few of the prerequisites. Accordingly, § 1 sketches simplicial sets and their realisations which are then applied in § 2 to construct Quillen's K-theory space and to introduce K-theory mod m. Sections Three, Four and Five introduce affine schemes, Henselian rings, Henselian pairs and their mod m K-theory. § 6 defines group cohomology and sketches Andrei Suslin's universal homotopy construction which is the backbone of the proof. § 7 sketches John Milnor's simplicial construction of the fibre of the map from the classifying space of a discrete Lie group to the classifying space of the Lie group with its classical topology and follows [265] in using it to compute the mod m algebraic K-theory of an Archimedean field. § 8 sketches the analogous result for commutative Banach algebras and § 9 outlines the alternative approach via excision in the case of the C^*-algebra $C(X)$.

Chapter 5 is the second instalment of upper triangular technology. It is taken from the paper by Jon Barker and me [27]. It evaluates the conjugacy class represented by the map $1 \wedge \psi^3 : bu \wedge bo \longrightarrow bu \wedge bo$ in the group of upper triangular matrices with coefficients in the 2-adic integers. The crux of the chapter is that it suffices to compute the effect of $1 \wedge \psi^3$ on homotopy modulo torsion with respect to a 2-adic basis coming from Mahowald's splitting of Chapter 3. There is a homotopy basis given in [58] with respect to which the matrix of $1 \wedge \psi^3$ is easy to compute – a fact which I learnt from Francis Clarke. Sections Two and Three carefully compare these two bases sufficiently to calculate the diagonal and the super-diagonal of the matrix. § 4 shows (by two methods) that any two matrices in the infinite upper triangular matrix with 2-adic coefficients with this diagonal and super-diagonal are conjugate. § 5 shows by an application, which goes back to [183], the manner in which the upper triangular technology can sometimes reduce homotopy theory to matrix algebra.

Chapter 6 contains various cohomological results related to real projective space and maps involving them. § 1 calculates the MU-theory and KU-theory of $\mathbb{RP}^n$ together with the effect of the Adams operations in these groups. It also calculates the effect of some of the Landweber-Novikov operations on $MU_*(\mathbb{RP}^n)$. § 2 contains a slightly different method for making these calculations, applied instead to $BP_*(\mathbb{RP}^n)$ in preparation for the applications of Chapter 7. § 3 applies these calculations to the study of the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$. The vanishing of the Whitehead product is equivalent to the classical Hopf invariant one problem and Theorem 6.3.2 is equivalent to the (then new) KU_* proof of the non-existence of classes of Hopf invariant one which I published in the midst of the book review [248]. Theorem 6.3.4 gives equivalent conditions, related to the Arf-Kervaire invariant one problem, for $[\iota_n, \iota_n]$ to be divisible by two. This result was proved in one direction in [30]. The converse is proved as a consequence of the main upper triangular technology result of Chapter 8 (Theorem 8.1.2), which is equivalent to the Arf-Kervaire invariant one reformulation of Chapter 1 § 1.8.9 and Theorem 1.8.10 originally conjectured in [30] (see Chapter 7, Theorem 7.2.2). § 4 contains results which relate e-invariants and Hopf invariants – first considered in Corollary D of [30]. In fact, Theorem 6.4.2 and Corollary 6.4.3 imply both Corollary D and its conjectured converse. § 5 contains a miscellany of results which relate the halving of the Whitehead product to MU_*-e-invariants. The material of Chapter 6 is based upon an unpublished 1984 manuscript concerning the MU-theory formulations of the results [30], enhanced by use of Chapter 8, Theorem 8.1.2 in several crucial places.

Chapter 7 applies BP-theory and its related J-theories to analyse the Arf-Kervaire invariant one problem. § 1 contains formulae for the J-theory Hurewicz image of $\Theta_{2^{n+1}-2} : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{2^{n+1}-2}$. In § 2 the Hurewicz images in J-theory and ju-theory are related and in Theorem 7.2.2 the conjecture of [30] is proved in its original ju-theory formulation. The material for Chapter 7 is taken from [251]. However, the main result (Theorem 7.2.2) is furnished with three proofs – an outline of the (straightforward but very technical) proof given in [251] and two much simpler proofs which rely on the use of Chapter 8, Theorem 8.1.2.

Chapter 8 is the third and final instalment of upper triangular technology. § 1 describes some historical background and how the method is used to prove the main technical results (Theorem 8.4.6, Theorem 8.4.7 and Corollary 8.4.8). Then the all important relation is proved, relating the bu-e-invariant of $\Theta_{8m-2} : \Sigma^\infty S^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-2}$ to the mod 2 Steenrod operations on the mapping cone of Θ_{8m-2}. § 2 computes the connective K-theory groups which are needed in the subsequent upper triangular calculations and the effect of the maps which correspond to the super-diagonal entries in the upper triangular matrix corresponding to the Adams operation ψ^3. § 3 describes the three types of mapping cone to which the method is applied in § 4. In addition to the mapping cone of Θ_{8m-2}, it is also necessary to apply the method to $\mathbb{RP}^{8m-1}$ – the mapping cone of the canonical map from S^{8m-2} to $\mathbb{RP}^{8m-2}$ – in order to estimate the 2-divisibility of some of the terms which appear in the upper triangular equations for the mapping cone of Θ_{8m-2}. In addition we apply the method to the mapping cone of the composition of Θ_{8m-2} with some maps of the form $\Sigma^\infty \mathbb{RP}^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-8r2}$ constructed by Hirosi Toda [276]. § 4 contains the final details of all these applications and the proofs of the main results – Theorem 8.4.6, Theorem 8.4.7 and Corollary 8.4.8.

The first eight chapters have been dedicated to classical stable homotopy theory in the form of the Arf-Kervaire invariant one problem. Therefore, to leaven the lump, Chapter 9 contains a brief overview of some current themes in stable homotopy theory. Much of this chapter is concerned with the $\mathbb{A}^1$-stable homotopy category of Fabien Morel and Vladimir Voevodsky ([281], [203]; see also [180]) because of its spectacular applications, its close relation to algebro-geometric themes in algebraic K-theory (e.g., higher Chow groups) and because several "classical results" with which I am very familiar seem to be given a new lease of life in the $\mathbb{A}^1$-stable homotopy category (see Chapter 9 § 9.2.15). For example, I am particularly interested in the potential of the upper triangular technology to establish relations between Adams operations in algebraic K-theory and Steenrod operations in motivic cohomology (see Chapter 9 § 9.2.15(vi)). The chapter closes with a very short scramble through some other interesting and fashionable aspects of stable homotopy theory currently under development and with some "late-breaking news" on recent Arf-Kervaire invariant activity, which was brought to my attention simultaneously on June 9, 2008 by Peter Landweber and Hadi Zare, a student of Peter Eccles at the University of Manchester and appears in the preprints ([16], [17], [18]).

I am very grateful for help and advice to Huajian Yang, my postdoc at McMaster University during 1996–98. It was during that period, when we were discussing the Arf-Kervaire invariant problem extensively, that Huajian convinced me of the utility of the Adams spectral sequence in connective K-theory, which led in particular to the results of Chapter 3, proved in 1998. In addition, I am very grateful to Jonathan Barker, my PhD student at the University of Southampton in 2003–2006, who took an interest in the "ψ^3 as an upper triangular matrix" project and worked hard in collaboration with me to find the matrix of Chapter 5, Theorem 5.1.2. I am indebted to Francis Clarke who, as a result of a conversation in a queue for coffee at a Glasgow conference in 2001, provided the all-important

guess as to the identity of the matrix we were looking for. In connection with Chapter 9, I am very grateful to Mike Hopkins for a very useful discussion, at the Fields Institute in May 2007, concerning alternative proofs of my classical result (Chapter 1, Theorem 1.3.3) which pointed David Gepner and me to the one which, suitably adapted (see [87]), yields the analogous results in the $\mathbb{A}^1$-stable homotopy category described in Chapter 9, § 9.2.15.

Finally, I am greatly indebted to my wife, Carolyn, for her patient forbearance during the writing of this book and for her invaluable, accurate proof-reading.

Victor Snaith
University of Sheffield
2008

Chapter 1

Algebraic Topology Background

They hunted till darkness came on, but they found
Not a button, or feather, or mark,
By which they could tell that they stood on the ground
Where the Baker had met with the Snark.

In the midst of the word he was trying to say,
In the midst of his laughter and glee,
He had softly and suddenly vanished away –
For the Snark was *a Boojum, you see.*

from "The Hunting of the Snark"
by Lewis Carroll [55]

The objective of this chapter is to sketch the historical and technical stable homotopy background which we shall need in the course of this book. § 1 deals with the history of the calculations of stable homotopy groups of spheres (the so-called "stable stems"). § 2 describes the framed manifold approach of Pontrjagin and Thom. § 3 introduces the classical stable homotopy category of spectra and § 4 describes the category's classical Adams spectral sequence. § 5 introduces the Snaith splittings and derives the Kahn-Priddy theorem, which is essential in order to be able to study the stable homotopy groups of spheres via the stable homotopy groups of $\mathbb{RP}^{\infty}$. § 6 recapitulates the properties of Steenrod's cohomology operations and § 7 does the same for the Dyer-Lashof algebra of homology operations. Finally § 8 describes several equivalent formulations of the Arf-Kervaire invariant one problem.

1.1 Stable homotopy groups of spheres

1.1.1. The history. The calculation of the stable homotopy groups of spheres is one of the most central and intractable problems in algebraic topology. In the 1950's Serre used his spectral sequence to study the problem [240]. In 1962 Toda used his triple products and the EHP sequence to calculate the first nineteen stems (that is, $\pi_j^S(S^0)$ for $0 \leq j \leq 19$) [275]. These methods were later extended by Mimura, Mori, Oda and Toda to compute the first thirty stems ([195], [196], [197], [212]). In the late 1950's the study of the classical Adams spectral sequence began [1]. According to ([145] Chapter 1), computations in this spectral sequence were still being pursued into the 1990's using the May spectral sequence and the lambda algebra. The best published results are May's thesis ([176], [177]) and the computation of the first forty-five stems by Barratt, Mahowald and Tangora ([29], [172]), as corrected by Bruner [53]. The use of the Adams spectral sequence based on Brown-Peterson cohomology theory (BP theory for short) was initiated by Novikov [211] and Zahler [297]. The BP spectral sequences were most successful at odd primes [186]. A comprehensive survey of these computations and the methods which have been used is to be found in Doug Ravenel's book [228].

We shall be interested in two-primary phenomena in the stable homotopy of spheres, generally acknowledged to be the most intractable case. What can be said at the prime $p = 2$? A seemingly eccentric, cart-before-the-horse method for computing stable stems was developed in 1970 by Joel Cohen [64]. Recall that for a generalised homology theory E_* and a stable homotopy spectrum X there is an Atiyah-Hirzebruch spectral sequence [72]

$$E^2_{p,q} = H_p(X; \pi_q(E)) \Longrightarrow E_{p+q}(X).$$

Cohen studied this spectral sequence with X an Eilenberg-Maclane spectrum and E equal to stable homotopy or stable homotopy modulo n. The idea was that in this case the spectral sequence is converging to zero in positive degrees and, since the homology of the Eilenberg-Maclane spectra are known, one can inductively deduce the stable stems. This strategy is analogous to the inductive computation of the cohomology of Eilenberg-Maclane spaces by means of the Serre spectral sequence [56]. Cohen was only able to compute a few low-dimensional stems before the method became too complicated. Therefore the method was discarded in favour of new methods which used the Adams spectral sequence. In 1972 Nigel Ray used the Cohen method with $X = MSU$ and $E = MSp$, taking advantage of knowledge of $H_*(MSU)$ and $MSp_*(MSU)$ to compute $\pi_j(MSp)$ for $0 \leq j \leq 19$ [229]. Again this method was discarded because David Segal had computed up to dimension thirty-one via the Adams spectral sequence and in [149] these computations were pushed to dimension one hundred.

In 1978 Stan Kochman and I studied the Atiyah-Hirzebruch spectral sequence method in the case where $X = BSp$ and E_* is stable homotopy. An extra ingredient was added in this paper; namely we exploited the Landweber-Novikov operations to study differentials. This improvement is shared by the case when $X = BP$ and E_* is stable homotopy, capitalising on the spareness of $H_*(BP)$ and using Quillen operations to compute the differentials. The result of this method is a computer-assisted calculation of the stable stems up to dimension sixty-four with particular emphasis on the two-primary part [145].

From ([228] Theorem 3.2.11) we have the following result.

Theorem 1.1.2. For the range $t - s \leq 13$ and $s \leq 7$ the group $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$ is generated as an $\mathbb{F}_2$-vector-space by the elements listed in the following table. Only the non-zero groups and elements are tabulated, the vertical coordinate is s and the horizontal is $t - s$.

$s \backslash t-s$	0	1	2	3	6	7	8	9	10	11
7	h_0^7									$P(h_1^2)$
6	h_0^6								$P(h_1^2)$	$P(h_0h_2)$
5	h_0^5							$P(h_1)$		$P(h_2)$
4	h_0^4					$h_0^3h_3$		h_1c_0		
3	h_0^3			h_1^3		$h_0^2h_3$	c_0	$h_1^2h_3$		
2	h_0^2		h_1^2	h_0h_2	h_2^2	h_0h_3	h_1h_3			
1	h_0	h_1		h_2		h_3				
0	1									

There are no generators for $t - s = 12, 13$ and the only generators in this range with $s > 7$ are powers of h_0.

Inspecting this table one sees that there can be no differentials in this range and we obtain the following table of values for the 2-Sylow subgroups of the stable stems $\pi_n^S(S^0) \otimes \mathbb{Z}_2$.

Corollary 1.1.3. For $n \leq 13$ the non-zero groups $\pi_n^S(S^0) \otimes \mathbb{Z}_2$ are given by the following table.

n	0	1	2	3	6	7	8	9	10	11
$\pi_n(S^0) \otimes \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$

For more recent computational details the reader is referred to [40], [41], [143], [144], [145], [146], [147], [148], [149], [150] and [151]. For example, according to [145], the 2-Sylow subgroup of $\pi_{62}^S(S^0)$ is $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$ with an element of Arf-Kervaire invariant one denoted by $A[62, 1]$ having order two.

1.2 Framed manifolds and stable homotopy groups

Definition 1.2.1. Let M^n be a compact C^∞ manifold without boundary and let $i : M^n \longrightarrow \mathbb{R}^{n+r}$ be an embedding. The normal bundle of i, denoted by $\nu(M,i)$, is the quotient of the pullback of the tangent bundle of $\mathbb{R}^{n+r}$ by the sub-bundle given by the tangent bundle of M

$$\nu(M,i) = \frac{i^*\tau(\mathbb{R}^{n+r})}{\tau(M)}$$

so that $\nu(M,i)$ is an r-dimensional real vector bundle over M^n.

If we give $\tau(\mathbb{R}^{n+r}) = \mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ the Riemannian metric obtained from the usual inner product in Euclidean space, the total space of the normal bundle $\nu(M,i)$ may be identified with the orthogonal complement of $\tau(M)$ in $i^*\tau(\mathbb{R}^{n+r})$. That is, the fibre at $z \in M$ may be identified with the subspace of vectors $(z,x) \in \mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ such that x is orthogonal to $i_*\tau(M)_z$ where i_* is the induced embedding of $\tau(M)$ into $\tau(\mathbb{R}^{n+r})$.

Lemma 1.2.2. If r is sufficiently large (depending only on n) and $i_1, i_2 : M^n \longrightarrow \mathbb{R}^{n+r}$ are two embeddings then $\nu(M,i_1)$ is trivial (i.e., $\nu(M,i_1) \cong M \times \mathbb{R}^r$) if and only if $\nu(M,i_2)$ is trivial.

Proof. For sufficiently large r, any two embeddings i_1 and i_2 are regularly homotopic and any two regular homotopies are homotopic through regular homotopies leaving the end-points fixed [110]. Here a regular homotopy means $H : M \times I \longrightarrow \mathbb{R}^{n+r}$ such that, for each $0 \leq t \leq 1$, the map $z \mapsto H(z,t)$ is an immersion and such that the differentials $H(-,t)_* : \tau(M) \longrightarrow \tau(\mathbb{R}^{n+r})$ define a homotopy. Therefore a regular homotopy from i_1 to i_2 defines a homotopy of vector bundles over M from $\nu(M,i_1)$ to $\nu(M,i_2)$. Furthermore any two homotopies defined in this manner are themselves homotopic relative to the end-points.

By homotopy invariance of vector bundles there is a bundle isomorphism $\nu(M,i_1) \cong \nu(M,i_2)$. $\square$

Definition 1.2.3. Let ξ be a vector bundle over a compact manifold M endowed with a Riemannian metric on the fibres. Then the Thom space is defined to be the quotient of the unit disc bundle $D(\xi)$ of ξ with the unit sphere bundle $S(\xi)$ collapsed to a point. Hence

$$T(\xi) = \frac{D(\xi)}{S(\xi)}$$

is a compact topological space with a basepoint given by the image of $S(\xi)$.

If M admits an embedding with a trivial normal bundle, as in Lemma 1.2.2, we say that M has a stably trivial normal bundle. Write M_+ for the disjoint union of M and a disjoint basepoint. Then there is a canonical homeomorphism

$$T(M \times \mathbb{R}^r) \cong \Sigma^r(M_+)$$

between the Thom space of the trivial r-dimensional vector bundle and the r-fold suspension of M_+, $(S^r \times (M_+))/(S^r \vee (M_+)) = S^r \wedge (M_+)$.

1.2.4. The Pontrjagin-Thom construction

Suppose that M^n is a manifold as in Definition 1.2.1 together with a choice of trivialisation of normal bundle $\nu(M, i)$. This choice gives a choice of homeomorphism

$$T(\nu(M, i)) \cong \Sigma^r(M_+).$$

Such a homeomorphism is called a framing of (M, i). Now consider the embedding $i : M^n \longrightarrow \mathbb{R}^{n+r}$ and identify the $n+r$-dimension sphere S^{n+r} with the one-point compactification $\mathbb{R}^{n+r} \bigcup\{\infty\}$. The Pontrjagin-Thom construction is the map

$$S^{n+r} \longrightarrow T(\nu(M, i))$$

given by collapsing the complement of the interior of the unit disc bundle $D(\nu(M, i))$ to the point corresponding to $S(\nu(M, i))$ and by mapping each point of $D(\nu(M, i))$ to itself.

Identifying the r-dimensional sphere with the r-fold suspension $\Sigma^r S^0$ of the zero-dimensional sphere (i.e., two points, one the basepoint) the map which collapses M to the non-basepoint yields a basepoint preserving map $\Sigma^r(M_+) \longrightarrow S^r$.

Therefore, starting from a framed manifold M^n, the Pontrjagin-Thom construction yields a based map

$$S^{n+r} \longrightarrow T(\nu(M, i)) \cong \Sigma^r(M_+) \longrightarrow S^r,$$

whose homotopy class defines an element of $\pi_{n+r}(S^r)$.

1.2.5. Framed cobordism

Two n-dimensional manifolds without boundary are called cobordant if their disjoint union is the boundary of some $n+1$-dimensional manifold. The first description of this equivalence relation was due to Poincaré ([217], see § 5 Homologies). This paper of Poincaré is universally considered to be the origin of algebraic topology. His concept of homology is basically the same as that of cobordism as later developed by Pontrjagin ([218], [219]), Thom [273] and others [260].

The notion of cobordism can be applied to manifolds with a specific additional structure on their stable normal bundle ($\nu(M^n, i)$ for an embedding of codimension $r \gg n$). In particular one can define the equivalence relation of framed cobordism between n-dimensional manifolds with chosen framings of their stable normal bundle. This yields a graded ring of framed cobordism classes Ω_*^{fr} where the elements in Ω_n^{fr} are equivalence classes of compact framed n-manifolds without boundary. The sum is induced by disjoint union and the ring multiplication by cartesian product of manifolds.

More generally one may extend the framed cobordism relation to maps of the form $f : M^n \longrightarrow X$ where M^n is a compact framed manifold without boundary and X is a fixed topological space. From such a map the Pontrjagin-Thom construction yields

$$S^{n+r} \longrightarrow T(\nu(M,i)) \cong \Sigma^r(M_+) \longrightarrow \Sigma^r(X_+)$$

whose homotopy class defines an element of $\pi_{n+r}(\Sigma^r(X_+))$.

The cobordism of maps f yields a graded ring of framed cobordism classes $\Omega_*^{\mathrm{fr}}(X)$. The Pontrjagin-Thom element associated to f does not depend solely on its class in $\Omega_n^{\mathrm{fr}}(X)$ but the image of f in the stable homotopy group

$$\pi_n^S((X_+)) = \lim_{\overrightarrow{r}} \ \pi_{n+r}(\Sigma^r(X_+)),$$

where the limit is taken over the iterated suspension map depends only on the framed cobordism class of f. Incidentally, once one chooses a basepoint in X, in the stable homotopy category of § 1.3.1 there is a stable homotopy equivalence of the form $X_+ \simeq X \vee S^0$ and therefore a non-canonical isomorphism $\pi_*^S(X_+) \cong \pi_*^S(X) \oplus \pi_*^S(S^0)$.

Pontrjagin ([218], [219]) was the first to study stable homotopy groups by means of framed cobordism classes, via the following result:

Theorem 1.2.6. The construction of § 1.2.5 induces an isomorphism of graded rings of the form

$$P : \Omega_*^{\mathrm{fr}} \stackrel{\cong}{\longrightarrow} \pi_*^S(S^0)$$

where the ring multiplication in $\pi_*^S(S^0)$ is given by smash product of maps.

More generally the construction of § 1.2.5 induces an isomorphism of graded groups of the form

$$P : \Omega_*^{\mathrm{fr}}(X) \stackrel{\cong}{\longrightarrow} \pi_*^S(X_+).$$

Sketch of proof. A complete proof may be found in ([260] pp. 18–23). To see that the map is well defined one uses the properties of regular homotopies in a manner similar to the proof of Lemma 1.2.2. To show that the map is surjective one has to recover a map $f : M^n \longrightarrow X$ from a homotopy class of maps of the form

$$S^{n+r} \longrightarrow \Sigma^r(X_+)$$

for some $r \gg n$. This is done by deforming the map to be transverse regular to $X \subset \Sigma^r(X_+)$ and taking M^n to be the inverse image of X. Transverse regularity means that a small tubular neighbourhood of M^n maps to a small neighbourhood of X of the form $X \times \mathbb{R}^r$ via a map which is an isomorphism on each fibre. This property induces a trivialisation of the normal bundle of M^n which gives the framing. □

1.3 The classical stable homotopy category

1.3.1. In this section we shall give a thumb-nail sketch of the stable homotopy category. Since we shall only be concerned with classical problems concerning homotopy groups we shall only need the most basic model of a stable homotopy category.

A stable homotopy category is required to be some sort of additive category into which one can put topological spaces in order to construct general homology and cohomology theories. More recently, spear-headed by Grothendieck, Quillen and Thomason [274], one has asked for a stable homotopy category in which one can perform the complete gamut of homological algebra and more, as in the $\mathbb{A}^1$-stable homotopy category of Morel-Voevodsky ([281], [203]). For brief reviews of the modern literature concerning stable homotopy with such enriched structures see, for example, [96]; see also Chapter 9.

The notion of a spectrum was originally due to Lima [162] and was formalised and published in [292]. A spectrum E is a sequence of base-pointed spaces and basepoint preserving maps (indexed by the positive integers)

$$E: \ \{\epsilon_n : \Sigma E_n \longrightarrow E_{n+1}\}$$

from the suspension of the nth space E_n to the $(n+1)$th space of E. An Ω-spectrum is the same type of data but given in terms of the adjoint maps

$$E: \ \{\mathrm{adj}(\epsilon_n) : E_n \longrightarrow \Omega E_{n+1}\}$$

from the nth space to the based loops on E_{n+1}. One requires that the connectivity of the $\mathrm{adj}(\epsilon_n)$'s increases with n. In many examples the $\mathrm{adj}(\epsilon_n)$'s are homotopy equivalences which are often the identity map.

A function $f : E \longrightarrow F$ of degree r, according to ([9] p. 140) in the stable homotopy category, is a family of based maps $\{f_n : E_n \longrightarrow F_{n-r}\}$ which satisfy the following relations for all n:

$$f_{n+1} \cdot \epsilon_n = \phi_{n-r} \cdot \Sigma f_n : \Sigma E_n \longrightarrow F_{n-r+1}$$

or equivalently

$$\Omega f_{n+1} \cdot \mathrm{adj}(\epsilon_n) = \mathrm{adj}(\phi_{n-r}) \cdot f_n : E_n \longrightarrow \Omega F_{n-r+1}$$

where $F = \{\phi_n : \Sigma F_n \longrightarrow F_{n+1}\}$. A morphism $f : E \longrightarrow F$ of degree r is the stable homotopy class of a function f. We shall not need the precise definition of this equivalence relation on functions; suffice it to say that $[E, F]_r$, the stable homotopy classes of morphisms from E to F of degree r in the stable homotopy category, form an abelian group. Summing over all degrees we obtain a graded abelian group $[E, F]_*$.

Forming the smash product spectrum $E \wedge F$ of two spectra E and F is technically far from straightforward although a serviceable attempt is made in

([9] pp. 158–190). Essentially the smash product $E \wedge F$ has an sth space which is constructed from the smash products of the spaces $E_n \wedge F_{s-n}$. For our purposes we shall need only the most basic notions of ring space and module spectra over ring spectra which are all the obvious translations of the notions of commutative rings and modules over them. However, one must bear in mind that a commutative ring spectrum is a generalisation of a graded commutative ring so that the map which involves switching $E_n \wedge E_m$ to $E_m \wedge E_n$ carries with it the sign $(-1)^{mn}$ (see [9] pp. 158–190)!

Example 1.3.2. (i) The category of topological spaces with base points is included into the stable homotopy category by means of the suspension spectrum $\Sigma^\infty X$. If X is a based space we define

$$(\Sigma^\infty X)_0 = X,\ (\Sigma^\infty X)_1 = \Sigma X,\ (\Sigma^\infty X)_2 = \Sigma^2 X, \ldots$$

with each ϵ_n being the identity map.

(ii) The fundamental suspension spectrum is the sphere spectrum $\Sigma^\infty S^0$ where $S^0 = \{0,1\}$ with 0 as base-point. Since $S^0 \wedge S^0 = S^0$ the suspension spectrum $\Sigma^\infty S^0$ is a commutative ring spectrum. Since $S^0 \wedge X = X$ for any based space every spectrum is canonically a module spectrum over the sphere spectrum.

(iii) If Π is an abelian group then the Eilenberg-Maclane space $K(\Pi, n)$ is characterised up to homotopy equivalence by the property that

$$\pi_i(K(\Pi, n)) \cong \begin{cases} \Pi & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\Omega K(\Pi, n+1) \simeq K(\Pi, n)$ and we may define the singular homology spectrum or Eilenberg-Maclane spectrum $\mathrm{H}\Pi$ by

$$(\mathrm{H}\Pi)_n = K(\Pi, n)$$

for $n \geq 0$. We shall be particularly interested in the case of singular homology modulo 2 $\mathrm{H}\mathbb{Z}/2$ which is also a commutative ring spectrum, because it is easy to construct non-trivial maps of the form $K(\mathbb{Z}/2, m) \wedge K(\mathbb{Z}/2, n) \longrightarrow K(\mathbb{Z}/2, m+n)$ by means of obstruction theory.

(iv) Homotopy classes of maps from a space X into $\mathbb{Z} \times BU$, whose base-point lies in the component $\{0\} \times BU$, classify complex vector bundles on X (see [26] or [116]). The tensor product of the reduced Hopf line bundle on S^2 with the universal bundle gives a map $\epsilon : S^2 \wedge (\mathbb{Z} \times BU) \longrightarrow \mathbb{Z} \times BU$ and the Bott Periodicity Theorem states that ([26] , [116]) the adjoint gives a homotopy equivalence

$$\mathrm{adj}(\epsilon) : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega^2(\mathbb{Z} \times BU).$$

This gives rise to the KU-spectrum or complex periodic K-theory spectrum defined for $n \geq 0$ by

$$\mathrm{KU}_{2n} = \mathbb{Z} \times BU \text{ and } \mathrm{KU}_{2n+1} = \Sigma(\mathbb{Z} \times BU).$$

Homotopy classes of maps from a space X into $\mathbb{Z} \times BO$ classify real vector bundles on X (see [26] or [116]). The real Bott Periodicity Theorem is a homotopy equivalence of the form

$$\epsilon : S^8 \wedge (\mathbb{Z} \times BO) \longrightarrow \mathbb{Z} \times BO$$

which, in a similar manner, yields a spectrum KO whose $8n$th spaces are each equal to $\mathbb{Z} \times BO$.

(v) A connective spectrum E is one which satisfies

$$\pi_r(E) = [\Sigma^\infty S^r, E]_0 = [\Sigma^\infty S^0, E]_r = 0$$

for $r < n_0$ for some integer n_0. Unitary and orthogonal connective K-theories, denoted respectively by bu and bo, are examples of connective spectra with $n_0 = 0$ ([9] Part III § 16).

For $m \geq 0$ let $(\mathbb{Z} \times BU)(2m, \infty)$ denote a space equipped with a map

$$(\mathbb{Z} \times BU)(2m, \infty) \longrightarrow \mathbb{Z} \times BU$$

which induces an isomorphism on homotopy groups

$$\pi_r((\mathbb{Z} \times BU)(2m, \infty)) \xrightarrow{\cong} \pi_r(\mathbb{Z} \times BU)$$

for all $r \geq 2m$ and such that $\pi_r((\mathbb{Z} \times BU)(2m, \infty)) = 0$ for all $r < 2m$. These spaces are constructed using obstruction theory and are unique up to homotopy equivalence. In particular there is a homotopy equivalence

$$(\mathbb{Z} \times BU)(2m, \infty) \simeq \Omega^2(\mathbb{Z} \times BU)(2m + 2, \infty)$$

which yields a spectrum bu given by

$$bu_{2m} = (\mathbb{Z} \times BU)(2m, \infty) \text{ and } bu_{2m+1} = \Sigma bu_{2m}.$$

The spectrum bo is constructed in a similar manner with

$$bo_{8m} = (\mathbb{Z} \times BO)(8m, \infty).$$

In addition one constructs closely related spectra bso and $bspin$ from $BSO = (\mathbb{Z} \times BO)(2, \infty)$ and $BSpin = (\mathbb{Z} \times BO)(3, \infty)$.

There are canonical maps of spectra $bu \longrightarrow$ KU and $bo \longrightarrow$ KO which induce isomorphisms on homotopy groups in dimensions greater than or equal to zero. These are maps of ring spectra. Similarly we have canonical maps of spectra of the form $bspin \longrightarrow bso \longrightarrow bo$.

(vi) The cobordism spectra ([9] p. 135) are constructed from Thom spaces (see [260] and Definition 1.2.3).

For example, let ξ_n denote the universal n-dimensional vector bundle over $BO(n)$ and let $MO(n)$ be its Thom space. The pullback of ξ_{n+1} via the canonical map $BO(n) \longrightarrow BO(n+1)$ is the vector bundle direct sum $\xi_n \oplus 1$ where 1 denotes the one-dimensional trivial bundle. The Thom space of $\xi_n \oplus 1$ is homeomorphic to $\Sigma MO(n)$ which yields a map

$$\epsilon_n : \Sigma MO(n) \longrightarrow MO(n+1)$$

and a resulting spectrum MO with $\mathrm{MO}_n = MO(n)$.

Replacing real vector bundles by complex ones gives MU with

$$\epsilon_{2n} : \Sigma^2 MU(n) \longrightarrow MU(n+1) = \mathrm{MU}_{2n+2} \text{ and } \mathrm{MU}_{2n+1} = \Sigma MU(n).$$

There are similar constructions associated to the families of classical Lie groups MSO, MSpin and MSp, where $\mathrm{MSp}_{4n} = MSp(n)$. These cobordism spectra are all connective spectra which are commutative ring spectra by means of the maps (for example, $MO(m) \wedge MO(n) \longrightarrow MO(m+n)$) induced by direct sum of matrices in the classical groups.

(vii) The following type of non-connective spectrum was introduced in [245]. Suppose that X is a homotopy commutative H-space and $B \in \pi_i(X)$ or $B \in \pi_i^S(X) = \pi_i(\Sigma^\infty X)$. Let X_+ denote the union of X with a disjoint base-point. Then there is a stable homotopy equivalence of the form

$$\Sigma^\infty X_+ \simeq \Sigma^\infty X \vee \Sigma^\infty S^0,$$

the wedge sum of the suspension spectra of X and S^0. Hence $B \in \pi_i(\Sigma^\infty X_+)$ and the multiplication in X induces $(X \times X)_+ \cong X_+ \wedge X_+ \longrightarrow X_+$ and thence

$$\epsilon : \Sigma^\infty S^i \wedge X_+ \stackrel{B \wedge 1}{\longrightarrow} \Sigma^\infty X_+ \wedge X_+ \longrightarrow \Sigma^\infty X_+.$$

These data define a spectrum $\Sigma^\infty X_+[1/B]$ in which the ith space is equal to X_+. This spectrum will, by construction, be stable homotopy equivalent (via the map ϵ) to its own ith suspension. In particular if B is the generator of $\pi_2(\mathbb{CP}^\infty)$ or $\pi_2(BU)$, the resulting spectra $\Sigma^\infty \mathbb{CP}^\infty_+[1/B]$ and $\Sigma^\infty BU_+[1/B]$ will have this type of stable homotopy periodicity of period 2.

In Example (iv) we met KU which has periodicity of period 2 and if we add a countable number of copies of MU together we may define

$$\mathrm{PMU} = \vee_{n>-\infty}^{\infty} \Sigma^{2n}\mathrm{MU}$$

which also has periodicity of period 2.

The following result was first proved in [245].

Theorem 1.3.3 ([245], [246])**.** There are stable homotopy equivalences of the form

$$\Sigma^\infty \mathbb{CP}^\infty_+[1/B] \simeq \mathrm{KU} \qquad \text{and} \qquad \Sigma^\infty BU_+[1/B] \simeq \mathrm{PMU}.$$

Proof. My original proof used the stable decompositions of [243] – sometimes called the Snaith splittings (see § 5) – and proceeds by first proving the PMU results and then deducing the KU result from it. Crucial use is made of the fact that the homotopy of MU and PMU is torsion free. Although the first proof of this was given in [245], the proof given in [246] is superior because it makes use of additional exponential properties of the stable splittings which were proved in [60] and [61].

A second proof, similar to one found by me and independently by Rob Arthan [23], is described in Chapter 9 § 9.2.15. I was reminded by Mike Hopkins of this second (actually it is the third!) proof when we were discussing, at the Fields Institute in May 2007, what modifications would work in the $\mathbb{A}^1$-stable homotopy category □

Example 1.3.4. Here are some ways to make new spectra from old.

(i) Given two spectra E and F we can form the smash product spectrum $E \wedge F$. We shall have quite a lot to say about the cases where E and F are various connective K-theory spectra yielding examples such as $bu \wedge bu$, $bo \wedge bo$ and $bu \wedge bo$. Each of these spectra is a left module spectrum over the left-hand factor and a right module over the right-hand one. For example the left bu-module structure on $bu \wedge bo$ is given by the map of spectra

$$bu \wedge (bu \wedge bo) = (bu \wedge bu) \wedge bo \stackrel{m\wedge 1}{\longrightarrow} bu \wedge bo$$

where $m : bu \wedge bu \longrightarrow bu$ is the multiplication in the ring spectrum bu.

(ii) Suppose that E is a spectrum and G is an abelian group. The Moore spectrum of type G ([9] p. 200, [228] p. 54) MG is a connective spectrum characterised by the following conditions on its homotopy and homology groups:

$$\begin{aligned}
&\pi_r(MG) = [\Sigma^\infty S^0, MG]_r = 0 \text{ for } r < 0,\\
&\pi_0(MG) \cong G \text{ and}\\
&H_r(MG; \mathbb{Z}) = \pi_r(MG \wedge \mathrm{H}\mathbb{Z}) = 0 \text{ for } r > 0.
\end{aligned}$$

The spectrum $E \wedge MG$ is referred to as E with coefficients in G. For example $E \wedge \Sigma^\infty \mathbb{RP}^2$ is the double suspension of E with coefficients in $\mathbb{Z}/2$.

(iii) Sometimes one wishes to concentrate on a limited aspect of a spectrum E such as, for example, the p-primary part of the homotopy groups. This can often be accomplished using the notions of localisation and completion of spaces or spectra. This appeared first in Alex Zabrodsky's method of "mixing homotopy types" [296] which he used to construct non-standard H-spaces. I first encountered this technique in the mimeographed MIT notes of Dennis Sullivan on geometric topology, part of which eventually appeared in [262]. Localisation of spectra and calculus of fractions is mentioned in Adams' book [9] but the final picture was first accomplished correctly by Pete Bousfield in [43].

Localisation (sometimes called completion) is related to the spectra with coefficients discussed in (ii). For example, p-local (or alternatively p-complete) connective K-theories are the subject of [168] (see also [10]) and frequently agree with connective K-theories with coefficients in the p-adic integers when applied to finite spectra.

(iv) Suppose that $f : E \longrightarrow F$ is a map of spectra of degree zero (if the degree is not zero then suspend or desuspend F). Then there is a mapping cone spectrum C_f called the cofibre of f which sits in a sequence of maps of spectra which generalise the Puppe sequence in the homotopy of spaces [257]

$$\cdots \longrightarrow \Sigma^{-1}C_f \longrightarrow E \xrightarrow{f} F \longrightarrow C_f \longrightarrow \Sigma E \xrightarrow{\Sigma f} \Sigma F \longrightarrow \Sigma C_f \longrightarrow \cdots$$

in which the spectrum to the right of any map is its mapping cone spectrum. This sequence gives rise to long exact homotopy sequences and homology sequences [9] some of which we shall examine in detail later, particularly in relation to j-theories and K-theory e-invariants defined via Adams operations (as defined in [5], for example).

The spectrum $\Sigma^{-1}C_f$ will sometimes be called the fibre spectrum of f.

One may construct new spectra from old as the fibre spectra of maps. For example there is a self-map of $\mathrm{MU}\mathbb{Z}_p$ – MU with p-adic coefficients – called the Quillen idempotent [225] (see also [9] Part II). The mapping cone of this homotopy idempotent is the Brown-Peterson spectrum BP ([228], [251]). There is one such spectrum for each prime.

When connective K-theory, say bu, is inflicted with coefficients in which p is invertible, then there is a self-map ψ^p called the pth Adams operation. The fibre of $\psi^p - 1$ is an example of a J-theory. For example we shall be very interested in

$$ju = \mathrm{Fibre}(\psi^3 - 1 : bu\mathbb{Z}_2 \longrightarrow bu\mathbb{Z}_2).$$

Here $\mathbb{Z}_p$ denotes the p-adic integers. We could replace $bu\mathbb{Z}_2$ by 2-localised bu in the sense of [43].

The spectrum jo is defined in a similar manner as ([30], see also Chapter 7 § 7.2.4)

$$jo = \mathrm{Fibre}(\psi^3 - 1 : bo\mathbb{Z}_2 \longrightarrow bspin\mathbb{Z}_2).$$

In Chapter 7 we shall also encounter 2-adic big J-theory and J'-theory which are defined by the fibre sequences

$$J \xrightarrow{\pi} BP \xrightarrow{\psi^3 - 1} BP \xrightarrow{\pi_1} \Sigma J$$

and

$$J' \xrightarrow{\pi'} BP \wedge BP \xrightarrow{\psi^3 \wedge \psi^3 - 1} BP \wedge BP \xrightarrow{\pi_1'} \Sigma J'.$$

Here ψ^3 in BP is induced by the Adams operation ψ^3 on $MU_*(-; \mathbb{Z}_2)$, which commutes with the Quillen idempotent which defines the summand BP.

Definition 1.3.5 (Generalised homology and cohomology theories)**.** We shall need the stable homotopy category because it is the correct place in which to study homology and cohomology of spaces and, more generally, spectra. Generalised homology and cohomology theories originate in [292].

If E and X are spectra, then we define the E-homology groups of X ([9] p. 196) by

$$E_n(X) = [\Sigma^\infty S^0, E \wedge X]_n$$

and write $E_*(X)$ for the graded group given by the direct sum over n of the $E_n(X)$'s. Sometimes we shall denote $E_*(\Sigma^\infty S^0)$ by $\pi_*(E)$, the homotopy of E.

We define the E-cohomology groups of X by

$$E^n(X) = [X, E]_{-n}$$

and we write $E^*(X)$ for the graded group given by the E-cohomology groups.

Example 1.3.6. Here are some tried and true homology and cohomology theories associated with some of the spectra from Example 1.3.2.

(i) The graded cohomology algebra given by the mod 2 Eilenberg-Maclane spectrum is isomorphic to the mod 2 Steenrod algebra (see Chapter 1, § 6)

$$(\mathrm{H}\mathbb{Z}/2)^*(\mathrm{H}\mathbb{Z}/2) = [\mathrm{H}\mathbb{Z}/2, \mathrm{H}\mathbb{Z}/2]_{-*} \cong \mathcal{A}$$

where the product in $\mathcal{A}$ corresponds to composition of maps of spectra. The graded mod 2 homology is a Hopf algebra isomorphic to the dual Steenrod algebra

$$\pi_*(\mathrm{H}\mathbb{Z}/2 \wedge \mathrm{H}\mathbb{Z}/2) \cong \mathcal{A}^*.$$

(ii) The complex cobordism theories MU_* and MU^* became very important with the discovery by Dan Quillen of a connection, described in detail in ([9] Part II) between formal group laws and $\pi_*(MU)$.

Theorem 1.3.7 ([189], [225]; see also ([9] p. 79))**.**

(i) $\pi_*(\mathrm{MU})$ is isomorphic to a graded polynomial algebra on generators in dimensions $2, 4, 6, 8, \ldots$.

(ii) Let L denote Lazard's ring ([9] p. 56) associated to the universal formal group law over the integers. Then there is a canonical isomorphism of graded rings

$$\theta : L \xrightarrow{\cong} \pi_*(MU).$$

Theorem 1.3.7(i) is proved using the classical mod p Adams spectral sequences. Theorem 1.3.7(ii) is suggested by the fact that the multiplication on $\mathbb{CP}^\infty$ gives rise to a formal group law

$$\pi_*(\mathrm{MU})[[x]] = MU^*(\mathbb{CP}^\infty) \longrightarrow MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) = \pi_*(\mathrm{MU})[[x_1, x_2]]$$

with coefficients in $\pi_*(\mathrm{MU})$.

Since Quillen's original discovery this theme has developed considerably and details may be found in [228] (see also [160], [161], [223], [163], [112]).

1.4 The classical Adams spectral sequence

1.4.1. We shall be particularly interested in the mod 2 classical Adams spectral sequence which was first constructed in [1] in order to calculate the stable homotopy groups of spaces. The complete background to the construction in that level of generality is described in the book [204]. The construction becomes easier when set up in the stable homotopy category and this is described in ([228] Chapter 3). The construction of an Adams spectral sequence based on a generalised homology theory was initiated in the case of MU by Novikov [211]. These generalised spectral sequences are discussed and explained in [9] and [228].

We shall sketch the construction of the classical mod 2 spectral sequence.

1.4.2. Mod 2 **Adams resolutions.** Let X be a connective spectrum such that $(\mathrm{H}\mathbb{Z}/2)^*(X)$ – which we shall often write as $H^*(X;\mathbb{Z}/2)$ – has finite type. This means that each $H^n(X;\mathbb{Z}/2)$ is a finite-dimensional $\mathbb{F}_2$-vector space. Also $H^*(X;\mathbb{Z}/2)$ is a left module over the mod 2 Steenrod algebra $\mathcal{A}$. Therefore one may apply the standard constructions of graded homological algebra ([109], [164]) to form the Ext-groups

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2).$$

In order to construct his spectral sequence Frank Adams imitated the homological algebra construction using spaces – in particular, mod 2 Eilenberg-Maclane spaces.

A (mod 2 classical) Adams resolution is a diagram of maps of spectra of the following form

$$\begin{tikzcd}
\cdots \arrow[r, "g_4"] & X_4 \arrow[r, "g_3"] \arrow[d, "f_4"] & X_3 \arrow[r, "g_2"] \arrow[d, "f_3"] & X_2 \arrow[r, "g_1"] \arrow[d, "f_2"] & X_1 \arrow[r, "g_0"] \arrow[d, "f_1"] & X_0 = X \arrow[d, "f_0"] \\
& K_4 & K_3 & K_2 & K_1 & K_0
\end{tikzcd}$$

in which each K_s is a wedge of copies of suspensions of the Eilenberg-Maclane spectrum $\mathrm{H}\mathbb{Z}/2$, each homomorphism

$$f_s^* : H^*(K_s;\mathbb{Z}/2) \longrightarrow H^*(X_s;\mathbb{Z}/2)$$

is surjective and each

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$$

is a fibring of spectra (that is, X_{s+1} is the fibre of f_s).

By Example 1.3.6(i) each $H^*(K_s;\mathbb{Z}/2)$ is a free $\mathcal{A}$-module and the long exact mod 2 cohomology sequences split into short exact sequences which splice together to give a free $\mathcal{A}$-resolution of $H^*(X;\mathbb{Z}/2)$:

$$\begin{aligned}\cdots \longrightarrow H^*(\Sigma^2 K_2;\mathbb{Z}/2) \longrightarrow H^*(\Sigma K_1;\mathbb{Z}/2)& \\ \longrightarrow H^*(K_0;\mathbb{Z}/2) \longrightarrow H^*(X_0;\mathbb{Z}/2) \longrightarrow 0.&\end{aligned}$$

The long exact homotopy sequences of the fibrings of spectra take the form

$$\cdots \longrightarrow \pi_*(X_{s+1}) \longrightarrow \pi_*(X_s) \longrightarrow \pi_*(X_{s+1}) \longrightarrow \pi_{*-1}(K_s) \longrightarrow \cdots$$

which fit together to give an exact couple which is one of the standard inputs to produce a spectral sequence [181]. In this case the spectral sequence is called the mod 2 Adams spectral sequence and satisfies the following properties.

Theorem 1.4.3. Let X be a connective spectrum such that X is of finite type. Then there is a convergent spectral sequence of the form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \pi_{t-s}(X) \otimes \mathbb{Z}_2$$

with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$.

Remark 1.4.4. In Theorem 1.4.3 the condition that X is of finite type is the spectrum analogue (see [9]) of a space being a CW complex with a finite number of cells in each dimension. It implies that $H^*(X;\mathbb{Z}/2)$ has finite type, but not conversely. The examples which will concern us are all of finite type having each homotopy group being finitely generated – explicitly they will be the finite smash products of connective K-theory spectra and suspension spectra of finite CW complexes.

1.5 Snaith splittings and the Kahn-Priddy theorem

1.5.1. Some personal history. In this section I shall sketch the proof of the Snaith splittings, which are decompositions in the stable homotopy category of the form [243]

$$\Sigma^\infty \Omega^n \Sigma^n X \simeq \vee_{k \geq 1} \Sigma^\infty \frac{F_k(X)}{F_{k-1}(X)}$$

for any reasonable connected space with base-point and for $1 \leq n \leq \infty$. This result was immediately useful in stable homotopy theory (for example, see [174]) because the components of the maps involved in this stable decomposition had not previously been available.

This last remark is not entirely true because one example of a map closely related to the two maps involved in splitting off the quadratic piece (that is, when $k = 2$) had cropped up in the proof of the Kahn-Priddy theorem [134]. The technical details of the Kahn-Priddy theorem circulated informally for a while before appearing in [8], [135] and [136]. The map used by Dan Kahn and Stewart Priddy was an example of the transfer map construction due to Jim Becker and Dan Gottlieb [32].

Shortly I am going to state the Kahn-Priddy theorem, with some of its historical background, but for now it will suffice to say that it allows us to lift 2-primary elements in the stable homotopy of spheres in a canonical manner to elements of the stable homotopy of $\mathbb{RP}^\infty$. For example this lifting, coming as it does from a splitting, preserves the order of elements. Even more importantly it was soon realised that the Kahn-Priddy lifting lowered the Adams spectral sequence filtration (see Theorem 1.4.3) so that, for example, an element in the stable homotopy of spheres detected by a secondary mod 2 cohomology operation would lift to an element detected by a primary mod 2 cohomology operation. This considerable convenience applies most importantly to the elements of Arf-Kervaire invariant one.

Returning to my discussion of the utility of the stable splittings, in this section I shall give my proof of the Kahn-Priddy theorem using the quadratic part of the stable splittings.

The stable splitting theorem was completed one rainy 1972 Sunday afternoon on a railway platform of the celebrated cathedral town of Ely in Cambridgeshire, England. On a Sunday there was a lot of time for finishing writing papers between consecutive trains to Cambridge – although I did not see anyone else there taking advantage of the opportunity. The details of the proof were a generalisation of an unstable splitting of $\Sigma\Omega\Sigma X$ due to Ioan James [117] (proved independently by John Milnor [193] using semi-simplicial techniques).

A couple of months later in May 1972 a conference took place at Oxford University in which Graeme Segal gave a sketch of how to prove the Kahn-Priddy theorem if one could construct operations in stable cohomotopy theory. Segal's construction of the requisite operations was not made rigorous, as far as I know, until twelve years later in [57] where it was referred to quite rightly as an "intriguing paper". I missed the conference (in those days UK conferences were far rarer and more unexpected than they are today – and the internet did not exist to advertise them nor funds to attend them! – so I was on holiday in Shropshire, not 40 miles away) but shortly thereafter I came up with the (previously unpublished) proof of the Kahn-Priddy theorem via Snaith splittings as a result of a conversation with Brian Sanderson, who had attended the conference.

Before embarking on proofs let us recall the motivation for the Kahn-Priddy theorem.

Let $O_{\infty}(\mathbb{R}) = \lim_{\overrightarrow{m}} O_m(\mathbb{R})$ denote the infinite orthogonal group given by the direct limit of the orthogonal groups of $n \times n$ matrices X with real entries and satisfying $X = X^{\mathrm{tr}}$, the transpose of X [8]. There is a very important, classical homomorphism – the stable J-homomorphism [5] – of the form

$$J : \pi_r(O_{\infty}(\mathbb{R})) \longrightarrow \pi_r^S(S^0) = \pi_r(\Sigma^{\infty} S^0)$$

as a consequence of the fact that an $n \times n$ orthogonal matrix yields a continuous homeomorphism of S^{n-1}. George Whitehead observed that J factorised through an even more stable J-homomorphism

$$J' : \pi_r^S(O_{\infty}(\mathbb{R})) = \pi_r(\Sigma^{\infty} O_{\infty}(\mathbb{R})) \longrightarrow \pi_r^S(S^0) = \pi_r(\Sigma^{\infty} S^0)$$

and conjectured that J' is surjective when $r > 0$. Independently, on the basis of calculations, Mark Mahowald conjectured that $\pi_r(\Sigma^{\infty}\mathbb{RP}^{\infty})$ maps surjectively onto the 2-primary part of $\pi_r(\Sigma^\infty S^0)$ via the composition of J' with the well-known map

$$\pi_r(\Sigma^\infty \mathbb{RP}^\infty) \longrightarrow \pi_r(\Sigma^\infty O_\infty(\mathbb{R}))$$

resulting from sending a line in projective space to the orthogonal reflection in its orthogonal hyperplane. Maholwald's conjecture was proved by Dan Kahn and Stewart Priddy ([134]; see also [135], [136] and [8]).

1.5.2. Sketch of proof of the stable splittings. Let $\mathcal{C}_{n,r}$ denote the space of r-tuples of linear embeddings $I^n \longrightarrow I^n$ of the n-dimensional unit cube into itself with parallel axes and disjoint interiors. This set of spaces comprises Mike Boardman's "little n-cubes operad" [38]. Let Λ denote the category of finite based sets $\mathbf{r} = \{0, 1, \ldots, r\}$ with base-point 0 and injective based maps. A morphism in Λ, $\phi : \mathbf{r} \longrightarrow \mathbf{s}$ determines a map of spaces

$$\phi : \mathcal{C}_{n,s} \longrightarrow \mathcal{C}_{n,r}$$

given by

$$(c_1, \ldots, c_s)\phi = (c_{\phi(1)}, \ldots, c_{\phi(r)})$$

and making the little cube operad into a contravariant functor. If X is a space then ϕ also determines a map

$$\phi : X^r \longrightarrow X^s$$

which sends the r-tuple $(x_1, \ldots, x_r)$ to $(x_{\phi(1)}, \ldots, x_{\phi(r)})$ filled out by putting the base-point into the remaining coordinates.

Define a space $\mathcal{C}_n X$ by

$$\mathcal{C}_n X = \coprod_{r \geq 0} \mathcal{C}_{n,r} \times X^r / \simeq$$

where $(c\phi, x) \simeq (c, \phi x)$ for $c \in \mathcal{C}_{n,s}$, $x \in X^r$ and $\phi : \mathbf{r} \longrightarrow \mathbf{s}$.

For a connected, base-pointed space with reasonable topology (e.g., a CW complex) $\mathcal{C}_n X$ gives a combinatorial model for an iterated loopspace [179]

$$\mathcal{C}_n X \simeq \Omega^n \Sigma^n X.$$

The space $\mathcal{C}_n X$ is filtered by the subspaces

$$F_k(X) = \coprod_{0 \leq r \leq k} \mathcal{C}_{n,r} \times X^r / \simeq$$

so that $F_{k-1}(X) \subset F_k(X)$ and the quotient $F_k(X)/F_{k-1}(X)$ is homeomorphic to the equivariant half-space product

$$\frac{F_k(X)}{F_{k-1}(X)} \cong \mathcal{C}_{n,k} \propto_{\Sigma_k} (X \wedge X \wedge \cdots \wedge X)$$

where Σ_k is the symmetric group permuting the little n-cubes and the coordinates in the k-fold smash product $X \wedge X \wedge \cdots \wedge X$.

The key to the Snaith splitting is the existence of combinatorial extensions

$$\pi_{n,k} : \Sigma^\infty \mathcal{C}_n X \longrightarrow \Sigma^\infty \frac{F_k(X)}{F_{k-1}(X)}$$

which are maps in the stable category extending the canonical quotient map

$$\Sigma^\infty F_k(X) \longrightarrow \Sigma^\infty \frac{F_k(X)}{F_{k-1}(X)}.$$

Given the existence of the combinatorial extensions it is straightforward to show that

$$\vee_{k=1}^{r} \ \pi_{n,k} : \ \Sigma^\infty F_r(X) \longrightarrow \vee_{k=1}^{r} \ \Sigma^\infty \frac{F_k(X)}{F_{k-1}(X)}$$

is a stable homotopy equivalence for $1 \leq r \leq \infty$.

Originally [243], I made the combinatorial extensions by some combinatorics and some point-set topology and they enjoyed a number of naturality properties as well as compatibility when n and r varied. Later Fred Cohen and Larry Taylor had the idea of using embeddings of equivariant half-smash products into Euclidean space together with the configuration space operad of k-tuples of points in Euclidean space to make the combinatorial extensions in a more functorial manner. This idea appears, elaborated a little to encompass "coefficient systems", in [60] and [61]. The advantage of this method of construction is that the maps can then be shown to be exponential – taking the loopspace product map to the smash product pairing of the $F_k(X)/F_{k-1}(X)$'s.

All we shall need is the original stable homotopy equivalence.

Theorem 1.5.3 ([243])**.** Let X be a connected CW complex. Then there is a stable homotopy equivalence of the form

$$\vee_{k=1}^{\infty} \ \pi_{n,k} : \ \Sigma^\infty \Omega^n \Sigma^n X \longrightarrow \vee_{k=1}^{\infty} \ \Sigma^\infty \frac{F_k(X)}{F_{k-1}(X)}$$

for $1 \leq n \leq \infty$. These equivalences are compatible as n varies and natural in maps of X.

The remainder of this section will be devoted to a sketch proof of the 2-primary Kahn-Priddy theorem.

1.5.4. The Whitehead product $[\iota_n, \iota_n]$**.** Choose a relative homeomorphism $h : (D^n, S^{n-1}) \longrightarrow (S^n, pt)$ then

$$[\iota_n, \iota_n] : S^{2n-1} \longrightarrow S^n$$

is given by the map sending $(x, y) \in S^{n-1} \times D^n \bigcup D^n \times S^{n-1} \cong S^{2n-1}$ to the appropriate one of $h(x)$ or $h(y)$ corresponding to the D^n-factor.

Define an involution τ on $S^{n-1} \times D^n \bigcup D^n \times S^{n-1}$ by $\tau(x,y) = (y,x)$ so that the orbit space satisfies

$$S^{2n-1}/\mathbb{Z}/2 \cong \Sigma^n \mathbb{RP}^{n-1}$$

and the map $[\iota_n, \iota_n]$ factorises through the orbit space to induce a map

$$w_n : \Sigma^n \mathbb{RP}^{n-1} \longrightarrow S^n.$$

Taking adjoints we obtain maps

$$\lambda_n = \mathrm{adj}([\iota_n, \iota_n]) : \Sigma S^{n-1} = S^n \longrightarrow \Omega^{n-1} S^n$$

and

$$k_n = \mathrm{adj}(w_n) : \Sigma \mathbb{RP}^{n-1} \longrightarrow \Omega^{n-1} S^n.$$

The map k_n is the subject of the Kahn-Priddy theorem ([134], [8], [135], [136]). If $\pi_{n-1} : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ is the standard quotient map then

$$k_n \cdot \Sigma \pi_{n-1} = \lambda_n : \Sigma S^{n-1} \longrightarrow \Omega^{n-1} S^n.$$

Consider the map which is the quadratic part of my stable homotopy decomposition of Theorem 1.5.3 in the case when $X = S^1$, the circle.

$$\pi_{n-1,2} : \Sigma^\infty \Omega^{n-1} S^n \longrightarrow \Sigma^\infty C_{n-1,2} \propto_{\Sigma_2} (S^1 \wedge S^1)$$

where $C_{n-1,2}$ is the space of pairs of distinct points in $\mathbb{R}^{n-1}$ [179].

In Theorem 1.5.3 $\mathcal{C}_{n-1,2}$ denoted the space of 2-tuples of linear embeddings $I^{n-1} \longrightarrow I^{n-1}$ of the $(n-1)$-dimensional unit cube into itself with parallel axes and disjoint interiors. However, mapping a cube to its centre gives an equivariant homotopy equivalence between the "little cubes" operad and the configuration space operad so we shall denote them both by the $\mathcal{C}_{n,r}$ in this discussion. The following result is well known.

Proposition 1.5.5. There is a homotopy equivalence

$$C_{n-1,2} \propto_{\Sigma_2} (S^1 \wedge S^1) \xrightarrow{\simeq} \Sigma \mathbb{RP}^{n-1}.$$

Proof. Let $T(E)$ denote the Thom space of E. Clearly there is a homeomorphism

$$C_{n-1,2} \propto_{\Sigma_2} (S^1 \wedge S^1) \cong \frac{C_{n-1,2} \times_{\Sigma_2} (\mathbb{R}^1 \times \mathbb{R}^1)}{C_{n-1,2} \times_{\Sigma_2} (\infty)},$$

which is the Thom space of the bundle over $C_{n-1,2}/\Sigma_2$ associated to the involution on $\mathbb{R}^1 \times \mathbb{R}^1$ which switches the coordinates. However, $C_{n-1,2}$ is the space of pairs of distinct points in $\mathbb{R}^{n-1}$. Sending $(z_1, z_2) \in C_{n-1,2}$ to multiples of $z_1 - z_2$ defines a fibring

$$\mathbb{R}^{n-2} \longrightarrow C_{n-1,2} \xrightarrow{\hat{\gamma}} \mathbb{RP}^{n-2}.$$

The homotopy equivalence $\hat{\gamma}$ induces a homotopy equivalence of Thom spaces between $C_{n-1,2} \propto_{\Sigma_2} (S^1 \wedge S^1)$ and

$$T(\mathbb{R} \oplus H/\mathbb{RP}^{n-2}) \cong \Sigma T(H/\mathbb{RP}^{n-2}).$$

Here H is the Hopf line bundle and the result follows because the Thom space $T(H/\mathbb{RP}^{n-2})$ is homeomorphic to $\mathbb{RP}^{n-1}$. □

Theorem 1.5.6. The composition

$$\Sigma^\infty \Sigma \mathbb{RP}^{n-1} \stackrel{\Sigma^\infty k_n}{\longrightarrow} \Sigma^\infty \Omega^{n-1} S^n \stackrel{\pi_{n-1,2}}{\longrightarrow} \Sigma^\infty \Sigma \mathbb{RP}^{n-1}$$

is a 2-local stable homotopy equivalence for $3 \leq n \leq \infty$.

Proof. The phrase "2-local stable homotopy equivalence" refers to the stable homotopy category in which spectra have been Bousfield-localised with respect to mod 2 singular homology; that is, localised in the sense of [43]. In this localised sense, it suffices to check that $\pi_{n-1,2} \cdot \Sigma^\infty k_n$ induces an isomorphism on $H_*(-;\mathbb{Z}/2)$ according to [43].

The space $\Omega^n S^n$ is an H-space with one component for each integer t, corresponding to the self-maps of S^n of degree t. The components are all homotopy equivalent by choosing a point denoted by [1] in the degree one component and adding it, in the H-space "addition", k times to give a homotopy equivalence between the degree t component and the degree $t+k$ component. The H-space product on mod 2 homology will be denoted by $a * b$ and $[-1]$ will denote a point in the minus one component and $[-2] = [-1] * [-1]$ and so on. The Dyer-Lashof operation Q^j moves the homology of the t-component to the $2t$-component. Hence $Q^j[1] * [-2]$ is in the homology of the zero degree component.

From [135]

$$(\mathrm{adj}(k_n))_*(b_j) = Q^j[1] * [-2] \in H_j(\Omega^n S^n; \mathbb{Z}/2)$$

where b_j generates $H_j(\mathbb{RP}^{n-1}; \mathbb{Z}/2)$ and $Q^j(x)$ is the jth Dyer-Lashof operation applied to x (see Chapter 1, § 7). Hence, applying the homology suspension σ_*, where $0 \neq \iota \in H_1(S^1; \mathbb{Z}/2)$,

$$Q^j(\iota) = \sigma_*(Q^j[1] * [-2]) = \sigma_*(\mathrm{adj}(k_n)_*(b_j)) = (k_n)_*(\Sigma b_j).$$

However, from [174] we have that

$$(\pi_{n-1,2})_*(Q^j(\iota)) = \Sigma b_j \in H_{j+1}(\Sigma \mathbb{RP}^{n-1}; \mathbb{Z}/2).$$ □

Corollary 1.5.7. The maps in Theorem 1.5.6 each induce an isomorphism on π_2, the group of order two.

1.5.8. The maps. For a space with base-point, denote by QX the infinite loopspace

$$QX = \lim_{\overrightarrow{n}} \ \Omega^n \Sigma^n X$$

which should be thought of as "the free infinite loopspace generated by X". This is because any map from the space X to an infinite loopspace Y extends to a canonical infinite loopspace map from QX to Y in the following manner. There is a structure map $D : QY \longrightarrow Y$ which comes from considering $Y = \Omega^k Y'$ and evaluating the maps from S^k to Y' on the first k-suspension coordinates. Examples of this type of evaluation map are

$$\Omega^k \Sigma^k \Omega^k Z \xrightarrow{\Omega^k \mathrm{eval}} \Omega^k Z$$

and D is the "limit as k goes to infinity" of these maps. Now given $f : X \longrightarrow Y$ the canonical infinite loopspace map is

$$\tilde{f} : QX \xrightarrow{Q(f)} QY \xrightarrow{D} Y.$$

Any map constructed in this manner induces on mod 2 homology a homomorphism of Hopf algebras which commutes with Dyer-Lashof operations (see Chapter 1, § 7), being an infinite loopspace map. For example, this would apply to the canonical infinite loopspace map

$$\tilde{f} : Q\mathbb{RP}^\infty \longrightarrow Q_0 S^0$$

made from a map $f : \mathbb{RP}^\infty \longrightarrow Q_0 S^0$ such as the adjoint of the map k_∞ of § 1.5.4.

Now consider the adjoint of the map obtained by taking the limit over n of $\pi_{n-1,2}$ in Theorem 1.5.6,

$$\mathrm{adj}(\pi_{\infty,2}) : QS^1 \longrightarrow Q\Sigma\mathbb{RP}^\infty.$$

This is *not* an infinite loopspace map because, being the second piece of my stable splitting, it is trivial on S^1 and the infinite canonical loopspace map which is trivial on S^1 is nullhomotopic. However, we can still determine a lot about the induced map in mod 2 homology.

Let $\iota \in H_1(S^1; \mathbb{Z}/2)$ denote the generator. Then as an algebra

$$H_*(QS^1; \mathbb{Z}/2) \cong \mathbb{Z}/2[Q^I(\iota)],$$

the polynomial algebra on all elements $Q^{i_1} Q^{i_2} \ldots Q^{i_s}(\iota)$ obtained by applying an admissible iterated Dyer-Lashof operation (see Chapter 1, § 7) to ι. Here (see Definition 1.7.3) admissible means that the sequence $I = (i_1, i_2, \ldots, i_s)$ satisfies $i_j \leq 2i_{j+1}$ for $1 \leq j \leq s-1$ and $i_u > i_{u+1} + i_{u+2} + \cdots + i_s + \deg(\iota) = i_{u+1} + i_{u+2} + \cdots + i_s + 1$ for $1 \leq u \leq s$. The weight of Q^I is defined to be 2^s.

Theorem 1.5.9. Suppose that $Q^{i_1}Q^{i_2}\ldots Q^{i_s}(\iota)$ is admissible. Then the homomorphism

$$\mathrm{adj}(\pi_{\infty,2})_* : H_*(QS^1;\mathbb{Z}/2) \longrightarrow H_*(Q\Sigma\mathbb{RP}^\infty;\mathbb{Z}/2)$$

satisfies, for $i_s, s > 0$,

$$\mathrm{adj}(\pi_{\infty,2})_*(Q^{i_1}Q^{i_2}\ldots Q^{i_s}(\iota)) \equiv Q^{i_1}Q^{i_2}\ldots Q^{i_{s-1}}(\Sigma b_{i_s})$$

modulo decomposables and elements $Q^J(\Sigma b_t)$ with Q^J of lower weight than Q^I and $t > 0$.

Proof. This follows from the definition of the combinatorial extension maps of § 1.5.2. On the kth part of the filtration F_kX the combinatorial extension of the map to the quadratic part of the splitting is based on sending an unordered k-tuple to the unordered $k!/2 \cdot (k-2)!$ consisting of the subsets of cardinality two.

Using the combinatorics of this map the partial computation is similar to that of either [132] or [135].

My original calculation followed a method described by Frank Adams in a 1971 Part III course at Cambridge University and alluded to in [8]. □

Theorem 1.5.10 (The Kahn-Priddy theorem [134]). Let $f : \mathbb{RP}^\infty \longrightarrow Q_0S^0$ be any map which induces an isomorphism on π_1 and let

$$\tilde{f} : Q\mathbb{RP}^\infty \longrightarrow Q_0S^0$$

denote the canonical infinite loopspace map of § 1.5.8. Then the composite

$$Q_0S^0 \xrightarrow{\Omega\,\mathrm{adj}(\pi_{\infty,2})} Q\mathbb{RP}^\infty \xrightarrow{\tilde{f}} Q_0S^0$$

is a 2-local homotopy equivalence.

Proof. As in the proof of Theorem 1.5.6 it suffices to prove that the induced map on mod 2 homology is an isomorphism. The map $\tilde{f}$ commutes with Dyer-Lashof operations and is a map of Hopf algebras.

As an algebra there is an isomorphism of the form

$$H_*(Q_0S^0;\mathbb{Z}/2) \cong \mathbb{Z}/2[Q^{i_1}Q^{i_2}\ldots Q^{i_t}[1] * [-2^t]]$$

where the sequences $(i_1,\ldots,i_t)$ run over admissible monomials in the iterated Dyer-Lashof operations. That is, $H_*(Q_0S^0;\mathbb{Z}/2)$ is the polynomial on the admissible monomials $Q^{i_1}Q^{i_2}\ldots Q^{i_t}[1]$ translated to the zero component of QS^0. This homology algebra is a subalgebra of the Laurent polynomial ring

$$H_*(QS^0;\mathbb{Z}/2) \cong H_*(Q_0S^0;\mathbb{Z}/2)[[1]^{\pm 1}]$$

where $[1]^{-1} = [-1]$ and the identity of the algebra is $[0]$.

The Dyer-Lashof operations act on $H_*(QS^0;\mathbb{Z}/2)$ preserving $H_*(Q_0S^0;\mathbb{Z}/2)$ in such a way that $Q^j([1]*[-1]) = Q^j[0] = 0$ for all $j > 0$ which yields the formula for $Q^j[-1]$ of Proposition 1.5.11 below.

Since Dyer-Lashof operations commute with the homology suspension homomorphism, Theorem 1.5.9 implies that

$$\Omega\,\mathrm{adj}(\pi_{\infty,2})_*(Q^{i_1}Q^{i_2}\ldots Q^{i_t}[1]*[-2^t]) \equiv Q^{i_1}Q^{i_2}\ldots Q^{i_{t-1}}(b_{i_t})$$

modulo decomposables. Therefore, since $\tilde{f}$ is an infinite loopspace map,

$$\tilde{f}_*(Q^{i_1}Q^{i_2}\ldots Q^{i_{t-1}}(b_{i_t})) = Q^{i_1}Q^{i_2}\ldots Q^{i_{t-1}}(Q^{i_t}[1]*[-2])$$

by Theorem 1.5.6. Therefore, by the Cartan formula for Dyer-Lashof operations § 1.7.2(v) and the formula of Proposition 1.5.11 below,

$$\tilde{f}_*(\Omega\,\mathrm{adj}(\pi_{\infty,2})_*(Q^{i_1}Q^{i_2}\ldots Q^{i_t}[1]*[-2^t])) \equiv Q^{i_1}Q^{i_2}\ldots Q^{i_{t-1}}Q^{i_t}[1]*[-2^t]$$

modulo decomposables. This completes the proof, by induction on dimension, since $H_*(QS^0;\mathbb{Z}/2)$ is a finite-dimensional $\mathbb{F}_2$-vector space in each dimension. □

Proposition 1.5.11 ([221] Lemma 2.1)**.** For $n \geq 0$,

$$Q^n[-1] = \sum_{\underline{\lambda}} (\lambda_1,\ldots,\lambda_n)\ (Q^1[1])^{\lambda_1} * (Q^2[1])^{\lambda_2} * \cdots * (Q^n[1])^{\lambda_n} * [-2\lambda - 2]$$

where the summation is taken over all sequences $\underline{\lambda} = \{\lambda_1,\ldots,\lambda_n\}$ of non-negative integers such that $n = \sum_{i=1}^n i\cdot\lambda_i,\ \ \lambda = \sum_{i=1}^n \lambda_i$ and $(\lambda_1,\ldots,\lambda_n)$ is the multinomial coefficient

$$(\lambda_1,\ldots,\lambda_n) = \frac{\lambda!}{\lambda_1!\ldots,\lambda_n!}.$$

Proof. By the Cartan formula § 1.7.2(v) for Dyer-Lashof operations,

$$0 = Q^n([-1]*[1]) = \sum_{i=0}^n Q^i[-1]*Q^{n-i}[1].$$

The result follows by induction. In fact, this summation is exactly that relating the elementary symmetric functions to the homogeneous symmetric functions (see [165] p. 4). □

Remark 1.5.12. There have been a number of alternative proofs of the Snaith splittings as well as papers verifying that the splitting maps can be assumed to preserve enriched structures (for example, E_n ring spectrum structures). For further details the reader is referred to [28], [57], [60], [61], [65], [132], [133], [134], [135] and [136]. The papers which started decompositions of loopspaces are [117] and [193].

There have been a number of results which used the stable splittings. For example, [62] was the basis of Nick Kuhn's proof of the Whitehead conjecture at odd primes – the analogue of the 2-adic version proved by Dan Kahn and Stewart Priddy (see § 1.5.1). For details on such applications the reader is referred to [144], [174], [245], [246], [247], [250], [44], [254] and secondary applications such as [69] and [274] (see also [54]).

For other proofs of the Kahn-Priddy theorem the reader is referred to [8], [28], [57], [132], [133], [134], [135], [136] and [238].

1.6 Cohomology operations

1.6.1. The Steenrod Algebra Modulo 2. Let $H^*(X, A; \mathbb{Z}/2)$ denote singular cohomology of the pair of topological spaces $A \subseteq X$ ([257], [259], [105]). The modulo 2 Steenrod operations are denoted by Sq^i for $i \geq 0$ which are characterised by the following axioms ([259] p. 2):

(i) For all $n \geq 0$,

$$Sq^i : H^n(X, A; \mathbb{Z}/2) \longrightarrow H^{n+i}(X, A; \mathbb{Z}/2)$$

is a natural homomorphism.

(ii) Sq^0 is the identity homomorphism.

(iii) If $\deg(x) = n$ then $Sq^n(x) = x^2$.

(iv) If $i > \deg(x)$ then $Sq^i(x) = 0$.

(v) The Cartan formula holds,

$$Sq^k(xy) = \sum_{i=0}^{k} Sq^i(x) Sq^{k-i}(y).$$

(vi) Sq^1 is the Bockstein homomorphism associated to the coefficient sequence

$$\cdots \longrightarrow H^n(X, A; \mathbb{Z}/4) \longrightarrow H^n(X, A; \mathbb{Z}/2) \xrightarrow{Sq^1} H^{n+1}(X, A; \mathbb{Z}/2) \longrightarrow \cdots .$$

(vii) The Adem relations hold. If $0 < a < 2b$ then

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \begin{pmatrix} b-1-j \\ a-2j \end{pmatrix} Sq^{a+b-j} Sq^j$$

where $[y]$ denotes the greatest integer less than or equal to y and

$$\begin{pmatrix} m \\ k \end{pmatrix} = \frac{m!}{k!(m-k)!}$$

is the usual binomial coefficient (modulo 2).

(viii) If $\delta : H^n(A;\mathbb{Z}/2) \longrightarrow H^{n+1}(X,A;\mathbb{Z}/2)$ is the coboundary homomorphism from the long exact cohomology sequence of the pair (X,A) then, for all $i \geq 0$,

$$\delta Sq^i = Sq^i \delta.$$

Example 1.6.2. Real projective space $\mathbb{RP}^n$. The cohomology ring of real projective n-spaces is $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$ where $\deg(x) = 1$. By induction, the axioms of § 1.6.1 imply that ([259] Lemma 2.4)

$$Sq^i(x^k) = \begin{pmatrix} k \\ i \end{pmatrix} x^{k+i}.$$

Definition 1.6.3. The Steenrod algebra $\mathcal{A}$. Let M be the graded $\mathbb{F}_2$ vector space with basis $\{Sq^0 = 1, Sq^1, Sq^2, Sq^3, \ldots\}$ with $\deg(Sq^i) = i$. Let $T(M)$ denote the tensor algebra of M. It is a connected, graded $\mathbb{F}_2$-algebra.

The modulo 2 Steenrod algebra, denoted by $\mathcal{A}$, is defined to be the quotient of $T(M)$ by the two-sided ideal generated by the Adem relations, if $0 < a < 2b$,

$$Sq^a \otimes Sq^b = \sum_{j=0}^{[a/2]} \begin{pmatrix} b-1-j \\ a-2j \end{pmatrix} Sq^{a+b-j} \otimes Sq^j.$$

A finite sequence of non-negative integers $I = (i_1, i_2, \ldots, i_k)$ is defined to have length k, written $l(I) = k$ and moment $m(I) = \sum_{s=1}^k \ s i_s$. A sequence I is called admissible if $i_1 \geq 1$ and $i_{s-1} \geq 2i_s$ for $2 \leq s \leq k$. Write $Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}$. Then Sq^0 and all the Sq^I with I admissible are called the admissible monomials of $\mathcal{A}$. The following result is proved by induction on the moment function $m(I)$.

Theorem 1.6.4 ([259] Chapter I, Theorem 3.1))**.** The admissible monomials form an $\mathbb{F}_2$-basis for the Steenrod algebra $\mathcal{A}$.

Definition 1.6.5. Let A be a graded $\mathbb{F}_2$-algebra with a unit $\eta : \mathbb{F}_2 \longrightarrow A$ and a co-unit $\epsilon : A \longrightarrow \mathbb{F}_2$. Therefore $\epsilon \cdot \eta = 1$. These homomorphisms preserve degree when $\mathbb{F}_2$ is placed in degree zero.

Then A is a Hopf algebra if:

(i) There is a comultiplication map

$$\psi : A \longrightarrow A \otimes A$$

which is a map of graded algebras when $A \otimes A$ is endowed with the algebra multiplication $(a \otimes a') \cdot (b \otimes b') = ab \otimes a'b'$ and

(ii) Identifying $A \cong A \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \otimes A$,

$$1 = (1 \otimes \epsilon) \cdot \psi = \psi \cdot (1 \otimes \epsilon) : A \longrightarrow A.$$

The comultiplication is associative if

$$(\psi \otimes 1) \cdot \psi = (1 \otimes \psi) \cdot \psi : A \longrightarrow A \otimes A \otimes A.$$

It is commutative if $\psi = T \cdot \psi$ where $T(a \otimes a') = a' \otimes a$ for all $a, a' \in \mathcal{A}$.

Theorem 1.6.6 ([259] Chapter II, Theorems 1.1 and 1.2)**.** The map of generators

$$\psi(Sq^k) = \sum_{i=0}^{k} Sq^i \otimes Sq^{k-i}$$

extends to a map of graded $\mathbb{F}_2$-algebras

$$\psi : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

making $\mathcal{A}$ into a Hopf algebra with a commutative, associative comultiplication.

Definition 1.6.7. Let $\mathcal{A}^*$ denote the dual vector space to $\mathcal{A}$ whose degree n subspace is $\mathcal{A}_n^* = \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}_n, \mathbb{F}_2)$. Let

$$\langle -, - \rangle : \mathcal{A}^* \times \mathcal{A} \longrightarrow \mathbb{F}_2$$

denote the evaluation pairing $\langle f, a \rangle = f(a)$.

Let $M_k = Sq^{I_k}$ where $I_k = (2^{k-1}, 2^{k-2}, \ldots, 2, 1)$ for any strictly positive integer k. This is an admissible monomial. Define $\xi_k \in \mathcal{A}^*_{2^k-1}$ by the following formulae for $\langle \xi_k, m \rangle$ where m runs through all admissible monomials:

$$\langle \xi_k, m \rangle = \begin{cases} 1 & \text{if } m = M_k, \\ 0 & \text{otherwise.} \end{cases}$$

The dual of a commutative coalgebra is a commutative algebra whose multiplication ψ^* is the dual of the comultiplication ψ. Similarly the dual of a Hopf algebra over $\mathbb{F}_2$ is again an $\mathbb{F}_2$-Hopf algebra. The following result describes the Hopf algebra $\mathcal{A}^*$.

Theorem 1.6.8 ([259] Chapter II, Theorems 2.2 and 2.3, [188])**.** The $\mathbb{F}_2$-Hopf algebra $\mathcal{A}^*$ is isomorphic to the polynomial algebra $\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots]$ with comultiplication given by

$$\phi^*(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i.$$

1.7 Homology operations

Definition 1.7.1. A based space X_0 is an infinite loopspace if there exists a sequence of based spaces $X_1, X_2, X_3, \ldots$ such that $X_i = \Omega X_{i+1}$, the space of loops in X_{i+1} which begin and end at the base-point, for each $i \geq 0$. A map of infinite loopspaces is defined in the obvious manner.

The principal example of an infinite loopspace is the space

$$QX = \lim_{\overrightarrow{n}} \Omega^n \Sigma^n X,$$

the limit over n of the space of based maps of the n-sphere into the n-fold suspension of X, $\Sigma^n X$.

Numerous other examples of infinite loopspaces occur throughout topology – for example, the classifying spaces for topological and algebraic K-theory (see [168], [10], [169]).

The mod p homology of an infinite loop space admits an algebra of homology operations which complement the operations on homology given by the duals of the Steenrod operations. The homology operations form a Hopf algebra which is usually called the Dyer-Lashof algebra after the paper [73]. However, the operations originated in the work of Araki and Kudo [21] (see also [209]).

We shall be concerned only with homology modulo 2. Most importantly, the map induced on mod 2 homology by an infinite loopspace map commutes with the Dyer-Lashof operations.

1.7.2. The Dyer-Lashof algebra modulo 2. Let $H_*(X;\mathbb{Z}/2)$ denote the singular homology modulo 2 of an infinite loopspace X. Hence $H_*(X;\mathbb{Z}/2)$ is an $\mathbb{F}_2$-Hopf algebra.

(i) For each $n \geq 0$ there is a linear map

$$Q^n : H_*(X;\mathbb{Z}/2) \longrightarrow H_{*+n}(X;\mathbb{Z}/2)$$

which is natural for maps of infinite loopspaces.

(ii) Q^0 is the identity map.

(iii) If $\deg(x) = n$ then $Q^n(x) = x^2$.

(iv) If $\deg(x) > n$ then $Q^n(x) = 0$.

(v) The Kudo transgression theorem holds:

$$Q^n \sigma_* = \sigma_* Q^n$$

where $\sigma_* : \tilde{H}_*(\Omega X;\mathbb{Z}/2) \longrightarrow H_{*+1}(X,\mathbb{Z}/2)$ is the homology suspension map.

(v) The multiplicative Cartan formula holds:

$$Q^n(xy) = \sum_{r=0}^{n} Q^r(x) Q^{n-r}(y).$$

(vi) The comultiplicative Cartan formula holds:

$$\psi(Q^n(x)) = \sum_{r=0}^{n} Q^r(x') \otimes Q^{n-r}(x'')$$

where the comultiplication is given by $\psi(x) = \sum x' \otimes x''$.

(vii) If $\chi : H_*(X;\mathbb{Z}/2) \longrightarrow H_*(X;\mathbb{Z}/2)$ is the canonical anti-automorphism of the Hopf algebra [190] then

$$\chi \cdot Q^n = Q^n \cdot \chi.$$

(viii) The Adem relations hold:

$$Q^r \cdot Q^s = \sum_i \left(\begin{array}{c} i-1 \\ 2i-r \end{array} \right) Q^{r+s-i} \cdot Q^i$$

if $r > 2s$.

(ix) Let $Sq^r_* : H_*(X;\mathbb{Z}/2) \longrightarrow H_{*-i}(X;\mathbb{Z}/2)$ denote the dual of the Steenrod operation Sq^r of § 1.6.1. Then the Nishida relations hold:

$$Sq^r_* \cdot Q^s = \sum_i \left(\begin{array}{c} s-r \\ r-2i \end{array} \right) Q^{s-r+i} \cdot Sq^i_*.$$

Definition 1.7.3. As with Steenrod operations in Definition 1.6.3 the Adem relations for iterated Dyer-Lashof operations lead to the notation of an admissible iterated operation. The element $Q^{i_1}Q^{i_2}\ldots Q^{i_s}(x)$ is called an admissible iterated Dyer-Lashof operation if the sequence $I = (i_1, i_2, \ldots, i_s)$ satisfies $i_j \leq 2i_{j+1}$ for $1 \leq j \leq s-1$ and $i_u > i_{u+1} + i_{u+2} + \cdots + i_s + \deg(x)$ for $1 \leq u \leq s$.

The weight of Q^I is defined to be 2^s.

Applications of the Dyer-Lashof operations may be found, for example, in ([143], [59], [169]).

1.8 The Arf-Kervaire invariant one problem

Definition 1.8.1. The Arf invariant of a quadratic form Let V be a finite-dimensional vector space over the field $\mathbb{F}_2$ of two elements. A quadratic form is a function $q : V \longrightarrow \mathbb{F}_2$ such that $q(0) = 0$ and

$$q(x+y) - q(x) - q(y) = (x, y)$$

is $\mathbb{F}_2$-bilinear (and, of course, symmetric). Notice that $(x, x) = 0$ so that $(-,-)$ is a symplectic bilinear form. Hence $\dim(V) = 2n$ and to say that q is non-singular means that there is an $\mathbb{F}_2$-basis of V, $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ say, such that $(a_i, b_j) = 0$ if $i \neq j$, $(a_i, b_i) = 1$ and $(a_i, a_j) = 0 = (b_i, b_j)$ otherwise.

In this case the Arf invariant of q is defined to be

$$c(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{F}_2.$$

Theorem 1.8.2 (Arf [22], see also [48] p. 52 and [235] p. 340)**.** The invariant $c(q)$ is independent of the choice of basis and two quadratic forms on V are equivalent if and only if their Arf invariants coincide.

1.8.3. The Arf-Kervaire invariant of a framed manifold. Michel Kervaire [138] defined an $\mathbb{F}_2$-valued invariant for compact, $(2l-2)$-connected $(4l-2)$-manifolds which are almost parallelisable and smooth in the complement of a point. Bill Browder [47] extended this definition to any framed, closed $(4l-2)$-manifold where, as in § 1.2.4, a framing of M is a homeomorphism $t : T(\nu) \xrightarrow{\cong} \Sigma^N(M_+)$, M_+ is the disjoint union of M and a basepoint and ν is the normal bundle of an embedding of M into $\mathbb{R}^{4l-2+N}$ and $T(\nu) = D(\nu)/S(\nu)$ is the Thom space of the normal bundle.

Browder also showed that the Arf invariant of a framed manifold was trivial unless $l = 2^s$ for some s. He did this by relating the Arf invariant of M to its class in the stable homotopy of spheres and the associated Adams spectral sequence.

Recall from Theorem 1.2.6 that the famous Pontrjagin-Thom construction forms the map

$$S^{4l-2+N} \cong \mathbb{R}^{4l-2+N}/(\infty) \stackrel{\text{collapse}}{\longrightarrow} D(\nu)/S(\nu) \cong \Sigma^N(M_+) \stackrel{\text{collapse}}{\longrightarrow} S^N$$

which yields an isomorphism between framed cobordism classes of $(4l-2)$-manifolds and the $(4l-2)$th stable homotopy group of spheres,

$$\pi^S_{4l-2}(S^0) = \pi_{4l-2}(\Sigma^\infty S^0).$$

Here is Bill Browder's definition, which was simplified by Ed Brown Jr. [50]. Given a framed manifold M^{2k} and $a \in H^k(M; \mathbb{Z}/2) \cong [M_+, K(\mathbb{Z}/2, k)]$ we compose with the Pontrjagin-Thom map to obtain an element of

$$\pi_{2k+N}(\Sigma^N K(\mathbb{Z}/2, k)) \cong \mathbb{F}_2.$$

This is a non-singular quadratic form $q_{M,t}$ on $H^k(M; \mathbb{Z}/2)$, depending on t, and the Arf-Kervaire invariant of (M, t) is $c(q_{M,t}) \in \mathbb{F}_2$.

Example 1.8.4. A Lie group has trivial tangent bundle so is frameable. There are framings of $S^1 \times S^1$, $S^3 \times S^3$ and $S^7 \times S^7$ which have Arf invariant one. As we shall see, there is an elegant way to construct a framed M^{30} of Arf invariant one (see Chapter 2, Proposition 2.3.4). Also in terms of $\pi^S_{62}(S^0)$ a framed manifold of Arf invariant one has been confirmed by the computer calculations of Stan Kochman [145] and there are also long calculations of Michael Barratt, John Jones and Mark Mahowald [30] asserting the existence in dimension 62.

That is the extent of existence results except for some failed attempts.

1.8.5. Equivalent formulations. (i) As in § 1.5.8, we write QS^0 for the infinite loopspace $\lim_{\overrightarrow{n}} \Omega^n S^n$, the limit over n of the space of based maps from S^n to itself. The components of this space are all homotopy equivalent, because it is an H-space, and there is one for each integer. The integer d is the degree of all the maps in the dth component $Q_d S^0$. The component with $d = 1$ is written SG, which has an H-space structure coming from composition of maps. Also SG is important in geometric topology because BSG classifies stable spherical fibrations and surgery on manifolds starts with the Spivak normal bundle, which is a spherical fibration.

Here is a description of the Arf-Kervaire invariant of a framed manifold represented as $\theta \in \pi_{4k+2}(\Sigma^\infty S^0)$. We may form the adjoint, which is a map $\mathrm{adj}(\theta) : S^{4k+2} \longrightarrow Q_0 S^0$. Adding a map of degree one yields $Q_0 S^0 \simeq SG$ and we may compose with the maps to SG/SO and thence to G/Top. However, famous work of Dennis Sullivan, part of his proof of the Hauptvermutung, shows that, at the prime 2,

$$G/Top \simeq \prod_k K(\mathbb{Z}_{(2)}, 4k) \times K(\mathbb{Z}/2, 4k+2).$$

Projecting to $K(\mathbb{Z}/2, 4k+2)$ yields an element of $\pi_{4k+2}(K(\mathbb{Z}/2, 4k+2)) \cong \mathbb{Z}/2$ which is the Arf-Kervaire invariant of θ. We shall need this description later (see Chapter 2 § 2.2.4).

(ii) The Kahn-Priddy Theorem yields a split surjection of the form

$$\pi_m(\Sigma^\infty \mathbb{RP}^\infty) \longrightarrow \pi_m(\Sigma^\infty S^0) \otimes \mathbb{Z}_2$$

for $m > 0$ – where $\mathbb{Z}_2$ denotes the 2-adic integers.

As explained in § 1.5.1, the Kahn-Priddy theorem follows from the existence of two maps. The first is the quadratic part of the Snaith splitting [243], which is a stable map

$$\Omega^{n-1} S^n = \Omega^{n-1}\Sigma^{n-1} S^1 \longrightarrow C_{n-1,2} \propto_{\Sigma_2} (S^1 \wedge S^1) \cong \Sigma\mathbb{RP}^{n-1}$$

and an easy map

$$\Sigma^n \mathbb{RP}^{n-1} = S^{2n-1}/\Sigma_2 \longrightarrow S^n.$$

Suppose that $\Theta : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^\infty$ is an S-map whose mapping cone is denoted by $\mathrm{Cone}(\Theta)$. Furthermore [48] the image of $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ under the Kahn-Priddy map has non-trivial Arf-Kervaire invariant if and only if the Steenrod operation ([251], [259])

$$Sq^{2^n} : H^{2^n-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow H^{2^{n+1}-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$$

is non-trivial.

Briefly, the reason for this is as follows. It is known that the splitting map in the Kahn-Priddy theorem lowers the Adams spectral sequence filtration [131]. The criterion used in [47] is that an element in the stable homotopy of spheres has Arf-Kervaire invariant one if and only if it is represented by $h^2_{2^n-1}$ on the $s = 2$ line of the Adams spectral sequence (see Theorem 1.1.2). These elements are in filtration two and therefore $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ must be in filtration one or zero in the Adams spectral sequence for $\mathbb{RP}^\infty$. It is easy to show that the filtration cannot be zero, since this would mean that the Hurewicz image of $[\Theta]$ in $H_*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is non-zero. In order to be in filtration one $[\Theta]$ has to be detected by a primary Steenrod operation on the mod 2 homology of its mapping cone and since the Steenrod algebra is generated by the Sq^{2^i}'s one of these must detect $[\Theta]$. The only possibility is Sq^{2^n}.

(iii) Here are two results from [247] which we shall prove in Chapter 2, Theorem 2.2.2 and 2.2.3:

By Theorem 1.2.6 a stable homotopy class in $\pi_{2^n-2}(\Sigma^\infty BO)$ may be considered as a pair (M, E) where M is a frameable (2^n-2)-manifold and E is a virtual vector bundle on M. These results use splittings constructed in [222] and [245] together with Dyer-Lashof operations and Steenrod operations in mod 2 homology of QX.

Let $b_i \in H_i(\mathbb{RP}^\infty; \mathbb{Z}/2)$ be a generator, then there is an algebra isomorphism $H_*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[b_1, b_2, b_3, \ldots]$ where the algebra multiplication is denoted by $x * y$.

Theorem 1.8.6. For $n \geq 3$ there exists a framed (2^n-2)-manifold with a non-zero Arf-Kervaire invariant if and only if $b_{2^{n-1}-1} * b_{2^{n-1}-1} \in H_{2^n-2}(BO; \mathbb{Z}/2)$ is stably spherical.

Theorem 1.8.7. $\mathrm{Arf}(M^{2^n-2}, E) = \langle [M], w_2(E)^{2^{n-1}-1} \rangle$.

The following result follows easily from Theorem 1.8.6 (the proofs will be given in Chapter 2, § 2.2.8 and § 2.2.9). The idea of making the requisite framed manifold as a quotient of a balanced product of a surface with $(\mathbb{RP}^7)^4$ goes back at least to Jim Milgram and was used by John Jones in the highly calculational [127]. Corollary 1.8.8 is a significant simplification of [127]; however, the latter gives the framing explicitly (on a slightly different manifold) .

Corollary 1.8.8. There exists a framed 30-manifold with non-trivial Arf invariant.

1.8.9. ***ju*$_*$-theory.** Now let bu denote 2-adic connective K-theory and define ju-theory by means of the fibration $ju \longrightarrow bu \xrightarrow{\psi^3 - 1} bu$, as in Example 1.3.4(iv). Hence ju_* is a generalised homology theory for which $ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$. Recall that, if $\iota \in ju_{2^{n+1}-2}(S^{2^{n+1}-2}) \cong \mathbb{Z}_2$ is a choice of generator, the associated ju-theory Hurewicz homomorphism

$$H_{ju} : \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty) \longrightarrow ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$$

is defined by $H_{ju}([\theta]) = \theta_*(\iota)$.

We are now ready to state the ju_*-formulation, which will be proved in Chapter 7, Theorem 7.2.2.

I first overheard the following theorem conjectured by Mark Mahowald in the late 1970's. The motivation for the conjecture was the fact that if $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ satisfies $2[\Theta] = 0$, then one can fairly easily prove the result using [4] – a fact which I learnt from John Jones. Therefore the difficulty is to remove the assumption that $[\Theta]$ has order two. It should be noted that all the elements of Arf-Kervaire one which are known to exist to date *do* have order two!

It was eventually conjectured in public by Michael Barratt, John Jones and Mark Mahowald in [30].

It was "proved" with a gap in [140], proved properly in [141] – using results of Haynes Miller, Doug Ravenel and Steve Wilson [186] – and proved differently in [251] using the Hurewicz homomorphism in BP-theory (see Chapter 7). The proofs which I shall give in Chapter 7 §§ 7.2.3–7.2.5 use, to differing degrees, the "upper triangular technology" of [252], [27] and [254] (see also Chapters 3 and 5) and are considerably simpler than previous ones.

Theorem 1.8.10. For $n \geq 1$ the image of $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ under the ju-theory Hurewicz homomorphism

$$H_{ju}([\Theta]) \in ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$$

is non-trivial if and only if Sq^{2^n} is non-trivial on $H^{2^n-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$.

In any case, $2H_{ju}([\Theta]) = 0$.

Chapter 2

The Arf-Kervaire Invariant via QX

"I know what you're thinking about," said Tweedledum; "but it isn't so, nohow." "Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't; it ain't. That's logic."

"Through the Looking Glass" by Lewis Carroll [55]

The objective of this chapter is to study $\pi_i(\Sigma^\infty X)$ by identifying it with $\pi_i(QX)$ where QX is the space introduced in Chapter 1 § 1.5.8. Since we are interested in the two-primary part of the stable homotopy groups of spheres we would like to concentrate on the case when $X = S^0$. However, for technical reasons concerned with the mod 2 homology of QX we have to consider the case when X is connected. Fortunately, by virtue of the Kahn-Priddy theorem of Chapter 1 Theorem 1.5.10, it is no restriction to consider only connected X since we can study the case when $X = \mathbb{RP}^\infty$.

Therefore our primary strategy will be to analyse invariants of elements of $\pi_i(\Sigma^\infty S^0)$ by lifting them to $\pi_i(\Sigma^\infty \mathbb{RP}^\infty)$. Then the question arises of which invariants to study. The simplest invariant of a map, for $i > 0$,

$$g : S^i \longrightarrow Q\mathbb{RP}^\infty$$

is its image under the mod 2 Hurewicz homomorphism

$$g_*(\iota) \in H_i(Q\mathbb{RP}^\infty; \mathbb{Z}/2)$$

where $\iota \in H_i(S^i; \mathbb{Z}/2)$ is a generator. The simple structure of the mod 2 homology of a sphere imposes the conditions that $g_*(\iota)$ will be a primitive element in the Hopf algebra $H_*(Q\mathbb{RP}^\infty; \mathbb{Z}/2)$ which, in addition, is annihilated by the homomorphisms which are dual to the Steenrod operations of Chapter 1 § 1.6.1.

As we shall see, the stable homotopy classes corresponding to framed manifolds having Arf-Kervaire invariant one, as in Chapter 1 § 1.8.5, will give rise to maps with non-zero Hurewicz image in mod 2 homology. To study these classes we may restrict ourselves, by a result of Bill Browder [47], to dimensions of the form

$i = 2^n - 2$. Accordingly I will begin by classifying Steenrod annihilated primitives in $H_{2^n-2}(QX;\mathbb{Z}/2)$ and $H_{2^n-1}(QX;\mathbb{Z}/2)$. Then this classification will be used to give the Arf-Kervaire invariant formulae which were promised in Chapter 1 Theorem 1.8.6 and Theorem 1.8.7 and to give a simple proof of the existence of a 30-dimensional framed manifold of Arf-Kervaire invariant one, as per Chapter 1 Corollary 1.8.8.

Almost all the material of this chapter is taken from the paper [247] which was written during a sabbatical visit to Aarhus in 1981, when I first developed an interest in the Arf-Kervaire invariant and learnt from Jorgen Tornehave some expertise in calculating it. At the end of the chapter (Theorem 2.3.6) I use the results to give an alternative proof of a result of Kee Lam [152] on the non-desuspendability of the Arf-Kervaire invariant one classes.

2.1 The Steenrod annihilated submodule

2.1.1. Homology modulo 2. Let $H_*(W;\mathbb{Z}/2)$ denote the singular homology modulo 2 of the space W and let

$$\langle -,-\rangle : H_*(W;\mathbb{Z}/2) \otimes H^*(W;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

denote the canonical non-singular pairing ([9], [257], [105]). The dual Steenrod operation Sq^t_* is the homology operation

$$Sq^t_* : H_n(W;\mathbb{Z}/2) \longrightarrow H_{n-t}(W;\mathbb{Z}/2)$$

characterised by the equation $\langle Sq^t_*(a),\alpha\rangle = \langle a, Sq^t(\alpha)\rangle$ for all $a \in H_*(W;\mathbb{Z}/2)$ and $\alpha \in H^*(W;\mathbb{Z}/2)$.

Denote by $H_*(W;\mathbb{Z}/2)_{\mathcal{A}}$ the $\mathcal{A}$-annihilated submodule

$$H_*(W;\mathbb{Z}/2)_{\mathcal{A}^*} = \{x \in H_*(W;\mathbb{Z}/2) \mid Sq^n_*(x) = 0 \text{ for all } n > 0\}.$$

Similarly we define the Steenrod annihilated submodule $M_{\mathcal{A}}$ for quotients and submodules M of homology which have well-defined actions induced by the Sq^t_*'s.

In this section we shall prove the following result, which will be used in § 2 to prove Theorems 2.2.2 and 2.2.3.

Theorem 2.1.2. Let $Y = QX = \lim_{\overrightarrow{n}} \Omega^n\Sigma^n X$ with X a connected CW complex. Then the $\mathcal{A}$-annihilated decomposables $\underline{Q}H_*(Y;\mathbb{Z}/2)_{\mathcal{A}}$ satisfy:

(i) For $n \geq 2$,

$$\underline{Q}H_{2^n-2}(Y;\mathbb{Z}/2)_{\mathcal{A}} \cong H_{2^n-2}(X;\mathbb{Z}/2)_{\mathcal{A}}.$$

(ii) For $n \geq 2$,

$$\underline{Q}H_{2^n-1}(Y;\mathbb{Z}/2)_{\mathcal{A}} \cong H_{2^n-1}(X;\mathbb{Z}/2)_{\mathcal{A}} \oplus Q^{2^{n-1}}(H_{2^{n-1}-1}(X;\mathbb{Z}/2))$$

where Q^i denotes the ith homology operation on $H_*(Y;\mathbb{Z}/2)$ ([21], [59], [73]) as in Chapter 1 § 1.7.2.

Theorem 2.1.2 will be proved in § 2.1.10 after some elementary preparatory lemmas. By ([21], [59], [73]) we may write $y \in \underline{Q}H_t(Y;\mathbb{Z}/2)$ as a sum

$$y = \sum_{\underline{r}} Q^{r_1}Q^{r_2}\dots Q^{r_s}(x(\underline{r}))$$

of non-trivial admissible monomials $Q^{r_1}Q^{r_2}\dots Q^{r_s}(x(\underline{r}))$. That is, as in Chapter 1, Definition 1.7.3, each sequence $\underline{r} = (r_1, r_2, \dots, r_s)$ satisfies $r_i \leq 2r_{i+1}$ for $1 \leq i \leq s-1$ and $r_u > r_{u+1} + r_{u+2} + \cdots + r_s + \deg(x(\underline{r}))$ for $1 \leq u \leq s$.

The proof of Theorem 2.1.2 will proceed by inductively applying $Sq_*^{2^j}$ to y for $j = 1, 2, \dots$. However, in general $Sq_*^{2^j}Q^{r_1}Q^{r_2}\dots Q^{r_s}(x(\underline{r}))$ may become very complicated when expressed as a sum of admissible monomials by means of the Adem and Nishida relations of Chapter 1 § 1.7.2(viii) and (ix) (see also [59] p. 5). By establishing the following inductive hypothesis we keep this process under control.

Definition 2.1.3. A monomial $Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j)$ if either:

(i) $s > j$ and $r_t \equiv -1$ (modulo 2^{j+2-t}) for $t = 1, 2, \dots, j+1$
or
(ii) $s \leq j$ and $r_t \equiv -1$ (modulo 2^{j+2-t}) for $t = 1, 2, \dots, s$ and $Sq_*^{2^m}(x) = 0$ for $m = 0, 1, \dots, j-s$.

Remark 2.1.4. If $Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j)$ and $s > 0$ then $Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j-1)$.

Lemma 2.1.5. If $y = Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j)$ then $Sq_*^t(y) = 0$ for $0 < t < 2^{j+1}$.

Proof. By the Adem relations ([259]; see also Chapter 1 § 1.6.1(vii)) it suffices to verify this for $t = 2^l$ with $0 \leq l \leq j$. If $s = 0$ this is immediate from condition $H(j)$. By the Nishida relations ([59] p. 5; see also Chapter 1 § 1.7.2(ix))

$$\begin{aligned} Sq_*^{2^l}(y) = {} & \binom{r_1 - 2^l}{2^l} Q^{r_1-2^l}Q^{r_2}\dots Q^{r_s}x \\ & + \sum_{0<t\leq 2^{l-1}} \binom{r_1 - 2^l}{2^l - 2t} Q^{r_1+t-2^l} Sq_*^t Q^{r_2}\dots Q^{r_s}x. \end{aligned}$$

Since $r_1 \equiv -1$ (modulo 2^{j+1}), by Definition 2.1.3, the binomial coefficient of the leading term is zero (see Chapter 5 § 2) while the remaining terms vanish by induction on s, by Remark 2.1.4. □

Lemma 2.1.6. If $y = Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j-1)$ then

$$Sq_*^{2^j}(y) = \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j}Q^{r_2}\dots Q^{r_s}x + Q^{r_1-2^{j-1}}Sq_*^{2^{j-1}}Q^{r_2}\dots Q^{r_s}x.$$

Proof. By the Nishida relations ([59] p. 5; see also Chapter 1 § 1.7.2(ix))

$$\begin{aligned} Sq_*^{2^j}(y) &= \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j} Q^{r_2} \ldots Q^{r_s} x \\ &\quad + \sum_{0<t\leq 2^{j-1}} \binom{r_1 - 2^j}{2^j - 2t} Q^{r_1+t-2^j} Sq_*^t Q^{r_2} \ldots Q^{r_s} x. \end{aligned}$$

By Lemma 2.1.5, $Sq_*^t Q^{r_2} \ldots Q^{r_s} x = 0$ for $t < 2^{j-1}$ which leaves only the two required terms. □

Corollary 2.1.7. If $s > j$ and $y = Q^{r_1} Q^{r_2} \ldots Q^{r_s} x$ satisfies condition $H(j-1)$ then

$$\begin{aligned} Sq_*^{2^j}(y) &= \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j} Q^{r_2} \ldots Q^{r_s} x \\ &\quad + \sum_{u=2}^{j+1} \binom{r_u - 2^{j-u+1}}{2^{j-u+1}} Q^{r_1-2^{j-1}} Q^{r_2-2^{j-2}} \ldots \\ &\quad \ldots Q^{r_{u-1}-2^{j-u+1}} Q^{r_u-2^{j-u+1}} \ldots Q^{r_s} x. \end{aligned}$$

Proof. By Lemma 2.1.6, when $j = 1$ we have

$$\begin{aligned} Sq_*^2(y) &= \binom{r_1 - 2}{2} Q^{r_1-2} Q^{r_2} \ldots Q^{r_s} x + Q^{r_1-1} Sq_*^1 Q^{r_2} \ldots Q^{r_s} x \\ &= \binom{r_1 - 2}{2} Q^{r_1-2} Q^{r_2} \ldots Q^{r_s} x + \binom{r_2 - 1}{1} Q^{r_1-1} Q^{r_2-1} \ldots Q^{r_s} x \end{aligned}$$

which begins the induction. The result now follows by induction, using Remark 2.1.4 and Lemma 2.1.6, because

$$\begin{aligned} Sq_*^{2^j}(y) &= \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j} Q^{r_2} \ldots Q^{r_s} x + Q^{r_1-2^{j-1}} Sq_*^{2^{j-1}} Q^{r_2} \ldots Q^{r_s} x \\ &= \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j} Q^{r_2} \ldots Q^{r_s} x + \binom{r_2 - 2^{j-1}}{2^{j-1}} Q^{r_1-2^{j-1}} Q^{r_2-2^{j-1}} Q^{r_3} \ldots Q^{r_s} x \\ &\quad + \sum_{u=2}^{j} \binom{r_{u+1} - 2^{j-u}}{2^{j-u}} Q^{r_1-2^{j-1}} Q^{r_2-2^{j-2}} Q^{r_3-2^{j-3}} \ldots \\ &\qquad \ldots Q^{r_u-2^{j-u}} Q^{r_{u+1}-2^{j-u}} \ldots Q^{r_s} x \\ &= \binom{r_1 - 2^j}{2^j} Q^{r_1-2^j} Q^{r_2} \ldots Q^{r_s} x \\ &\quad + \sum_{u=2}^{j+1} \binom{r_u - 2^{j-u+1}}{2^{j-u+1}} Q^{r_1-2^{j-1}} Q^{r_2-2^{j-2}} \ldots Q^{r_{u-1}-2^{j-u+1}} Q^{r_u-2^{j-u+1}} \ldots Q^{r_s} x \end{aligned}$$

as required. □

Corollary 2.1.8. If $s \leq j$ and $y = Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j-1)$ then

$$\begin{aligned}Sq_*^{2^j}(y) = &\binom{r_1 - 2^j}{2^j} Q^{r_1-2^j}Q^{r_2}\dots Q^{r_s}x\\ &+ \sum_{u=2}^{s}\binom{r_u - 2^{j-u+1}}{2^{j-u+1}} Q^{r_1-2^{j-1}}Q^{r_2-2^{j-2}}\dots Q^{r_{u-1}-2^{j-u+1}}Q^{r_u-2^{j-u+1}}\dots Q^{r_s}x\\ &+ Q^{r_1-2^{j-1}}Q^{r_2-2^{j-2}}\dots Q^{r_s-2^{j-s+1}}Sq_*^{2^{j-s+1}}(x).\end{aligned}$$

Proof. The induction is similar to that in the proof of Corollary 2.1.7, starting with the case when $s = j$ which follows from Lemma 2.1.6 by induction on s. $\square$

Lemma 2.1.9. If $y = Q^{r_1}Q^{r_2}\dots Q^{r_s}x$ satisfies condition $H(j-1)$ and is admissible (that is, $r_i \leq 2r_{i+1}$ for $i = 1, 2, \dots, s-1$; see the explanation following Theorem 2.1.2) then each term of Corollaries 2.1.7 and 2.1.8 is admissible.

Proof. If $r_m \leq 2r_{m+1}$ then $r_m - 2^\epsilon \leq 2r_{m+1}$ so that a change from (r_m, r_{m+1}) to $(r_m - 2^\epsilon, r_{m+1})$ is admissible. Also $r_m - 2^\epsilon \leq 2(r_{m+1} - 2^{\epsilon-1})$ so that a change to $(r_m - 2^\epsilon, r_{m+1} - 2^{\epsilon-1})$ is also admissible. Consider the change from (r_m, r_{m+1}) to $(r_m - 2^{j-m}, r_{m+1} - 2^{j-m})$ for $m \leq s$ so that $r_m \equiv -1$ (modulo 2^{j-m+1}) and $r_{m+1} \equiv -1$ (modulo 2^{j-m}). Suppose this change is inadmissible; then $2r_{m+1} \geq r_m > 2r_{m+1} - 2^{j-m}$ which contradicts the above congruences. $\square$

2.1.10. The proof of Theorem 2.1.2. Suppose that $y = \sum_{\underline{r}} Q^{r_1}Q^{r_2}\dots Q^{r_s}x(\underline{r})$ is in $\underline{Q}H_{2^n-\epsilon}(Y;\mathbb{Z}/2)_{\mathcal{A}}$ with $\epsilon = 1$ or 2. We may assume that the monomials of y are admissible and linearly independent. Since Adem and Nishida relations on $\underline{Q}H_*$ respect length we may assume that each monomial has the same length equal to $s \leq n$. In addition each $\underline{r} = (r_1, r_2, \dots, r_s)$ must have $2^{n-1} \leq r_1 \leq 2^n - \epsilon - 1$ since X is connected and $Q^{2^n-\epsilon}(1) = 0$.

For $j \leq n-1$ assume that each monomial in y satisfies condition $H(j-1)$ and consider the equation $Sq_*^{2^j}(y) = 0$ in $\underline{Q}H_{2^n-\epsilon-2^j}(Y;\mathbb{Z}/2)$. The expression for $Sq_*^{2^j}(y)$ is the sum over $\underline{r}$ of the formulae of Corollaries 2.1.7 and 2.1.8 in which, by Lemma 2.1.9, each term is admissible.

We shall show below that when $\epsilon = 2$ and $0 < s$ all the monomials in the sum are indecomposable and hence evidently linearly independent. This will complete the proof in dimension $2^n - 2$ because the only possible monomials remaining are those for which $s = 0$, as required.

If $r_u \equiv -1$ (modulo 2^{j-u+1}) the binomial coefficient

$$\binom{r_u - 2^{j-u+1}}{2^{j-u+1}}$$

is even if and only if $r_u \equiv -1$ (modulo 2^{j-u+2}) (see Chapter 5 § 2) while

$$Q^{r_1-2^{j-1}}Q^{r_2-2^{j-2}}\dots Q^{r_s-2^{j-s+1}}Sq_*^{2^{j-s+1}}(x(\underline{r}))$$

is decomposable if and only if $Sq_*^{2^{j-s+1}}(x(\underline{r})) = 0$. Hence each non-zero monomial in y satisfies condition $H(j)$ and $y = 0$ (since we are assuming $0 < s$) by induction on j.

Similarly in dimension $2^n - 1$ we shall show that only the monomial $Q^{2^{n-1}}x$ (when $s = 1$ and $\deg(x) = 2^{n-1} - 1$) contributes a decomposable term to $Sq_*^{2^j}(y)$. The proof in dimension $2^n - 1$ is then completed by observing that $y = v + Q^{2^{n-1}}x$ ($v \in H_{2^n-1}(X;\mathbb{Z}/2)$, $x \in H_{2^{n-1}-1}(X;\mathbb{Z}/2)$) is in $\underline{Q}H_{2^n-1}(Y;\mathbb{Z}/2)_{\mathcal{A}}$ if and only if v is $\mathcal{A}$-annihilated.

To complete the proof, therefore, we must examine for decomposability the monomials in the formulae of Corollaries 2.1.7 and 2.1.8. The monomial $Q^{r_1} Q^{r_2} \ldots Q^{r_s}x(\underline{r})$ satisfies

$$r_u > r_{u+1} + \cdots + r_s + \deg(x(\underline{r}))$$

for $u = 1, \ldots, s$.

Consider first the $(2^n - 2)$-dimensional case. One finds that $Q^{r_1 - 2^j} Q^{r_2} \ldots Q^{r_s}x(\underline{r})$ can only be decomposable if $2r_1 - 2^j \leq 2^n - 2 < 2r_1$ so that $r_1 = 2^{n-1} + 2^{j-1} - 1$ which is impossible if $r_1 \equiv -1$ (modulo 2^j). Similarly if $s > j$ and $u \geq 2$ then $Q^{r_1-2^{j-1}}Q^{r_2-2^{j-2}} \ldots Q^{r_{u-1}-2^{j-u+1}}Q^{r_u-2^{j-u+1}} \ldots Q^{r_s}x(\underline{r})$ is decomposable only if $r_u - 2^{j-u+1} \leq r_{u+1} + \cdots + r_s + \deg(x(\underline{r})) < r_u$. Since $Q^{r_1}Q^{r_2} \ldots Q^{r_s}x(\underline{r})$ satisfies condition $H(j-1)$ this means that $r_{u+1} + \cdots + r_s + \deg(x(\underline{r})) = r_u + u - 1 - 2^{j-u+1}$ from which $2^n - 2 = r_1 + r_2 + \cdots + 2r_u + u - 1 - 2^{j-u+1}$ which implies $-2 \equiv -2 - 2^{j-u+1}$ (modulo 2^{j-u+2}) which is impossible. If $s \leq j \leq n-1$ then $Q^{r_1-2^{j-1}} \ldots Q^{r_s-2^{j-s+1}}Sq_*^{2^{j-s+1}}(x(\underline{r}))$ is always indecomposable if $Sq_*^{2^{j-s+1}}(x(\underline{r})) \neq 0$.

For the $(2^n - 1)$-dimensional case a similar argument shows that decomposable terms occur only when $s = 1$ and that these come only from monomials of the form $Q^{2^{n-1}}x$ with $\deg(x) = 2^{n-1} - 1$. This completes the proof of Theorem 2.1.2. $\square$

Example 2.1.11. (i) If M is a connected closed compact (2^n-2)-dimensional manifold then $\underline{Q}H_{2^n-2}(QM;\mathbb{Z}/2)_{\mathcal{A}}$ is either zero or $\mathbb{Z}/2$ generated by the fundamental class of M, denoted by $[M]$.

(ii) Let $0 \neq b_j \in H_j(\mathbb{RP}^\infty;\mathbb{Z}/2)$ so that

$$Sq_*^a(b_j) = \binom{j-a}{a} b_{j-a}.$$

Therefore $\underline{Q}H_{2^n-2}(Q\mathbb{RP}^\infty;\mathbb{Z}/2)_{\mathcal{A}} = 0$ for $n \geq 2$. Also

$$\underline{Q}H_{2^n-1}(Q\mathbb{RP}^\infty;\mathbb{Z}/2)_{\mathcal{A}} \cong \mathbb{Z}/2\langle b_{2^{n-1}-1}\rangle \oplus \mathbb{Z}/2\langle Q^{2^{n-2}}(b_{2^{n-2}-1})\rangle$$

because

$$\binom{2^m-1-a}{a} \equiv 0 \quad \text{(modulo 2)}.$$

(iii) Let $O(2)$ denote the orthogonal group of 2×2 real matrices W satisfying $WW^{\mathrm{tr}} = I$, where W^{tr} is the transpose, and let D_8 denote the dihedral subgroup of order eight generated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the diagonal matrices. The inclusion of the diagonal matrices gives a chain of groups

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \subset D_8 \subset O(2).$$

The homology modulo 2 of these groups is well known (see [221], [222], [250]). Write $b_i * b_j$ for the image of $b_i \otimes b_j \in H_{i+j}(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}/2)$ in $H_{i+j}(G; \mathbb{Z}/2)$ for $G = D_8$ or $G = O(2)$. By an argument similar to that of § 2.1.10 one finds that

$$H_{2^n-2}(BG; \mathbb{Z}/2)_{\mathcal{A}} = \mathbb{Z}/2\langle b_{2^{n-1}-1} * b_{2^{n-1}-1} \rangle$$

when $G = D_8$ or $G = O(2)$. Hence

$$\underline{Q}H_{2^n-2}(QBG; \mathbb{Z}/2)_{\mathcal{A}} \cong \mathbb{Z}/2\langle b_{2^{n-1}-1} * b_{2^{n-1}-1} \rangle$$

in these cases.

Definition 2.1.12. Since QX is an H-space its homology $H_*(QX; \mathbb{Z}/2)$ is a Hopf algebra (see Chapter 1 Definition 1.6.5). The m-dimensional primitives of this Hopf algebra are given by

$$\underline{P}H_*(QX; \mathbb{Z}/2) = \{x \in H_m(QX; \mathbb{Z}/2) \mid \psi(x) = x \otimes 1 + 1 \otimes x\}$$

where ψ is the comultiplication induced by the diagonal map $QX \longrightarrow QX \times QX$.

We are going to turn our attention now to $\underline{P}H_*(QX; \mathbb{Z}/2)_{\mathcal{A}}$ in the cases of Example 2.1.11.

Firstly, from the exact sequence of Milnor-Moore [190] we have the following result:

Lemma 2.1.13. Let X be connected and $Y = QX$. Then there is an exact sequence of the form

$$0 \longrightarrow \underline{P}H_m(Y; \mathbb{Z}/2)_{\mathcal{A}} \xrightarrow{i} \underline{P}H_{2m}(Y; \mathbb{Z}/2)_{\mathcal{A}} \xrightarrow{\pi} \underline{Q}H_{2m}(Y; \mathbb{Z}/2)_{\mathcal{A}}$$

where $i(x) = x^2$ and $\pi(y) \equiv y$ (modulo decomposables).

Proof. Without the subscript this would be part of the Milnor-Moore exact sequence referred to above. Therefore we must first show that $\mathrm{Ker}(\pi) \subseteq \mathrm{Im}(i)$ because the rest of the result is clear. Suppose that $x \in \underline{P}H_{2m}(Y; \mathbb{Z}/2)_{\mathcal{A}}$ and $\pi(x) = 0$. Therefore $x = y^2$ with y primitive. We must show that $Sq^a_*(y) = 0$ for all $a > 0$. However, this is immediate since $Sq^{2a}_*(y^2) = Sq^a_*(y)^2$ and squaring is injective on $H_*(Y; \mathbb{Z}/2)$. The same argument shows that $Sq^a_*(y)^2 = 0$ if y is a Steenrod annihilated primitive and in this case $Sq^{2a+1}_*(y^2) = 0$ by the Cartan formula (Chapter 1, Definition 1.6.1(v)). $\square$

Example 2.1.14. (i) Let $X = \mathbb{RP}^\infty$ and let $N_m(x_1, x_2, \ldots)$ denote the mth Newton polynomial in variables $x_1, x_2, \ldots$. From Example 2.1.11(ii) and Lemma 2.1.13 we know that $\underline{P}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ is generated by squares of elements in $\underline{P}H_{2^{n-1}-1}(QX;\mathbb{Z}/2)_{\mathcal{A}}$.

Since $\underline{P}H_{2^{n-1}-1}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ injects into $\underline{Q}H_{2^{n-1}-1}(QX;\mathbb{Z}/2)_{\mathcal{A}}$, Example 2.1.11(ii) implies that any element of $\underline{P}H_{2^{n-1}-1}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ may be written in the form

$$\lambda N_{2^{n-1}-1} + \mu Q^{2^{n-2}}(N_{2^{n-2}-1})$$

for some $\lambda, \mu \in \mathbb{Z}/2$ where $N_j = N_j(b_1, b_2, \ldots) \equiv (-1)^j j b_j$ (modulo decomposables).

By induction, using the formula

$$N_m - b_1 N_{m-1} + b_2 N_{m-2} - \cdots + (-1)^m m b_m = 0$$

we may evaluate $Sq^a_*(N_j)$ and we find that

$$Sq^a_*(N_{2^n-1}) = 0, \text{ if } a > 1$$

and

$$Sq^1_*(N_{2^n-1}) = N_{2^n-2} = (N_{2^{n-1}-1})^2$$

so that $\lambda = \mu$.

Hence

$$\underline{P}H_{2^n-2}(Q\mathbb{RP}^\infty;\mathbb{Z}/2)_{\mathcal{A}} = \mathbb{Z}/2\langle (N_{2^{n-1}-1})^2 + (Q^{2^{n-2}} N_{2^{n-2}-1})^2 \rangle.$$

(ii) Let $X = BD_8$ or $X = BO(2)$. From Example 2.1.11(iii) the image of $\underline{P}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ in $\underline{Q}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ is generated by $\lambda b_{2^{n-1}-1} * b_{2^{n-1}-1}$ for some $\lambda \in \mathbb{Z}/2$. Arguing as in Example 2.1.11(i) any element of $\underline{P}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ has the form

$$p_n + v^2 + (Q^{2^{n-2}}(x))^2$$

where $p_n \in \underline{P}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ is either zero or hits

$$b_{2^{n-1}-1} * b_{2^{n-1}-1} \in \underline{Q}H_{2^n-2}(QX;\mathbb{Z}/2)_{\mathcal{A}}$$

and $v \in \underline{P}H_{2^{n-1}-1}(QX;\mathbb{Z}/2)$, $x \in \underline{P}H_{2^{n-2}-1}(QX;\mathbb{Z}/2)_{\mathcal{A}}$ satisfy $Sq^1_*(v) = x^2$.

2.2 The Arf-Kervaire invariant and $\pi_*(\Sigma^\infty BO)$

2.2.1. By Chapter 1 Theorem 1.2.6, a stable homotopy class in $\pi_{2^n-2}(\Sigma^\infty BO)$ may be considered as a pair (M, E) where M is a $(2^n - 2)$-dimensional, frameable manifold and E is a virtual real vector bundle on M. In this section we shall explain the fact, mentioned in Chapter 1 § 1.8.5(iii), that (M, E) has an Arf-Kervaire invariant which lies in $\mathbb{Z}/2$. $\mathrm{Arf}(M, E)$ depends on the frameability of M

but not on the choice of framing, which is a considerable computational advantage (see the application given in § 2.3.1).

For $n \geq 3$ finding such pairings with $\mathrm{Arf}(M, E) \not\equiv 0$ (modulo 2) is equivalent to the classical problem (see Chapter 1 § 1.8.3) of finding framed manifolds of Arf-Kervaire invariant one. This problem has been extensively studied ([138], [47], [183], [50], [127], [129], [247], [140], [141], [251], [254]).

In this section we shall prove the following two results, which were promised in Chapter 1 § 1.8.6 and § 1.8.7.

Theorem 2.2.2. For $n \geq 3$ there exists a framed $(2^n - 2)$-manifold with non-zero Arf-Kervaire invariant if and only if $b_{2^{n-1}-1} * b_{2^{n-1}-1} \in H_{2^n-2}(BO; \mathbb{Z}/2)$ is stably spherical (that is, the Hurewicz image of an element of $\pi_{2^n-2}(\Sigma^\infty BO)$).

Theorem 2.2.3. $\mathrm{Arf}(M, E) = \langle [M], w_2(E)^{2^{n-1}-1} \rangle$.

In Proposition 2.3.4 we illustrate Theorem 2.2.3 when $n = 5$ in constructing a 30-dimensional framed manifold with non-zero Arf-Kervaire invariant.

2.2.4. Throughout this section we shall work in the 2-local homotopy and stable homotopy category (see Chapter 1, Example 1.3.4(iii) and [43]).

A closed, compact framed manifold N^{2^n-2} is equivalent to an element $\Theta_N \in \pi_{2^n-2}(\Sigma^\infty S^0)$ (see Chapter 1, Theorem 1.2.6). The Arf-Kervaire invariant of N has the following definition when $n \geq 3$. Form the adjoint

$$\mathrm{adj}(\Theta_N) : S^{2^n-2} \longrightarrow Q_0 S^0.$$

Here $Q_i S^0$ is the component of $QS^0 = \lim_{\overrightarrow{n}} \Omega^n S^n$ consisting of maps of degree i. Translating $\mathrm{adj}(\Theta_N)$ from $Q_0 S^0$ to $Q_1 S^0 = SG$ gives

$$\hat{\Theta}_N = \mathrm{adj}(\Theta_N) * [1] : S^{2^n-2} \longrightarrow SG.$$

Form the composite of this map with the canonical maps π and k to give

$$S^{2^n-2} \xrightarrow{\hat{\Theta}_N} SG \xrightarrow{\pi} G/O \xrightarrow{k} G/Top \simeq \prod_m K(G_m, 2m)$$

where $G_{2k} = \mathbb{Z}_{(2)}$, the 2-local integers, and $G_{2k+1} = \mathbb{Z}/2$. The homotopy group of such maps is $\mathbb{Z}/2$ and the Arf-Kervaire invariant of N^{2^n-2} is given by the homotopy class of this composite as an element of $\mathbb{Z}/2$.

Let $A : BO \longrightarrow G/O$ be a solution of the Adams conjecture ([5], [32], [52]). Let $A : BO(2) \longrightarrow G/O$ also denote the restriction of the solution of the Adams conjecture. There is a lifting A' of the restriction of A to BD_8 which factors through SG and we obtain the following homotopy commutative diagram.

$$\begin{tikzcd}
QBD_8 \arrow[r] & Q(BO(2)) \arrow[r] & Q(BO) \\
Q(SG) \arrow[r] \arrow[u, "QA'"] & Q(G/O) \arrow[r, "1"] \arrow[u, "QA"] & Q(G/O) \arrow[u, "QA"] \\
SG \arrow[r, "\pi"] \arrow[u, "D"] & G/O \arrow[u, "D"] &
\end{tikzcd}$$

In the above diagram D denotes the structure map of an infinite loopspace [59] (see also Chapter 1 § 1.5.8).

Proposition 2.2.5 ([222]). The maps $D \cdot QA'$ and $D \cdot QA$ in § 2.2.4 are both split surjections.

2.2.6. In addition there are splittings ([166], [167])

$$SG \simeq J \times \mathrm{Coker}\, J, \qquad G/O \simeq BSO \times \mathrm{Coker}\, J$$

such that π may be identified with a map of the form

$$\pi = \pi' \times 1 : J \times \mathrm{Coker}\, J \longrightarrow BSO \times \mathrm{Coker}\, J.$$

The Arf-Kervaire classes for $n \geq 3$,

$$k_{2^n-2} \in H^{2^n-2}(G/\,\mathrm{Top}; \mathbb{Z}/2)$$

are carried by the $Coker J$ factor ([52], [166], [167]). Hence, when $n \geq 3$, if $\hat{\Theta}_N$ in § 2.2.4 has non zero Arf-Kervaire invariant we may find a map

$$\Theta' : S^{2^n-2} \longrightarrow Q(BO)$$

such that $k \cdot \pi \cdot \Theta'$ is non-trivial.

However, by transversality, Θ' is equivalent to a frameable manifold M^{2^n-2} together with a map (that is, a virtual vector bundle; see [116]) $E : M \longrightarrow BO$. Given the pair (M, E) the map Θ' is recovered as sketched in Chapter 1 Theorem 1.2.6. There is a map $S^{2^n-2} \longrightarrow QM$ whose adjoint S-map induces a map which in homology modulo 2 hits the fundamental class of M ([116] p. 215) and the composite

$$S^{2^n-2} \longrightarrow QM \xrightarrow{QE} Q(BO)$$

is Θ'. By the Arf-Kervaire invariant of the pair (M^{2^n-2}, E), written $\mathrm{Arf}(M, E)$, we shall mean the homotopy class of

$$S^{2^n-2} \longrightarrow QM \xrightarrow{D \cdot Q(A \cdot E)} G/O \longrightarrow G/\,\mathrm{Top}\,.$$

Conversely, if $n \geq 3$, given a pair (M^{2^n-2}, E) with non-zero Arf-Kervaire invariant we may lift the Coker J-component of the resulting homotopy class in $\pi_{2^n-2}(G/O)$ to give an element of $\pi_{2^n-2}(SG)$ with non-trivial Arf-Kervaire invariant. Hence we have shown the following result:

Proposition 2.2.7. For $n \geq 3$ there exists a framed (2^n-2)-dimensional manifold with non-zero Arf-Kervaire invariant if and only if there exists a pair (M^{2^n-2}, E) consisting of a frameable manifold M and a virtual vector bundle on M with $\mathrm{Arf}(M, E)$ non-zero.

2.2.8. Proof of Theorem 2.2.2. Given $\Phi \in \pi_{2^n-2}(\Sigma^\infty BO)$ with Hurewicz image

$$\Phi_*(\iota) = b_{2^{n-1}-1} * b_{2^{n-1}-1} \in H_{2^n-2}(BO; \mathbb{Z}/2)$$

the stable splitting of BO ([245] p. 21) yields a stable homotopy class $\Theta \in \pi_{2^n-2}(\Sigma^\infty BO(2))$ with

$$\Theta_*(\iota) = b_{2^{n-1}-1} * b_{2^{n-1}-1} \in H_{2^n-2}(BO(2); \mathbb{Z}/2).$$

By Example 2.1.14(ii), the adjoint

$$g = \mathrm{adj}(\Theta) : S^{2^n-2} \longrightarrow QBO(2)$$

has Hurewicz image

$$p_n + v^2 + (Q^{2^{n-2}}(x))^2 \in H_{2^n-2}(QBO(2); \mathbb{Z}/2)$$

where $p_n \neq 0$ and p_n, v, x are as in Example 2.1.14(ii).

By ([184] § 2.10(b)) the map

$$k_{2^n-2} : G/O \longrightarrow G/\,\mathrm{Top} \longrightarrow K(\mathbb{Z}/2, 2^n-2)$$

is a primitive cohomology class. In fact, by ([169] Chapter 7), it is a two-fold loop map. Hence

$$\langle (D \cdot QA)_*(v^2 + (Q^{2^{n-2}}(x))^2), k_{2^n-2} \rangle = 0$$

and

$$\langle (D \cdot QA)_*(p_n), k_{2^n-2} \rangle = \langle (D \cdot QA)_*(b_{2^{n-1}-1} * b_{2^{n-1}-1}), k_{2^n-2} \rangle.$$

This last Kronecker product is equal to

$$\langle b_{2^{n-1}-1} * b_{2^{n-1}-1}, A^*(k_{2^n-2}) \rangle$$

since $b_{2^{n-1}-1} * b_{2^{n-1}-1}$ originates in $H_{2^n-2}(BO(2); \mathbb{Z}/2)$ and this is non-zero by the formula for $\overline{\Delta} A^*(k_{2^n-2})$ given in ([47] p. 164(i)). Hence Φ represents a pair (M, E) with $\mathrm{Arf}(M, E) \neq 0$ and the result follows from Proposition 2.2.7.

Conversely given $\Theta \in \pi_{2^n-2}(SG)$ with non-zero Arf-Kervaire invariant we may lift $\pi \cdot \Theta$ to Θ' through $QBO(2)$ by Proposition 2.2.5. The calculation above shows that $\mathrm{Arf}(\Theta)$ equals the coefficient of $b_{2^{n-1}-1} * b_{2^{n-1}-1}$ in the Hurewicz image

$$(\mathrm{adj}(\Theta'))_*(\iota) \in H_{2^n-2}(BO(2); \mathbb{Z}/2)$$

of the S-map $\mathrm{adj}(\Theta')$. However, by Example 2.1.14(ii), this is just the image of p_n in

$$\underline{Q}H_{2^n-2}(Q(BO(2)); \mathbb{Z}/2)_{\mathcal{A}} \cong H_{2^n-2}(BO(2); \mathbb{Z}/2)_{\mathcal{A}} = \langle b_{2^{n-1}-1} * b_{2^{n-1}-1} \rangle$$

so that $b_{2^{n-1}-1} * b_{2^{n-1}-1}$ is stably spherical, as required. $\square$

2.2.9. Proof of Theorem 2.2.3. The Arf-Kervaire invariant of (M, E) is the value of the homotopy class of the composite

$$S^{2^n-2} \xrightarrow{\lambda} QM \xrightarrow{D \cdot Q(A \cdot E)} G/O \xrightarrow{k_{2^n-2}} K(\mathbb{Z}/2, 2^n - 2)$$

where $\mathrm{adj}(\lambda)$ has S-map Hurewicz image equal to $[M] \in H_{2^n-2}(M; \mathbb{Z}/2)$. Hence $\lambda_*(\iota)$ lies in $\underline{P}H_{2^n-2}(QM; \mathbb{Z}/2)_{\mathcal{A}}$ and hits $[M]$ modulo decomposables, by Example 2.1.11(i). The argument of § 2.2.8 shows that $\mathrm{Arf}(M, E)$ is the coefficient of $b_{2^{n-1}-1} * b_{2^{n-1}-1}$ in $E_*[M] \in H_{2^n-2}(BO; \mathbb{Z}/2)$. However, this coefficient is easily seen to be equal to $\langle [M], w_2(E)^{2^{n-1}-1} \rangle$, as required. $\square$

2.3 Applications to the Arf-Kervaire invariant one problem

In this section we apply the formulae of this chapter to give a simple proof that there exists a 30-dimensional framed manifold of Arf-Kervaire invariant one, as promised in Chapter 1 § 1.8.8, and to prove a result of Kee Lam [152] using Hurewicz images in $Q\mathbb{RP}^\infty$.

2.3.1. Application to $\mathrm{Arf}(N^{30}, E)$. Let $M^2 = \mathbb{RP}^2 \#(S^1 \times S^1)$, the connected sum of $\mathbb{RP}^2$ and a 2-dimensional torus. Following [127] we use M^2 to construct a surface with a free D_8-action on it. The dihedral group D_8 may be realised as the subgroup of the symmetric group Σ_4 generated by $\sigma_1 = (1, 2)$ and $\tau = (1, 3)(2, 4)$. Hence, if $\sigma_2 = (3, 4)$ then

$$D_8 = \{\sigma_1, \sigma_2, \tau \mid \sigma_1^2 = \sigma_2^2 = \tau^2 = 1,\ \sigma_1\tau = \tau\sigma_2\}.$$

Let

$$D_8 \longrightarrow X \longrightarrow X/D_8 = M^2$$

be the D_8-principal bundle classified by the map

$$F : \mathbb{RP}^2 \#(S^1 \times S^1) \xrightarrow{\text{collapse}} \mathbb{RP}^2 \vee (S^1 \times S^1) \xrightarrow{f \vee g} BD_8$$

where f is given by

$$f : \mathbb{RP}^2 \subseteq \mathbb{RP}^\infty = B\langle \sigma_1 \rangle \longrightarrow BD_8$$

and g is given by

$$g : S^1 \times S^1 \subset \mathbb{RP}^\infty \times \mathbb{RP}^\infty = B\langle \tau, \sigma_1\sigma_2 \rangle \longrightarrow BD_8.$$

Both f and g are compositions of the canonical maps.

I learnt of the following result from Jim Milgram who was probably the first to notice it and independently from Jorgen Tornehave. A framing on this 30-dimensional balanced product construction $X \times_{D_8} (S^7)^4$ is used in [127].

Lemma 2.3.2. Let D_8 act on a cartesian product Y^4 by means of its permutation action. If Y is a 7-dimensional frameable manifold then $X \times_{D_8} Y^4$ is frameable.

Proof. Set $N^{30} = X \times_{D_8} Y^4$; then the tangent bundle of N is the sum of the pullback of the tangent bundle τ_M of M^2 with the tangent bundle along the fibres. The latter is stably $7\xi^4$ where $\xi^4 = X \times_{D_8} \mathbb{R}^4$ where D_8 permutes the Euclidean coordinates in $\mathbb{R}^4$. Hence it suffices to show that

$$0 = 7\xi^4 + \tau_M \in \tilde{KO}(M^2).$$

The Mayer-Vietoris sequence for M^2 takes the form

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\partial} \tilde{KO}^0(\mathbb{RP}^2) \oplus \tilde{KO}^0(S^1 \times S^1) \xrightarrow{\phi} \tilde{KO}^0(M^2) \longrightarrow \cdots$$

where

$$\tilde{KO}^0(\mathbb{RP}^2) \cong \mathbb{Z}/4 \text{ and } \tilde{KO}^0(S^1 \times S^1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Comparison with other long exact sequences shows that $\partial(1) = (2, 0, 0, 1)$. Since $\tau_{\mathbb{RP}^2}$ represents $3 \in \mathbb{Z}/4 \cong \tilde{KO}^0(\mathbb{RP}^2)$ and $\tau_{S^1 \times S^1}$ is stably trivial we have $\tau_M = \phi(3, 0, 0, 0)$. Finally $\xi^4 = \phi(1, 0, 0, 1)$ so that

$$7\xi^4 + \tau_M = \phi(\partial(1)) = 0,$$

as required. □

2.3.3. Define

$$E : X \times_{D_8} (\mathbb{RP}^7)^4 \longrightarrow BO(4)$$

to be the following composition of canonical maps:

$$X \times_{D_8} (\mathbb{RP}^7)^4 \xrightarrow{\hat{F} \times i^4} ED_8 \times_{D_8} BO(1)^4 = BD_8 \propto O(1)^4 \xrightarrow{j} BO(4).$$

Here $D_8 \propto O(1)^4$ is the semi-direct product of D_8 with the diagonal matrices $O(1)^4$ and j is induced by the inclusion of this group into the orthogonal group $O(4)$ of 4×4 real orthogonal matrices. The map $\hat{F}$ covers F, the classifying map of M^2 and $i : \mathbb{RP}^7 \subset \mathbb{RP}^\infty = BO(1)$ is the canonical inclusion.

Proposition 2.3.4. $\mathrm{Arf}(N^{30}, E) = 1$.

Proof. We must compute the coefficient of $b_{15} * b_{15}$ in $E_*[N^{30}]$. One checks easily that

$$F_*[M^2] = b_2 * b_0 + b_1 * b_1 \in H_2(BD_8; \mathbb{Z}/2).$$

Hence $E_*[N^{30}]$ is the image of

$$\begin{aligned}(b_2 * b_0 + b_1 * b_1) \otimes b_7^{\otimes 4} \in H_{30}(C_*(ED_8) \otimes_{D_8} H_*(\mathbb{RP}^\infty; \mathbb{Z}/2)^{\otimes 4}) \\ \cong H_{30}(BD_8 \propto O(1)^4; \mathbb{Z}/2).\end{aligned}$$

By definition of the Dyer-Lashof operations in $H_*(BO; \mathbb{Z}/2)$, which are computed in [221], this image is

$$Q^9(b_7) * b_7 * b_7 + Q^8(b_7) * Q^8(b_7)$$

which, by [221], equals

$$b_{15} * b_{15} \ + \ (\text{terms of weight } \geq 3 \text{ in the } b_i)$$

which completes the calculation. □

2.3.5. The desuspension of the Arf-Kervaire invariant one classes [152]. As we saw in Chapter 1 § 1.8.5(ii) formulation of the Arf-Kervaire one problem is to construct stable maps

$$\Theta_{2^n-2} : \Sigma^\infty S^{2^n-2} \longrightarrow \Sigma^\infty \mathbb{RP}^\infty$$

which satisfy certain properties (see also [247]).

The question arises, if Θ_{2^n-2} exists then how far does it desuspend? The following result is a remark by Kee Lam [152] which we shall prove using Theorem 2.1.2 in the form of Example 2.1.14(i).

Theorem 2.3.6. If $n > 2$ then Θ_{2^n-2} cannot be realised as an unstable map

$$\Theta : S^{2^n-1} \longrightarrow \Sigma\mathbb{RP}^\infty.$$

Proof. Form the adjoint of Θ, which we shall also denote by Θ,

$$\Theta : S^{2^n-2} \longrightarrow \Omega\Sigma\mathbb{RP}^\infty.$$

Consider the maps in homology modulo 2,

$$H^{2^n-2}(S^{2^n-2}; \mathbb{Z}/2) \xrightarrow{\Theta_*} H^{2^n-2}(\Omega\Sigma\mathbb{RP}^\infty; \mathbb{Z}/2) \longrightarrow H^{2^n-2}(Q\mathbb{RP}^\infty; \mathbb{Z}/2).$$

Under Θ_*, the generator of the homology of the sphere maps to a primitive in the coalgebra $H^*(\Omega\Sigma\mathbb{RP}^\infty; \mathbb{Z}/2)$. As an algebra this is the tensor algebra of $H_*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ [117], [108], [63]. However, it is well known that the primitives in such a tensor algebra are generated by iterated Lie brackets and by taking squares, starting with the primitives in $H_*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ [190]. This means that the image in the commutative algebra $H^*(Q\mathbb{RP}^\infty; \mathbb{Z}/2)$ must be a multiple of

$$b_1^{2^n-2} \in H^{2^n-2}(\Omega^\infty\Sigma^\infty\mathbb{RP}^\infty; \mathbb{Z}/2)$$

which contradicts the main result of [247] except when $n = 2$.

However, the discussions of Chapter 1 § 1.8.5(i) and of § 2.2.4 of this chapter show that for Θ_{2^n-2} to have Arf-Kervaire invariant one it must have a non-zero Hurewicz image in mod 2 homology. However, this Hurewicz image is described in Example 2.1.14(i) and is equal to $b_1^{2^n-2}$ only when $n = 2$. $\square$

Chapter 3

The Upper Triangular Technology

Topology, the pinnacle of human thought. In four centuries it may be useful.

"The First Circle" by Alexander Solzhenitzin [242]

The object of this chapter is to establish the basic result which relates the upper triangular group to operations in connective K-theory. This result will identify a certain group of operations with the infinite upper triangular group with entries in the 2-adic integers. This identification will be canonical up to inner automorphisms. The 2-adic integers enter here because we are going to work in the stable homotopy category of 2-localised spectra in the sense of [43] (see also Chapter 1 § 1.3.4(iv)). Frank Adams was rather fond of dicta and one of his favorites was something to the effect that it is preferable to have a modest example of something mathematical providing it was *canonical.* That sort of sentiment is the motivation behind my upper triangular technology – that it is better to have only a subset of the operations in connective K-theory providing it is canonical. However, I shall go further and ask for "rigidity". In our case that means turning the 2-adic integral group-ring into a ring of K-theory operations in order that manipulating them becomes as rigid as group-ring algebra.

The basis of this section is the 2-local decomposition of smash products like $bu \wedge bu$ and $bo \wedge bo$ which were first established around 1970 by Mark Mahowald, Don Anderson, Jim Milgram and others. These decompositions are explained in [9]. In this chapter I shall give my approach, which uses a few more properties of the Snaith splitting of $\Omega^2 S^3$ than did the original proofs.

3.1 Connective K-theory

3.1.1. Let bu and bo denote the stable homotopy spectra representing 2-local unitary and orthogonal connective K-theory respectively (see Chapter 1, Examples 1.3.2(v) and 1.3.4(iii)). Thus the smash product, $bu \wedge bo$ is a left bu-module spectrum and so we may consider the ring of left bu-module endomorphisms of degree zero in the stable homotopy category of spectra [9] (see also Chapter 1 § 1.3.1), which we shall denote by $\mathrm{End}_{\text{left-}bu\text{-mod}}(bu \wedge bo)$. The group of units in this ring will be denoted by $Aut_{\text{left-}bu\text{-mod}}(bu \wedge bo)$, the group of homotopy classes of left bu-module homotopy equivalences and left $\mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ denote the subgroup of left bu-module homotopy equivalences which induce the identity map on $H_*(bu \wedge bo; \mathbb{Z}/2)$.

Let $U_\infty\mathbb{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty\mathbb{Z}_2$ if $X_{i,j} \in \mathbb{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j < i$ and $X_{i,i}$ is a 2-adic unit. This upper triangular group is *not* equal to the direct limit $\lim_{\overrightarrow{n}} U_n\mathbb{Z}_2$ of the finite upper triangular groups.

Our main result (proved in § 3.2.2) is the following:

Theorem 3.1.2. There is an isomorphism of the form

$$\psi : \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo) \xrightarrow{\cong} U_\infty\mathbb{Z}_2.$$

3.1.3. There is a similar calculation of the group $\mathrm{Aut}^0_{\text{left-}bo\text{-mod}}(bo \wedge bo)$ which I shall leave to the reader. In fact, the appearance of bo in Theorem 3.1.2 is just for convenience. The main use of this result will be to realise the 2-adic group-ring, $\mathbb{Z}_2[U_\infty\mathbb{Z}_2]$, as a subring of the left-bu module endomorphisms of

$$bu \wedge bu \simeq bu \wedge bo \wedge \Sigma^{-2}\mathbb{CP}^2.$$

This uses the equivalence $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$ first noticed by Reg Wood and independently by Don Anderson (both unpublished). This equivalence is easy to prove, once noticed, and a proof is given in Chapter 5 § 5.5.1.

The preference for bu over bo is that, if F is an algebraically closed field of characteristic different from 2, then there is a homotopy equivalence of ring spectra $bu \simeq \underline{KF}\mathbb{Z}_2$ between 2-adic connective K-theory and the algebraic K-theory spectrum of F with coefficients in the 2-adic integers ([264] [265]; see also Chapter 4, Corollaries 4.6.10 and 4.7.10). As explained in Remark 3.3.5, Theorem 3.1.2 implies that the 2-adic group-ring, $\mathbb{Z}_2[U_\infty\mathbb{Z}_2]$, may be considered as a ring of operations on 2-adic algebraic K-theory, an observation which I hope to develop subsequently.

This chapter is organised in the following manner. In §§ 3.1.4–3.1.6 we recall the decomposition of $bu \wedge bu$ and $bu \wedge bo$ together with several related facts about Steenrod algebra structure. In § 2 we prove Theorem 3.1.2. In § 3 we explain the application of Theorem 3.1.2 to the construction of operations on algebraic K-theory and on Chow groups. We conclude the chapter with some remarks about the importance of the matrix $1 \wedge \psi^3$ in $U_\infty\mathbb{Z}_2$. We shall establish the identity of this matrix in Chapter 5.

I am very grateful to Michael Crabb and Richard Kane for helpful conversations. The results of this chapter first appeared in [252].

3.1.4. In this section we recall the splitting of $bu \wedge bu$ ([9], [175]; see also [185]).

Let X be a finite spectrum and let DX denote its S-dual. Following the account of ([9] pp. 190–196) this means that we have a map in the stable homotopy category (see Chapter 1 § 1.3.1)

$$e : DX \wedge X \longrightarrow S^0$$

such that, if W is a finite spectrum and Z is arbitrary, there is an isomorphism ([9] Proposition 5.4 p. 195)

$$T : [W, Z \wedge DX]_* \stackrel{\cong}{\longrightarrow} [W \wedge X, Z]_*$$

given by $T[f] = [(1_Z \wedge e) \cdot (f \wedge 1_X)]$.

Hence, setting $W = S^0$ and $Z = X$, we have

$$\mu = T^{-1}(1_X) : S^0 \longrightarrow X \wedge DX.$$

Setting $Z = H\mathbb{Z}/2$, the mod 2 Eilenberg-Maclane spectrum of Chapter 1 § 1.3.2(ii), we obtain an isomorphism

$$U : H^{-*}(DX; \mathbb{Z}/2) \stackrel{\cong}{\longrightarrow} H_*(X; \mathbb{Z}/2) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}(H^*(X; \mathbb{Z}/2), \mathbb{Z}/2)$$

whose composition, U, is given by

$$U(\alpha)(\beta) = \mu^*(\beta \otimes \alpha) \in H^*(S^0; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for all $\alpha \in H^{-*}(DX; \mathbb{Z}/2)$, $\beta \in H^*(X; \mathbb{Z}/2)$.

Let $\mathcal{A}$ denote the mod 2 Steenrod algebra [259] (see also Chapter 1, Definition 1.6.3) then, for $m > 0$,

$$U(Sq^m(\alpha))(\beta) = \mu^*(\beta \otimes Sq^m(\alpha)) = \sum_{a=1}^{m} \mu^*(Sq^a(\beta) \otimes Sq^{m-a}(\alpha)),$$

since $Sq^m(\mu^*(\beta \otimes \alpha)) = 0$.

Let χ denote the canonical anti-automorphism ([259] pp. 25–26). We have

$$U(Sq^m(\alpha))(\beta) = \sum_{a=1}^{m} U(Sq^{m-a}(\alpha))(Sq^a(\beta)).$$

For $m = 1$, $U(Sq^1(\alpha))(\beta) = U(\alpha)(Sq^1(\beta)) = U(\alpha)(\chi(Sq^1)(\beta))$. If, by induction, we have $U(Sq^n(\alpha))(\beta) = U(\alpha)(\chi(Sq^n)(\beta))$ for all $n < m$ then

$$U(Sq^m(\alpha))(\beta) = \sum_{a=1}^{m} U(\alpha)(\chi(Sq^{m-a})(Sq^a(\beta)) = U(\alpha)(\chi(Sq^m)(\beta)),$$

since $\sum_{a=0}^{m} \chi(Sq^{m-a})Sq^a = 0$.

Therefore, as a left $\mathcal{A}$-module, $H^{-*}(DX;\mathbb{Z}/2)$ is isomorphic to $H_*(X;\mathbb{Z}/2)$ where the left action by Sq^a corresponds to $\chi(Sq^a)_*$, composition with $\chi(Sq^a)$ (cf. [247]; see also Chapter 2). However, $\chi(Sq^1) = Sq^1$ and $\chi(Sq^{01}) = Sq^{01}$, because these are primitives in the Hopf algebra, $\mathcal{A}$. If we set $B = E(Sq^1, Sq^{01})$, the exterior algebra on Sq^1 and Sq^{01}, then $H^{-*}(DX;\mathbb{Z}/2)$ is isomorphic as a left B-module to $H_*(X;\mathbb{Z}/2)$ on which Sq^1 and Sq^{01} acts via Sq^1_* and Sq^{01}_*, respectively.

3.1.5. Consider the second loopspace of the 3-sphere, $\Omega^2 S^3$. There is an algebra isomorphism [244] of the form

$$H_*(\Omega^2 S^3;\mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots]$$

where $\xi_t = Q_1^{t-1}(\iota)$ has degree $2^t - 1$ and ξ_t is primitive. Here $\iota = \xi_1$ is the image of the generator of $H_1(S^1;\mathbb{Z}/2)$. The right action of Sq^1 and Sq^{01}, via their duals Sq^1_* and Sq^{01}_*, on $H_*(\Omega^2 S^3;\mathbb{Z}/2)$ is given by [244]

$$(\xi_t)Sq^{01} = \begin{cases} \xi_{t-2}^4 & \text{if } t \geq 3, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\xi_t)Sq^{1} = \begin{cases} \xi_{t-1}^2 & \text{if } t \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

These are the same formulae which give the right action on $H_*(bu;\mathbb{Z}/2)$ ([9] pp. 340–342). Since $B = E(Sq^1, Sq^{01})$ is a commutative ring, we may consider $H_*(\Omega^2 S^3;\mathbb{Z}/2)$ to be a *left* B-module via the formulae, $Sq^1(\xi_t) = (\xi_t)Sq^1$ and $Sq^{01}(\xi_t) = (\xi_t)Sq^{01}$.

In order to apply these observations to $\Omega^2 S^3$ we would prefer it to be a finite complex. However, there exists a model for $\Omega^2 S^3$ which is filtered by finite complexes ([51], [243], see also Chapter 1 § 1.5.2)

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \cdots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

and there is a stable homotopy equivalence, an example of the Snaith splitting of Chapter 1, Theorem 1.5.3, of the form

$$\Omega^2 S^3 \simeq \vee_{k\geq 1} F_k/F_{k-1}.$$

In addition, by ([60]; see also [249]), this stable homotopy equivalence may be assumed to be multiplicative in the sense that the H-space product on $\Omega^2 S^3$ induces a graded homotopy-ring structure

$$\{F_k/F_{k-1} \wedge F_l/F_{l-1} \longrightarrow F_{k+l}/F_{k+l-1}\}$$

on $\vee_{k\geq 1} F_k/F_{k-1}$. To obtain a graded homotopy-ring with identity we add an extra base-point by defining $F_0 = S^0$, $F_j = *$ for $j < 0$ and replacing $\vee_{k\geq 1} F_k/F_{k-1}$ by $\vee_{k\geq 0} F_k/F_{k-1}$.

The geometrical construction of the homology operation Q_1 (see [21] and [73]) shows that $\xi_1 = \iota \in H_1(F_1; \mathbb{Z}/2)$ and that $\xi_t \in H_{2^t-1}(F_{2^{t-1}}/F_{2^{t-1}-1}; \mathbb{Z}/2)$, in terms of the induced splitting of $H_*(\Omega^2 S^3; \mathbb{Z}/2)$, so that there is an algebra isomorphism of B-modules of the form

$$H_*(\vee_{k\geq 0} F_{4k}/F_{4k-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^4, \xi_2^2, \xi_3, \xi_4, \ldots].$$

Next, write $H_*(\Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2) = \mathbb{Z}/2\langle 1\rangle \oplus \mathbb{Z}/2\langle x\rangle$ for the mod 2 homology of the double-desuspension of the complex projective plane. Hence Sq^1_* and Sq^{01}_* act trivially on $x \in H_2(\Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2)$. Therefore we may define an isomorphism of right B-modules

$$\Phi : H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \xrightarrow{\cong} \mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$$

by the formula, for $\epsilon = 0, 1$,

$$\Phi((\xi_1^4)^{\epsilon_1}(\xi_2^2)^{\epsilon_2}\xi_3^{\epsilon_3}\xi_4^{\epsilon_4}\ldots\xi_t^{\epsilon_t} \otimes x^{\epsilon}) = \xi_1^{4\epsilon_1+2\epsilon}(\xi_2^2)^{\epsilon_2}\xi_3^{\epsilon_3}\xi_4^{\epsilon_4}\ldots\xi_t^{\epsilon_t}.$$

Here the right B-module structure on $\mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$ is that given by the formulae introduced previously.

On the other hand, there is an isomorphism of algebras with right B-module structure ([9] p. 340)

$$H_*(bu; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots],$$

where these ξ_i's are the canonical Milnor generators of the dual Steenrod algebra ([259] pp. 19–22, see also Chapter 1, Definition 1.6.7). Therefore we have a canonical isomorphism of graded, right B-algebras

$$\Phi : H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \xrightarrow{\cong} H_*(bu; \mathbb{Z}/2).$$

If

$$\lambda = Sq^1 \quad \text{or} \quad Sq^{01}, \quad f \in H^*(bu; \mathbb{Z}/2)$$

and

$$\alpha \in H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2)$$

then $(\Phi(\alpha))\lambda = \Phi((\alpha)\lambda)$ and

$$\langle \lambda(f), \Phi(\alpha)\rangle = (\Phi(\alpha))\lambda(f) = \Phi((\alpha)\lambda)(f) = \langle f, \Phi(\alpha)\lambda)\rangle.$$

However, if we interpret α as belonging to

$$H^{-*}(\vee_{k\geq 0}D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2)$$

then $(\alpha)\lambda$ becomes the left translate of α by λ, $\lambda(\alpha)$, for $\lambda = Sq^1$ or Sq^{01}. Identifying $H_*(bu; \mathbb{Z}/2)$ with the dual of the left B-module, $H^*(bu; \mathbb{Z}/2)$, we have $f(\Phi(\lambda(\alpha))) = \lambda(f)(\Phi(\alpha))$. This means that the adjoint of Φ,

$$\mathrm{adj}(\Phi) \in \mathrm{Hom}(H^{-*}(\vee_{k\geq 0}D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2), \mathbb{Z}/2)$$

given by $\mathrm{adj}(\Phi)(\alpha \otimes f) = f(\Phi(\alpha))$ satisfies, if $\lambda = Sq^1$ or Sq^{01},

$$\begin{aligned}\mathrm{adj}(\Phi)(\lambda(\alpha \otimes f)) &= \mathrm{adj}(\Phi)(\lambda(\alpha) \otimes f + \alpha \otimes \lambda(f)) \\ &= f(\Phi(\lambda(\alpha))) + \lambda(f)(\Phi(\alpha)) \\ &= 0 \\ &= \lambda(\mathrm{adj}(\Phi)(\alpha \otimes f)).\end{aligned}$$

Therefore we have a canonical family of maps, for $k \geq 0$,

$$\mathrm{adj}(\Phi)_k \in \mathrm{Hom}_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2), \mathbb{Z}/2)$$

such that $\mathrm{adj}(\Phi) = \sum_{k \geq 1}(\mathrm{adj}(\Phi)_k)$.

The analysis of the right B-module $H_*(bu; \mathbb{Z}/2)$, in ([9] Proposition 16.4 and pp. 340–342), shows that each left B-module of the form $H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2)$ satisfies the conditions of ([9] p. 353). That is, up to direct sums with projectives these B-modules are equivalent to finite sums of $\Sigma^a I^b$, the ath suspension of the bth tensor power of the augmentation ideal, with $a+b$ even. Here a and b may be negative. The same result holds if we smash with a finite number of copies of bu. For such modules as the left variable, $\mathrm{Ext}_B^{s,t}(-,-)$ vanishes for $s > 0$ and $t-s$ odd and the related Adams spectral sequence collapses. In particular the Adams spectral sequence

$$\begin{aligned}E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}&(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2)^{\otimes^2}, \mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge bu \wedge bu) \otimes \mathbb{Z}_2\end{aligned}$$

has

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2), \mathbb{Z}/2)$$

and $E_2^{s,t} \cong E_\infty^{s,t}$, by ([9] Lemma 17.12 p. 361).

In addition, by the 2-local version of ([9] pp. 354–355; see [9] p. 358–359), the Hurewicz homomorphisms yield an injection of the form

$$\begin{gathered}\pi_t(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge bu \wedge bu) \otimes \mathbb{Z}_2 \\ \downarrow \\ \mathrm{Ext}_B^{0,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2), \mathbb{Z}/2) \\ \oplus \\ H_t(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge bu \wedge bu) \otimes \mathbb{Q}_2\end{gathered}$$

where $\mathbb{Q}_2$ denotes the field of 2-adic rationals.

The collapsing of the spectral sequence ensures that there exists at least one element

$$\mathrm{adj}(\lambda)_k \in \pi_*(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge bu \wedge bu) \otimes \mathbb{Z}_2$$

whose mod 2 Hurewicz image is $\mathrm{adj}(\Phi)_k$. Such an element corresponds, via S-duality with $W = S^0$, $Z = bu \wedge bu$, to a (2-local) S-map of the form

$$\lambda_k : F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu \wedge bu$$

whose induced map in mod 2 cohomology is equal to Φ_k^*, the k-component of the dual of Φ.

Now λ_k^* is a left $\mathcal{A}$-module homomorphism

$$\lambda_k^* : H^*(bu \wedge bu; \mathbb{Z}/2) \longrightarrow H^*(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2)$$

while Φ_k^* is a left B-module homomorphism

$$\Phi_k^* : H^*(bu; \mathbb{Z}/2) \longrightarrow H^*(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2).$$

Here we have identified B with its dual Hopf algebra, B_*. The relation between λ_k^* and Φ_k^* is described in the following manner.

There is a left $\mathcal{A}$-module isomorphism, $H^*(bu; \mathbb{Z}/2) \cong \mathcal{A} \otimes_B \mathbb{Z}/2$ ([9] Proposition 16.6 p. 335), and an isomorphism ([9] p. 338)

$$\psi : \mathcal{A} \otimes_B H^*(bu; \mathbb{Z}/2) \xrightarrow{\cong} H^*(bu; \mathbb{Z}/2) \otimes H^*(bu; \mathbb{Z}/2)$$

given by $\psi(a \otimes_B b) = \sum (a' \otimes_B 1) \otimes a''(b)$ where the diagonal of $a \in \mathcal{A}$ satisfies $\Delta(a) = \sum a' \otimes a''$ and $b \in H^*(bu; \mathbb{Z}/2)$. On the other hand, Φ_k^* induces a left $\mathcal{A}$-module homomorphism

$$\phi_k^* : \mathcal{A} \otimes_B H^*(bu; \mathbb{Z}/2) \longrightarrow H^*(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2)$$

given by $\phi_k^*(a \otimes_B b) = a(\Phi_k^*(b))$. These homomorphisms satisfy

$$\phi_k^* = \lambda_k^* \cdot \psi.$$

Now consider the composition

$$\begin{aligned} L = (m \wedge 1)\left(\sum_{k \geq 0} 1 \wedge \lambda_k\right) : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) &\longrightarrow bu \wedge bu \wedge bu \\ &\longrightarrow bu \wedge bu \end{aligned}$$

where $m : bu \wedge bu \longrightarrow bu$ is the bu-product. This map induces an isomorphism on mod 2 cohomology. To see this it suffices to show that the composition

$$\begin{aligned} \left(\sum_{k \geq 0} 1 \otimes \lambda_k^*\right)(m^* \otimes 1)\psi &: \mathcal{A} \otimes_B H^*(bu; \mathbb{Z}/2) \\ &\longrightarrow H^*(\vee_{k \geq 0} F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2; \mathbb{Z}/2) \end{aligned}$$

is an isomorphism. For $a \in \mathcal{A}$ and $b \in H^*(bu;\mathbb{Z}/2) \cong \mathcal{A} \otimes_B \mathbb{Z}/2$ define $\psi'(a \otimes_B b) \in H^*(bu;\mathbb{Z}/2) \otimes (\mathcal{A} \otimes_B H^*(bu;\mathbb{Z}/2))$ by $\psi'(a \otimes_B b) = \sum (a' \otimes_B 1) \otimes (a'' \otimes_B b)$ where $\Delta(a) = \sum a' \otimes a''$. We find that

$$\begin{aligned}(1 \otimes \psi)\psi'(a \otimes_B b) &= (1 \otimes \psi)\left(\sum (a' \otimes_B 1) \otimes (a'' \otimes_B b)\right) \\ &= \sum (a' \otimes_B 1) \otimes (a_1 \otimes_B 1) \otimes a_2(b),\end{aligned}$$

where $(1 \otimes \Delta)\Delta(a) = \sum a' \otimes a_1 \otimes a_2$, so that

$$(1 \otimes \psi)\psi' = (\Delta \otimes 1)\psi = (m^* \otimes 1)\psi.$$

From this identity we have

$$\begin{aligned}\left(\sum_{k\geq 0} 1 \otimes \lambda_k^*)(m^* \otimes 1\right)\psi(a \otimes_B b) &= \left(\sum_{k\geq 0} 1 \otimes \phi_k^*\right)\psi'(a \otimes_B b) \\ &= \sum (a' \otimes_B 1) \otimes a''\left(\sum_{k\geq 0} \Phi_k^*(b)\right) \\ &= \psi\left(1_{\mathcal{A}} \otimes_B \left(\sum_{k\geq 0} \Phi_k^*\right)\right)(a \otimes_B b).\end{aligned}$$

Since $\sum_{k\geq 0} \Phi_k^*$ is an isomorphism of left B-modules this composition is an isomorphism of left $\mathcal{A}$-modules.

To recapitulate, we have proved part (i) of the following result, part (ii) being proved in a similar manner.

Theorem 3.1.6. In the notation of § 3.1.4, there are 2-local homotopy equivalences of left-bu-module spectra of the form

(i) $L : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \longrightarrow bu \wedge bu$

and

(ii) $\hat{L} : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge bo$.

Theorem 3.1.6 should be compared with the odd primary analogue which is described in detail in [137].

3.1.7. Comparison with mod 2 cohomology. It is very easy to compare the 2-local splitting of left bu-module spectra

$$L : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \xrightarrow{\simeq} bu \wedge bu$$

with a corresponding splitting for mod 2 cohomology.

There is a unique, non-trivial map of spectra, $\iota : bu \longrightarrow H\mathbb{Z}/2$, and we wish to construct a homotopy commutative diagram of the form

$$\begin{array}{ccc} \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) & \xrightarrow{L} & bu \wedge bu \\ \Big\downarrow \vee_{k\geq 0}\iota\wedge 1\wedge 1 & & \Big\downarrow \iota\wedge 1 \\ \vee_{k\geq 0} H\mathbb{Z}/2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) & \xrightarrow{L'} & H\mathbb{Z}/2 \wedge bu \end{array}$$

in which L' is a homotopy equivalence.

However, the Adams spectral sequence

$$\begin{gathered} \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes H^*(H\mathbb{Z}/2 \wedge bu; \mathbb{Z}/2), \mathbb{Z}/2) \\ \Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge H\mathbb{Z}/2 \wedge bu) \end{gathered}$$

has

$$E_2^{s,t} \cong \mathrm{Ext}_B^{s,t}(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes \mathcal{A}, \mathbb{Z}/2)$$

which is zero if s is non-zero, since $\mathcal{A}$ is a free B-module [190]. Also, composition with ι corresponds to the canonical map on $E_2^{0,*}$,

$$\begin{gathered} \mathrm{Hom}_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes \mathcal{A} \otimes_B \mathbb{Z}/2, \mathbb{Z}/2) \\ \downarrow \\ \mathrm{Hom}_B(H^{-*}(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2) \otimes \mathcal{A}, \mathbb{Z}/2) \end{gathered}$$

given by composition with $\iota^* : \mathcal{A} \longrightarrow \mathcal{A} \otimes_B \mathbb{Z}/2$. Hence there exists

$$\mathrm{adj}(\lambda')_k \in \pi_0(D(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \wedge H\mathbb{Z}/2 \wedge bu)$$

such that

$$\iota \cdot \mathrm{adj}(\lambda)_k \simeq \mathrm{adj}(\lambda')_k$$

and therefore

$$\iota \cdot \lambda_k \simeq \lambda'_k : F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow H\mathbb{Z}/2 \wedge bu.$$

We see that we may set L' equal to

$$\begin{gathered} L' = (m' \wedge 1)(\textstyle\sum_{k\geq 0} 1 \wedge \lambda_k) : \vee_{k\geq 0} H\mathbb{Z}/2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \\ \longrightarrow H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge bu \longrightarrow H\mathbb{Z}/2 \wedge bu \end{gathered}$$

where $m' : H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \longrightarrow H\mathbb{Z}/2$ is the cup-product.

Also L' induces an isomorphism in mod 2 homology, since it is a homomorphism between free, graded modules of finite type over the polynomial ring, $\mathcal{A}_* \cong \mathbb{Z}/2[\xi_1, \xi_2, \ldots]$, and $\mathbb{Z}/2 \otimes_{\mathcal{A}_*} (L')_* = \mathbb{Z}/2 \otimes_{(\mathcal{A}\otimes_B \mathbb{Z}/2)_*} (L)_*$ is an isomorphism ([154] pp. 603–605).

3.2 The role of the upper triangular group

3.2.1. In this section I am going to prove Theorem 3.1.2, which will be accomplished in § 3.2.2 after some preparatory discussion. Let us begin with some motivation from homotopy theory. Let $\psi^3 : bu \longrightarrow bu$ denote the Adams operation. In order to understand the map

$$1 \wedge (\psi^3 - 1) : bu \wedge bu \longrightarrow bu \wedge bu$$

we observe that it is a left bu-module map and therefore we ought to study all such maps. The 2-local splitting of $bu \wedge bu$ of Theorem 3.1.6(i) implies that we need only study left bu-module maps of the form

$$\phi_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu \wedge (F_{4l}/F_{4l-1}) \wedge \Sigma^{-2}\mathbb{CP}^2$$

for each pair, $k, l \geq 0$. In addition, the factor $\Sigma^{-2}\mathbb{CP}^2$ will only be a nuisance so we shall study

$$1 \wedge (\psi^3 - 1) : bu \wedge bo \longrightarrow bu \wedge bo$$

first. By virtue of the 2-local splitting of Theorem 3.1.6(ii)

$$L'' : bu \wedge bo \stackrel{\simeq}{\longrightarrow} \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})$$

we are led to study the corresponding left bu-module maps, $\{\phi''_{k,l}\}$, of the form

$$\phi''_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1}).$$

A left bu-module map of this form is, in turn, determined by its restriction to $S^0 \wedge (F_{4k}/F_{4k-1})$. This restriction is a homotopy element of the form

$$[\phi''_{k,l}] \in \pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2.$$

This homotopy group is calculated by means of the (collapsed) Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_B^{s,t}(H^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \otimes H^*(F_{4l}/F_{4l-1}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2. \end{aligned}$$

Recall from ([9] p. 332) that Σ^a is the (invertible) B-module given by $\mathbb{Z}/2$ in degree a, $\Sigma^{-a} = \mathrm{Hom}(\Sigma^a, \mathbb{Z}/2)$ and I is the augmentation ideal, $I = \ker(\epsilon : B \longrightarrow \mathbb{Z}/2)$. Hence, if $b > 0$, $I^{-b} = \mathrm{Hom}(I^b, \mathbb{Z}/2)$, where I^b is the b-fold tensor product of I. These duality identifications may be verified using the criteria of ([9] p. 334 Theorem 16.3) for identifying $\Sigma^a I^b$.

In ([9] p. 341) it is shown that the B-module given by

$$H^{-*}(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \cong H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$$

is stably equivalent to $\Sigma^{2^{r-1}+1} I^{2^{r-1}-1}$ when $0 < 4k = 2^r$.

Therefore $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-(2^{r-1}+1)}I^{1-2^{r-1}}$ when $0 < 4k = 2^r$. If k is not a power of 2 we may write $4k = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$ with $2 \leq r_1 < r_2 < \cdots < r_t$. In this case

$$H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2) \cong \otimes_{j=r_1}^{r_t} H_*(F_{2^j}/F_{2^j-1};\mathbb{Z}/2)$$

which is stably equivalent to

$$\Sigma^{2^{r_1-1}+1+2^{r_2-1}+1+\cdots+2^{r_t-1}+1}I^{2^{r_1-1}-1+2^{r_2-1}-1+\cdots+2^{r_t-1}-1} = \Sigma^{2k+\alpha(k)}I^{2k-\alpha(k)},$$

where $\alpha(k)$ equals the number of 1's in the dyadic expansion of k. Similarly, $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-2k-\alpha(k)}I^{\alpha(k)-2k}$.

Next we observe that $\mathrm{Ext}_B^{s,t}(\Sigma^a M,\mathbb{Z}/2) \cong \mathrm{Ext}_B^{s,t-a}(M,\mathbb{Z}/2)$. Also the short exact sequence

$$0 \longrightarrow I \otimes M \longrightarrow B \otimes M \longrightarrow M \longrightarrow 0$$

yields a long exact sequence of the form

$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}_B^{s,t}(B\otimes M,\mathbb{Z}/2) \longrightarrow \mathrm{Ext}_B^{s,t}(I\otimes M,\mathbb{Z}/2) \longrightarrow \mathrm{Ext}_B^{s+1,t}(M,\mathbb{Z}/2) \\ \longrightarrow \mathrm{Ext}_B^{s+1,t}(B\otimes M,\mathbb{Z}/2) \longrightarrow \cdots \end{aligned}$$

so that, if $s > 0$, there is an isomorphism

$$\mathrm{Ext}_B^{s,t}(I\otimes M,\mathbb{Z}/2) \xrightarrow{\cong} \mathrm{Ext}_B^{s+1,t}(M,\mathbb{Z}/2).$$

Therefore, for $s > 0$,

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)}I^{2l-2k-\alpha(l)+\alpha(k)},\mathbb{Z}/2) \\ &\cong \mathrm{Ext}_B^{s+2l-2k-\alpha(l)+\alpha(k),t-2l+2k-\alpha(l)+\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2). \end{aligned}$$

Now $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b]$ where $a \in \mathrm{Ext}_B^{1,1}$, $b \in \mathrm{Ext}_B^{1,3}$ and the contributions to $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$ come from the groups $\{E_2^{s,s} \mid s \geq 0\}$. This corresponds to $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/2,\mathbb{Z}/2)$ when $u = s + 2l - 2k - \alpha(l) + \alpha(k)$ and $v = s - 2l + 2k - \alpha(l) + \alpha(k)$, which implies that $v - u = 4(k-l)$. This implies that $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/2,\mathbb{Z}/2) = 0$ if $l > k$ or, equivalently, that each $E_2^{s,s}$ is zero when $l > k$. Therefore $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2 = 0$ if $l > k$.

Now suppose that $l \leq k$. If $\mathrm{Ext}_B^{u,v}(\mathbb{Z}/2,\mathbb{Z}/2)$ is non-zero, then it is cyclic of order two generated by $a^{(3u-v)/2}b^{(v-u)/2}$ and when $u = s+2l-2k-\alpha(l)+\alpha(k), v = s - 2l + 2k - \alpha(l) + \alpha(k)$ this monomial is equal to $a^{s+4l-4k-\alpha(l)+\alpha(k)}b^{2(k-l)}$. Furthermore, in order for this group to be non-zero we must have $s \geq 4(k-l) + \alpha(l) - \alpha(k)$ which implies that $s \geq 0$ if $k = l$ and $s \geq 4(k-l) + \alpha(l) - \alpha(k) \geq 2(k-l) + 1$ if $k > l$. The last inequality is seen by writing $l = 2^{\alpha_1} + \cdots + 2^{\alpha_r}$ with $0 \leq \alpha_1 < \cdots < \alpha_r$ and $k - l = 2^{\epsilon_1} + \cdots + 2^{\epsilon_q}$ with $0 \leq \epsilon_1 < \cdots < \epsilon_q$. Then $\alpha(l) = r$ and $\alpha(l + 2^{\epsilon_q}) \leq r + 1$ so that, by induction, $\alpha(k) \leq r + q$ which yields

$$2(k-l) + \alpha(l) - \alpha(k) \geq 2(k-l) - q \geq \sum_{j=1}^{q}(2^{\epsilon_j+1} - 1) \geq 1.$$

Suppose now that $k > l$ and consider the non-trivial homotopy classes of left-bu-module maps

$$\phi''_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

which induce the zero map on mod 2 homology. In the spectral sequence these maps are represented by elements of $E_2^{s,s} = E_\infty^{s,s}$ with $s > 0$ since being represented in $E_\infty^{0,*}$ is equivalent to being detected by the induced map in mod 2 homology. By the preceding discussion, the only other possibility is that $\phi''_{k,l}$ is represented in $E_\infty^{\epsilon+4(k-l)+\alpha(l)-\alpha(k),\epsilon+4(k-l)+\alpha(l)-\alpha(k)}$ for some $\epsilon \geq 0$. Since multiplication by two on $\pi_0(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2$ corresponds to multiplication by $a \in Ext_B^{1,1}(\mathbb{Z}/2, \mathbb{Z}/2)$ in the spectral sequence, we see that

$$\phi''_{k,l} = \gamma 2^\epsilon \iota_{k,l},$$

for some 2-adic unit γ and positive integer ϵ, where

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

is represented in the spectral sequence by a generator of

$$E_2^{4(k-l)+\alpha(l)-\alpha(k),4(k-l)+\alpha(l)-\alpha(k)}.$$

When $k = l$ a similar argument shows that

$$\phi''_{k,k} = \gamma 2^\epsilon \iota_{k,k}$$

where $\iota_{k,k}$ denotes the identity map of $bu \wedge (F_{4k}/F_{4k-1})$. In particular, if $\phi''_{k,k}$ induces the identity map on mod 2 homology then $\epsilon = 0$. $\square$

3.2.2. Proof of Theorem 3.1.2. Recall that $\mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ is the group, under composition, of homotopy classes of 2-local homotopy equivalences of

$$\vee_{k \geq 0}\ bu \wedge (F_{4k}/F_{4k-1})$$

given by left bu-module maps which induce the identity map on $H_*(-;\mathbb{Z}/2)$. The discussion of § 3.2.1 shows that the elements of this group are in one-one correspondence with the matrices in $U_\infty\mathbb{Z}_2$. More specifically, the discussion shows that there is a bijection of *sets*

$$\psi : U_\infty(\mathbb{Z}_2) \longrightarrow \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$$

given by

$$\psi(X) = \sum_{l \leq k} X_{l,k}\iota_{k,l} : bu \wedge (\vee_{k\geq 0} F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (\vee_{k\geq 0} F_{4k}/F_{4k-1}).$$

Here $\iota_{k,l}$ is chosen as in § 3.2.1.

We shall obtain the isomorphism of Theorem 3.1.2 by choosing the $\iota_{k,l}$'s to make ψ an isomorphism of groups *with the convention* that a composition of maps $f \cdot g$ implies that g is applied *first* followed by f. In [252] I seem to have unwisely used the opposite convention!

However, in order to ensure that ψ is an isomorphism of groups we must choose the $\iota_{k,l}$ more carefully. In fact, I claim that we may choose the $\iota_{k,k}$'s to be the identity maps. Then choose each $\iota_{k+1,k}$ as in § 3.2.1 and define

$$\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \cdots \iota_{k,k-1}$$

for all $k - l \geq 2$. With our convention for the composition of maps, we must have $\iota_{k,l} \cdot \iota_{s,t} = 0$ unless $l \leq k = t \leq s$ in which case $\iota_{k,l} \cdot \iota_{s,k} = \iota_{s,l}$. Then ψ is an isomorphism of groups, since we have

$$\begin{aligned}\psi(X) \cdot \psi(Y) &= \Big(\sum_{l \leq k} X_{l,k}\iota_{k,l}\Big)\Big(\sum_{t \leq s} Y_{t,s}\iota_{s,t}\Big) = \Big(\sum_{l \leq k = t \leq s} X_{l,k}Y_{t,s}\iota_{k,l} \cdot \iota_{s,t}\Big) \\ &= \Big(\sum_{l \leq k \leq s} X_{l,k}Y_{k,s}\iota_{s,l}\Big) = \Big(\sum_{l \leq s}(XY)_{l,s}\iota_{s,l}\Big) = \psi(XY)\end{aligned}$$

as required.

It remains to verify the claim. For $k > l > m$ we need to know the relation between the composition $\iota_{l,m} \cdot \iota_{k,l}$ and $\iota_{k,m}$. Set $s(k,l) = 4(k-l) + \alpha(l) - \alpha(k)$.

The element $\iota_{k,l}$ is represented by the generator of [4]

$$\mathrm{Ext}_B^{s(k,l),s(k,l)}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)}I^{2l-2k-\alpha(l)+\alpha(k)}, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

and $\iota_{l,m}$ by that of $\mathrm{Ext}_B^{s(l,m),s(l,m)}(\Sigma^{2m-2l+\alpha(m)-\alpha(l)}I^{2m-2l-\alpha(m)+\alpha(l)}, \mathbb{Z}/2) \cong \mathbb{Z}/2$ while $\iota_{k,m}$ is represented by a generator of

$$\mathrm{Ext}_B^{s(k,m),s(k,m)}(\Sigma^{2m-2k+\alpha(m)-\alpha(k)}I^{2m-2k-\alpha(m)+\alpha(k)}, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

The composition, $\iota_{l,m} \cdot \iota_{k,l}$, is represented by the product of the representatives under the pairing induced by the tautological B-module isomorphism,

$$\Sigma^a I^b \otimes \Sigma^{a'} I^{b'} \cong \Sigma^{a+a'} I^{b+b'}$$

for suitable positive integers a, a', b, b'. Via the dimension-shifting isomorphisms described in § 3.2.1, the pairing

$$\mathrm{Ext}_B^{s,s}(\Sigma^a I^b, \mathbb{Z}/2) \otimes \mathrm{Ext}_B^{s',s'}(\Sigma^{a'} I^{b'}, \mathbb{Z}/2) \longrightarrow \mathrm{Ext}_B^{s+s',s+s'}(\Sigma^{a+a'} I^{b+b'}, \mathbb{Z}/2)$$

may be identified with the product

$$\begin{aligned}&\mathrm{Ext}_B^{s+b,s-a}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes \mathrm{Ext}_B^{s'+b',s'-a'}(\mathbb{Z}/2, \mathbb{Z}/2) \\ &\qquad \longrightarrow \mathrm{Ext}_B^{s+s'+b+b',s+s'-a-a'}(\mathbb{Z}/2, \mathbb{Z}/2)\end{aligned}$$

which is an isomorphism whenever both sides are non-trivial. Therefore, since $s(k,l) + s(l,m) = s(k,m)$, this is true in our case and there exists a 2-adic unit $u_{k,l,m} \in \mathbb{Z}_2^*$ such that

$$\iota_{l,m} \cdot \iota_{k,l} = u_{k,l,m} \iota_{k,m}.$$

This relation justifies the choice of $\iota_{k,l}$'s when $k - l \geq 2$ and completes the proof of Theorem 3.1.2 $\square$

3.3 An application to algebraic K-theory

3.3.1. As in § 3.1.1 let bu and bo denote the stable homotopy spectra representing 2-local (sometimes, for example [168], referred to as 2-adically completed) unitary and orthogonal connective K-theory respectively. Hence bu is a commutative ring spectrum with multiplication and unit maps $m : bu \wedge bu \longrightarrow bu$ and $\eta : S^0 \longrightarrow bu$, respectively. Also bo is a commutative ring spectrum and a two-sided bu-module.

Suppose now that E is a connective, right-bu-module spectrum. Hence we have a multiplication $\mu : E \wedge bu \longrightarrow E$ such that

$$\mu \cdot (1 \wedge m) \simeq \mu \cdot (\mu \wedge 1) : E \wedge bu \wedge bu \longrightarrow E.$$

Form the compositions

$$\begin{aligned} L_E : E \wedge bu = E \wedge S^0 \wedge bu &\stackrel{1\wedge\eta\wedge 1}{\longrightarrow} E \wedge bu \wedge bu \\ &\stackrel{1\wedge L^{-1}}{\longrightarrow} \vee_{k\geq 0} E \wedge bu \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \\ &\stackrel{\vee_{k\geq 0}\mu\wedge 1\wedge 1}{\longrightarrow} \vee_{k\geq 0} E \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2) \end{aligned}$$

and

$$\begin{aligned} \hat{L}_E : E \wedge bo = E \wedge S^0 \wedge bo &\stackrel{1\wedge\eta\wedge 1}{\longrightarrow} E \wedge bu \wedge bo \\ &\stackrel{1\wedge \hat{L}^{-1}}{\longrightarrow} \vee_{k\geq 0} E \wedge bu \wedge F_{4k}/F_{4k-1} \\ &\stackrel{\vee_{k\geq 0}\mu\wedge 1}{\longrightarrow} \vee_{k\geq 0} E \wedge F_{4k}/F_{4k-1} \end{aligned}$$

where L and $\hat{L}$ are the 2-local equivalences of Theorem 3.1.6.

Theorem 3.3.2. The maps L_E and $\hat{L}_E$ of § 3.3.1 are 2-local homotopy equivalences.

Proof. We must show that L_E and $\hat{L}_E$ induce isomorphisms in mod 2 homology. The two cases are similar. However, this is easily seen for L_E from the discussion of § 3.1.5. Identify $H_*(\vee_{k\geq 0}(F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2); \mathbb{Z}/2)$ and $H_*(bu; \mathbb{Z}/2)$ with $\mathbb{Z}/2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots]$ as in § 3.1.5. Then the construction of L ensures that

$$(L^{-1})_*(1 \otimes z) = 1 \otimes z + \sum_\alpha b_\alpha \otimes c_\alpha$$

where each $b_\alpha \in H_*(bu;\mathbb{Z}/2)$ has strictly positive degree. Hence

$$(L_E)_*(a \otimes z) = a \otimes z + \sum_\alpha \mu_*(a \otimes b_\alpha) \otimes c_\alpha$$

and induction on the degree of z shows that $(L_E)_*$ is an isomorphism. $\square$

3.3.3. Operations in algebraic K-theory. The rest of this section will be concerned with the potential application of Theorem 3.3.2 when E is the spectrum of 2-localised algebraic K-theory. This application uses a crucial result of Andrei Suslin ([264], [265]) which relates connective K-theory to the algebraic K-theory of an algebraically closed field. Although this result is not central to my main theme it is an important result with a beautifully elegant proof. Therefore, for the interested reader's convenience, I have included in Chapter 4 a sketch of algebraic K-theory and of Suslin's theorem.

Let F be an algebraically closed field of characteristic different from 2; then there is a homotopy equivalence of ring spectra $bu \simeq \underline{KF}\mathbb{Z}_2$ between 2-adic connective K-theory and the algebraic K-theory spectrum of F with coefficients in the 2-adic integers ([264], [265]). Let X be a scheme – in the sense of algebraic geometry (see [77] [104]) – over $\mathrm{Spec}(F)$ so that the algebraic K-theory spectrum of X with coefficients in the 2-adic integers, $\underline{KX}\mathbb{Z}_2$, is a right-$\underline{KF}\mathbb{Z}_2$-module spectrum. Setting $E = \underline{KX}\mathbb{Z}_2$ in Theorem 3.3.2 we obtain:

Corollary 3.3.4. There are 2-local homotopy equivalences of the form

$$L_{KX} : \underline{KX}\mathbb{Z}_2 \wedge \underline{KF}\mathbb{Z}_2 \longrightarrow \vee_{k\geq 0}\underline{KX}\mathbb{Z}_2 \wedge (F_{4k}/F_{4k-1} \wedge \Sigma^{-2}\mathbb{CP}^2)$$

and

$$\hat{L}_{KX} : \underline{KX}\mathbb{Z}_2 \wedge bo \longrightarrow \vee_{k\geq 0}\underline{KX}\mathbb{Z}_2 \wedge F_{4k}/F_{4k-1}.$$

Remark 3.3.5. The splitting of Theorem 3.1.6 may be used to give a family of well-behaved operations in connective K-theory. At odd primes this is developed in [137], for example. In a similar manner the splittings of Corollary 3.3.4 may be used to give a family of operations on the algebraic K-theory of F-schemes.

More precisely, for $k \geq 0$ let

$$\underline{KX}\mathbb{Z}_2(k) = \underline{KX}\mathbb{Z}_2 \wedge F_{4k}/F_{4k-1}$$

so that $\underline{KX}\mathbb{Z}_2(0) = \underline{KX}\mathbb{Z}_2$. Then we may define maps of spectra of the form

$$Q^n : \underline{KX}\mathbb{Z}_2(k) \longrightarrow \underline{KX}\mathbb{Z}_2(k+n)$$

to be given by the components of the composition

$$\begin{aligned}\vee_{n\geq 0}Q^n : \underline{KX}\mathbb{Z}_2 \wedge F_{4k}/F_{4k-1} &\xrightarrow{1\wedge\eta\wedge 1} \underline{KX}\mathbb{Z}_2 \wedge bo \wedge F_{4k}/F_{4k-1} \\ &\xrightarrow{\hat{L}_{KX}\wedge 1} \vee_{n\geq 0}\underline{KX}\mathbb{Z}_2 \wedge F_{4n}/F_{4n-1} \wedge F_{4k}/F_{4k-1} \\ &\xrightarrow{1\wedge m} \vee_{n\geq 0}\underline{KX}\mathbb{Z}_2 \wedge F_{4n+4k}/F_{4n+4k-1}.\end{aligned}$$

Here $m : F_{4n}/F_{4n-1} \wedge F_{4k}/F_{4k-1} \longrightarrow F_{4n+4k}/F_{4n+4k-1}$, as in § 3.1.5, is induced by the loopspace multiplication on $\Omega^2 S^3$ via the Snaith splitting. The construction of the Q^n's imitates that of ([137] p. 20).

In the case when X is a regular scheme of finite type over F these operations should induce interesting operations on Chow theory by virtue of the isomorphism ([226] Theorem 5.19, see also [35], [37], [67], [82], [88], [139], [199], [213], [271])

$$H^p(X; \underline{\underline{K}}_p) \cong A^p(X).$$

Operations in connective K-theory have been thoroughly examined before ([9], [137], [175], [185]). The difference between my approach and previous ones is to view the Q^n's as lying in $\mathbb{Z}_2[U_\infty \mathbb{Z}_2]$ in order to control better the relations such as that between $Q^n Q^m$ and Q^{n+m} (cf. [137] p. 98).

Incidentally, using equivariant intersection cohomology theory, Steenrod operations on Chow theory have been constructed in [45] while similar operations are constructed in [280] using motivic cohomology. In addition, it should be pointed out that the modern cohomological viewpoint on Chow groups – and more generally Spencer Bloch's higher Chow groups of smooth schemes over a field – is as motivic cohomology groups (see [83], [130], [155], [180], [268], [283], [291]).

Chapter 4

A Brief Glimpse of Algebraic K-theory

K-theory is the linear algebra of algebraic topology.

Michael Atiyah

The contents of this chapter began life as a short postgraduate course on the K-theory of Banach algebras given at the University of Southampton early in 2003. It seems appropriate for inclusion here because of the application to algebraic K-theory given in Chapter 3 § 3.3.3 which uses the rigidity results of Andrei Suslin given in [264] and [265]. In addition to a sketch of the proof of the rigidity results we shall sketch the construction of the algebraic K-theory (and K-theory mod n) for rings (that is, affine schemes – see § 4.3.1). The extension of algebraic K-theory to schemes in general is described in detail in many sources (for example [226], [227], [83], [130]) and Suslin's results are described in a less dilettante manner in ([264], [265]; see also [142]).

My first objective is to explain how the K-theory of topological algebras can become much simpler when inflicted with coefficients modulo m. This seems to me an essential deep fact, due originally to Suslin and then generalised by Fischer and Prasolov independently, which should be borne in mind when studying things like the Baum-Connes conjecture concerning the assembly map

$$K_i^{\mathrm{top}}(BG) \longrightarrow K_i^{\mathrm{top}}(C^*_{\mathrm{red}}G)$$

for a torsion free discrete group G [198].

However, I have another motive for going back to reconsider Suslin's theorem. In the 1970's I showed how to construct unitary cobordism from topological K-theory [245] and went on to propose an algebraic cobordism for schemes by imitating my construction with algebraic K-theory replacing topological K-theory (see Chapter 1, Theorem 1.3.3). The process involved localising certain stable homotopy groups and, incidentally, introduced "Bott periodic K-theory" (see [69]) which was later shown by Bob Thomason [274] to coincide with étale K-theory. At the time of writing [245] it was clear, by studying the K-theory localisation sequences, that my definition of algebraic cobordism led to groups which were too large. Recently other (better) definitions of algebraic cobordism have appeared due to Voevodsky ([281], [287], [203]) and Levine-Morel [160] which prompted me to see if, in light of results not available in the 1970's, I could get from algebraic K-theory to something resembling these new algebraic cobordisms. Eventually I hope to elaborate on the contents of this chapter to show how to go from the algebraic K-theory of the ring $C(X)$ of complex-valued continuous functions on X to the unitary cobordism of X (more precisely to $(MU \wedge bu)^*(X; \mathbb{Z}/m)$) but right now that construction would take us even further off topic (however, see the assertions and conjectures in Chapter 9 § 9.2.15).

4.1 Simplicial sets and their realisations

Definition 4.1.1. Let Δ denote the category whose objects are the non-negative integers $\{\mathbf{n} \mid n \geq 0\}$ where $\mathbf{n}$ equals the totally ordered set of $n+1$ integers $\mathbf{n} = \{0, 1, 2, \ldots, n\}$ and a morphism $\theta : \mathbf{m} \longrightarrow \mathbf{n}$ is an order-preserving map of finite sets from $\{0, 1, 2, \ldots, m\}$ to $\{0, 1, 2, \ldots, n\}$. Let Δ^{op} denote the opposite category – that is, same objects but

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}}(\mathbf{m}, \mathbf{n}) = \mathrm{Hom}_{\Delta}(\mathbf{n}, \mathbf{m}).$$

A simplicial set is a (covariant) functor ([92] p. 3)

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathrm{Sets}.$$

Example 4.1.2. Let Y be a topological space. We shall now describe the associated simplicial set $S(Y)$. The topological standard n-simplex $|\Delta^n| \subset \mathbb{R}^{n+1}$ is the topological subspace

$$|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}.$$

If $\theta \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(\mathbf{n}, \mathbf{m})$ is a morphism in Δ^{op}, which means that as a map of sets we have

$$\theta : \{0, \ldots, m\} \longrightarrow \{0, \ldots, n\},$$

define

$$\theta_* : |\Delta^m| \longrightarrow |\Delta^n|$$

by the formula

$$\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n)$$

where

$$s_i = \begin{cases} 0 & \text{if } \theta^{-1}(i) \text{ is empty,} \\ \sum_{j \in \theta^{-1}(i)} t_j & \text{if } \theta^{-1}(i) \text{ is nonempty.} \end{cases}$$

Define the functor $S(Y)$ by $S(Y)(\mathbf{n}) = \mathrm{Map}(|\Delta^n|, Y)$ and

$$S(Y)(\theta) = (f \mapsto f \cdot \theta_*) : S(Y)(\mathbf{n}) = \mathrm{Map}(|\Delta^n|, Y) \longrightarrow \mathrm{Map}(|\Delta^m|, Y) = S(Y)(\mathbf{m}).$$

Definition 4.1.3. If X is a simplicial set then the (geometric) realisation of X, denoted by $|X|$, is the topological space given by the disjoint union of all the spaces $|\Delta^n| \times X(n)$ divided out by the equivalence relation

$$(\theta_*(x), z) \simeq (x, X(\theta)(z))$$

for all $\theta \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(\mathbf{n}, \mathbf{m})$, $x \in |\Delta^m|$ and $z \in X(n)$ where θ_* is as in Example 4.1.2.

4.1.4. The classical description of a simplicial set X. Among the morphisms in $Hom_\Delta(\mathbf{m}, \mathbf{n})$ we have the following examples:

$$\begin{aligned} d^i &: \mathbf{n-1} \longrightarrow \mathbf{n} \quad 0 \leq i \leq n \quad \text{(cofaces)},\\ s^j &: \mathbf{n+1} \longrightarrow \mathbf{n} \quad 0 \leq j \leq n \quad \text{(codegeneracies)} \end{aligned}$$

given by the injective, order-preserving map such that

$$d^i : \{0, \ldots, n-1\} \mapsto \{0, \ldots, i-1, i+1, \ldots, n\}$$

and the surjective, order-preserving map such that

$$s^j : \{0, \ldots, n+1\} \mapsto \{0, \ldots, j-1, j, j, j+1, \ldots, n\}.$$

These maps satisfy the following list of cosimplicial identities:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{if } i < j,\\ s^j d^i &= d^i s^{j-1} && \text{if } i < j,\\ s^j d^j &= 1 = s^j d^{j+1}, &&\\ s^j d^i &= d^{i-1} s^j && \text{if } i > j+1,\\ s^j s^i &= s^i s^{j+1} && \text{if } i \leq j. \end{aligned}$$

These are a set of generators and relations between the morphisms of Δ. Thus, in order to specify the simplicial set X, it suffices to give a sequence of sets $X(n)$ – the n-simplices of X – for each non-negative integer n together with maps of sets

$$\begin{aligned} d_i &: X(n) \longrightarrow X(n-1) \quad 0 \leq i \leq n \quad \text{(faces)},\\ s_j &: X(n) \longrightarrow X(n+1) \quad 0 \leq j \leq n \quad \text{(degeneracies)} \end{aligned}$$

which satisfy the simplicial identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j,\\ d_i s_j &= s_{j-1} d_i && \text{if } i < j,\\ d_j s_j &= 1 = d_{j+1} s_j, &&\\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1,\\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j. \end{aligned}$$

This is the classical manner in which to specify a simplicial set [178].

The functors $Y \mapsto S(Y)$ and $X \mapsto |X|$ give inverse equivalences between the category **Top** of topological spaces and the category of simplicial sets ([92] Proposition 2.2 p. 7, [178]). In fact the realisation functor is left adjoint to the singular functor in the sense that there is a bijection

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \overset{\cong}{\leftrightarrow} \mathrm{Hom}_{\mathbf{SSets}}(X, S(Y))$$

which is natural in simplicial sets X and topological spaces Y. Hence corresponding to the identity map of $S(Y)$ we have a canonical map of the form

$$|S(Y)| \longrightarrow Y$$

for all topological spaces Y and this map is a weak homotopy equivalence ([92], [178]).

The homology of a simplicial set X is defined by taking the homology of the chain complex whose i-chains are given by the free abelian group on the i-simplices $X(i)$ with a differential given by $d = \sum_j (-1)^j d_j$ and with this definition one finds that $H_*(S(Y); \mathbb{Z}) \cong H_*(Y; \mathbb{Z})$, the singular homology of the space Y, and that the simplicial set BG of § 4.4.1 has homology equal to the group cohomology of G which is also isomorphic to $H_*(BG; \mathbb{Z})$.

4.2 Quillen's higher algebraic K-theory

4.2.1. We shall now recall Quillen's original definition of the algebraic K-groups of a ring A via the space $BGL(A)^+$ ([226], [227]). There are many equivalent but more sophisticated definitions currently in use – due to Waldhausen and others – but for our purposes $BGL(A)^+$ will suffice.

Let $f : X \longrightarrow Y$ be a map of connected CW complexes with basepoint. We call f acyclic if the equivalent conditions are satisfied:

(i) The induced map is an isomorphism

$$H_*(X; f^*(\mathcal{L})) \xrightarrow{\cong} H_*(Y; \mathcal{L})$$

for any local coefficient system (i.e., $\mathbb{Z}[\pi_1(Y)]$-module) on Y.

(ii) The homotopy fibre F of f is acyclic (i.e., $\tilde{H}_*(F; \mathbb{Z}) = 0$) where, if $y_0 \in Y$ is the basepoint,

$$F = \{(x, p) \in X \times \mathrm{Map}([0, 1], Y) \mid p(0) = y_0,\ p(1) = f(x)\}.$$

If f is acyclic then there is an induced isomorphism of the form

$$f_\# : \pi_1(X)/N \xrightarrow{\cong} \pi_1(Y)$$

where $N \lhd \pi_1(X)$ is a perfect normal subgroup – i.e., $N = [N, N]$, the commutator subgroup of N. Conversely, by the theory of covering spaces, given a connected

CW complex X and a perfect normal subgroup $N \lhd \pi_1(X)$ there is an acyclic map, which is unique up to homotopy, $f : X \longrightarrow Y$ such that $f_{\#}$ is surjective with kernel N.

Now let A be a ring with an identity element. Let $GL_n(A)$, the $n \times n$ general linear group, denote the group of invertible $n \times n$ matrices with entries in A. Sending an $n \times n$ matrix U to the $(n+1) \times (n+1)$ matrix $U \oplus 1$ consisting of U in the top left n rows and columns, a 1 in the $(n+1, n+1)$th entry and zeroes everywhere else gives an embedding of $GL_n(A) \subset GL_{n+1}(A)$ and the infinite general linear group is the union

$$GL(A) = \bigcup_n GL_n(A),$$

a discrete group. Let $BGL(A)$ denote the classifying space of $GL(A)$ as in § 4.7.1. Hence $\pi_1(BGL(A)) = GL(A)$ containing a normal subgroup $E(A)$ generated by the elementary matrices and $E(A)$ is perfect, by a result of J.H.C. Whitehead [31]. Therefore there is an acylic map, unique up to homotopy, of the form

$$f : BGL(A) \longrightarrow BGL(A)^+$$

which induces on fundamental groups the canonical quotient

$$GL(A) \longrightarrow GL(A)/E(A).$$

In [31] one finds Whitehead's definition of the first K-group of A, $K_1(A) = GL(A)/E(A)$ and the zero-th K-group $K_0(A)$ as the Grothendieck group of finitely generated projective A-modules.

Hence this is consistent with Quillen's definition

$$K_i(A) = \pi_i(K_0(A) \times BGL(A)^+)$$

for all $i \geq 0$. In fact, this definition of $K_2(A)$ also agrees with the original definition given in [192].

4.2.2. Algebraic K-theory modulo m [49]. The n-dimensional Moore space for $\mathbb{Z}/m$, which will be denoted by $M^n(\mathbb{Z}/m)$, is defined as the mapping cone of the self-map of degree m on the $(n-1)$-dimensional sphere

$$M^n(\mathbb{Z}/m) = S^{n-1} \bigcup_m e^m$$

for $n \geq 2$. If $[Z, X]$ denotes the set of based homotopy classes of maps from Z to X, then $\pi_i(X) = [S^i, X]$ and we define $\pi_i(X; \mathbb{Z}/m)$ by

$$\pi_i(X; \mathbb{Z}/m) = [M^i(\mathbb{Z}/m), X]$$

for $i \geq 2$. The long exact Puppe sequence [257] becomes

$$\cdots \longrightarrow \pi_{i+1}(X; \mathbb{Z}/m) \longrightarrow \pi_i(X) \stackrel{(m\cdot -)}{\longrightarrow} \pi_i(X) \longrightarrow \pi_i(X; \mathbb{Z}/m) \longrightarrow \cdots$$

which ends with $\longrightarrow \pi_1(X) \stackrel{(m\cdot -)}{\longrightarrow} \pi_1(X)$ if $\pi_1(X)$ is abelian. This applies if we take $X = BGL(A)^+$ so, following Browder [49], we define

$$K_i(A; \mathbb{Z}/m) = \pi_i(BGL(A)^+; \mathbb{Z}/m)$$

for $i \geq 2$. Hence we obtain universal coefficient short exact sequences

$$0 \longrightarrow K_i(A) \otimes \mathbb{Z}/m \longrightarrow K_i(A; \mathbb{Z}/m) \longrightarrow \mathrm{Tor}(\mathbb{Z}/m, K_{i-1}(A)) \longrightarrow 0$$

where $\mathrm{Tor}(\mathbb{Z}/m, K_{i-1}(A)) = \{x \in K_{i-1}(A) \mid mx = 0\}$.

For the rings, we shall be interested in $\mathrm{Tor}(\mathbb{Z}/m, K_0(A)) = 0$ so that we shall be consistent if we define

$$K_1(A; \mathbb{Z}/m) = K_1(A) \otimes \mathbb{Z}/m$$

and refrain from defining $K_0(A; \mathbb{Z}/m)$.

4.3 The correspondence between affine schemes and commutative rings

4.3.1. Let R be a commutative ring. Associated to R is the affine scheme $\mathrm{Spec}(R)$ called the spectrum of R. The spectrum of R consists of a topological space whose points are the proper prime ideals of R,

$$\mathrm{Spec}(R) = \{P \mid P \lhd R,\ P \text{ prime}\}$$

together with the Zariski topology on this set. To specify a topology we have to give a collection of closed subsets of $\mathrm{Spec}(R)$ – in this case they will be called $V(S)$ where $S \subset R$ is any subset defined by

$$V(S) = \{P \in \mathrm{Spec}(R) \mid S \subset P\}.$$

Note that $S \subseteq T$ implies that $V(T) \subseteq V(S)$ and that $V(S) = V(I_S)$ where I_S is the ideal generated by S. To give a topology the empty set and the whole set must be closed – which is true because $V(\text{empty set}) = \mathrm{Spec}(R)$ and $V(R)$ is empty because R is not a proper ideal of itself – then finite unions and arbitrary intersections of closed sets must also be closed, which is true because ([104] Lemma 2.1 p. 70)

$$\bigcap_{\alpha \in \mathcal{A}} V(S_\alpha) = V(\bigcup_{\alpha \in \mathcal{A}} S_\alpha), \quad V(S_1) \bigcup V(S_2) = V(I_{S_1} \cdot I_{S_2}).$$

Each element of R can be thought of as a "function" on $\mathrm{Spec}(R)$ which sends a prime ideal P to an element of the residue field at P, $\kappa(P)$ which is defined to be the field of fractions of the integral domain R/P. An element $r \in R$ corresponds to the function

$$r : P \mapsto r(P) = r + P \in R/P \subseteq \kappa(P).$$

These "functions" are called the regular functions on $\mathrm{Spec}(R)$ – they are just the elements of R and are not really functions; for example r is not determined by the values $r(P)$ as one sees by taking $R = K[X]/(X^2)$ with K a field and considering the functions corresponding to 0 and X. Therefore

$$V(S) = \{P \in \mathrm{Spec}(R) \mid r(P) = 0 \text{ for all } r \in S\}.$$

The structure of $\mathrm{Spec}(R)$ does not stop there! It is a topological space together with a canonical sheaf of commutative rings on it. A presheaf of rings on a topological space X is a functorial assignment $U \mapsto \mathcal{O}(U)$ of a commutative ring $\mathcal{O}(U)$ to each open set U where U is open if and only if $X - U$ is closed. Functoriality means that for each inclusion $U \subset U'$ of open sets we have a homomorphism of rings

$$\mathrm{res}_{U',U} : \mathcal{O}(U') \longrightarrow \mathcal{O}(U)$$

such that $\mathrm{res}_{U,U} = 1_{\mathcal{O}(U)}$ and $\mathrm{res}_{U'',U} = \mathrm{res}_{U',U} \cdot \mathrm{res}_{U'',U'}$ for $U \subset U' \subset U''$. The presheaf $\mathcal{O}$ is a sheaf if, for $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ and a family of elements $r_\alpha \in \mathcal{O}(U_\alpha)$ such that

$$\mathrm{res}_{U_\alpha, U_\alpha \bigcap U_\beta}(r_\alpha) = \mathrm{res}_{U_\beta, U_\alpha \bigcap U_\beta}(r_\beta)$$

for all α, β, there exists a unique element $r \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_\alpha}(r) = r_\alpha$ for all $\alpha \in \mathcal{A}$.

In the case of $X = \mathrm{Spec}(R)$ the sheaf $U \mapsto \mathcal{O}_X(U)$ is the sheaf of regular functions on X – or the structure sheaf of X – and is defined in the following manner [104]. If $U = \mathrm{Spec}(R) - V(S)$ then $\mathcal{O}_X(\mathrm{Spec}(R) - V(S))$ consists of all functions

$$s : U \longrightarrow \prod_{P \in U} R(R-P)^{-1},$$

where s is locally a quotient of elements of R. Usually the localisation $R(R-P)^{-1}$ is denoted by $R_{(P)}$. Explicitly the local condition means that for each $P \in U$ there exists an open U' such that $P \in U' \subseteq U$ and elements $r, r' \in R$ with $r' \notin Q$ for each $Q \in U'$ and $s(Q) = r/r' \in R_{(Q)}$. One finds that $\mathcal{O}_X(\mathrm{Spec}(R) - V(\{r\})) = Rr^{-1}$ and $\mathcal{O}_X(\mathrm{Spec}(R)) = R$.

We shall only need affine schemes. However, in general a scheme $(X, \mathcal{O}_X)$ is a topological space together with a sheaf of rings such that each point $x \in X$ has an open neighbourhood U such that U together with the restriction of $\mathcal{O}_X$ to U is isomorphic to an affine scheme $\mathrm{Spec}(R)$.

Next we define a morphism of schemes. From ([78] Definition I-39 p. 29) a morphism between schemes $X \longrightarrow Y$ is a continuous map of topological spaces $\psi : X \longrightarrow Y$ together with a map of sheaves of rings on Y,

$$\psi^{\#} : \mathcal{O}_Y \longrightarrow \psi_* \mathcal{O}_X$$

where $\psi_* \mathcal{O}_X(U) = \mathcal{O}_X(\psi^{-1}(U))$. This map is required to satisfy the condition that for any $P \in X$ and any neighbourhood U of $Q = \psi(P) \in Y$ a regular function $s \in \mathcal{O}_Y(U)$ vanishes at Q ($s(Q) = 0 \in R_{(Q)}$ in the affine case) if and only if $\psi^{\#}(s) \in \psi_* \mathcal{O}_X(U) = \mathcal{O}_X(\psi^{-1}(U))$ vanishes at P.

The local ring at $P \in X$ is defined as

$$\mathcal{O}_{X,P} = \lim_{\substack{\rightarrow \\ P \in U}} \mathcal{O}_X(U)$$

where the limit is taken over all open sets containing P. This is a local ring whose maximal ideal $\mathcal{M}_{X,P}$ consists of all functions which vanish at P. When $X = \mathrm{Spec}(R)$ we have

$$\mathcal{O}_{\mathrm{Spec}(R),P} = R_{(P)} \text{ and } \mathcal{M}_{\mathrm{Spec}(R),P} = P \cdot R_{(P)}.$$

The condition on $\psi^{\#}$ has an equivalent reformulation in terms of the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$. On passing to the limit any map of sheaves induces a homomorphism between these local rings. In particular $\psi^{\#}$ induces

$$\psi^{\#} : \mathcal{O}_{Y,Q} \longrightarrow \lim_{\substack{\rightarrow \\ Q \in U}} \mathcal{O}_X(\psi^{-1}(U)) \longrightarrow \mathcal{O}_{X,P}$$

and the local condition is equivalent to this being a *local homomorphism*

$$\psi^{\#}(\mathcal{M}_{Y,Q}) \subseteq \mathcal{M}_{X,P}.$$

Theorem 4.3.2 ([78] Theorem I-40 p. 30)**.** For any scheme X and any commutative ring R, the morphisms

$$(\psi, \psi^{\#}) : X \longrightarrow \mathrm{Spec}(R)$$

are in one-one correspondence with the homomorphisms of rings

$$\phi : R \longrightarrow \mathcal{O}_X(X)$$

given by the formula

$$\phi = \psi^{\#}(\mathrm{Spec}(R)) : R = \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \longrightarrow \mathcal{O}_X(X).$$

Sketch Proof. To construct the inverse map suppose we are given ϕ. To construct ψ we need to specify the prime ideal $\phi(P)$ for each element $P \in X$. We have a map of rings

$$\tilde{\phi} : R \xrightarrow{\phi} \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,P}$$

and we set $\psi(P) = \tilde{\phi}^{-1}(\mathcal{M}_{X,P})$. To define $\psi^{\#}$ it is sufficient to define $\psi^{\#}(U)$ when $U = \mathrm{Spec}(R) - V(r)$ ([78] Proposition I-12). However, we have

$$\phi : R \longrightarrow \mathcal{O}_X(X)$$

which we can localise to obtain

$$\psi^{\#}(U) : Rr^{-1} = \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R) - V(r)) \longrightarrow \mathcal{O}_X(X)\phi(r)^{-1} \longrightarrow \mathcal{O}_X(\psi^{-1}(U)).$$

$\square$

Corollary 4.3.3 ([78] Corollary I-41 p. 30)**.** The category of affine schemes is equivalent to the opposite category of commutative rings with identity.

4.4 Henselian rings

Definition 4.4.1 ([187] I § 4 pp. 32–39)**.** Let A be a local ring with maximal ideal $\mathcal{M} \lhd A$ and residue field $k = A/\mathcal{M}$. Write $a \mapsto \overline{a}$ for either of the canonical homomorphisms $A \longrightarrow k$ or $A[T] \longrightarrow k[T]$.

Let B be a commutative ring with unit. Two polynomials $f(T), g(T) \in B[T]$ are *strictly coprime* if the two principal ideals $(f(T))$ and $(g(T))$ are coprime in $B[T]$ (i.e., $B[T]f(T) + B[T]g(T) = B[T]$). For example, $f(T)$ and $g(T) = T - a$ are coprime if $f(a) \neq 0$ but they are strictly coprime if and only if $f(a)$ is a unit in B.

Now suppose that A is complete (i.e., the canonical homomorphism $A \to \lim_{\overleftarrow{n}} A/\mathcal{M}^n$ is an isomorphism); then Hensel's Lemma states that: if $f(T) \in A[T]$ is a monic polynomial such that $\overline{f(T)} = g_0(T)h_0(T)$ with $g_0(T), h_0(T) \in k[T]$ monic and coprime, then there exist monic polynomials $g(T), h(T) \in A[T]$ such that $f(T) = g(T)h(T)$ and $\overline{g(T)} = g_0(T)$, $\overline{h(T)} = h_0(T)$. In general, any local ring A with this property is called Henselian.

Lemma 4.4.2.

(i) In Definition 4.4.1 the $g(T), h(T)$ in the factorisations are strictly coprime.

(ii) If $f(T), g(T) \in A[T]$ with $f(T)$ monic are such that $\overline{f(T)}, \overline{g(T)} \in k[T]$ are coprime then $f(T), g(T)$ are strictly coprime in $A[T]$.

(iii) In Definition 4.4.1 the factorisation $f(T) = g(T)h(T)$ is unique.

Proof. Clearly (i) is a special case of (ii). To prove (ii) let M denote the A-module given by

$$M = A[T]/(A[T]f(T) + A[T]g(T)).$$

Since $f(T)$ is monic, M is a finitely generated A-module. Therefore by Nakayama's Lemma $M = 0$ if and only if $\mathcal{M}M = M$. Since $\overline{f(T)}, \overline{g(T)}$ are coprime we have

$$k[T] = k[T]\overline{f(T)} + k[T]\overline{g(T)}$$

and so

$$A[T] = \mathcal{M}A[T] + A[T]f(T) + A[T]g(T),$$

which implies that $\mathcal{M}M = M$.

To prove (iii), suppose that $f(T) = g(T)h(T) = g_1(T)h_1(T)$ with $g(T)$, $h(T)$, $g_1(T)$, $h_1(T)$ all monic and $\overline{g(T)} = \overline{g_1(T)}$, $\overline{h(T)} = \overline{h_1(T)}$. Then, by part (ii), $g(T)$ and $h_1(T)$ are strictly coprime in $A[T]$ so there exist $r(T), s(T) \in A[T]$ such that

$$1 = g(T)r(T) + h_1(T)s(T).$$

Therefore

$$g_1(T) = g_1(T)g(T)r(T) + g_1(T)h_1(T)s(T) = g_1(T)g(T)r(T) + g(T)(T)s(T)$$

so that $g(T)$ divides $g_1(T)$. However $g(T)$ and $g_1(T)$ are both monic of the same degree and therefore are equal. □

Theorem 4.4.3 ([187] Theorem 4.2 p. 32)**.** For A, $\mathcal{M}$ and k as in Definition 4.4.1 the following are equivalent:

(i) A is Henselian.

(ii) Any finite, commutative A-algebra B is a direct product of local rings

$$B \cong \prod_i B_i$$

where the B_i are necessarily isomorphic to the localisations $B_{\mathcal{M}_i}$ as the $\mathcal{M}_i$ run through the maximal ideals of B.

(iii) If $f : Y \longrightarrow X = \mathrm{Spec}(A)$ is quasi-finite and separated then

$$Y = Y_0 \bigcup Y_1 \bigcup \cdots \bigcup Y_n \quad \text{(disjoint union)}$$

where $f(Y_0)$ does not contain $\mathcal{M}$ and Y_i is finite over X and is the spectrum of a local ring for all $i \geq 1$.

(iv) If $f : Y \longrightarrow X = \mathrm{Spec}(A)$ is étale and there is a point $y \in Y$ such that $f(y) = \mathcal{M}$ and $k(y) = k = A/\mathcal{M}$ then f has a section $s : X \longrightarrow Y$.

(v) Let $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$ and suppose there exists $a = (a_1, \ldots, a_n) \in k^n$ such that $\overline{f}(a) = 0$ for $1 \leq i \leq n$ and

$$\det\left(\left(\frac{\partial \overline{f_i}}{\partial T_j}\right)(a)\right) \neq 0.$$

Then there exists $b \in A^n$ with $\overline{b} = a$ and $f_i(b) = 0$ for $1 \leq i \leq n$.

(vi) Let $f(T) \in A[T]$ be such that $\overline{f(T)} = g_0(T)h_0(T) \in k[T]$ with $g_0(T)$ monic and $g_0(T)$, $h_0(T)$ coprime. Then $f(T) = g(T)h(T) \in A[T]$ with $g(T)$ monic and $\overline{g(T)} = g_0(T)$, $\overline{h(T)} = h_0(T)$.

Proposition 4.4.4 ([187] pp. 34–35)**.**

(i) Any complete local ring is Henselian.

(ii) If A is Henselian so is any finite local A-algebra B and any quotient ring A/I.

(iii) If A is Henselian then the functor $B \mapsto B \otimes_A k$ induces an equivalence between the category of finite étale A-algebras and the category of finite étale k-algebras.

4.4.5. Henselianisation. Let A be a Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field k as in § 4.4.1. Then a local homomorphism of local rings $i : A \longrightarrow A^h$ is called the Henselianisation of A if A^h is Henselian and every local homomorphism from A to a Henselian ring factorises uniquely through i.

Three constructions of the Henselianisation of A are given in ([187] pp. 36–37). For example one may construct A^h as the intersection of all local Henselian subrings S of the completion

$$\hat{A} = \lim_{\overleftarrow{n}} A/\mathcal{M}^n$$

such that the maximal ideal $\hat{\mathcal{M}} \lhd \hat{A}$ intersects S in the maximal ideal of S.

Now suppose that B is a Noetherian A-algebra together with a homomorphism of A-algebras $u : B \longrightarrow A$ which is the identity on $A \subseteq B$. Then $u^{-1}(\mathcal{M}) \lhd B$ is a maximal ideal, since $B/u^{-1}(\mathcal{M}) \cong A/\mathcal{M} = k$. Therefore we may form the local ring $B(B - u^{-1}(\mathcal{M}))^{-1}$ and the Henselianisation of this local ring, written merely as B^h, is called the *Henselianisation of* B over u.

4.5 Modulo m K-theory of Henselian pairs

Proposition 4.5.1. Let (R, I) be a Henselian pair and let m be an integer which is invertible in R/I. Then the following are equivalent:

(i) $K_*(R; \mathbb{Z}/m) \longrightarrow K_*(R/I; \mathbb{Z}/m)$ is an isomorphism,

(ii) $H_*(GL(R); \mathbb{Z}/m) \longrightarrow H_*(GL(R/I); \mathbb{Z}/m)$ is an isomorphism,

(iii) $H_*(GL(R, I); \mathbb{Z}/m) = 0$ where, as in § 4.6.5,

$$GL(R, I) = \mathrm{Ker}(GL(R) \longrightarrow GL(R/I)).$$

Proof. Since $I \subseteq \mathrm{Rad}(R)$ the homomorphism $K_0(R) \longrightarrow K_0(R/I)$ is injective ([31] Ch. IX § 1). Since (R, I) is a Henselian pair one sees (cf. [231] Ch. XI § 2) that the map on idempotents $\mathrm{Idemp}(M_n(R)) \longrightarrow \mathrm{Idemp}(M_n(R/I))$ is surjective and hence we have an isomorphism

$$K_0(R) \xrightarrow{\cong} K_0(R/I).$$

Therefore (i) is equivalent to the map on homotopy modulo m,

$$\pi_*(BGL(R)^+;\mathbb{Z}/m) \longrightarrow \pi_*(BGL(R/I)^+;\mathbb{Z}/m)$$

being an isomorphism. When $* = 1$ we have a surjection

$$K_1(R) \cong R^* \longrightarrow K_1(R/I) \cong (R/I)^*,$$

since $I \subseteq \mathrm{Rad}(R)$, with kernel $1 + I$ which is uniquely m-divisible and so

$$K_1(R;\mathbb{Z}/m) = K_1(R) \otimes \mathbb{Z}/m \longrightarrow K_1(R/I;\mathbb{Z}/m) = K_1(R/I) \otimes \mathbb{Z}/m$$

is an isomorphism. Hence, by the modulo m Whitehead Theorem in algebraic topology [257] parts (i) and (ii) are equivalent.

Consider the Hochschild-Serre spectral sequence

$$E^2_{p,q} = H_p(GL(R/I); H_q(GL(R,I);\mathbb{Z}/m)) \Longrightarrow H_{p+q}(GL(R);\mathbb{Z}/m).$$

The action of $GL(R/I)$ on $H_q(GL(R,I);\mathbb{Z}/m)$ is induced by the conjugation action of $GL(R)$ and is trivial ([265] Proposition 1.3) so that, if m is a prime,

$$E^2_{p,q} \cong H_p(GL(R/I);\mathbb{Z}/m) \otimes H_q(GL(R,I);\mathbb{Z}/m)$$

and a standard argument shows that (ii) and (iii) are equivalent. To get from the case when m is prime to the general case is a simple exercise with the universal coefficient theorems [257]. $\square$

4.6 The universal homotopy construction

4.6.1. Here is the dictionary between the affine A-schemes and the commutative A-algebras which we are going to use.

affine schemes	commutative A-algebras
$GL_n = \mathrm{Spec}(\mathcal{O}_n)$	$\mathcal{O}_n = A[T_{i,j}]_{1\le i,j\le n}(\det(T_{i,j})^{-1})$
$c : GL_n \to \mathrm{Spec}(A)$	$c =$ constants : $\mathrm{A} \to \mathcal{O}_\mathrm{n}$
$X_{n,i} = GL_n \times \cdots \times GL_n = \mathrm{Spec}(\mathcal{O}_{\mathrm{n,i}})$	$\mathcal{O}_{n,i} = \mathcal{O}_n \otimes_A \cdots \otimes_A \mathcal{O}_n$ (i factors)
$X_{n,i} \to \mathrm{Spec}(A)$	$A \to \mathcal{O}_{n,i},\ a \mapsto c(a) \otimes 1 \otimes \cdots \otimes 1$
unit section $u : \mathrm{Spec}(A) \to X_{n,i}$	$T_{i,j} \mapsto \delta_{ij}$ $u : \mathcal{O}_{n,i} \to A$
Henselianisation along u $X^h_{n,i}$ $u : \mathrm{Spec}(A) \to X^h_{n,i}$	Henselianisation over u (§ 4.6.5) $\mathcal{O}^h_{n,i}$ $u : \mathcal{O}^h_{n,i} \to A$

4.6.2. Let G be a discrete group and suppose that A and B are $\mathbb{Z}[G]$-modules. We recall some homological algebra. Suppose that

$$\cdots \longrightarrow P_n \xrightarrow{d} \cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

is a projective $\mathbb{Z}[G]$-resolution of A. Hence each P_i is a projective $\mathbb{Z}[G]$-module and the $\mathbb{Z}[G]$-homomorphisms of the resolution satisfy

$$dd = 0, \quad \epsilon d = 0$$

and the resolution is exact (i.e., the kernel of each map equals the image of its predecessor). The group, $\mathrm{Ext}^i_{\mathbb{Z}[G]}(A, B)$, is defined to be the ith homology group of the chain complex

$$\cdots \xleftarrow{d^*} \mathrm{Hom}_{\mathbb{Z}[G]}(P_n, B) \xleftarrow{d^*} \cdots \xleftarrow{d^*} \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, B) \longleftarrow 0.$$

Hence

$$\mathrm{Ext}^i_{\mathbb{Z}[G]}(A, B) = \frac{\ker(d^* : \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i+1}, B))}{\mathrm{im}(d^* : \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i-1}, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, B))}$$

where we adopt the convention that $P_i = 0$ if $i < 0$. Up to a canonical isomorphism, this definition is independent of the choice of resolution.

The ith cohomology group, $H^i(G; B)$, is defined to be given by

$$H^i(G; B) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, B)$$

where G acts trivially on $\mathbb{Z}$, the integers.

A canonical projective resolution of $\mathbb{Z}$ is given by the bar resolution

$$\cdots \xrightarrow{d_2} B_2G \xrightarrow{d_1} B_1G \xrightarrow{d_0} B_0G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Here B_nG is the free left $\mathbb{Z}[G]$-module on G^n. If $(g_1, \ldots, g_n) \in G^n$ we write $[g_1 \mid g_2 \mid \cdots \mid g_n]$ for the corresponding $\mathbb{Z}[G]$-basis element of B_nG. We write $[\ ]$ for the basis element of B_0G. The $\mathbb{Z}[G]$-homomorphisms of this resolution are given by

$$\begin{aligned} &\epsilon(g_1[\ ]) = 1, && \text{and} \\ &d_n([g_1 \mid g_2 \mid \cdots \mid g_{n+1}]) && = g_1[g_2 \mid \cdots \mid g_{n+1}] \\ & && \quad \textstyle\sum_{i=1}^n (-1)^i [g_1 \mid \cdots \mid g_i g_{i+1} \mid \cdots \mid g_{n+1}] \\ & && \quad (-1)^{n+1}[g_1 \mid g_2 \mid \cdots \mid g_n]. \end{aligned}$$

One may show that the bar resolution is exact by constructing a contracting homotopy

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} B_0G \xrightarrow{s_0} B_1G \xrightarrow{s_1} \cdots$$

given by the formulae

$$\begin{aligned} \eta(1) &= [\ ], && \text{and} \\ s_n(g_1[g_2 \mid \cdots \mid g_n]) &= [g_1 \mid g_2 \mid \cdots \mid g_n] && \text{for } n \geq 0. \end{aligned}$$

One readily verifies the following identities

$$\begin{aligned} 1 &= \epsilon\eta, \\ 1 &= \eta\epsilon + d_0 s_0 && \text{and} \\ 1 &= s_{n-1}d_{n-1} + d_n s_n && \text{for } n \geq 1. \end{aligned}$$

The ith homology group $H_i(G; B)$ is defined to be given by homology of the chain complex

$$\cdots \xrightarrow{d\otimes 1} P_n \otimes_{\mathbb{Z}[G]} B \xrightarrow{d\otimes 1} \cdots \xrightarrow{d\otimes 1} P_0 \otimes_{\mathbb{Z}[G]} B \longrightarrow 0$$

where $\{P_*, d\}$ is a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

Hence

$$H_i(G; B) = \frac{\ker(d \otimes 1 : P_i \otimes_{\mathbb{Z}[G]} B \longrightarrow P_{i-1} \otimes_{\mathbb{Z}[G]} B)}{\operatorname{im}(d \otimes 1 : P_{i+1} \otimes_{\mathbb{Z}[G]} B \longrightarrow P_i \otimes_{\mathbb{Z}[G]} B)}.$$

For any group G and integer m let $C_*(G, \mathbb{Z}/m)$ denote the standard chain complex for computing $H_*(G; \mathbb{Z}/m)$. In other words $C_*(G, \mathbb{Z}/m) = B_*G \otimes_{\mathbb{Z}[G]} \mathbb{Z}/m$ where G acts trivially on $\mathbb{Z}/m$. Let $\tilde{C}_*(G, \mathbb{Z}/m)$ denote the corresponding reduced complex in which $C_0(G, \mathbb{Z}/m)$ is replaced by zero.

4.6.3. Consider the following morphisms of affine schemes over A

$$p_j^i : X_{n,i} \longrightarrow X_{n,i-1}$$

given by

$$p_j^i(g_1, \ldots, g_i) = \begin{cases} (g_2, \ldots, g_i) & \text{if } j = 0, \\ (g_1, \ldots, g_{j-1}, g_j g_{j+1}, \ldots g_i) & \text{if } 1 \leq j \leq i-1, \\ (g_1, \ldots, g_{i-1}) & \text{if } j = i. \end{cases}$$

Being a morphism of affine schemes over A means that p_j^i commutes with the canonical maps to $\operatorname{Spec}(A)$. A morphism of affine schemes in one direction is equivalent to a map of A-algebras in the other direction, by Corollary 4.3.3,

$$(p_j^i)^* : \mathcal{O}_{n,i-1} \to \mathcal{O}_{n,i}$$

such that $(p_j^i)^*(c(a) \otimes 1 \otimes \cdots \otimes 1) = c(a) \otimes 1 \otimes \cdots \otimes 1 \otimes 1$ for all $a \in A$. In this case, for $u_i \in \mathcal{O}_n$,

$$\begin{aligned} &(p_j^i)^*(u_1 \otimes u_2 \otimes \cdots \otimes u_{i-1}) \\ &= \begin{cases} 1 \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_{i-1} & \text{if } j = 0, \\ u_1 \otimes u_2 \otimes \cdots \otimes u_{j-1} \otimes \Delta(u_j) \otimes u_{j+1} \otimes \cdots \otimes u_{i-1} & \text{if } 1 \leq j \leq i-1, \\ u_1 \otimes u_2 \otimes \cdots \otimes u_{i-1} \otimes 1 & \text{if } j = i \end{cases} \end{aligned}$$

where $\Delta(T_{a,b}) = \sum_{v=1}^{n} T_{a,v} \otimes T_{v,b}$. Also $(p_j^i)^*$ is an A-algebra homomorphism which is the identity on A and so is Δ. The homomorphism Δ respects localisation by inverting $\det(T_{i,j})$ because of the following formula:

$$\begin{aligned}
&\Delta(\det(T_{i,j})) \\
&= \Delta(\textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{sign}(\sigma)} T_{1,\sigma(1)} T_{2,\sigma(2)} \ldots T_{n,\sigma(n)}) \\
&= \textstyle\sum_{\sigma, a(i)} (-1)^{\mathrm{sign}(\sigma)}\, T_{1,a(1)} T_{2,a(2)} \ldots T_{n,a(n)} \otimes T_{a(1),\sigma(1)} T_{a(2),\sigma(2)} \ldots T_{a(n),\sigma(n)} \\
&= \textstyle\sum_{\sigma, a \in S_n} (-1)^{\mathrm{sign}(\sigma)}\, T_{1,a(1)} T_{2,a(2)} \ldots T_{n,a(n)} \otimes T_{a(1),\sigma(1)} T_{a(2),\sigma(2)} \ldots T_{a(n),\sigma(n)} \\
&= \textstyle\sum_{\sigma, a \in S_n} (-1)^{\mathrm{sign}(a)}\, T_{1,a(1)} T_{2,a(2)} \ldots T_{n,a(n)} \otimes \det(T_{i,j}) \\
&= \det(T_{i,j}) \otimes \det(T_{i,j}).
\end{aligned}$$

Lemma 4.6.4. The A-algebra $\mathcal{O}_n$ with coproduct $\Delta : \mathcal{O}_n \longrightarrow \mathcal{O}_n \otimes_A \mathcal{O}_n$, unit $A \longrightarrow \mathcal{O}_n$ given by the inclusion and augmentation $u : \mathcal{O}_n \longrightarrow A$ given as in the table in § 4.6.1 is a Hopf algebra.

Proof. Associativity $(1 \otimes_A \Delta)\Delta = (\Delta \otimes_A 1)\Delta$ is clear as is the fact that Δ is a homomorphism of A-algebras. Also $(u \otimes 1)\Delta(T_{i,j}) = \sum_{a=1}^{n} \delta_{ia} T_{a,j} = T_{i,l}$ so that $(u \otimes 1)\Delta = 1$ and similarly $(1 \otimes u)\Delta = 1$. □

4.6.5. It is clear when $j = 0$ and $j = i$ that $u = u(p_j^i)^* : \mathcal{O}_{n,i-1} \to A$. Also $u(\Delta(T_{a,b})) = 0$ unless $a = b$ so that $u = u(p_j^i)^*$ for all $1 \le j \le i-1$, too. Therefore $(p_j^i)^*$ induces a homomorphism of A-algebras (see § 4.6.3)

$$(p_j^i)^* : \mathcal{O}_{n,i-1}^h \to \mathcal{O}_{n,i}^h$$

which commute with u.

Set

$$\mathcal{M}_{n,i}^h = \mathrm{Ker}(u : \mathcal{O}_{n,i}^h \longrightarrow A).$$

This ideal lies in the radical of $\mathcal{O}_{n,i}^h$ – the (Jacobson) radical is the intersection of all the maximal ideals so that elements of it are in the kernel to every ring homomorphism to a field.

Set $GL(R, I) = \mathrm{Ker}(GL(R) \longrightarrow GL(R/I))$ and define $GL_n(R, I)$ similarly.

Now passing to general linear groups we have induced homomorphisms of groups

$$(p_j^i)^* : GL(\mathcal{O}_{n,i-1}^h, \mathcal{M}_{n,i-1}^h) \longrightarrow GL(\mathcal{O}_{n,i}^h, \mathcal{M}_{n,i}^h)$$

and

$$(p_j^i)^* : \tilde{C}_*(GL(\mathcal{O}_{n,i-1}^h, \mathcal{M}_{n,i-1}^h), \mathbb{Z}/m) \longrightarrow \tilde{C}_*(GL(\mathcal{O}_{n,i}^h, \mathcal{M}_{n,i}^h), \mathbb{Z}/m).$$

We have morphisms of affine schemes over $\mathrm{Spec}(A)$

$$X_{n,i}^h \longrightarrow X_{n,i} \xrightarrow{pr_k} GL_n$$

which preserve the unit section. The corresponding ring homomorphism

$$pr_k^* : \mathcal{O}_n \longrightarrow \mathcal{O}_{n,i}$$

is given by $pr_k^*(a) = 1^{\otimes k-1} \otimes a \otimes 1^{\otimes n-k}$. The ring homomorphisms

$$\mathcal{O}_n \xrightarrow{pr_k^*} \mathcal{O}_{n,i} \longrightarrow \mathcal{O}_{n,i}^h$$

respect the unit homomorphisms down to A and so we have induced maps

$$\mathcal{M}_n \xrightarrow{pr_k^*} \mathcal{M}_{n,i} \longrightarrow \mathcal{M}_{n,i}^h.$$

There is a canonical "tautological" matrix in $\alpha \in GL_n(\mathcal{O}_n)$ whose (i,j)th entry is $T_{i,j}$. Furthermore the unit map $u : \mathcal{O}_n \longrightarrow A$ sends α to the identity matrix in $GL_n(A)$ so that $\alpha \in GL_n(\mathcal{O}_n, \mathcal{M}_n)$. Therefore

$$\alpha_k = pr_k^*(\alpha) \in GL_n(\mathcal{O}_{n,i}^h, \mathcal{M}_{n,i}^h)$$

for each k in the range $1 \le k \le i$.

Define $u_{n,i} \in \tilde{C}_i(GL_n(\mathcal{O}_{n,i}^h, \mathcal{M}_{n,i}^h), \mathbb{Z}/m)$ by $u_{n,0} = 0$ and for $n \ge 1$,

$$u_{n,i} = [\alpha_1|\alpha_2|\ldots|\alpha_i] \otimes 1 \in B_i GL_n(\mathcal{O}_{n,i}^h, \mathcal{M}_{n,i}^h) \otimes \mathbb{Z}/m.$$

Now consider the boundary

$$\begin{aligned} d(u_{n,i}) \quad &= [\alpha_2|\ldots|\alpha_i] \otimes 1 - [\alpha_1\alpha_2|\ldots|\alpha_i] \otimes 1 + [\alpha_1|\alpha_2\alpha_3|\ldots|\alpha_i] \otimes 1 \\ &\qquad \cdots + (-1)^{i+1}[\alpha_1|\alpha_2|\ldots|\alpha_{i-1}] \otimes 1 \end{aligned}$$

bearing in mind that $\alpha_j \in GL\mathcal{O}_{n,i-1}^h$ is the element obtained by putting the tautological matrix in the jth position tensor entry in $\mathcal{O}_n^{\otimes i-1}$. The map

$$(p_0^i)^* : \mathcal{O}_{n,i-1}^h \longrightarrow \mathcal{O}_{n,i}^h$$

is induced by inserting a 1 in the first tensor entry so that $(p_0^i)^*(\alpha_j) = \alpha_{j+1}$ for $1 \le j \le i-1$ and so

$$[\alpha_2|\ldots|\alpha_i] \otimes 1 = (p_0^i)^*([\alpha_1|\alpha_2|\ldots|\alpha_{i-1}] \otimes 1) = (p_0^i)^*(u_{n,i-1}).$$

Now consider $(p_1^i)^* : \mathcal{O}_{n,i-1}^h \longrightarrow \mathcal{O}_{n,i}^h$ which is induced by Δ on the first tensor factor. Hence $(p_1^i)^*(\alpha_j) = \alpha_{j+1}$ for $2 \le j \le i-1$. On the other hand $(p_1^i)^*(\alpha_1)$ is an $n \times n$ matrix having as its (i,j)th entry

$$(\Delta \otimes I^{\otimes i-2})(T_{i,j} \otimes I^{\otimes i-2}) = \sum_{a=1}^{n} T_{i,a} \otimes T_{a,j} \otimes I^{\otimes i-2} = (\alpha_1\alpha_2)_{i,j}$$

so that

$$[\alpha_1\alpha_2|\ldots|\alpha_i] \otimes 1 = (p_1^i)^*([\alpha_1|\alpha_2|\ldots|\alpha_{i-1}] \otimes 1) = (p_1^i)^*(u_{n,i-1}).$$

Arguing similarly for each term in $d(u_{n,i})$ we have shown the following result.

Lemma 4.6.6.

$$d(u_{n,i}) = \sum_{j=0}^{i} (-1)^j (p_j^i)^*(u_{n,i-1}).$$

Proposition 4.6.7. If $\tilde{H}_i(GL(\mathcal{O}^h_{n,i}, \mathcal{M}^h_{n,i}); \mathbb{Z}/m) = 0$ then, for each $i \geq 0$, there exist chains

$$c_{n,i} \in \tilde{C}_{i+1}(GL(\mathcal{O}^h_{n,i}, \mathcal{M}^h_{n,i}), \mathbb{Z}/m)$$

such that

$$d(c_{n,i}) = u_{n,i} - \sum_{j=0}^{i} (-1)^j (p_j^i)^*(c_{n,i-1}).$$

Proof. Set $c_{n,0} = 0$. Hence the formula is true for $i = 0$ and now assume that it is true for $c_{n,0}, \ldots, c_{n.i-1}$. The result will follow once we show that

$$d(u_{n,i} - \sum_{j=0}^{i} (-1)^j (p_j^i)^*(c_{n,i-1})) = 0,$$

since we are assuming the triviality of homology modulo m. However, by Lemma 4.6.6,

$$\begin{aligned}
&d(u_{n,i} - \textstyle\sum_{j=0}^{i} (-1)^j (p_j^i)^*(c_{n,i-1})) \\
&= \textstyle\sum_{j=0}^{i} (-1)^j (p_j^i)^*(u_{n,i-1}) - \sum_{j=0}^{i} (-1)^j (p_j^i)^*(d(c_{n,i-1})) \\
&= \textstyle\sum_{j=0}^{i} (-1)^j (p_j^i)^*(u_{n,i-1}) - \sum_{j=0}^{i} (-1)^j (p_j^i)^*(u_{n,i-1}) \\
&\qquad + \textstyle\sum_{j=0}^{i} (-1)^j (p_j^i)^*(\sum_{k=0}^{i-1} (-1)^k (p_k^{i-1})^*(c_{n,i-2})) \\
&= \textstyle\sum_{j=0}^{i} (-1)^j (p_j^i)^*(\sum_{k=0}^{i-1} (-1)^k (p_k^{i-1})^*(c_{n,i-2})) \\
&= \textstyle\sum_{j=0}^{i} \sum_{k=0}^{i-1} (-1)^{j+k} (p_j^i)^*((p_k^{i-1})^*(c_{n,i-2})) \\
&= 0
\end{aligned}$$

since $(p_j^i)^*(p_k^{i-1})^* = (p_k^i)^*(p_{j-1}^{i-1})^*$ for $k \leq j$. □

Theorem 4.6.8. Suppose that (R, I) is a Henselian pair and R is an A-algebra where A and m are as in § 4.6.5 and Proposition 4.6.7. Then

$$K_*(R; \mathbb{Z}/m) \longrightarrow K_*(R/I; \mathbb{Z}/m)$$

is an isomorphism.

Proof. By Proposition 4.5.1 it suffices to show that $\tilde{H}_*(GL(R, I); \mathbb{Z}/m) = 0$. Now

$$\tilde{H}_*(GL(R, I); \mathbb{Z}/m) \cong \lim_{\vec{n}} \tilde{H}_*(GL_n(R, I); \mathbb{Z}/m)$$

so that it suffices to show that the chain map

$$i_* : \tilde{C}_*(GL_n(R, I); \mathbb{Z}/m) \longrightarrow \tilde{C}_*(GL(R, I); \mathbb{Z}/m)$$

is nullhomotopic. Now, for $i \geq 1$, $\tilde{C}_i(GL_n(R,I);\mathbb{Z}/m)$ is the free $\mathbb{Z}/m$-module on a basis consisting of elements of the form $[\beta_1|\beta_2|\ldots|\beta_i] \otimes 1$ where $\beta_j \in GL_n(R,I)$ for $1 \leq j \leq i$. Suppose that the (s,t)th entry of β_j is $\beta_j(s,t)$ so that

$$\beta_j(s,s) \in 1+I, \qquad \beta_j(s,t) \in I \text{ if } s \neq t.$$

The matrices $\beta_1, \beta_2, \ldots, \beta_i$ define a morphism $\mathrm{Spec}(R) \longrightarrow X_{n,i}$ of affine schemes over $\mathrm{Spec}(A)$ which sends the closed subscheme $\mathrm{Spec}(R/I) \longrightarrow \mathrm{Spec}(R)$ to the unit section. In terms of A-algebras this means that we have a homomorphism of A-algebras (i.e., equal to the identity on the subalgebra A) of the form

$$\mathcal{O}_{n,i} \longrightarrow R$$

given by sending $T_{s,t}$ in the jth tensor factor to $\beta_j(s,t)$. The unit section property means that the two maps

$$\mathcal{O}_{n,i} \longrightarrow R \longrightarrow R/I$$

and

$$\mathcal{O}_{n,i} \xrightarrow{u} A \longrightarrow A/\mathcal{M} \longrightarrow R/I$$

are equal. This follows from the congruences satisfied by the $\beta_j(s,t)$'s together with the fact that $A \longrightarrow R$ is a local homomorphism.

By the universal property of Henselianisation the homomorphism extends to

$$\phi_\beta : \mathcal{O}^h_{n,i} \longrightarrow R$$

having the analogous property when composed with reduction modulo I.

Define a chain homotopy

$$s : \tilde{C}_i(GL_n(R,I);\mathbb{Z}/m) \longrightarrow \tilde{C}_{i+1}(GL(R,I);\mathbb{Z}/m)$$

by the formula $s([\beta_1|\beta_2|\ldots|\beta_i] \otimes 1) = \phi_\beta(c_{n,i})$. Then we obtain, arguing as in the proofs of Lemma 4.6.6 and Proposition 4.6.7,

$$\begin{aligned}
&(sd+ds)([\beta_1|\beta_2|\ldots|\beta_i] \otimes 1) \\
&= s([\beta_2|\ldots|\beta_i] \otimes 1) - s([\beta_1\beta_2|\ldots|\beta_i] \otimes 1) + s([\beta_1|\beta_2\beta_3|\ldots|\beta_i] \otimes 1) \\
&\qquad \ldots (-1)^{i+1} s([\beta_1|\beta_2|\ldots|\beta_{i-1}] \otimes 1) + \phi_\beta(d(c_{n,i})) \\
&= s([\beta_2|\ldots|\beta_i] \otimes 1) - s([\beta_1\beta_2|\ldots|\beta_i] \otimes 1) + s([\beta_1|\beta_2\beta_3|\ldots|\beta_i] \otimes 1) \\
&\qquad \ldots (-1)^{i+1} s([\beta_1|\beta_2|\ldots|\beta_{i-1}] \otimes 1) \\
&\qquad + \phi_\beta(u_{n,i} - \textstyle\sum_{j=0}^{i} (-1)^j (p^i_j)^*(c_{n,i-1})) \\
&= \phi_\beta(u_{n,i}) \\
&= [\beta_1|\beta_2|\ldots|\beta_i] \otimes 1.
\end{aligned}$$

Therefore i_* is trivial on homology because, by the usual argument, any i-dimensional cycle z will satisfy

$$i_*(z) = s(d(z)) + d(s(z)) = d(s(z))$$

so that

$$0 = i_*[z] \in \tilde{H}_i(GL(R, I); \mathbb{Z}/m). \qquad \square$$

Corollary 4.6.9. If B is an A-algebra where A satisfies the conditions of § 4.6.5, Lemma 4.6.6 and Proposition 4.6.7 then so does B.

Corollary 4.6.10. If (R, I) is a Henselian pair and R is an F-algebra where F is a field such that $HCF(m, \mathrm{char}(F)) = 1$, then $K_*(R; \mathbb{Z}/m) \longrightarrow K_*(R/I; \mathbb{Z}/m)$ is an isomorphism.

Proof. This follows from a theorem of Ofer Gabber ([81]; see also [90]) which states that $A = F$ satisfies the conditions of Lemma 4.6.6 and Proposition 4.6.7 when the characteristic of F does not divide m. $\square$

4.7 K-theory of Archimedean fields

4.7.1. If G is a topological group we shall use the notation BG^{top} to denote the classifying space of G considered as a topological group [116] and we shall use BG to denote the classifying space for G as a discrete group. Recall that for any discrete group we have the following canonical model for BG. Let EG denote the geometrical realisation of the simplicial set whose m-simplices are $(m+1)$-tuples of elements of G with face and degeneracy operators corresponding to omitting and repeating the corresponding element, respectively (see § 4.1.3).

This space is contractible with a free action by G and thus EG/G is a model for BG. Hence a model for BG is the geometrical realisation of the simplicial set whose m-simplices are m-tuples of elements of G of the form

$$[g_1, \ldots, g_m] = \langle e, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_m \rangle \text{ (modulo } G)$$

with face and degeneracy operators given by

$$d_i([g_1, \ldots, g_m]) = \begin{cases} [g_2, \ldots, g_m] \text{ if } i = 0, \\ [g_1, \ldots, g_i g_{i+1}, \ldots, g_m] \text{ if } 1 \leq i \leq m-1, \\ [g_1, \ldots, g_{m-1}] \text{ if } i = m, \end{cases}$$

and

$$s_i[g_1, \ldots, g_m] = [g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_m].$$

4.7.2. BG_ϵ**.** Let G be a Lie group with a finite number of connected components. We are going to be mainly concerned with $SL_n\mathbb{C}$ and $GL_n\mathbb{C}$. Fix a left-invariant Riemannian metric on G and denote by G_ϵ the ϵ-ball centred at the identity element of G.

Denote by BG_ϵ the realisation of the simplicial set whose m-simplices are m-tuples $[g_1, \ldots, g_m]$ of elements of G such that $G_\epsilon \bigcap g_1 G_\epsilon \bigcap g_2 G_\epsilon \cdots \bigcap g_m G_\epsilon$ is non-empty. This is made into a simplicial set using the faces and degeneracies of § 4.7.1.

Recall ([116], [257], [258]) that the classifying space BG^{top} is a topological space such that principal G^{top}-bundles over X are in one-one correspondence with homotopy classes of maps from X to BG^{top}. Also there is a universal principal G^{top} bundle $G^{\mathrm{top}} \longrightarrow EG^{\mathrm{top}} \longrightarrow BG^{\mathrm{top}}$ in which EG^{top} is a free G^{top}-space which is contractible.

Theorem 4.7.3. If ϵ is sufficiently small then

$$BG_{\epsilon} \longrightarrow BG \longrightarrow BG^{\mathrm{top}}$$

is a fibration.

Proof. We have a commutative diagram in which the vertical maps, which are homotopy equivalences, are as in § 4.1.4.

$$\begin{array}{ccccc}
|S(G^{\mathrm{top}})| & \longrightarrow & |S(EG^{\mathrm{top}})| & \longrightarrow & |S(BG^{\mathrm{top}})| \\
\downarrow & & \downarrow & & \downarrow \\
G^{\mathrm{top}} & \longrightarrow & EG^{\mathrm{top}} & \longrightarrow & BG^{\mathrm{top}}
\end{array}$$

The action by G on $|S(EG^{\mathrm{top}})|$ is free so we may divide out by this action to obtain a fibration

$$|S(G^{\mathrm{top}})|/G \longrightarrow |S(EG^{\mathrm{top}})|/G \longrightarrow |S(BG^{\mathrm{top}})|.$$

Therefore, up to homotopy, we have a fibration [194]

$$|S(G^{\mathrm{top}})|/G \longrightarrow BG \longrightarrow BG^{\mathrm{top}}.$$

Now suppose that ϵ is sufficiently small that G_{ϵ} is geodesically convex ([99] § 5.2) and therefore every iterated intersection $g_0G_{\epsilon} \bigcap g_1G_{\epsilon} \bigcap \cdots \bigcap g_mG_{\epsilon}$ is contractible. Denote by $X_{\bullet}$ the simplicial space whose m-simplices are given by the disjoint union

$$X_m = \bigcup_{g_0,\ldots,g_m \in G} g_0G_{\epsilon} \bigcap g_1G_{\epsilon} \bigcap \cdots \bigcap g_mG_{\epsilon}.$$

A bisimplicial set $Y_{\bullet\bullet}$ is a simplicial object in the category of simplicial sets – so it has two grading indices. A simplicial set may be considered as a bisimplicial set which is constant in one of the grading directions. A bisimplicial set $Y_{\bullet\bullet}$ has a realisation $|Y_{\bullet\bullet}|$ which may be obtained by realising first in one grading direction and then in the other.

Denote by $Y_{\bullet\bullet}$ the bisimplicial set given by $Y_{pq} = S(X_q)(p)$. Denote by $S(G^{\mathrm{top}})_{\epsilon}$ the subobject of $S(G^{\mathrm{top}})$ whose m-simplices consist of the singular m-simplices which lie in gG_{ϵ} for some $g \in G$ and denote by E_{ϵ} the simplicial set whose m-simplices consist of $(m+1)$-tuples $(g_0, \ldots, g_m)$ of elements of G such that $g_0 G_\epsilon \bigcap g_1 G_\epsilon \bigcap \cdots \bigcap g_m G_\epsilon$ is non-empty.

We have two canonical maps of bisimplicial sets

$$S(G^{\mathrm{top}})_\epsilon \xleftarrow{\phi} Y_{\bullet\bullet} \xrightarrow{\psi} E_\epsilon$$

where we consider $S(G^{\mathrm{top}})_\epsilon$ as constant in the q-grading and E_ϵ as constant in the p-grading.

Proposition 4.7.5(i)–(ii) yields homotopy equivalences of the form

$$|S(G^{\mathrm{top}})_\epsilon| \xleftarrow{\simeq} |Y_{\bullet\bullet}| \xrightarrow{\simeq} |E_\epsilon|.$$

Furthermore G acts freely on all these spaces in such a way that ϕ and ψ are G-equivariant. Hence these maps are G-homotopy equivalences. Dividing out by the action of G yields homotopy equivalences of the form

$$|S(G^{\mathrm{top}})_\epsilon|/G \xleftarrow{\simeq} |Y_{\bullet\bullet}|/G \xrightarrow{\simeq} |E_\epsilon|/G = BG_\epsilon.$$

By the same argument, Proposition 4.7.5(iii) yields a homotopy equivalence of the form

$$|S(G^{\mathrm{top}})_\epsilon|/G \xrightarrow{\simeq} |S(G^{\mathrm{top}})|/G$$

such that the composition

$$|S(G^{\mathrm{top}})_\epsilon|/G \xrightarrow{\simeq} |S(G^{\mathrm{top}})|/G \longrightarrow BG$$

equals the inclusion $BG_\epsilon \subset BG$. This completes the proof of Theorem 4.7.3 $\square$

Remark 4.7.4. An inspection of the proof of Theorem 4.7.3 shows that if $\delta < \epsilon$, the composition

$$BG_\delta \subseteq BG_\epsilon \longrightarrow |S(G^{\mathrm{top}})|/G$$

is homotopic to the constructed equivalence $BG_\delta \xrightarrow{\simeq} |S(G^{\mathrm{top}})|/G$ so that the inclusion induces a homotopy equivalence

$$BG_\delta \simeq BG_\epsilon.$$

Recall that a map of simplicial or bisimplicial sets is an equivalence if the induced map on realisations is a homotopy equivalence.

Proposition 4.7.5.

(i) ϕ is an equivalence.

(ii) ψ is an equivalence.

(iii) The inclusion

$$|S(G^{\mathrm{top}})_\epsilon| \subseteq |S(G^{\mathrm{top}})|$$

is a homotopy equivalence.

Proof. Part (ii) amounts to the well-known simplicial approximation theorem ([257] Ch. 4 § 4).

Parts (i) and (ii) can be proved using Theorem A and Theorem B of [226] (see also [92] and [178]) which reduces (i) to showing that $Y_{p\bullet} \longrightarrow S(G^{\mathrm{top}})_\epsilon(p)$ is an equivalence for each p and that $Y_{\bullet q} \longrightarrow E_\epsilon(q)$ is an equivalence for each q. Both these facts are verified by a further application of Theorem A and Theorem B of [226]. □

Theorem 4.7.6. Let $k = \mathbb{R}, \mathbb{C}$ denote either the field of real or complex numbers. Then, if ϵ is small enough, the embedding

$$BGL_n(k)_\epsilon \subset BGL_n(k) \subset BGL(k)$$

induces the zero homomorphism on reduced homology modulo m, $\tilde{H}_*(-;\mathbb{Z}/m)$.

Proof. This proof is very similar to that of Theorem 4.6.8.

Denote by $\mathcal{O}^{\mathrm{cont}}_{n,i}$ the ring of germs of continuous k-valued functions on the i-fold cartesian product $GL_n(k) \times GL_n(k) \times \cdots \times GL_n(k)$ in some neighbourhood of the identity. Such a germ is the equivalence class of a function in a neighbourhood of the identity where two functions are equivalent if they agree on some, possibly smaller, neighbourhood of the identity. Hence $GL_r(\mathcal{O}^{\mathrm{cont}}_{n,i})$ may be identified with the germs of continuous $GL_r(k)$-valued functions on the i-fold cartesian product $GL_n(k) \times GL_n(k) \times \cdots \times GL_n(k)$ in some neighbourhood of the identity. Therefore every chain $c \in C_q(GL_r(\mathcal{O}^{\mathrm{cont}}_{n,i});\mathbb{Z}/m)$ defines a continuous map from some neighbourhood of the identity in $GL_n(k) \times GL_n(k) \times \cdots \times GL_n(k)$ to $C_q(GL_r(k);\mathbb{Z}/m)$.

Let $\mathcal{O}^h_{n,i}$ be the Henselianisation introduced in § 4.6.1. The ring $\mathcal{O}^{\mathrm{cont}}_{n,i}$ is also Henselian [231] so that there exists a canonical homomorphism

$$\mathcal{O}^h_{n,i} \longrightarrow \mathcal{O}^{\mathrm{cont}}_{n,i}$$

and therefore the image of $c_{n,i}$ of Proposition 4.6.7 yields a chain

$$c^{\mathrm{cont}}_{n,i} \in \tilde{C}_{i+1}(GL(\mathcal{O}^{\mathrm{cont}}_{n,i}), \mathbb{Z}/m).$$

For any $N > 0$ we can find an $\epsilon > 0$ such that $c^{\mathrm{cont}}_{n,i}$ is defined on the i-fold cartesian product $GL_n(k)_\epsilon \times GL_n(k)_\epsilon \times \cdots \times GL_n(k)_\epsilon$ for each $0 \leq i \leq N$. Thus for each $0 \leq i \leq N$ we get a chain homotopy map

$$s_i : C_i(GL_n(k)_\epsilon;\mathbb{Z}/m) \longrightarrow C_{i+1}(GL_r(k);\mathbb{Z}/m) \longrightarrow C_{i+1}(GL(k);\mathbb{Z}/m).$$

It is clear (cf. Theorem 4.6.8 (proof)) that s_* gives a chain homotopy in dimensions less than N between the canonical chain map and zero. Hence the canonical map is zero on homology in all dimensions $i > 0$, which completes the proof since the homologies of $C_*(GL_n(k)_\epsilon;\mathbb{Z}/m)$ and $C_*(GL(k);\mathbb{Z}/m)$ are $H_*(BGL_n(k)_\epsilon;\mathbb{Z}/m)$ and $H_*(BGL(k);\mathbb{Z}/m)$, respectively. □

4.7.7. $SL(k)$**.** For technical reasons – the fact that $BSL(k)$ is simply connected but $BGL(k)$ is not – we shall need to work with the special linear groups $SL_n(k)$ and $SL(k)$ given by the kernels of the determinant map. There is a homomorphism

$$GL(k) \longrightarrow SL(k)$$

which sends X to

$$\begin{pmatrix} \det(X)^{-1} & 0 \\ 0 & X \end{pmatrix}$$

which is conjugate to a left inverse to the inclusion of $SL(k)$ into $GL(k)$. Therefore

$$H_*(BSL(k);\mathbb{Z}/m) \longrightarrow H_*(BGL(k);\mathbb{Z}/m)$$

is injective, which implies the following result:

Corollary 4.7.8. Let $k = \mathbb{R}, \mathbb{C}$ denote either the field of real or complex numbers. Then, if ϵ is small enough, the embedding

$$BSL_n(k)_\epsilon \subset BSL_n(k) \subset BSL(k)$$

induces the zero homomorphism on reduced homology modulo m, $\tilde{H}_*(-;\mathbb{Z}/m)$.

Corollary 4.7.9. Let $k = \mathbb{R}, \mathbb{C}$ denote either the field of real or complex numbers. Then, if ϵ is small enough, $\tilde{H}_*(BSL_n(k)_\epsilon;\mathbb{Z}/m) = 0$ for $0 \leq i \leq (n-1)/2$.

Proof. By the universal coefficient theorem [257] we may assume that m is a prime.

Consider the Serre spectral sequence for the fibration [257]

$$BSL_n(k)_\epsilon \longrightarrow BSL_n(k) \longrightarrow BSL_n(k)^{\mathrm{top}}$$

which takes the form

$$E^2_{s,t} = H_s(BSL_n(k)^{\mathrm{top}};\mathbb{Z}/m) \otimes H_t(BSL_n(k)_\epsilon;\mathbb{Z}/m) \Longrightarrow H_{s+t}(BSL_n(k);\mathbb{Z}/m),$$

because $BSL_n(k)^{\mathrm{top}}$ is simply connected and therefore the local coefficient system is trivial. Now the edge homomorphism in this spectral sequence

$$H_s(BSL_n(k);\mathbb{Z}/m) \longrightarrow E^2_{s,0}$$

is surjective [194]. Surjectivity implies that none of the differentials

$$d_r : E^2_{s,t} \cong E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

are non-zero when $t = 0$.

Consider the least integer t_0 for which $\tilde{H}_{t_0}(BSL_n(k)_\epsilon;\mathbb{Z}/m) \neq 0$. We must have $E^2_{0,t_0} = E^r_{0,t_0}$ for all $r \geq 2$ and so

$$H_{t_0}(BSL_n(k)_\epsilon;\mathbb{Z}/m) \longrightarrow H_{t_0}(BSL_n(k);\mathbb{Z}/m)$$

must be injective. However, by the homological stability results of ([263], [264]), $H_{t_0}(BSL_n(k);\mathbb{Z}/m) \longrightarrow H_{t_0}(BSL(k);\mathbb{Z}/m)$ is an isomorphism if $t_0 \leq (n-1)/2$, which contradicts Corollary 4.7.8. □

Corollary 4.7.10. Let $k = \mathbb{R}, \mathbb{C}$ denote either the field of real or complex numbers. Then the following is true.

(i) On homology modulo m,

$$H_*(BSL(k)^+; \mathbb{Z}/m) \longrightarrow H_*(BSL(k)^{\mathrm{top}}; \mathbb{Z}/m)$$

is an isomorphism.

(ii) On homotopy modulo m,

$$\pi_*(BSL(k)^+; \mathbb{Z}/m) \longrightarrow \pi_*(BSL(k)^{\mathrm{top}}; \mathbb{Z}/m)$$

is an isomorphism.

(iii) On algebraic K-theory modulo m,

$$K_*(k; \mathbb{Z}/m) \longrightarrow \pi_*(BGL(k)^{\mathrm{top}}; \mathbb{Z}/m)$$

is an isomorphism for $* \geq 1$.

(iv) $\pi_j(BGL(\mathbb{C})^{\mathrm{top}}; \mathbb{Z}/m) \cong \mathbb{Z}/m$ for $j = 2s \geq 2$ and is zero for $j \geq 1$ odd.

Part (i) implies part (ii) by the modulo m Hurewicz Theorem. Part (i) follows from Corollary 4.7.9 and the Serre spectral sequence when $n \to \infty$. Part (iii) follows from the homotopy equivalences

$$BGL(k)^{\mathrm{top}} \simeq BSL(k)^{\mathrm{top}} \times (Bk^*)^{\mathrm{top}}, \ BGL(k) \simeq BSL(k) \times Bk^*$$

together with part (ii) and the facts that $Bk^* \longrightarrow (Bk^*)^{\mathrm{top}}$ induces an isomorphism on modulo m homology and homotopy. Part (iv) is a consequence of Bott periodicity [26]. □

4.8 Commutative Banach algebras

4.8.1. In this section we shall recall the extension of Suslin's theorem (Corollary 4.7.10) from the real or complex fields to the case of a general commutative Banach algebra A. This generalisation is to be found in [220] but the case when A is a commutative C^*-algebra (recall that, by the Gelfand-Naimark Theorem any such is isomorphic to the continuous complex-valued functions on the maximal ideal space of A) was also proved in [79]. Both proofs make essential use of the universal homotopy construction of Proposition 4.6.7.

Let A be a commutative Banach algebra and $0 < \epsilon \leq 1/n$. Define $U(n, \epsilon) \subset GL_n(A)$ to be the neighbourhood of the identity given by the $n \times n$ matrices of the form $1 + Z$ with $\rho(Z_{i,j}) < \epsilon$ for all $1 \leq i, j \leq n$ where $\rho(w) = \inf_m ||w^m||^{1/m}$.

Theorem 4.8.2 ([220] Lemma 1)**.** Theorem 4.7.3 remains true if G and G_ϵ are replaced by $BGL_n(A)$ and $U(n, \epsilon)$ respectively. That is,

$$BU(n, \epsilon) \longrightarrow BGL_n(A) \longrightarrow BGL_n(A)^{\mathrm{top}}$$

is a fibration for ϵ small enough.

4.8.3. Next one uses the Henselian local ring $\mathcal{O}_{n,i}$ of germs of real analytic functions on $GL_n \times \cdots \times GL_n$ in a neighbourhood of the identity and its maximal ideal $\mathcal{M}_{n,i}$ as in Theorem 4.7.6 to prove the following result.

Theorem 4.8.4 ([220] Proposition 1)**.** Let F denote the fibre of $BGL(A) \longrightarrow BGL(A)^{\text{top}}$. Then

$$\tilde{H}_*(F; \mathbb{Z}/m) = 0.$$

Theorem 4.8.5 ([220] Theorem 1)**.** Let A be a commutative Banach algebra and m an arbitrary positive integer. Then the following is true.

(i) On homology modulo m,

$$H_*(BSL(A)^+; \mathbb{Z}/m) \longrightarrow H_*(BSL(A)^{\text{top}}; \mathbb{Z}/m)$$

is an isomorphism.

(ii) On homotopy modulo m,

$$\pi_*(BSL(A)^+; \mathbb{Z}/m) \longrightarrow \pi_*(BSL(A)^{\text{top}}; \mathbb{Z}/m)$$

is an isomorphism.

(iii) On algebraic K-theory modulo m,

$$K_*(A; \mathbb{Z}/m) \longrightarrow \pi_*(BGL(A)^{\text{top}}; \mathbb{Z}/m)$$

is an isomorphism for $* \geq 1$.

Proof. In order to follow the proofs of Corollaries 4.7.8–4.7.10 we just have to show that $BA^* \longrightarrow (BA^*)^{\text{top}}$ induces an isomorphism on homotopy modulo m. This follows from the fact that the mth power map $A^* \longrightarrow A^*$ is a Serre fibration with fibre X which satisfies $\pi_0(X) = X$ ([220] Proposition 2). □

The following result is analogous to Proposition 4.5.1.

Theorem 4.8.6 ([220] Theorem 2)**.** Let A be a not necessarily commutative Banach algebra and m an arbitrary positive integer. Suppose that $I \lhd A$ is a closed ideal such that $I \subseteq \text{Rad}(A)$. Then

$$K_*(A; \mathbb{Z}/m) \longrightarrow K_*(A/I; \mathbb{Z}/m)$$

is an isomorphism for $* \geq 1$.

4.9 The case of $C(X)$ via excision

4.9.1. In this section let X be a finite CW complex and let $C(X)$ denote the C^*-algebra of continuous complex-valued functions on X. By Theorem 4.8.6 with $A = C(X)$ – first proved in [79] and [220] – we know that

$$K_0(C(X)) \times BGLC(X)^+ \longrightarrow K_0^{\text{top}}(X) \times BGLC(X)^{\text{top}}$$

induces an isomorphism on homology and homotopy modulo m. By [272] a finite-dimensional, complex vector bundle on X is equivalent to a finitely generated projective module over $C^*(X)$ so that $K_0(C(X)) \cong K_0^{\mathrm{top}}(X)$.

Now suppose that $X = Y \bigcup e^n$ is obtained from a finite CW complex by attaching an n-cell. In this case I will briefly sketch how to use excision to prove Theorem 4.8.6 for $A = C(X)$. We shall proceed by induction on the number of cells in X and on the dimension of X. By Corollary 4.7.10 and additivity of K-theory we have the result for a finite set of points.

Next we prove Theorem 4.8.6 for $A = C(X)$ when X is the n-dimensional disc or sphere. Then we use some Mayer-Vietoris exact sequences in K-theory.

First we recall the exact sequence of a Serre subcategory ([227] Theorem 4) of a small abelian category $\mathcal{A}$. Recall that if $\mathcal{A}$ is an abelian category then a subcategory $\mathcal{B}$ is a Serre subcategory if it is a full subcategory and for all exact sequences in $\mathcal{A}$

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

$A \in \mathcal{B}$ if and only if $A', A'' \in \mathcal{B}$. In this case there is a quotient category $\mathcal{A}/\mathcal{B}$ and a K-theory exact sequence, if $\mathcal{A}$ is a small abelian category, of the form

$$\cdots \longrightarrow K_i(\mathcal{B}) \longrightarrow K_i(\mathcal{A}) \longrightarrow K_i(\mathcal{A}/\mathcal{B}) \longrightarrow K_{i-1}(\mathcal{B}) \longrightarrow \cdots .$$

Now let us recall the excision exact sequences of ([269], [270]). In [269] it is shown that every C^*-algebra satisfies excision. Since we are studying only K-theory modulo m we could appeal instead to [290] which shows that K-theory modulo m always satisfies excision when m is invertible.

Continuing for the moment without coefficients modulo m, excision means that if A is a C^*-algebra which is given as a two-sided ideal $A \lhd R$ of a $\mathbb{Q}$-algebra R, then there is a long exact K-theory sequence of the form

$$\cdots \longrightarrow K_i(A) \longrightarrow K_i(R) \longrightarrow K_i(R/A) \longrightarrow K_{i-1}(A) \longrightarrow \cdots .$$

This sequence is also exact for K-theory with finite or rational coefficients.

Restriction induces an embedding $C(X/Y) \subset C(X)$. The image of $C(X/Y)$ consists of all functions which are constant on Y. This is a C^*-subalgebra but not a two-sided ideal. However, if $C_0(X/Y)$ consists of functions which are zero at the point corresponding to Y, then $C_0(X/Y)$ is a two-sided ideal in both $C(X/Y)$ and $C(X)$. Therefore we have two long exact sequences

$$\cdots \longrightarrow K_i(C_0(X/Y)) \longrightarrow K_i(C(X/Y)) \longrightarrow K_i(\mathbb{C}) \longrightarrow K_{i-1}(C_0(X/Y)) \longrightarrow \cdots$$

and

$$\begin{aligned} \cdots \longrightarrow K_i(C_0(X/Y)) \longrightarrow K_i(C(X)) \longrightarrow K_i(C(X)/C_0(X/Y)) \\ \longrightarrow K_{i-1}(C_0(X/Y)) \longrightarrow \cdots . \end{aligned}$$

There is a natural map from the upper sequence to the lower which results in a Mayer-Vietoris exact sequence of the form

$$\begin{aligned}\cdots \longrightarrow K_i(C(X/Y)) \longrightarrow K_i(C(X)) \oplus K_i(\mathbb{C}) \longrightarrow K_i(C(X)/C_0(X/Y)) \\ \longrightarrow K_{i-1}(C(X/Y)) \longrightarrow \cdots .\end{aligned}$$

We have a restriction map $C(X)/C_0(X/Y) \longrightarrow C(Y)$ which is surjective. Also, if we put a vertical distance in the n-cell and let X_t denote the union of Y with a collar of height t $(0 \leq t \leq 1)$ so that $X_0 = Y$ and $X_1 = X$. We have maps $C(X_t) \longrightarrow C(X)/C_0(X/Y)$ and there is an isomorphism of the form

$$\varinjlim_{t} C(X_t) \xrightarrow{\cong} C(X)/C_0(X/Y)$$

which induces an isomorphism

$$K_*(C(Y)) \cong \varinjlim_{t} K_*(C(X_t)) \xrightarrow{\cong} K_*(C(X)/C_0(X/Y))$$

which is inverse to the map given by restriction

$$K_i(C(X)/C_0(X/Y)) \longrightarrow K_i(C(Y)).$$

By the universal coefficient short exact sequence of § 4.2.2 the same is true in dimensions greater than zero for K-theory modulo m.

Since X/Y is homeomorphic to the n-sphere S^n we complete the induction by comparing the two exact sequences – with a little care in low dimensions –

$$\begin{aligned}\cdots \longrightarrow K_i(C(S^n);\mathbb{Z}/m) \longrightarrow K_i(C(X);\mathbb{Z}/m) \oplus K_i(\mathbb{C};\mathbb{Z}/m) \\ \longrightarrow K_i(C(Y);\mathbb{Z}/m) \longrightarrow K_{i-1}(C(S^n);\mathbb{Z}/m) \longrightarrow \cdots\end{aligned}$$

and

$$\begin{aligned}\cdots \longrightarrow K_i^{\mathrm{top}}(C(S^n);\mathbb{Z}/m) \longrightarrow K_i^{\mathrm{top}}(C(X);\mathbb{Z}/m) \oplus K_i^{\mathrm{top}}(\mathbb{C};\mathbb{Z}/m) \\ \longrightarrow K_i^{\mathrm{top}}(C(Y);\mathbb{Z}/m) \longrightarrow K_{i-1}^{\mathrm{top}}(C(S^n);\mathbb{Z}/m) \longrightarrow \cdots .\end{aligned}$$

Chapter 5

The Matrix Corresponding to $1 \wedge \psi^3$

Unfortunately no one can be told what the Matrix is. You have to see it for yourself.

from Morpheus in "The Matrix"

The objective of this chapter is to determine the conjugacy class of the map $1 \wedge \psi^3$ in the upper triangular group $U_\infty \mathbb{Z}_2$ in the sense of Chapter 3, Theorem 3.1.2. § 1 recapitulates the background and states the main result (Theorem 5.1.2). § 2 contains the central calculations in which the effect of $1 \wedge \psi^3$ is estimated with respect to the $\mathbb{Z}_2$-module basis coming from the version of the Mahowald splitting given in Chapter 3. § 3 uses these calculations to determine the diagonal and super-diagonal elements in the matrix. § 4 gives two proofs of the result (Theorem 5.4.2) that any two matrices with this diagonal and super-diagonal are conjugate – in particular this gives Theorem 5.1.2. § 5 contains two applications. The first describes the analogous upper triangular result in which $bu \wedge bo$ is replaced by $bu \wedge bu$ and the second uses some elementary matrix algebra to prove a result concerning the map $\phi_n : bo \longrightarrow bo$ given by $\phi_n = (\psi^3 - 1)(\psi^3 - 9) \ldots (\psi^3 - 9^{n-1})$, which is similar to a result of [185] concerning the Adams filtration of ϕ_n.

5.1 The fundamental result of upper triangular technology

5.1.1. Once again let bu and bo denote the stable homotopy spectra representing unitary and orthogonal connective K-theory respectively, each localised with respect to mod 2 singular homology in the sense of [43] (sometimes also referred to as 2-adically completed connective K-theory spectra). Recall from Chapter 3 § 3.1.1 that the main result of Chapter 3 (Theorem 3.1.2; see also [252]) is the existence of an isomorphism of groups

$$\psi : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo).$$

Here $U_\infty \mathbb{Z}_2$ is the group of upper triangular matrices with coefficients in the 2-adic integers and $\mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ denotes the group of left bu-module automorphisms of $bu \wedge bo$ in the stable homotopy category of 2-local spectra, which induce the identity on mod 2 singular homology. For the reader's convenience, the details of this isomorphism are briefly recapitulated from Chapter 3 (see also [252]) in the next section.

This isomorphism is defined up to inner automorphisms of $U_\infty \mathbb{Z}_2$. Given an important automorphism in $\mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ one is led to ask: "What is its conjugacy class in $U_\infty \mathbb{Z}_2$?" By far the most important such automorphism is $1 \wedge \psi^3$, where $\psi^3 : bo \longrightarrow bo$ denotes the Adams operation (for example, see [168] and related chapter in [10]).

The following is the main result of this chapter, which is proved by combining the discussion of § 5.3.5 with Theorem 5.4.2. This result, and most of the material of this section, appeared first in [27]. However, in that paper the proof given for Theorem 5.1.2 really only showed that the matrix for $1 \wedge \psi^3$ could be conjugated to have the required form in the first N columns for any arbitrarily large positive integer N. It requires only a simple, fairly standard compactness remark to give the full version of Theorem 5.1.2. In § 4 I have added the compactness argument which was omitted from [27] and an alternative proof which was suggested by Francis Clarke and used by my student Jonathan Barker in his University of Southampton PhD thesis.

Theorem 5.1.2. Under the isomorphism ψ the automorphism $1 \wedge \psi^3$ corresponds to an element in the conjugacy class of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

By techniques which are described in ([9] pp. 338–360) and reiterated in § 3, Theorem 5.1.2 reduces to calculating the effect of $1 \wedge \psi^3$ on $\pi_*(bu \wedge bo)$ modulo torsion. The difficulty arises because, in order to identify $\psi^{-1}(1 \wedge \psi^3)$, one must compute the map on homotopy modulo torsion in terms of an unknown 2-adic basis defined in terms of the splitting of $bu \wedge bo$, which was described in Chapter 3 (see § 5.2.2 and § 5.3.1 below). On the other hand a very convenient 2-adic basis is defined in [58] and the crucial fact is that $1 \wedge \psi^3$ acts on the second basis by the matrix of Theorem 5.1.2. This fact was pointed out to me by Francis Clarke in 2001 and led to the confident prediction appearing as a footnote in ([252] p. 1273). Verifying the prediction has proved a little more difficult than first imagined!

Once one has Theorem 5.1.2, a number of homotopy problems become merely a matter of matrix algebra. In § 5 we give an example concerning the maps $1 \wedge$

$(\psi^3 - 1)(\psi^3 - 9)\ldots(\psi^3 - 9^{n-1})$ where we prove a vanishing result (Theorem 5.5.4) which is closely related to the main theorem of [185], as explained in Remark 5.5.5.

In Chapter 8 I shall explain how Theorem 5.1.2 gives a simple method for calculating the left unit map

$$(\eta \wedge 1 \wedge 1)_* : \pi_*(bo \wedge X) \longrightarrow \pi_*(bo \wedge bo \wedge X)$$

in 2-adic connective K-theory. This map is fundamental in determining connective K-theory operations which are analogous to the all-important Quillen operations in BP-theory, which appear in [137] for example. I shall use this method – referred to here as "upper triangular technology" – to study elements of Arf-Kervaire invariant one in $\pi_*(\Sigma^\infty \mathbb{RP}^\infty)$.

5.2 2-adic homotopy of $bu \wedge bo$

5.2.1. Let bu and bo denote the stable homotopy spectra representing unitary and orthogonal connective K-theory respectively, each 2-localised as in § 5.1.1. We shall begin by recalling the 2-local homotopy decomposition of $bu \wedge bo$ which is one of a number of similar results which were discovered by Mark Mahowald in the 1970's. These results may be proved in several ways (for example, see ([9] pp. 190–196), [175] and [185]). For notational reasons we shall refer to the proof which appears in Chapter 3 (see also [252] § 2).

Consider the second loopspace of the 3-sphere, $\Omega^2 S^3$. As explained in Chapter 1 § 1.5.1, there exists a model for $\Omega^2 S^3$ which is filtered by finite complexes ([51], [243])

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \cdots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

and there is a stable homotopy equivalence, an example of a Snaith splitting, of the form

$$\Omega^2 S^3 \simeq \vee_{k \geq 1} F_k / F_{k-1}.$$

There is a 2-local homotopy equivalence of left-bu-module spectra (see [252] Theorem 2.3(ii)) of the form

$$\hat{L} : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo.$$

The important fact about this homotopy equivalence is that its induced map on mod 2 homology is a specific isomorphism which is described in Chapter 3, Theorem 3.1.6 (see also [252] § 2.2).

From this decomposition we obtain left-bu-module spectrum maps of the form

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

where $\iota_{k,k} = 1$, $\iota_{k,l} = 0$ if $l > k$ and, as explained in Chapter 3 § 3.2.2 (see also [252] § 3.1), $\iota_{k,l}$ is defined up to multiplication by a 2-adic unit when $k > l$.

Consider the ring of left bu-module endomorphisms of degree zero in the stable homotopy category of spectra [9], which we shall denote by $\mathrm{End}_{\text{left-}bu\text{-mod}}(bu \wedge bo)$. The group of units in this ring will be denoted by $\mathrm{Aut}_{\text{left-}bu\text{-mod}}(bu \wedge bo)$, the group of homotopy classes of left bu-module homotopy equivalences and let $\mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ denote the subgroup of left bu-module homotopy equivalences which induce the identity map on $H_*(bu \wedge bo; \mathbb{Z}/2)$.

Let $U_\infty \mathbb{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty \mathbb{Z}_2$ if $X_{i,j} \in \mathbb{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j < i$ and $X_{i,i}$ is a 2-adic unit. This upper triangular group is *not* equal to the direct limit $\lim_{\vec{n}} U_n \mathbb{Z}_2$ of the finite upper triangular groups. The main result of Chapter 3 (see also [252]) is the existence of an isomorphism of groups

$$\psi : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo).$$

By the Mahowald decomposition of $bu \wedge bo$ the existence of ψ is equivalent to an isomorphism of the form

$$\psi : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(\vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})).$$

If we choose $\iota_{k,l}$ to satisfy $\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \ldots \iota_{k,k-1}$ for all $k - l \geq 2$ then, for $X \in U_\infty \mathbb{Z}_2$, we define (Chapter 3 § 3.2.2; see also [252] § 3.2)

$$\psi(X) = \sum_{l \leq k} X_{l,k} \iota_{k,l}$$

where

$$\sum_{l \leq k} X_{l,k} \iota_{k,l} : bu \wedge (\vee_{k \geq 0} F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (\vee_{k \geq 0} F_{4k}/F_{4k-1}).$$

The ambiguity in the definition of the $\iota_{k,l}$'s implies that ψ is defined up to conjugation by a diagonal matrix in $U_\infty \mathbb{Z}_2$.

5.2.2. Bases for $\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}$. Let $G_{s,t}$ denote the 2-adic homotopy group modulo torsion

$$G_{s,t} = \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

so

$$G_{*,*} = \oplus_{s,t} \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}.$$

From [9] or Chapter 3 (see also [252])

$$G_{s,t} \cong \begin{cases} \mathbb{Z}_2 & \text{if } s \text{ even, } s \geq 4t, \\ 0 & \text{otherwise} \end{cases}$$

and if $\tilde{G}_{s,t}$ denotes $\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2$, then $\tilde{G}_{s,t} \cong G_{s,t} \oplus W_{s,t}$ where $W_{s,t}$ is a finite, elementary abelian 2-group.

In [58] a $\mathbb{Z}_2$-basis is given for $G_{*,*}$ consisting of elements lying in the subring $\mathbb{Z}_2[u/2, v^2/4]$ of $\mathbb{Q}_2[u/2, v^2/4]$. One starts with the elements

$$c_{4k} = \Pi_{i=1}^{k} \left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}} \right), \qquad k = 1, 2, \ldots$$

and "rationalises" them, after the manner of ([9] p. 358), to obtain elements of $\mathbb{Z}_2[u/2, v^2/4]$. In order to describe this basis we shall require a few well-known preparatory results about 2-adic valuations.

Proposition 5.2.3. For any integer $n \geq 0$, $9^{2^n} - 1 = 2^{n+3}(2s+1)$ for some $s \in \mathbb{Z}$.

Proof. We prove this by induction on n, starting with $9 - 1 = 2^3$. Assuming the result is true for n, we have

$$\begin{aligned} 9^{2^{(n+1)}} - 1 \quad &= (9^{2^n} - 1)(9^{2^n} + 1) \\ &= (9^{2^n} - 1)(9^{2^n} - 1 + 2) \\ &= 2^{n+3}(2s+1)(2^{n+3}(2s+1) + 2) \\ &= 2^{n+4}(2s+1)\underbrace{(2^{n+2}(2s+1) + 1)}_{\text{odd}} \end{aligned}$$

as required. □

Proposition 5.2.4. For any integer $l \geq 0$, $9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ for some $s \in \mathbb{Z}$, where $\nu_2(l)$ denotes the 2-adic valuation of l.

Proof. Write $l = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_k}$ with $0 \leq e_1 < e_2 < \cdots < e_k$ so that $\nu_2(l) = e_1$. Then, by Proposition 5.2.3,

$$\begin{aligned} 9^l - 1 &= 9^{2^{e_1} + 2^{e_2} + \cdots + 2^{e_k}} - 1 \\ &= ((2s_1 + 1)2^{e_1+3} + 1) \ldots ((2s_k + 1)2^{e_k+3} + 1) - 1 \\ &\equiv (2s_1 + 1)2^{e_1+3} \ (\text{modulo} 2^{e_1+4}) \\ &= 2^{e_1+3}(2t+1) \end{aligned}$$

as required. □

Proposition 5.2.5. For any integer $l \geq 1$, $\prod_{i=1}^{l} (9^l - 9^{i-1}) = 2^{\nu_2(l!)+3l}(2s+1)$ for some $s \in \mathbb{Z}$.

Proof. By Proposition 5.2.4 we have

$$\begin{aligned} \prod_{i=1}^{l} (9^l - 9^{i-1}) = \prod_{i=1}^{l} (9^{l-i+1} - 1)9^{i-1} &= \prod_{i=1}^{l} 2^{\nu_2(l-i+1)+3}(2t_i + 1)9^{i-1} \\ &= (2t+1)2^{\nu_2(l!)+3l}, \end{aligned}$$

as required. □

Proposition 5.2.6. For any integer $l \geq 0$, $2^{\nu_2(l!)+3l} = 2^{4l-\alpha(l)}$ where $\alpha(l)$ is equal to the number of 1's in the dyadic expansion of l. In particular, $\alpha(l-1) = \alpha(l) - 1 + \nu_2(l)$ for all $l \geq 2$.

Proof. Write $l = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_k}$ with $0 \leq e_1 < e_2 < \cdots < e_k$ so that $\alpha(l) = k$.

$9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ for some $s \in \mathbb{Z}$, where $\nu_2(l)$ denotes the 2-adic valuation of l. Then

$$\begin{array}{ccclclclcl}
\nu_2(l!) & = & & 2^{\alpha_1-1} & + & 2^{\alpha_2-1} & + & \cdots & + & 2^{\alpha_k-1} \\
& & + & 2^{\alpha_1-2} & + & 2^{\alpha_2-2} & + & \cdots & + & 2^{\alpha_k-2} \\
& & & \vdots & & & & & & \vdots \\
& & + & 1 & + & 2^{\alpha_2-\alpha_1} & + & \cdots & + & 2^{\alpha_k-\alpha_1} \\
& & & & + & 1 & + & \cdots & + & 2^{\alpha_k-\alpha_2} \\
& & & & & & & & + & 1
\end{array}$$

because the first row counts the multiples of 2 less than or equal to l, the second row counts the multiples of 4, the third row counts multiples of 8 and so on. Adding by columns we obtain

$$\nu_2(l!) = 2^{\alpha_1} - 1 + 2^{\alpha_2} - 1 + \cdots + 2^{\alpha_k} - 1 = l - k$$

which implies that $2^{3l+\nu_2(l!)} = 2^{3l+l-\alpha(l)} = 2^{4l-\alpha(l)}$, as required.

In particular $\alpha(l) - 1 + \nu_2(l) = l - \nu_2(l!) - 1 + \nu_2(l) = l - 1 - \nu_2((l-1)!) = \alpha(l-1)$. $\square$

5.2.7. Bases continued. Consider the elements $c_{4k} = \Pi_{i=1}^{k}\left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}}\right)$, introduced in § 5.2.2, for a particular $k = 1, 2, \ldots$. For completeness write $c_0 = 1$ so that $c_{4k} \in \mathbb{Q}_2[u/2, v^2/4]$. Since the degree of the numerator of c_{4k} is $2k$, Proposition 5.2.6 implies that

$$f_{4k} = 2^{4k-\alpha(k)-2k} c_{4k} = 2^{2k-\alpha(k)} \Pi_{i=1}^{k}\left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}}\right)$$

lies in $\mathbb{Z}_2[u/2, v^2/4]$ but $2^{4k-\alpha(k)-2k-1}c_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$. Similarly

$$(u/2)f_{4k} = 2^{4k-\alpha(k)-2k-1}uc_{4k} \in \mathbb{Z}_2[u/2, v^2/4]$$

but $2^{4k-\alpha(k)-2k-2}uc_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$ and so on. This process is the "rationalisation yoga" referred to in § 5.2.2. One forms $u^j c_{4k}$ and then multiplies by the smallest positive power of 2 to obtain an element of $\mathbb{Z}_2[u/2, v^2/4]$.

By Proposition 5.2.6, starting with $f_{4l} = 2^{4l-\alpha(l)-2l}c_{4l}$ this process produces the following set of elements of $\mathbb{Z}_2[u/2, v^2/4]$:

$$\begin{gathered}
f_{4l},\ (u/2)f_{4l},\ (u/2)^2 f_{4l}, \ldots,\ (u/2)^{2l-\alpha(l)} f_{4l}, \\
u(u/2)^{2l-\alpha(l)} f_{4l},\ u^2(u/2)^{2l-\alpha(l)} f_{4l},\ u^3(u/2)^{2l-\alpha(l)} f_{4l}, \ldots.
\end{gathered}$$

As explained in ([9] p. 352 et seq), the Hurewicz homorphism defines an injection of graded groups of the form

$$\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \longrightarrow \mathbb{Q}_2[u/2, v^2/4]$$

which, by the main theorem of [58], induces an isomorphism between $\frac{\pi_*(bu\wedge bo)\otimes\mathbb{Z}_2}{\text{Torsion}}$ and the free graded $\mathbb{Z}_2$-module whose basis consists of the elements of $\mathbb{Z}_2[u/2, v^2/4]$ listed above for $l = 0, 1, 2, 3, \ldots$.

From this list we shall be particularly interested in the elements whose degree is a multiple of 4. Therefore denote by

$$g_{4m,4l} \in \mathbb{Z}_2[u/2, v^2/4]$$

for $l \leq m$ the element produced from f_{4l} in degree $4m$. Hence, for $m \geq l$, $g_{4m,4l}$ is given by the formula

$$g_{4m,4l} = \begin{cases} u^{2m-4l+\alpha(l)}\left[\frac{u^{2l-\alpha(l)}f_{4l}}{2^{2l-\alpha(l)}}\right] & \text{if } 4l - \alpha(l) \leq 2m, \\ \left[\frac{u^{2(m-l)}f_{4l}}{2^{2(m-l)}}\right] & \text{if } 4l - \alpha(l) > 2m. \end{cases}$$

Lemma 5.2.8. In the notation of § 5.2.2, let Π denote the projection

$$\Pi : \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong G_{*,*} \longrightarrow G_{*,k} = \oplus_m \, G_{m,k}.$$

Then $\Pi(g_{4k,4i}) = 0$ for all $i < k$.

Proof. Since $G_{m,k}$ is torsion free it suffices to show that $\Pi(g_{4k,4i})$ vanishes in $G_{*,k} \otimes \mathbb{Q}_2$. When $i < k$, by definition

$$g_{4k,4i} \in u^{2k-2i}\frac{\pi_{4i}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2 \subset \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2.$$

However Π projects onto $\oplus_s \frac{\pi_s(bu\wedge F_{4k}/F_{4k-1})\otimes\mathbb{Z}_2}{\text{Torsion}}$ and commutes with multiplication by u so the result follows from the fact that the homotopy of $bu \wedge F_{4k}/F_{4k-1}$ is trivial in degrees less than $4k$ (see Chapter 3 § 3.2.1 or [252] § 3). $\square$

5.2.9. Recall from § 5.2.2 that $G_{4k,k} \cong \mathbb{Z}_2$ for $k = 0, 1, 2, 3, \ldots$ so we may choose a generator z_{4k} for this group as a module over the 2-adic integers (with the convention that $z_0 = f_0 = 1$). Let $\tilde{z}_{4k}$ be any choice of an element in the 2-adic homotopy group $\tilde{G}_{4k,k} \cong G_{4k,k} \oplus W_{4k,k}$ whose first coordinate is z_{4k}.

Lemma 5.2.10. Let B denote the exterior subalgebra of a $\mathbb{Z}/2$ Steenrod algebra generated by Sq^1 and $Sq^{0,1}$ (see Chapter 1 § 6). In the collapsed Adams spectral sequence (see Chapter 1 § 4, [9] or [252])

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(H^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 \end{aligned}$$

the homotopy class $\tilde{z}_{4k}$ is represented either in $E_2^{0,4k}$ or $E_2^{1,4k+1}$.

Proof. Recall from § 5.2.2 that

$$\pi_{4k}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 = \tilde{G}_{4k,k} \cong \mathbb{Z}_2 \oplus W_{4k,k}.$$

The following behaviour of the filtration coming from the spectral sequence is well known, being explained in [9]. The group $\tilde{G}_{4k,k}$ has a filtration

$$\cdots \subset F^i \subset \cdots F^2 \subset F^1 \subseteq F^0 = \tilde{G}_{4k,k}$$

with $F^i/F^{i+1} \cong E_2^{i,4k+i}$ and $2F^i \subseteq F^{i+1}$. Also $2 \cdot W_{4k,k} = 0$, every non-trivial element of $W_{4k,k}$ being represented in $E_2^{0,4k}$. Furthermore for $i = 1, 2, 3, \ldots$ we have $2F^i = F^{i+1}$ and $F^1 \cong \mathbb{Z}_2$.

Now suppose that $\tilde{z}_{4k}$ is represented in $E_2^{j,4k+j}$ for $j \geq 2$; then $\tilde{z}_{4k} \in F^j$. From the multiplicative structure of the spectral sequence there exists a generator $\hat{z}_{4k}$ of F^1 such that $2^j \hat{z}_{4k}$ generates F^{j+1} and therefore $2^j\gamma\hat{z}_{4k} = 2\tilde{z}_{4k}$ for some 2-adic integer γ. Hence $2(2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k}) = 0$ and so $2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k} \in W_{4k,k}$ which implies the contradiction that the generator z_{4k} is divisible by 2 in $G_{4k,k}$. □

Theorem 5.2.11. In the notation of § 5.2.7 and § 5.2.9,

$$z_{4k} = \Sigma_{i=0}^{k} 2^{\beta(k,i)} \lambda_{4k,4i} g_{4k,4i} \in \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

with $\lambda_{s,t} \in \mathbb{Z}_2$, $\lambda_{4k,4k} \in \mathbb{Z}_2^*$ and

$$\beta(k,i) = \begin{cases} 4(k-i) - \alpha(k) + \alpha(i) & \text{if } 4i - \alpha(i) > 2k, \\ 2k - \alpha(k) & \text{if } 4i - \alpha(i) \leq 2k. \end{cases}$$

Proof. From [58], as explained in § 5.2.7, a $\mathbb{Z}_2$-module basis for $G_{4k,*}$ is given by $\{g_{4k,4l}\}_{0 \leq l \leq k}$. Hence there is a relation of the form

$$z_{4k} = \lambda_{4k,4k} g_{4k,4k} + \tilde{\lambda}_{4k,4(k-1)} g_{4k,4(k-1)} + \cdots + \tilde{\lambda}_{4k,0} g_{4k,0}$$

where $\tilde{\lambda}_{4k,4i}$ and $\lambda_{4k,4k}$ are 2-adic integers. Applying the projection $\Pi : G_{4k,*} \longrightarrow G_{4k,k}$ we see that $z_{4k} = \Pi(z_{4k}) = \lambda_{4k,4k}\Pi(g_{4k,4k})$, by Lemma 5.2.8. Hence, if $\lambda_{4k,4k}$ is not a 2-adic unit, then z_{4k} would be divisible by 2 in $G_{4k,k}$ and this is impossible since z_{4k} is a generator, by definition.

Multiplying the relation

$$z_{4k} = \lambda_{4k,4k}\Pi(g_{4k,4k}) = \lambda_{4k,4k}\Pi(f_{4k}) \in G_{4k,k}$$

by $(u/2)^{2k-\alpha(k)}$ we obtain

$$(u/2)^{2k-\alpha(k)} z_{4k} = \lambda_{4k,4k}\Pi((u/2)^{2k-\alpha(k)} f_{4k}),$$

which lies in $G_{8k-2\alpha(k),k}$, by the discussion of § 5.2.7. Therefore, in $G_{8k-2\alpha(k),k} \otimes \mathbb{Q}_2$ we have the relation

$$(u/2)^{2k-\alpha(k)} z_{4k} = (u/2)^{2k-\alpha(k)} f_{4k} + \sum_{i=0}^{k-1} \tilde{\lambda}_{4k,4i} (u/2)^{2k-\alpha(k)} g_{4k,4i}.$$

Since the left-hand side of the equation lies in $G_{8k-2\alpha(k),k}$, the $\mathbb{Q}_2$ coefficients must all be 2-adic integers once we re-write the right-hand side in terms of the basis of § 5.2.7.

For $i = 0, 1, \ldots, k-1$,

$$\begin{aligned}
&(u/2)^{2k-\alpha(k)} g_{4k,4i} \\
&= \begin{cases} \frac{u^{2k-\alpha(k)+2k-4i+\alpha(i)+2i-\alpha(i)}}{2^{2k-\alpha(k)+2i-\alpha(i)}} f_{4i} & \text{if } 4i-\alpha(i) \leq 2k, \\ \frac{u^{2k-\alpha(k)+2k-2i}}{2^{2k-\alpha(k)+2k-2i}} f_{4i} & \text{if } 4i-\alpha(i) > 2k \end{cases} \\
&= \begin{cases} \frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}} f_{4i} & \text{if } 4i-\alpha(i) \leq 2k, \\ \frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}} f_{4i} & \text{if } 4i-\alpha(i) > 2k. \end{cases}
\end{aligned}$$

Now we shall write $(u/2)^{2k-\alpha(k)} g_{4k,4i}$ as a power of 2 times a generator derived from f_{4i} in § 5.2.7 (since we did not define any generators called $g_{4k+2,4i}$ the generator in question will be $g_{8k-2\alpha(k),4i}$ only when $\alpha(k)$ is even).

Assume that $4i - \alpha(i) \leq 2k$ so that $2i - \alpha(i) \leq 4k - 2i - \alpha(k)$ and

$$\frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}} f_{4i} = \frac{1}{2^{2k-\alpha(k)}} u^{4k-4i-\alpha(k)+\alpha(i)} (u/2)^{2i-\alpha(i)} f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{2k-\alpha(k)}$ in the 2-adic integers, as required.

Finally assume that $4i - \alpha(i) > 2k$. We have $2i - \alpha(i) \leq 4k - 2i - \alpha(k)$ also. To see this observe that $\alpha(i) + \alpha(k-i) - \alpha(k) \geq 0$ because, by Proposition 5.2.6, this equals the 2-adic valuation of the binomial coefficient $\binom{k}{i}$. Therefore

$$\alpha(k) - \alpha(i) \leq \alpha(k-i) \leq k - i < 4(k-i).$$

Then, as before,

$$\frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}} f_{4i} = \frac{1}{2^{4k-4i-\alpha(k)+\alpha(i)}} u^{4k-4i-\alpha(k)+\alpha(i)} (u/2)^{2i-\alpha(i)} f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{4k-4i-\alpha(k)+\alpha(i)}$ in the 2-adic integers, as required. □

Theorem 5.2.12.

(i) In the collapsed Adams spectral sequence and the notation of Lemma 5.2.10, $\tilde{z}_{4k}$ may be chosen to be represented in $E_2^{0,4k}$.

(ii) In fact, $\tilde{z}_{4k}$ may be taken to be the smash product of the unit η of the bu-spectrum with the inclusion of the bottom cell j_k into F_{4k}/F_{4k-1},

$$S^0 \wedge S^{4k} \stackrel{\eta\wedge j_k}{\longrightarrow} bu \wedge F_{4k}/F_{4k-1}.$$

Proof. For part (i), suppose that $\tilde{z}_{4k}$ is represented in $E_2^{1,4k+1}$. By Lemma 5.2.10 we must show that this leads to a contradiction. From Chapter 3 or [252] we know that on the $s = 1$ line the non-trivial groups are precisely $E_2^{1,4k+1}, E_2^{1,4k+3}, \ldots, \ldots$ $\ldots, \ldots, E_2^{1,8k+2-2\alpha(k)}$ which are all of order two. From the multiplicative structure of the spectral sequence, if a homotopy class w is represented in $E_2^{j,4k+2j-1}$ and $E_2^{j,4k+2j+1}$ is non-zero, then there is a homotopy class w' represented in $E_2^{j,4k+2j+1}$ such that $2w' = uw$. Applied to $\tilde{z}_{4k}$ this implies that the homotopy element $u^{2k-\alpha(k)+1}\tilde{z}_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$. Hence $u^{2k-\alpha(k)+1}z_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$ in $G_{*,*}$, which contradicts the proof of Theorem 5.2.11.

For part (ii) consider the Adams spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_B^{s,t}(H^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \Longrightarrow \pi_{t-s}(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2. \end{aligned}$$

We have an isomorphism

$$E_2^{0,t} \cong \mathrm{Hom}(\frac{H^t(F_{4k}/F_{4k-1}; \mathbb{Z}/2)}{A^t}, \mathbb{Z}/2)$$

where

$$A^t = Sq^1 H^{t-1}(F_{4k}/F_{4k-1}; \mathbb{Z}/2) + Sq^{0,1} H^{t-3}(F_{4k}/F_{4k-1}; \mathbb{Z}/2).$$

The discussion of the homology groups $H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$ given in ([9] p. 341; see also § 5.3.1 below) shows that $E_2^{0,4k} \cong \mathbb{Z}/2$, generated by the Hurewicz image of $\eta \wedge j_k$. Therefore the generator of $E_2^{0,4k}$ represents $\eta \wedge j_k$. Since there is only one non-zero element in $E_2^{0,4k}$ it must also represent $\tilde{z}_{4k}$, by part (i), which completes the proof. □

5.3 The matrix

5.3.1. Consider the left-bu-module spectrum map of § 5.2.1,

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

when $l > k$. This map is determined up to homotopy by its restriction, via the unit of bu, to (F_{4k}/F_{4k-1}). By S-duality this restriction is equivalent to a map of the form

$$S^0 \longrightarrow D(F_{4k}/F_{4k-1}) \wedge bu \wedge (F_{4l}/F_{4l-1}),$$

where DX denotes the S-dual of X. Maps of this form are studied by means of the (collapsed) Adams spectral sequence (see Chapter 3 or [252] § 3.1), where B is as in Lemma 5.2.10,

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_B^{s,t}(H^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \otimes H^*(F_{4l}/F_{4l-1}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2. \end{aligned}$$

Recall from ([9] p. 332) that Σ^a is the (invertible) B-module given by $\mathbb{Z}/2$ in degree a, $\Sigma^{-a} = \mathrm{Hom}(\Sigma^a, \mathbb{Z}/2)$ and I is the augmentation ideal, $I = \ker(\epsilon : B \longrightarrow \mathbb{Z}/2)$. Hence, if $b > 0$, $I^{-b} = \mathrm{Hom}(I^b, \mathbb{Z}/2)$, where I^b is the b-fold tensor product of I. These duality identifications may be verified using the criteria of ([9] p. 334 Theorem 16.3) for identifying $\Sigma^a I^b$.

In ([9] p. 341) it is shown that the B-module given by

$$H^{-*}(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \cong H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$$

is stably equivalent to $\Sigma^{2^{r-1}+1} I^{2^{r-1}-1}$ when $0 < 4k = 2^r$.

Therefore $H^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-(2^{r-1}+1)} I^{1-2^{r-1}}$ when $0 < 4k = 2^r$. If k is not a power of two we may write $4k = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$ with $2 \leq r_1 < r_2 < \cdots < r_t$. In this case

$$H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2) \cong \otimes_{j=r_1}^{r_t} H_*(F_{2^j}/F_{2^j-1}; \mathbb{Z}/2)$$

which is stably equivalent to $\Sigma^{2k+\alpha(k)} I^{2k-\alpha(k)}$, where $\alpha(k)$ equals the number of 1's in the dyadic expansion of k, as in Proposition 5.2.6.

Similarly, $H^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-2k-\alpha(k)} I^{\alpha(k)-2k}$. From this, for all $s > 0$, one easily deduces a canonical isomorphism ([252] p. 1267) of the form

$$\begin{aligned} E_2^{s,t} &\cong \mathrm{Ext}_B^{s,t}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)} I^{2l-2k-\alpha(l)+\alpha(k)}, \mathbb{Z}/2) \\ &\cong \mathrm{Ext}_B^{s+2l-2k-\alpha(l)+\alpha(k), t-2l+2k-\alpha(l)+\alpha(k)}(\mathbb{Z}/2, \mathbb{Z}/2). \end{aligned}$$

Also there is an algebra isomorphism of the form

$$\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]$$

where $a \in \mathrm{Ext}_B^{1,1}$, $b \in \mathrm{Ext}_B^{1,3}$. As explained in Chapter 3 (see also[252] p. 1270) $i_{k,l}$ is represented in

$$\begin{aligned} &E_2^{4(k-l)+\alpha(l)-\alpha(k), 4(k-l)+\alpha(l)-\alpha(k)} \\ &\cong \mathrm{Ext}_B^{2k-2l, 6k-6l}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 = \langle b^{2k-2l} \rangle. \end{aligned}$$

Proposition 5.3.2. For $l < k$, in the notation of § 2, the homomorphism

$$(\iota_{k,l})_* : G_{4k,k} \longrightarrow G_{4k,l}$$

satisfies $(\iota_{k,l})_*(z_{4k}) = \mu_{4k,4l} 2^{2k-2l-\alpha(k)+\alpha(l)} u^{2k-2l} z_{4l}$ for some 2-adic unit $\mu_{4k,4l}$.

Proof. Let $\tilde{z}_{4k} \in \tilde{G}_{4k,k}$ be as in § 5.2.9 so that, proved in a similar manner to Lemma 5.2.10, $2\tilde{z}_{4k}$ is represented in $E_2^{1,4k+1}$ in the spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 \end{aligned}$$

where, from § 5.3.1, we have

$$\begin{aligned} &E_2^{1,4k+1} \\ &\cong \mathrm{Ext}_B^{1+2k-\alpha(k),4k+1-2k-\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2 = \langle a^{2k+1-\alpha(k)} \rangle. \end{aligned}$$

The multiplicative pairing between these spectral sequences shows that

$$(\iota_{k,l})^*(2\tilde{z}_{4k}) \in \tilde{G}_{4k,l}$$

is represented in the spectral sequence

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}_B^{s,t}(H^*(F_{4l}/F_{4l-1};\mathbb{Z}/2),\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2 \end{aligned}$$

by the generator of $E_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)}$ because $a^{2k+1-\alpha(k)}b^{2k-2l}$ is the generator of

$$\begin{aligned} &E_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)} \\ &\cong \mathrm{Ext}_B^{1+4k-2l-\alpha(k),1+8k-6l-\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2). \end{aligned}$$

Since multiplication by a and b in the spectral sequence corresponds to multiplication by 2 and u respectively on homotopy groups, we have the following table of representatives in $\pi_*(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2$.

homotopy element	representative	dimension
$2z_{4l}$	$a^{2l-\alpha(l)+1}$	$4l$
$(u/2)(2z_{4l})$	$a^{2l-\alpha(l)}b$	$4l+2$
$(u/2)^2(2z_{4l})$	$a^{2l-\alpha(l)-1}b^2$	$4l+4$
$\vdots$	$\vdots$	$\vdots$
$(u/2)^{2l-\alpha(l)}(2z_{4l})$	$ab^{2l-\alpha(l)}$	$8l-2\alpha(l)$
$u(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+1}$	$8l-2\alpha(l)+2$
$u^2(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+2}$	$8l-2\alpha(l)+4$
$\vdots$	$\vdots$	$\vdots$

Therefore there are two cases for $(\iota_{k,l})_*(2\tilde{z}_{4k})$. If $2k-2l \geq 2l-\alpha(l)+1$ then b^{2k-2l} represents $u^{2k-2l-(2l-\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l} = u^{2k-4l+\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l}$ and, up to multiplication by 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to

$$2^{1+2k-\alpha(k)}u^{2k-4l+\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l},$$

as required. On the other hand, if $2k-2l \leq 2l-\alpha(l)$ then

$$a^{2l-\alpha(l)+1-(2k-2l)}b^{2k-2l} = a^{4l-2k-\alpha(l)+1}b^{2k-2l}$$

represents $(u/2)^{2k-2l}(2\tilde{z}_{4l})$ which shows that, up to 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to

$$2^{1+2k-\alpha(k)-(4l-2k-\alpha(l)+1)}(u/2)^{2k-2l}(2\tilde{z}_{4l}) = 2^{4k-\alpha(k)-4l+\alpha(l)}(u/2)^{2k-2l}(2\tilde{z}_{4l}),$$

as required. □

Proposition 5.3.3. Let $\psi^3 : bo \longrightarrow bo$ denote the Adams operation, as usual. Then, in the notation of § 5.2.7,

$$(1\wedge\psi^3)_*(g_{4k,4k}) = \begin{cases} 9^k g_{4k,4k} + 9^{k-1}2^{\nu_2(k)+3}g_{4k,4k-4} & \text{if } k \geq 3, \\ 9^2 g_{8,8} + 9\cdot 2^3 g_{8,4} & \text{if } k = 2, \\ 9g_{4,4} + 2g_{4,0} & \text{if } k = 1, \\ g_{0,0} & \text{if } k = 0. \end{cases}$$

Proof. The map $(1\wedge\psi^3)_*$ fixes u, multiplies v by 9 and is multiplicative. Therefore

$$\begin{aligned} &(1\wedge\psi^3)_*(c_{4k}) \\ &= \Pi_{i=1}^k\big(\tfrac{9v^2-9^{i-1}u^2}{9^k-9^{i-1}}\big) \\ &= 9^{k-1}\big(\tfrac{(9v^2-9^ku^2+9^ku^2-u^2)\Pi_{i=2}^k(v^2-9^{i-2}u^2)}{\Pi_{i=1}^k(9^k-9^{i-1})}\big) \\ &= 9^{k-1}\big(\tfrac{(9v^2-9^ku^2)\Pi_{i=2}^k(v^2-9^{i-2}u^2)}{\Pi_{i=1}^k(9^k-9^{i-1})}\big) + 9^{k-1}\big(\tfrac{(9^ku^2-u^2)\Pi_{i=2}^k(v^2-9^{i-2}u^2)}{\Pi_{i=1}^k(9^k-9^{i-1})}\big) \\ &= 9^k\big(\tfrac{(v^2-9^{k-1}u^2)\Pi_{i=2}^k(v^2-9^{i-2}u^2)}{\Pi_{i=1}^k(9^k-9^{i-1})}\big) + 9^{k-1}\big(\tfrac{(9^ku^2-u^2)\Pi_{i=2}^k(v^2-9^{i-2}u^2)}{\Pi_{i=1}^k(9^k-9^{i-1})}\big) \\ &= 9^kc_{4k} + 9^{k-1}(9^k-1)\big(\tfrac{u^2\Pi_{i=1}^{k-1}(v^2-9^{i-1}u^2)}{(9^k-1)\Pi_{i=1}^{k-1}(9^k-9^{i-1})}\big) \\ &= 9^kc_{4k} + 9^{k-1}u^2c_{4k-4}. \end{aligned}$$

Hence, for $k \geq 1$, we have

$$\begin{aligned} &(1\wedge\psi^3)_*(f_{4k}) \\ &= 2^{2k-\alpha(k)}(1\wedge\psi^3)_*(c_{4k}) \\ &= 2^{2k-\alpha(k)}9^kc_{4k} + 9^{k-1}u^22^{2k-\alpha(k)-2k+2+\alpha(k-1)+2k-2-\alpha(k-1)}c_{4k-4} \\ &= 9^kf_{4k} + 9^{k-1}u^22^{2-\alpha(k)+\alpha(k-1)}f_{4k-4} \\ &= 9^kf_{4k} + 9^{k-1}u^22^{\nu_2(k)+1}f_{4k-4}, \end{aligned}$$

which yields the result, by the formulae of § 5.2.7. □

Proposition 5.3.4. When $k > l$,

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = \begin{cases} 9^l g_{4k,4l} + 9^{l-1} g_{4k,4l-4} \\ \qquad \text{if } 4l - \alpha(l) \leq 2k, \\ 9^l g_{4k,4l} + 9^{l-1} 2^{4l-\alpha(l)-2k} g_{4k,4l-4} \\ \qquad \text{if } 4l - \alpha(l) - \nu_2(l) - 3 \\ \qquad\quad \leq 2k < 4l - \alpha(l), \\ 9^l g_{4k,4l} + 9^{l-1} 2^{3+\nu_2(k)} g_{4k,4l-4} \\ \qquad \text{if } 2k < 4l - \alpha(l) - \nu_2(l) - 3 \\ \qquad\quad < 4l - \alpha(l). \end{cases}$$

Proof. Suppose that $4l - \alpha(l) \leq 2k$ then, by Proposition 5.3.3 (proof),

$$\begin{aligned} & (1 \wedge \psi^3)_*(g_{4k,4l}) \\ & = (1 \wedge \psi^3)_*(u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l-\alpha(l)} f_{4l}}{2^{2l-\alpha(l)}}\right]) \\ & = u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l-\alpha(l)}(9^l f_{4l} + 9^{l-1} u^2 2^{\nu_2(l)+1} f_{4l-4})}{2^{2l-\alpha(l)}}\right] \\ & = 9^l g_{4k,4l} + 9^{l-1} u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l-\alpha(l)} u^2 2^{\nu_2(l)+1} f_{4l-4}}{2^{2l-\alpha(l)}}\right] \\ & = 9^l g_{4k,4l} + 9^{l-1} u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l+2-\alpha(l)} 2^{\nu_2(l)+1} f_{4l-4}}{2^{2l-\alpha(l)}}\right]. \end{aligned}$$

Then, since $\nu_2(l) = 1 + \alpha(l-1) - \alpha(l)$,

$$\begin{aligned} 4(l-1) - \alpha(l-1) \quad & = 4l - \alpha(l) + \alpha(l) - \alpha(l-1) - 4 \\ & = 4l - \alpha(l) - 3 - \nu_2(l) < 2k \end{aligned}$$

so that

$$\begin{aligned} g_{4k,4l-4} \quad & = u^{2k-4l+4+\alpha(l-1)} \left[\tfrac{u^{2l-2-\alpha(l-1)} f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\ & = u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l+2-\alpha(l)} f_{4l-4}}{2^{2l-2-\alpha(l)+\alpha(l)-\alpha(l-1)}}\right] \\ & = u^{2k-4l+\alpha(l)} \left[\tfrac{u^{2l+2-\alpha(l)} f_{4l-4}}{2^{2l-\alpha(l)-\nu_2(l)-1}}\right]. \end{aligned}$$

Therefore, for $0 < l < k$ suppose that $4l - \alpha(l) \leq 2k$, then

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1} g_{4k,4l-4}.$$

Similarly, for $0 < l < k$ if $4l - \alpha(l) > 2k$ then, by Proposition 5.3.3 (proof),

$$\begin{aligned} & (1 \wedge \psi^3)_*(g_{4k,4l}) \\ & = (1 \wedge \psi^3)_*(\left[\tfrac{u^{2(k-l)} f_{4l}}{2^{2(k-l)}}\right]) \\ & = \left[\tfrac{u^{2(k-l)}(9^l f_{4l} + 9^{l-1} u^2 2^{\nu_2(k)+1} f_{4k-4})}{2^{2(k-l)}}\right] \\ & = 9^l g_{4k,4l} + 9^{l-1} \left[\tfrac{u^{2k-2l+2} 2^{\nu_2(k)+1} f_{4k-4}}{2^{2(k-l)}}\right]. \end{aligned}$$

This situation splits into two cases given by

(i) $4l - \alpha(l) - \nu_2(l) - 3 \leq 2k < 4l - \alpha(l)$ or

(ii) $2k < 4l - \alpha(l) - \nu_2(l) - 3 < 4l - \alpha(l)$.

In case (i) $4l - 4 - \alpha(l-1) = 4l - \alpha(l) - \nu_2(l) - 3 \leq 2k$ and so again we have

$$\begin{aligned} g_{4k,4l-4} &= u^{2k-4l+4+\alpha(l-1)}\left[\tfrac{u^{2l-2-\alpha(l-1)}f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\ &= \tfrac{u^{2k-2l+2}f_{4l-4}}{2^{2l-1-\nu_2(l)-\alpha(l)}} \\ &= \tfrac{u^{2k-2l+2}2^{1+\nu_2(l)}f_{4l-4}}{2^{2k-2l+4l-\alpha(l)-2k}} \end{aligned}$$

so that

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}2^{4l-\alpha(l)-2k} g_{4k,4l-4}.$$

In case (ii)

$$g_{4k,4l-4} = \left[\tfrac{u^{2k-2l+2}f_{4k-4}}{2^{2(k-l+2)}}\right]$$

so that

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}2^{3+\nu_2(k)} g_{4k,4l-4}. \qquad \square$$

5.3.5. In the notation of § 5.2.1, suppose that $A \in U_\infty \mathbb{Z}_2$ satisfies

$$\psi(A) = [1 \wedge \psi^3] \in \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo).$$

Therefore, by definition of ψ and the formula of Theorem 5.2.11,

$$\begin{aligned} \textstyle\sum_{l \leq k} A_{l,k}(\iota_{k,l})_*(z_{4k}) &= (1 \wedge \psi^3)_*(z_{4k}) \\ &= \Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}(1 \wedge \psi^3)_*(g_{4k,4i}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\textstyle\sum_{l \leq k} A_{l,k}(\iota_{k,l})_*(z_{4k}) \\ &= A_{k,k} z_{4k} + \textstyle\sum_{l<k} A_{l,k}\mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}z_{4l} \\ &= A_{k,k}\Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}g_{4k,4i} \\ &\quad + \textstyle\sum_{l<k}\ \Sigma_{i=0}^l\ A_{l,k}\ \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}2^{\beta(l,i)}\lambda_{4l,4i}g_{4l,4i}. \end{aligned}$$

In order to determine the $A_{k,l}$'s it will suffice to express $u^{2k-2l}g_{4l,4i}$ as a multiple of $g_{4k,4i}$ and then to equate coefficients in the above expressions. By definition

$$\begin{aligned} u^{2k-2l}g_{4l,4i} &= \begin{cases} u^{2k-2l}u^{2l-4i+\alpha(i)}\left[\tfrac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i - \alpha(i) \leq 2l, \\ u^{2k-2l}\left[\tfrac{u^{2(l-i)}f_{4i}}{2^{2(l-i)}}\right] & \text{if } 4i - \alpha(i) > 2l \end{cases} \\ &= \begin{cases} \tfrac{u^{2k-2i}f_{4i}}{2^{2i-\alpha(i)}} & \text{if } 4i - \alpha(i) \leq 2l, \\ \tfrac{u^{2k-2i}f_{4i}}{2^{2l-2i}} & \text{if } 4i - \alpha(i) > 2l \end{cases} \end{aligned}$$

while

$$g_{4k,4i} = \begin{cases} u^{2k-4i+\alpha(i)}\left[\frac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i-\alpha(i) \leq 2k, \\ \left[\frac{u^{2(k-i)}f_{4i}}{2^{2(k-i)}}\right] & \text{if } 4i-\alpha(i) > 2k. \end{cases}$$

From these formulae we find that

$$u^{2k-2l}g_{4l,4i} = \begin{cases} g_{4k,4i} & \text{if } 4i-\alpha(i) \leq 2l \leq 2k, \\ 2^{4i-\alpha(i)-2l}g_{4k,4i} & \text{if } 2l < 4i-\alpha(i) \leq 2k, \\ 2^{2k-2l}g_{4k,4i} & \text{if } 2l < 2k < 4i-\alpha(i). \end{cases}$$

Now let us calculate $A_{l,k}$.

When $k = 0$ we have

$$z_0 = (1 \wedge \psi^3)_*(z_0) = A_{0,0}(\iota_{0,0})_*(z_0) = A_{0,0}z_0$$

so that $A_{0,0} = 1$.

When $k = 1$ we have

$$\begin{aligned} &\textstyle\sum_{l\leq 1} A_{l,1}(\iota_{1,l})_*(z_4) \\ &= (1 \wedge \psi^3)_*(z_4) \\ &= \lambda_{4,4}(1 \wedge \psi^3)_*(g_{4,4}) + 2\lambda_{4,0}(1 \wedge \psi^3)_*(g_{4,0}) \\ &= \lambda_{4,4}(9g_{4,4} + 2g_{4,0}) + 2\lambda_{4,0}g_{4,0} \end{aligned}$$

and

$$\begin{aligned} &\textstyle\sum_{l\leq 1} A_{l,k}(\iota_{1,l})_*(z_4) \\ &= A_{1,1}z_4 + A_{0,1}\mu_{1,0}2g_{4,0} \\ &= A_{1,1}(2\lambda_{4,0}g_{4,0} + \lambda_{4,4}g_{4,4}) + A_{0,1}\mu_{1,0}2g_{4,0} \end{aligned}$$

which implies that $A_{1,1} = 9$ and $A_{0,1} = \mu_{1,0}^{-1}(\lambda_{4,4} - 8\lambda_{4,0})$ so that $A_{0,1} \in \mathbb{Z}_2^*$.

When $k = 2$ we have

$$\begin{aligned} &\textstyle\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) \\ &= (1 \wedge \psi^3)_*(z_8) \\ &= (1 \wedge \psi^3)_*(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\ &= \lambda_{8,8}(9^2g_{8,8} + 9 \cdot 2^3g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4} + g_{8,0}) + 2^3\lambda_{8,0}g_{8,0} \end{aligned}$$

and

$$\begin{aligned} &\textstyle\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) \\ &= A_{2,2}z_8 + A_{1,2}(\iota_{2,1})_*(z_8) + A_{0,2}(\iota_{2,0})_*(z_8) \\ &= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\ &\qquad + A_{1,2}(\mu_{8,4}2^2u^2z_4) + A_{0,2}(\mu_{8,0}2^3u^4z_0) \\ &= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\ &\qquad + A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}u^2g_{4,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0} \\ &= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\ &\qquad + A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\lambda_{8,8}(9^2 g_{8,8} + 9\cdot 2^3 g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4} + g_{8,0}) + 2^3\lambda_{8,0}g_{8,0}\\ &= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0})\\ &\qquad + A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3 g_{8,0}\end{aligned}$$

which yields

$$\begin{aligned}&9^2 = A_{2,2},\\ &\lambda_{8,8}\cdot 9 + \lambda_{8,4}(9-9^2) = A_{1,2}\mu_{8,4}\lambda_{4,4},\\ &\lambda_{8,4} + \lambda_{8,0}(1-9^2) = A_{1,2}\mu_{8,4}\lambda_{4,0} + A_{0,2}\mu_{8,0}.\end{aligned}$$

Hence $A_{1,2}\in\mathbb{Z}_2^*$.

Now assume that $k\geq 3$ and consider the relation derived above:

$$\begin{aligned}&\Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}(1\wedge\psi^3)_*(g_{4k,4i})\\ &= A_{k,k}\Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}g_{4k,4i}\\ &\quad + \sum_{l<k}\ \Sigma_{i=0}^l\ A_{l,k}\ \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}2^{\beta(l,i)}\lambda_{4l,4i}g_{4l,4i}.\end{aligned}$$

The coefficient of $g_{4k,4k}$ on the left side of this relation is equal to $\lambda_{4k,4k}9^k$ and on the right side it is $A_{k,k}\lambda_{4k,4k}$ so that $A_{k,k}=9^k$ for all $k\geq 3$. From the coefficient of $g_{4k,4k-4}$ we obtain the relation

$$\begin{aligned}&\lambda_{4k,4k}9^{k-1}2^{\nu_2(k)+3} + 2^{3+\nu_2(k)}\lambda_{4k,4k-4}9^{k-1}\\ &= 9^k 2^{3+\nu_2(k)}\lambda_{4k,4k-4}\\ &\quad + A_{k-1,k}\ \mu_{4k,4k-4}2^{2-\alpha(k)+\alpha(k-1)}2^2\lambda_{4k-4,4k-4}2^{3+\nu_2(k)}\lambda_{4k,4k-4}9^{k-1}\\ &= 9^k 2^{3+\nu_2(k)}\lambda_{4k,4k-4}\\ &\quad + A_{k-1,k}\ \mu_{4k,4k-4}2^{3+\nu_2(k)}\lambda_{4k-4,4k-4}\end{aligned}$$

which shows that $A_{k-1,k}\in\mathbb{Z}_2^*$ for all $k\geq 3$. This means that we may conjugate A by the matrix

$$D = \mathrm{diag}(1, A_{1,2}, A_{1,2}A_{2,3}, A_{1,2}A_{2,3}A_{3,4},\ldots)\in U_\infty\mathbb{Z}_2$$

to obtain

$$DAD^{-1} = C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \cdots\\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \cdots\\ 0 & 0 & 9^2 & 1 & c_{3,5} & \cdots\\ 0 & 0 & 0 & 9^3 & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\end{pmatrix}.$$

In the next section we examine whether we can conjugate this matrix further in $U_\infty\mathbb{Z}_2$ to obtain the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

5.4 The matrix reloaded

5.4.1. Let $B, C \in U_\infty\mathbb{Z}_2$ denote the upper triangular matrices which occurred in § 5.3.5,

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \ldots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \ldots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The following result is the main result of this section. Along with the discussion of § 5.3.5 it completes the proof of Theorem 5.1.2. This result comes in two flavours – Theorem 5.4.2 and Theorem 5.4.3 – the first of which is just an existence statement and the second gives the conjugating matrix explicitly. The proof of Theorem 5.4.2, given in § 5.4.4, Lemma 5.4.5 together with § 5.4.6, is the original one found by Jonathan Barker and myself and used in [27]. The proof of Theorem 5.4.3, given in § 5.4.7, comes from the PhD thesis of Jonathan Barker and follows a suggestion by Francis Clarke.

Theorem 5.4.2. There exists an upper triangular matrix $U \in U_\infty\mathbb{Z}_2$ such that $U^{-1}CU = B$.

Theorem 5.4.3. Let $V = (v_{i,j})_{i,j\geq 1}$ be the upper triangular matrix whose entries satisfy

$$v_{1,j} = \begin{cases} 1 & \text{if } j = 1 \text{ or } j = 2, \\ 0 & \text{if } j > 2 \end{cases}$$

and

$$v_{i+1,j} = \left(\sum_{r=i}^{j-2} v_{i,r} c_{i,r} \right) + v_{i,j-1} + (9^{j-1} - 9^{i-1}) v_{i,j}.$$

Then $V \in U_\infty \mathbb{Z}_2$ and $VC = BV$.

5.4.4. Proof of Theorem 5.4.2 – the induction. Let U have the form

$$\begin{pmatrix} 1 & u_{1,2} & u_{1,3} & u_{1,4} & \cdots \\ 0 & 1 + (9-1)u_{1,2} & u_{2,3} & u_{2,4} & \cdots \\ 0 & 0 & 1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3} & u_{3,4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Then $(UB)_{j,j} = U_{j,j} B_{j,j} = C_{j,j} U_{j,j} = (CU)_{j,j}$ and, in fact, $(UB)_{j,j+1} = (CU)_{j,j+1}$ for all j, too. For any $1 < s < j$ we have

$$(UB)_{j-s,j} = u_{j-s,j} 9^{j-1} + u_{j-s,j-1}$$

and

$$\begin{aligned} (CU)_{j-s,j} &= 9^{j-s-1} u_{j-s,j} + u_{j-s+1,j} \\ &\quad + c_{j-s,j-s+2} u_{j-s+2,j} + \cdots + c_{j-s,j} u_{j,j}. \end{aligned}$$

In order to prove Theorem 5.4.2 it suffices to verify that we are able to solve for the $u_{i,j}$ in the equations $(UB)_{s,t} = (CU)_{s,t}$ for all $s \leq t$ inductively in such in a manner that, for every k, the first k-columns of the equality $UB = CU$ is achieved after a finite number of steps. The proof is then completed by the compactness argument which is given in § 5.4.6. Lemma 5.4.5 provides a method which proceeds inductively on the columns of U. □

Lemma 5.4.5. For $j \geq 3$ and $1 < s < j$, $u_{j-s,j-1}$ may be written as a linear combination of

$$u_{j-2,j-1}, u_{j-3,j-1}, \ldots, u_{j-s+1,j-1} \text{ and } u_{j-1,j}, u_{j-2,j}, \ldots, u_{1,j}.$$

Proof. We shall prove the result by induction on j. Consider the case $j = 3$; we have the following equation:

$$\begin{aligned} & u_{3-s,3} 9^2 + u_{3-s,2} \\ & = 9^{2-s} u_{3-s,3} + u_{4-s,3} + c_{3-s,5-s} u_{5-s,3} + \cdots + c_{3-s,3} u_{3,3} \end{aligned}$$

for $1 < s < 3$; that is, $s = 2$. Hence substituting $s = 2$ gives

$$\begin{aligned} & u_{1,3} 9^2 + u_{1,2} = u_{1,3} + u_{2,3} + c_{1,3} u_{3,3} \\ & \implies u_{1,2} = (1 - 9^2) u_{1,3} + u_{2,3} + c_{1,3}(1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3}) \\ & \implies (1 - (9-1)c_{1,3}) u_{1,2} = (1 - 9^2) u_{1,3} + (1 + (9^2 - 9)) c_{1,3} u_{2,3} + c_{1,3} \end{aligned}$$

and since $(1-(9-1)c_{1,3})$ is a 2-adic unit we can write $u_{1,2}$ as a $\mathbb{Z}_2$-linear combination of $u_{1,3}$ and $u_{2,3}$ as required.

We now need to show that if the lemma is true for $j = 3, 4, \ldots, k-1$ then it is also true for $j = k$. This means we need to solve

$$\begin{aligned}
&u_{k-s,k}9^{k-1} + u_{k-s,k-1} \\
&= 9^{k-s-1}u_{k-s,k} + u_{k-s+1,k} + c_{k-s,k-s+2}u_{k-s+2,k} \\
&\qquad\qquad + \cdots + c_{k-s,k}u_{k,k}
\end{aligned}$$

for $u_{k-s,k-1}$ for $1 < s < k$. This equation may be rewritten

$$\begin{aligned}
&u_{k-s,k-1} \\
&= (9^{k-s-1} - 9^{k-1})u_{k-s,k} + u_{k-s+1,k} + c_{k-s,k-s+2}u_{k-s+2,k} \\
&\qquad\qquad + \cdots + c_{k-s,k}u_{k,k} \\
&= (9^{k-s-1} - 9^{k-1})u_{k-s,k} + u_{k-s+1,k} \\
&\qquad\qquad + c_{k-s,k-s+2}u_{k-s+2,k} + \cdots \\
&\cdots + c_{k-s,k}(1 + (9-1)u_{1,2} + (9^2-9)u_{2,3} \\
&\qquad\qquad + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k}).
\end{aligned}$$

Now consider the case $s = k - 1$:

$$\begin{aligned}
&u_{1,k-1} \\
&= (1 - 9^{k-1})u_{1,k} + u_{2,k} + c_{1,3}u_{3,k} + \cdots \\
&\cdots c_{1,k}\underbrace{(1 + (9-1)u_{1,2} + (9^2-9)u_{2,3} + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k})}_{B}.
\end{aligned}$$

By repeated substitutions the bracket B may be rewritten as a linear combination of $u_{1,k-1}, u_{2,k-1}, \ldots, u_{k-2,k-1}$ and $u_{k-1,k}$. The important point to notice about this linear combination is that the coefficient of $u_{1,k-1}$ will be an even 2-adic integer. Hence, we can move this term to the left-hand side of the equation to obtain a 2-adic unit times $u_{1,k-1}$ being equal to a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{2,k-1}, u_{3,k-1}, \ldots, u_{k-2,k-1}$ as required.

Now consider $s = k - 2$:

$$\begin{aligned}
&u_{2,k-1} \\
&= (9 - 9^{k-1})u_{2,k} + u_{3,k} + c_{2,4}u_{4,k} + \cdots \\
&\cdots + c_{2,k}\underbrace{(1 + (9-1)u_{1,2} + (9^2-9)u_{2,3} + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k})}_{B'}.
\end{aligned}$$

As before B' can be written as a linear combination of $u_{1,k-1}$, $u_{2,k-1}$, $\ldots$, $u_{k-2,k-1}$, $u_{k-1,k}$ and from the case $s = k-1$, $u_{1,k-1}$ may be replaced by a linear combination of $u_{1,k}, \ldots, u_{k-1,k}$ and $u_{2,k-1}, \ldots, u_{k-2,k-1}$. Again the important observation is that the coefficient of $u_{2,k-1}$ is an even 2-adic integer, hence this term can be moved to the left-hand side of the equation to yield a 2-adic unit times $u_{2,k-1}$ being equal to a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{3,k-1}, \ldots, u_{k-2,k-1}$ as required.

Clearly this process may be repeated for $s = k-3, k-4, \ldots, 2$ to get a 2-adic unit times $u_{k-s,k-1}$ as a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{k-s+1,k-1}, \ldots, u_{k-2,k-1}$ as required. □

5.4.6. Proof of Theorem 5.4.2 – the compactness argument. The argument of § 5.4.4 and Lemma 5.4.5 proves that $1 \wedge \psi^3$ corresponds to a conjugacy class in $U_\infty \mathbb{Z}_2$ of the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

only up to any finite approximation – that is, so that the first N columns has this form for arbitrary N – and that Theorem 5.4.2 is true in the same finite approximation sense. This is usually adequate for topological applications such as those of Chapter 8. The following compactness argument completes the proof of Theorem 5.4.2.

With the p-adic topology $\mathbb{Z}_p \cong 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*$ if p is odd or $\mathbb{Z}_2 \cong 1 + 4\mathbb{Z}_2 \subset \mathbb{Z}_2^*$ is a compact, Hausdorff topological group. With the discrete topology on $\mathbb{F}_p^*$ or $(\mathbb{Z}/4)^*$ so it $\mathbb{Z}_p^* = \mathbb{Z}_p \times \mathbb{F}_p^*$ or $\mathbb{Z}_2^* \cong \mathbb{Z}_2 \times (\mathbb{Z}/4)^*$. Now $U_\infty \mathbb{Z}_2$ is a countable union of a product of these compact spaces so it is a compact, Hausdorff topological group by Tychonoff's Theorem. The conjugacy class of $X \in U_\infty \mathbb{Z}_2$ is a closed compact subset because it is isomorphic to the coset space $U_\infty \mathbb{Z}_2 / Z(X)$ where $Z(X)$ is the centraliser of X. This is compact because it is a quotient of a compact space and in a Hausdorff, compact space it must also be closed.

The discussion of § 5.4.4 and Lemma 5.4.5 constructs a sequence of elements $U_n C U_n^{-1} = C_n$ in the conjugacy class of C, which means that there is a convergent subsequence whose diagonal entries are constant and so are the superdiagonal ones – so let us suppose that $C_n \to \tilde{C}$. Then the entries in the C_n's must converge to those of $\tilde{C}$. Off the diagonal and the superdiagonal these must be zero because we can arrange that the first n columns of C_n have zeroes off the diagonal and superdiagonal. But the limit $\tilde{C}$ must be in the conjugacy class, because it is closed, as required. □

5.4.7. Proof of Theorem 5.4.3. We shall prove by induction that $v_{i,i} \in \mathbb{Z}_2^*$, the 2-adic units, for $i \geq 1$, which is sufficient to prove that V is invertible and therefore lies in $U_\infty \mathbb{Z}_2$.

For $i = 1$, $v_{1,1} = 1$ which is clearly in $\mathbb{Z}_2^*$. Now assume $v_{i,i} \in \mathbb{Z}_2^*$ for all $1 \leq i < n$ where n is an integer. We wish to show that this implies $v_{n,n} \in \mathbb{Z}_2^*$. By definition

$$v_{n,n} = v_{n-1,n-1} + (9^{n-1} - 9^{n-2})v_{n-1,n}.$$

Since $9^{n-1} - 9^{n-2} = 9^{n-2}(9-1) \in 2\mathbb{Z}_2$ and, by the induction hypothesis, $v_{n-1,n-1} \in \mathbb{Z}_2^*$ it follows that

$$v_{n,n} \in \mathbb{Z}_2^* + 2\mathbb{Z}_2 = \mathbb{Z}_2^*.$$

Hence, by induction, $v_{i,i} \in \mathbb{Z}_2^*$ for $i \geq 1$, as required.

To complete the proof we shall verify that $(VC)_{i,j} = (BV)_{i,j}$ for $j > i$. The entries of VC and BV, for $i \geq j$ are given by the formulae

$$\begin{aligned} (VC)_{i,j} &= v_{i,i}c_{i,j} + v_{i,i+1}c_{i+1,j} + \cdots \\ &\qquad +v_{i,j-2}c_{j-2,j} + v_{i,j-1} + v_{i,j}9^{j-1} \\ &= \left(\sum_{r=i}^{j-2} v_{i,r}c_{i,r}\right) + v_{i,j-1} + v_{i,j}9^{j-1} \end{aligned}$$

and

$$(BV)_{i,j} = 9^{i-1}v_{i,j} + v_{i+1,j}.$$

On the other hand for our choice of V we find that for all $i, j \geq 1$ and $i \geq j$:

$$\begin{aligned} &(VC)_{i,j} \\ &= \left(\sum_{r=i}^{j-2} v_{i,r}c_{i,r}\right) + v_{i,j-1} + v_{i,j}9^{j-1} \\ &= v_{i+1,j} - (9^{j-1} - 9^{i-1})v_{i,j} + v_{i,j}9^{j-1} \\ &= v_{i+1,j} - 9^{j-1}v_{i,j} + 9^{i-1}v_{i,j} + v_{i,j}9^{j-1} \\ &= v_{i+1,j} + 9^{i-1}v_{i,j} \\ &= (BV)_{i,j} \end{aligned}$$

as required. □

5.5 Applications

5.5.1. ***bu*** $\wedge$ ***bu*.** Theorem 5.1.2 implies that, in the 2-local stable homotopy category, there exists an equivalence $C' \in \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ such that

$$C'(1 \wedge \psi^3)C'^{-1} = \sum_{k \geq 0} 9^k \iota_{k,k} + \sum_{k \geq 1} \iota_{k,k-1}$$

where $\iota_{k,l}$ is as in § 5.2.1, considered as a left bu-endomorphism of $bu \wedge bo$ via the equivalence $\hat{L}$ of § 5.2.1.

In Chapter 3 and in [252] use is made of an equivalence of the form $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$, first noticed by Reg Wood (as remarked in [9]) and independently by Don Anderson (both unpublished). This is easy to construct. By definition

$$bu^0(\Sigma^{-2}\mathbb{CP}^2) \cong bu^2(\mathbb{CP}^2) \cong [\mathbb{CP}^2, BU]$$

and from the cofibration sequence $S^0 \longrightarrow \Sigma^{-2}\mathbb{CP}^2 \longrightarrow S^2$ we see that

$$bu^0(\Sigma^{-2}\mathbb{CP}^2) \cong \mathbb{Z} \oplus \mathbb{Z},$$

fitting into the exact sequence

$$0 \longrightarrow bu^0(S^2) \longrightarrow bu^0(\Sigma^{-2}\mathbb{CP}^2) \longrightarrow bu^0(S^0) \longrightarrow 0.$$

Choosing any stable homotopy class $x : \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu$, restricting to the generator of $bu^0(S^0)$, yields an equivalence of the form

$$bo \wedge (\Sigma^{-2}\mathbb{CP}^2) \stackrel{c\wedge x}{\longrightarrow} bu \wedge bu \stackrel{\mu}{\longrightarrow} bu$$

in which c denotes complexification and μ is the product.

In the 2-local stable homotopy category there is a map

$$\Psi : \Sigma^{-2}\mathbb{CP}^2 \longrightarrow \Sigma^{-2}\mathbb{CP}^2$$

which satisfies $\Psi^*(z) = \psi^3(z)$ for all $z \in bu^0(\Sigma^{-2}\mathbb{CP}^2)$. For example, take Ψ to be 3^{-1} times the double desuspension of the restriction to the four-skeleton of the CW complex $\mathbb{CP}^\infty = BS^1$ of the map induced by $z \mapsto z^3$ on S^1, the circle. With this definition there is a homotopy commutative diagram in the 2-local stable homotopy category

$$\begin{tikzcd} bo \wedge \Sigma^{-2}\mathbb{CP}^2 \arrow[r, "\psi^3\wedge\Psi"'] \arrow[d, "\simeq"] & bo \wedge \Sigma^{-2}\mathbb{CP}^2 \arrow[d, "\simeq"'] \\ bu \arrow[r, "\psi^3"] & bu \end{tikzcd}$$

in which the vertical maps are given by the Anderson-Wood equivalence.

Now suppose that we form the smash product with $\Sigma^{-2}\mathbb{CP}^2$ of the 2-local left bu-module equivalence

$$bu \wedge bo \simeq \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1})$$

to obtain a left bu-module equivalence of the form

$$bu \wedge bu \simeq \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2.$$

For $l \leq k$ set $\kappa_{k,l} = \iota_{k,l} \wedge \Psi$ to give

$$\begin{array}{c} bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2 \\ \downarrow\ \kappa_{k,l} = \iota_{k,l} \wedge \Psi \\ bu \wedge (F_{4l}/F_{4l-1}) \wedge \Sigma^{-2}\mathbb{CP}^2. \end{array}$$

Then we obtain the following result.

Theorem 5.5.2. In the notation of § 5.5.1, in the 2-local stable homotopy category, there exists $C' \in \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo)$ such that

$$1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$$

satisfies

$$(C' \wedge 1)(1 \wedge \psi^3)(C' \wedge 1)^{-1} = \sum_{k \geq 0} 9^k \kappa_{k,k} + \sum_{k \geq 1} \kappa_{k,k-1}.$$

5.5.3. $\mathrm{End}_{\textbf{left-}bu\textbf{-mod}}(bu \wedge bo)$**.** In this section we shall apply Theorem 5.1.2 to study the ring of left-bu-module homomorphisms of $bu \wedge bo$. As usual we shall work in the 2-local stable homotopy category. Let $\tilde{U}_\infty \mathbb{Z}_2$ denote the ring of upper triangular, infinite matrices with coefficients in the 2-adic integers. Therefore the group $U_\infty \mathbb{Z}_2$ is a subgroup of the multiplicative group of units of $\tilde{U}_\infty \mathbb{Z}_2$. Choose a left-$bu$-module homotopy equivalence of the form

$$\hat{L} : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo,$$

as in § 5.2.1. For any matrix $A \in \tilde{U}_\infty \mathbb{Z}_2$ we may define a left-bu-module endomorphism of $bu \wedge bo$, denoted by λ_A, by the formula

$$\lambda_A = \hat{L} \cdot (\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}.$$

Incidentally here and throughout this section we shall use the convention that a composition of maps starts with the right-hand map, which is the convention used in the definition of the isomorphism ψ of § 5.2.1. When $A \in U_\infty \mathbb{Z}_2$ we have the relation $\lambda_A = \psi(A)$. For $A, B \in \tilde{U}_\infty \mathbb{Z}_2$ we have (see also Chapter 3 § 3.2.2)

$$\begin{aligned}
\lambda_A \cdot \lambda_B &= (\hat{L} \cdot (\textstyle\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}) \cdot (\hat{L} \cdot (\textstyle\sum_{0 \leq t \leq s} B_{t,s} \iota_{s,t}) \cdot \hat{L}^{-1}) \\
&= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq t \leq s} A_{l,t} B_{t,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\
&= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq s} (AB)_{l,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\
&= \lambda_{AB}.
\end{aligned}$$

By Theorem 5.1.2 there exists $H \in U_\infty \mathbb{Z}_2$ such that

$$1 \wedge \psi^3 = \lambda_{HBH^{-1}}$$

for

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Hence, for any integer $u \geq 1$, we have $1 \wedge (\psi^3 - 9^{u-1}) = \lambda_{HB_u H^{-1}}$ where $B_u = B - 9^{u-1} \in \tilde{U}_\infty \mathbb{Z}_2$ and 9^{u-1} denotes 9^{u-1} times the identity matrix. Following [185] write $\phi_n : bo \longrightarrow bo$ for the composition $\phi_n = (\psi^3 - 1)(\psi^3 - 9) \ldots (\psi^3 - 9^{n-1})$. Write

$$X_n = B_1 B_2 \ldots B_n \in \tilde{U}_\infty \mathbb{Z}_2.$$

Theorem 5.5.4.

(i) In the notation of § 5.5.3,

$$1 \wedge \phi_n = \lambda_{HX_n H^{-1}}$$

for $n \geq 1$.

(ii) The first n-columns of X_n are trivial.

(iii) Let $C_n = Cone(\hat{L} : \vee_{0 \leq k \leq n-1} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo)$, which is a left-$bu$-module spectrum. Then in the 2-local stable homotopy category there exists a commutative diagram of left-bu-module maps of the form

$$\begin{tikzcd} bu \wedge bo \arrow[rr, "1 \wedge \phi_n"] \arrow[dr, "\pi_n"'] & & bu \wedge bo \\ & C_n \arrow[ur, "\hat{\phi}_n"'] & \end{tikzcd}$$

where π_n is the cofibre of the restriction of $\hat{L}$. Also $\hat{\phi}_n$ is determined up to homotopy by this diagram.

(iv) More precisely, for $n \geq 1$ we have

$$(X_n)_{s,s+j} = 0 \text{ if } j < 0 \text{ or } j > n$$

and the other entries are given by the formula

$$(X_n)_{s,s+t} = \sum_{1 \leq k_1 < k_2 < \cdots < k_t \leq n} A(k_1) A(k_2) \ldots A(k_t)$$

where

$$\begin{aligned} A(k_1) &= \textstyle\prod_{j_1 = n-k_1+1}^{n} (9^{s-1} - 9^{j_1 - 1}), \\ A(k_2) &= \textstyle\prod_{j_2 = n-k_2+1}^{n-k_1-1} (9^{s} - 9^{j_2 - 1}), \\ A(k_3) &= \textstyle\prod_{j_3 = n-k_3+1}^{n-k_2-1} (9^{s+1} - 9^{j_3 - 1}), \\ &\vdots \\ A(k_t) &= \textstyle\prod_{j_t = 1}^{n-k_t-1} (9^{s+t+1} - 9^{j_t - 1}). \end{aligned}$$

Proof. Part (i) follows immediately from the discussion of § 5.5.3. Part (ii) follows from part (iv), but it is simpler to prove it directly. For part (ii) observe that the B_i commute, being polynomials in the matrix B, so that $X_n = X_{n-1}B_n$. Since $(B_n)_{s,t}$ is zero except when $t = s, s+1$ we see that $(X_n)_{i,j} = (X_{n-1})_{i,j}(B_n)_{j,j} + (X_{n-1})_{i,j-1}(B_n)_{j-1,j}$ is zero by induction if $j < n$. When $j = n$ by induction we have $(X_n)_{i,j} = (X_{n-1})_{i,n}(B_n)_{n,n}$ which is trivial because $(B_n)_{n,n} = 9^{n-1} - 9^{n-1}$. In view of the decomposition of $bu \wedge bo$, part (iii) amounts to showing that HX_nH^{-1} corresponds to a left-bu-module endomorphismm of $\vee_{0 \leq k}\ bu \wedge (F_{4k}/F_{4k-1})$ which is trivial on each summand $bu \wedge (F_{4k}/F_{4k-1})$ with $k \leq n-1$. The (i,j)th entry in this matrix is the multiple of $\iota_{j-1,i-1} : bu \wedge (F_{4j-4}/F_{4j-5}) \longrightarrow bu \wedge (F_{4i-4}/F_{4i-5})$ given by the appropriate component of the map. The first n columns are zero if and only if the map has no non-trivial components whose domain is $bu \wedge (F_{4j-4}/F_{4j-5})$ with $j \leq n$. Since H is upper triangular and invertible, the first n columns of X_n vanish if and only if the same is true for HX_nH^{-1}. Finally the formulae of part (iv) result from the fact that B_j has $9^{m-1} - 9^{j-1}$ in the (m,m)th entry, 1 in the $(m, m+1)$th entry and zero elsewhere. □

Remark 5.5.5. Theorem 5.5.4 is closely related to the main result of [185]. Following [185] let $bo^{(n)} \longrightarrow bo$ denote the map of 2-local spectra which is universal for all maps $X \longrightarrow bo$ which are trivial with respect to all higher $\mathbb{Z}/2$-cohomology operations of order less than n (compare this with [185] Theorem B). Milgram shows that ϕ_{2n} factorises through a map of the form $\theta_{2n} : bo \longrightarrow \Sigma^{8n} bo^{(2n-\alpha(n))}$ and that ϕ_{2n+1} factorises through a map of the form $\theta_{2n+1} : bo \longrightarrow \Sigma^{8n+4} bsp^{(2n-\alpha(n))}$ and then uses the θ_m's to produce a left-bo-module splitting of $bo \wedge bo$. Using the Anderson-Wood homotopy equivalence $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$ mentioned in § 5.5.1 and in [252], one may pass from the splitting of $bu \wedge bo$ to that of $bo \wedge bo$ (and back again). In the light of this observation, the existence of the diagram of Theorem 5.5.4 should be thought of as the upper triangular matrix version of the proof that the θ_n's exist. The advantage of the matrix version is that Theorem 5.5.4(iv) gives us every entry in the matrix X_n, not just the zeroes in the first n columns.

Chapter 6

Real Projective Space

"If there's no meaning in it," said the King, "that saves a world of trouble, you know, as we needn't try to find any. And yet I don't know," he went on; "I seem to see some meaning in them, after all."

from "Alice in Wonderland – Alice's evidence" by Lewis Carrol [55]

The objective of this chapter is to present the cohomological calculations (in MU_*, KU_* and BP_*) which will be needed in this and later chapters for the study of maps of the form

$$g : \Sigma^\infty S^{2^{k+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{2^{k+1}-2}$$

and related Whitehead product maps. In [30] it is shown that if g_* is non-zero on jo_*-theory, which was introduced in Chapter 1, Example 1.3.4(iv), then g is detected by Sq^{2^k}. On the other hand detection by jo_*-theory is equivalent to the KU_*-e-invariant (defined by means of $\psi^3 - 1$) being $\frac{(3^{2^k}-1)(2w+1)}{4}$. The calculations of this chapter will eventually be used to prove both this and the converse result (conjectured in [30]) in Chapter 8, Theorem 8.1.2.

§ 1 calculates the MU-theory and KU-theory of $\mathbb{RP}^n$ together with the effect of the Adams operations in these groups. It also calculates the effect of some of the Landweber-Novikov operations on $MU_*(\mathbb{RP}^n)$. § 2 contains a slightly different method for making these calculations, applied instead to $BP_*(\mathbb{RP}^n)$ in preparation for the applications of Chapter 7. § 3 applies these calculations to the study of the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$. The vanishing of the Whitehead product is equivalent to the classical Hopf invariant one problem and Theorem 6.3.2 is equivalent to the (then new) KU_* proof of the non-existence of classes of Hopf invariant one which I published in the midst of the book review [248]. Theorem 6.3.4 gives equivalent conditions, related to the Arf-Kervaire invariant one problem, for $[\iota_n, \iota_n]$ to be divisible by two. This result was proved in one direction in [30]. The converse is proved as a consequence of the main upper triangular technology result of Chapter 8 (Theorem 8.1.2), which is equivalent to the Arf-Kervaire invariant one reformulation of Chapter 1 § 1.8.9 and Theorem

1.8.10 originally conjectured in [30] (see Chapter 7, Theorem 7.2.2). § 4 contains results which relate e-invariants and Hopf invariants – first considered in Corollary D of [30]. In fact Theorem 6.4.2 and Corollary 6.4.3 imply both Corollary D and its conjectured converse. § 5 contains a miscellany of results which relate the halving of the Whitehead product to MU_*-e-invariants. The material of Chapter 6 is based upon an unpublished 1984 manuscript concerning the MU-theory formulations of the results of [30], enhanced by use of Chapter 8, Theorem 8.1.2 in several crucial places.

I started considering these problems after conversations with Paul Selick. The use of MU-theory materialised after several useful conversations with Andy Baker, Peter Eccles and Nigel Ray during my visit to Manchester University in 1984, under the auspices of the London Mathematical Society.

6.1 MU-theory and KU-theory of $\mathbb{RP}^n$

Let MU and KU denote the unitary cobordism and periodic unitary K-theory spectra, respectively, as introduced in Chapter 1, Example 1.3.2(iv) and (vi) (see also Chapter 1, Theorem 1.3.3). In this section, for completeness, I will review the calculation of $MU^*(\mathbb{RP}^n)$, $MU_*(\mathbb{RP}^n)$, $KU^*(\mathbb{RP}^n)$ and $KU_*(\mathbb{RP}^n)$ together with certain formulae which will describe the homology operations which we shall need later. Nigel Ray showed me how to compute $MU_*(\mathbb{RP}^n)$ by a method from one of his unpublished works (dated 1969).

6.1.1. Recall ([9] Part II) that $MU^*(\mathbb{CP}^\infty) \cong MU^*(pt)[[x]]$ where $x \in MU^2(\mathbb{CP}^\infty)$ is the first Conner-Floyd class of the Hopf line bundle. If $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty$ is the product then

$$m^*(x) = x \otimes 1 + 1 \otimes x + \sum_{i,j \geq 1} a_{i,j} x^i \otimes x^j$$

where $a_{i,j} \in MU^{2-2i-2j}(pt) = \pi_{2i+2j-2}(MU)$. The power series $m^*(x)$ is called the formal group law associated with MU-theory. It is a famous theory of Dan Quillen ([225]; see also [9] Part II and Chapter 1 Theorem 1.3.7) that $m^*(x)$ is the universal formal group law and that $\pi_*(MU) = MU_*(pt) \cong \mathbb{Z}[a_{1,1}, a_{1,2}, \ldots]/\simeq$, a quotient of the polynomial ring on the $a_{i,j}$'s whose precise description is to be found in ([9] p. 56).

Let $\pi : \mathbb{RP}^\infty \longrightarrow \mathbb{CP}^\infty$ be the standard map, then $m \cdot (\pi \times \pi) \cdot \Delta$ is trivial, where $\Delta : \mathbb{RP}^\infty \longrightarrow \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is the diagonal map. If we set $w = \pi^*(x) \in MU^2(\mathbb{RP}^\infty)$ then

$$0 = 2w + \sum_{i,j \geq 1} a_{i,j} w^{i+j}.$$

Note that the leading terms of this relation are

$$0 = [2]w = 2w + a_{1,1} w^2 + \cdots .$$

Here $[k]w$ is the formal group law notation for the result of applying the k-fold product map to w [294]. The Atiyah-Hirzebruch spectral sequence

$$E_2^{s,t} = H^2(\mathbb{RP}^{2n+1}; MU^t(pt)) \Longrightarrow MU^{s+t}(\mathbb{RP}^{2n+1})$$

collapses for dimensional reasons.

Proposition 6.1.2. As an $MU^*(pt)$-module

(a) $MU^*(\mathbb{RP}^{2n+1}) \cong \dfrac{MU^*(pt)[w]}{\langle w^{n+1},\ 2w + \sum_{i,j\geq 1}\ a_{i,j}w^{i+j}\rangle} \oplus MU^*(pt)\langle\tau\rangle$

where the second summand is free on $\tau \in MU^{2n+1}(\mathbb{RP}^{2n+1})$.

(b) $MU^*(\mathbb{RP}^{2n}) \cong \dfrac{MU^*(pt)[w]}{\langle w^{n+1},\ 2w + \sum_{i,j\geq 1}\ a_{i,j}w^{i+j}\rangle}$.

Proof. Part (a) follows easily from part (b), which is proved by calculating the group orders using the collapsed Atiyah-Hirzebruch spectral sequence to obtain the order of $MU^*(\mathbb{RP}^{2n})$ (see the proof of the BP analogue in § 6.2.2). □

6.1.3. The Conner-Floyd map is a natural transformation of the form

$$\gamma : MU^*(X) \longrightarrow KU^*(X)$$

([68]; see also [245] Part II § 9). In Chapter 1, Theorem 1.3.3, periodic MU-theory is constructed by "localising" the suspension spectrum of BU and KU-theory is constructed by doing the same to $\mathbb{CP}^\infty$. The determinant map induces a surjection $PMU \longrightarrow KU$ which, in turn, induces the Conner-Floyd map ([245] Part II § 9). Since the formal group associated to KU is $x \otimes 1 + x \otimes 1 + Bx \otimes x$ where B is the Bott element and γ respects the formal group laws we see that the Conner-Floyd map sends $a_{1,1}$ to the Bott element and $a_{i,j}$ to zero otherwise. Here $KU^*(X)$ is treated as being $\mathbb{Z}/2$-graded, by Bott periodicity. Hence, as is well known [26], we obtain the reduced K-theory of real projective spaces.

Proposition 6.1.4.

$$\begin{aligned}
\tilde{KU}^0(\mathbb{RP}^{2n+1}) &\cong \mathbb{Z}/2^n\langle\hat{w}\rangle,\\
\tilde{KU}^1(\mathbb{RP}^{2n+1}) &\cong \mathbb{Z}\langle\hat{\tau}\rangle,\\
\tilde{KU}^0(\mathbb{RP}^{2n}) &\cong \mathbb{Z}/2^n\langle\hat{w}\rangle,\\
\tilde{KU}^1(\mathbb{RP}^{2n}) &= 0
\end{aligned}$$

where $\hat{w}^2 + 2\hat{w} = 0$ and $\hat{w} = q(w),\ \hat{\tau} = \gamma(\tau)$.

Proof. This proof uses the collapsed Atiyah-Hirzebruch spectral sequence and surjectivity of the Conner-Floyd map. □

6.1.5. We have classes $\beta_{2m+1} \in MU_{2m+1}(\mathbb{RP}^t)$ for $t \geq 2m+1$ represented by the canonical inclusion of the manifold $\mathbb{RP}^{2m+1}$ into $\mathbb{RP}^t$ (compare framed cobordism in Chapter 1 § 1.2.5; see also [225] and [260]). Since $\mathbb{RP}^{2n+1}$ is a complex manifold we have Poincaré duality isomorphisms ([9] Part III)

$$D : MU^j(\mathbb{RP}^{2n+1}) \xrightarrow{\cong} MU_{2n+1-j}(\mathbb{RP}^{2n+1})$$

given by $D(x) = x \bigcap \beta_{2n+1}$ satisfying $D(w^m) = \beta_{2n-2m+1}$. The inverse isomorphism is

$$\hat{D} : MU_{2n+1-j}(\mathbb{RP}^{2n+1}) \xrightarrow{\cong} MU^j(\mathbb{RP}^{2n+1})$$

given by the slant product $\hat{D}(z) = \pi^*(\Lambda_{\mathbb{RP}^{2n+1}})/z$ where, Δ denoting the diagonal,

$$\Lambda_{\mathbb{RP}^{2n+1}} \in MU^{2n+1}(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2n+1}, \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2n+1} - \Delta)$$

is the Thom class of the tangent bundle and $\pi^*(\Lambda_{\mathbb{RP}^{2n+1}})$ is its image in

$$MU^{2n+1}(\mathbb{RP}^{2n+1} \wedge \mathbb{RP}^{2n+1})$$

(see [9] Part III).

From § 6.1.1 we have relations for $0 \leq m \leq n-1$,

$$0 = 2\beta_{2m+1} + a_{1,1}\beta_{2m-1} + \sum_{i,j \geq 1,\ i+j \geq 3} a_{i,j}\beta_{2m+3-2i-2j}$$

in $MU_{2m+1}(\mathbb{RP}^{2n+1})$.

Proposition 6.1.6.

(a) As an $MU_*(pt)$-module, $MU_*(\mathbb{RP}^{2n+1})$ is the direct sum of a free module on generators 1 and β_{2n+1} with a module generated by $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ subject to the relations of § 6.1.5.

(b) As an $MU_*(pt)$-module, $MU_*(\mathbb{RP}^{2n})$ is the direct sum of a free module on the generator 1 with a module generated by $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ subject to the relations of § 6.1.5.

Proof. The collapsed Atiyah-Hirzebruch spectral sequence shows that $MU_*(\mathbb{RP}^{2n})$ embeds into $MU_*(\mathbb{RP}^{2n+1})$ so that part (b) follows from part (a). Since the Poincaré duality isomorphism D is a homomorphism of $MU_*(pt)$-modules, part (a) follows from Proposition 6.1.2(b). □

Proposition 6.1.7.

$$\begin{aligned}
\tilde{KU}_{-1}(\mathbb{RP}^{2n+1}) &\cong \mathbb{Z}\langle\hat{\beta}_{2n+1}\rangle \oplus \mathbb{Z}/2^n\langle\hat{\beta}_{2n-1}\rangle, \\
\tilde{KU}_{-1}(\mathbb{RP}^{2n}) &\cong \mathbb{Z}/2^n\langle\hat{\beta}_{2n-1}\rangle, \\
\tilde{KU}_0(\mathbb{RP}^m) &= 0
\end{aligned}$$

where $B^{n+1}\hat{\beta}_{2n+1} = \gamma(\beta_{2n+1})$ and $B^n\hat{\beta}_{2n-1} = \gamma(\beta_{2n-1})$ and $B \in KU_2(pt)$ is the Bott periodicity element.

Proof. This follows at once from Proposition 6.1.6 using the homology Conner-Floyd map

$$\gamma : MU_*(\mathbb{RP}^m) \longrightarrow KU_*(\mathbb{RP}^m)$$

([68]; see also [245] Part II § 9). Alternatively it may be deduced from Proposition 6.1.4 using Poincaré duality as in the proof of Proposition 6.1.6. □

Lemma 6.1.8. Let $\mathbb{Z}_{(2)} = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ \mathrm{HCF}(a,b) = 1,\ b \text{ odd}\}$ denote the integers localised at 2. Let k be an odd integer and let

$$\psi^k : MU_*(X; \mathbb{Z}_{(2)}) \longrightarrow MU_*(X; \mathbb{Z}_{(2)})$$

denote the MU-theory Adams operation (see [9] Part II; [245] pp. 59/60). Then, for $1 \leq j \leq n$,

$$\psi^k(\beta_{2n+1-2j}) = k^{n+1-j}\beta_{2n+1-2j} \in MU_{2n+1-2j}(\mathbb{RP}^{2n}; \mathbb{Z}_{(2)}).$$

Proof. If suffices to calculate in $MU_{2n+1-2j}(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)})$. By § 6.1.5,

$$\psi^k(\hat{D}(\beta_{2n+1-2j})) = \psi^k(w)^j$$

and $k\psi^k(w) = w$ since k is odd ([245] pp. 59/60). If Λ_w is the MU-theory Thom class of the Hopf bundle, then

$$\Lambda_{\mathbb{RP}^{2n+1}} = (\Lambda_w)^{n+1}$$

and $\psi^k(\Lambda_w) = \rho_k(w)\Lambda_w$ where $k\rho_k(w)w = k^*(w) = w$. Here k^* is the map induced by the kth power map on $\mathbb{RP}^\infty = B\mathbb{Z}/2$, which is equal to the identity. Hence

$$\begin{aligned}
\tfrac{w^j}{k^j} &= \psi^k(\hat{D}(\beta_{2n+1-2j})) \\
&= \psi^k(\Lambda_w^{n+1}/\beta_{2n+1-2j}) \\
&= (\rho_k(w)^{n+1}\Lambda_w^{n+1})/\psi^k(\beta_{2n+1-2j}) \\
&= \rho_k(w)^{n+1}\hat{D}(\psi^k(\beta_{2n+1-2j})).
\end{aligned}$$

Therefore, since $1 \leq j$,

$$\hat{D}(\psi^k(\beta_{2n+1-2j})) = w^j/k^j\rho_k(w)^{n+1} = k^{n+1-j}w^j$$

and the result follows from the formulae of § 6.1.5. □

Corollary 6.1.9. Let k be an odd integer. The KU-theory Adams operation

$$\psi^k : KU_{-1}(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)}) \longrightarrow KU_{-1}(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)})$$

satisfies $\psi^k(\hat{\beta}_{2n-1}) = \hat{\beta}_{2n-1}$.

Proof. This follows from the fact that $\psi^k(B) = kB$ in K-theory and that $\psi^k(\gamma(z)) = \gamma(\psi^k(z))$ where γ is the homology Conner-Floyd map (see § 6.1.3 and Proposition 6.1.7). □

Proposition 6.1.10.

(a) If $k > 1$ is an odd integer then

$$\hat{D}(\psi^k(\beta_{2n+1})) \equiv 1 + \frac{(\rho_k(w)-1)(1-k^{n+1})}{k^n(1-k)} \text{ (modulo } 2^n)$$

in $MU^0(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)})$, where $\rho_k(w)$ is as in Lemma 6.1.8(proof).

(b) If k is an odd integer then, in $KU_{2n+1}(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)})$,

$$\psi^k(\hat{\beta}_{2n+1}) = \hat{\beta}_{2n+1} + \frac{(1-k^{n+1})}{2k^n}\hat{\beta}_{2n-1}.$$

Proof. For part (a), from the proof of Lemma 6.1.8, if $\rho_k(w) = 1 + \hat{\rho}w$, then $\hat{\rho}w^2 = w\frac{(1-k)}{k}$. We have

$$\begin{aligned}
\hat{D}(\psi^k(\beta_{2n+1})) &= (1+\hat{\rho}w)^{-n-1} \\
&\qquad \text{as in Lemma 6.1.8 with j=0,} \\
&= 1 + \textstyle\sum_{j=1}^{n} (-1)^n(\hat{\rho}w)^j \begin{pmatrix} n+j \\ j \end{pmatrix} \\
&\qquad \text{as } w^{n+1} = 0 \text{ by Proposition 6.1.2(b),} \\
&= 1 + \left(\textstyle\sum_{j=1}^{n} (-1)^n \ \frac{(1-k)^{j-1}}{k^{j-1}} \begin{pmatrix} n+j \\ j \end{pmatrix} \right) \hat{\rho}w \\
&\equiv 1 + \left(\textstyle\sum_{j=1}^{\infty} (-1)^n \ \frac{(1-k)^{j-1}}{k^{j-1}} \begin{pmatrix} n+j \\ j \end{pmatrix} \right) \hat{\rho}w \\
&\qquad \text{(modulo } 2^n) \\
&\equiv 1 + \hat{\rho}w\left(\tfrac{(1+x)^{-n-1}-1}{x}\right)_{x=(1-k)/k} \\
&\equiv 1 + (\rho_k(w)-1)\tfrac{(1-k^{n+1})}{k^n(1-k)}
\end{aligned}$$

which proves (a).

To prove part (b) we apply γ to (a) using the facts that $2^n\hat{w} = 0$, by Proposition 6.1.4, and that $\gamma(\rho_k(w)) = 1 + \frac{(k-1)}{2}\hat{w}$ so that

$$\hat{D}(\psi^k(\hat{\beta}_{2n+1})) = 1 + \frac{(1-k^{n+1})}{2k^n}\hat{w} \in KU_*(\mathbb{RP}^{2n+1}; \mathbb{Z}_{(2)}),$$

which proves (b). □

6.1.11. If $E = (e_1, e_2, \ldots)$ is a finitely non-zero sequence of integers $e_j \geq 0$ and $|E| = \sum_j \; je_j$ we have Landweber-Novikov operations

$$S_E : MU_* \longrightarrow MU_{*-2|E|}$$

dual to the stable MU-cohomology operations of ([9] Part I).

We write S_n for $S_{(n,0,0,0,\ldots)}$. If μ is the product on the MU-spectrum we have a commutative diagram, resulting from the Cartan formula.

$$\begin{tikzcd} MU \wedge MU \arrow[r, "\mu"] \arrow[d, "\sum_a S_a \wedge S_{m-a}"] & MU \arrow[d, "S_m"] \\ MU \wedge MU \arrow[r, "\mu"] & MU \end{tikzcd}$$

If $\hat{D}$ is the homomorphism of § 6.1.5 then this diagram implies

$$\begin{array}{ll} S_m(\hat{D}(z)) & = S_m(\Lambda_w^{n+1}/z) \\ & = \sum_{a=0}^m \; (S_a(\Lambda_w^{n+1})/S_{m-a}(z)) \\ & = \sum_{a=0}^m \left(\begin{array}{c} n+1 \\ a \end{array} \right) (w^a \Lambda_w^{n+1}/S_{m-a}(z)), \\ & \sum_{a=0}^m \left(\begin{array}{c} n+1 \\ a \end{array} \right) w^a \hat{D}(S_{m-a}(z)). \end{array}$$

Proposition 6.1.12. If $0 \leq m \leq t$ then

$$S_m(\beta_{2t+1}) = (-1)^m \left(\begin{array}{c} m+t \\ m \end{array} \right) \beta_{2t-2m+1} \in MU_*(\mathbb{RP}^{2n+1}).$$

Proof. It suffices, by naturality, to consider the case when $t = n$. Since $\hat{D}(\beta_{2n+1}) = 1$ we have $S_m(\hat{D}(\beta_{2n+1})) = 0$ for $m > 0$. Therefore § 6.1.11 implies that

$$0 = \sum_{a=0}^m \left(\begin{array}{c} n+1 \\ a \end{array} \right) w^a \hat{D}(S_{m-a}(\beta_{2n+1})).$$

By induction $\hat{D}(S_j(\beta_{2n+1})) = \lambda_j w^j$ and

$$1 = \sum_{m=0}^{n+1} \sum_{a=0}^m \left(\begin{array}{c} n+1 \\ a \end{array} \right) \lambda_{m-a} w^m.$$

Therefore, as $w^{n+1} = 0$, λ_j is the coefficient of w^j in $(1+w)^{-n-1}$ which is

$$(-1)^j \left(\begin{array}{c} j+n \\ j \end{array} \right),$$

which gives the required result, by § 6.1.5. □

Remark 6.1.13. Let $\nu_2(m)$ denote the 2-adic valuation of the integer m. Consider the formula

$$S_{u+1}(\beta_{4u+3}) = (-1)^{u+1} \binom{3u+2}{u+1} \beta_{2u+1}.$$

Since $\nu_2(m!) = m - \alpha(m)$ (see Chapter 5 Proposition 5.2.6), where $\alpha(m)$ is the number of 1's in the dyadic expansion of m, one easily finds that

$$\nu_2 \binom{3u+2}{u+1} = 1 \iff u+1 = 2^j \text{ for some } j.$$

This is related to halving the Whitehead product (see Theorem 6.3.4 and also § 5 and Chapter 7 § 7.2.3).

Lemma 6.1.14. In $MU_*(-;\mathbb{Z}_{(2)})$ and $MU^*(-)$ for each n,

$$\sum_{|E|=n} (k^{|E|}\psi^k S_E - S_E\psi^k) = 0.$$

Proof. It suffices to check this in MU-cohomology. Set $\Gamma = \sum_E S_E$ and $\Phi = \sum_E k^{|E|}S_E$. Then $\Gamma\psi^k$ and $\psi^k\Phi$ are both natural ring homomorphisms so it suffices to show, for $x \in MU^2(\mathbb{CP}^\infty)$ as in § 6.1.1, that $\psi^k(\Phi(x)) = \Gamma(\psi^k(x))$. However,

$$\begin{aligned}
\Gamma(\psi^k(x)) &= \textstyle\sum_E S_E(\frac{k^*(x)}{k}) \\
&= \textstyle\frac{1}{k}\sum_E S_E(k^*(x)) \\
&= \textstyle\frac{1}{k}k^*(x + x^2 + x^3 + \cdots) \\
&= \textstyle\frac{k^*(x)}{k} + k(\frac{k^*(x)}{k})^2 + k^2(\frac{k^*(x)}{k})^3 + \cdots \\
&= \psi^k(x + kx^2 + k^2x^3 + \cdots) \\
&= \psi^k(\Phi(x)),
\end{aligned}$$

as required. $\square$

Lemma 6.1.15. The following diagram commutes:

$$\begin{array}{ccc}
MU_j(X) & \xrightarrow{S_n} & MU_{j-2n}(X) \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\
H_j(X;\mathbb{Z}/2) & \xrightarrow{Sq^{2n}_*} & H_{j-2n}(X;\mathbb{Z}/2)
\end{array}$$

where T is induced by the Thom class.

Proof. We prove the dual result in cohomology. Set $Sq = \sum_{n\geq 0} Sq^{2n}$ and $S = \sum_{n\geq 0} S_n$; then TS and SqT are natural ring homomorphisms so that, as in

Lemma 6.1.14, we have only to check $T(S(x)) = Sq(T(x))$ for $x \in MU^2(\mathbb{CP}^\infty)$. However, $T(S(x)) = T(x + x^2)$ while

$$Sq(T(x)) = Sq(c_1) = c_1 + c_1^2 = T(x) + T(x)^2,$$

as required. □

6.2 BP-theory of $\mathbb{RP}^n$

6.2.1. Let BP denote the 2-adic Brown-Peterson spectrum ([9] pp. 109–116; [294]) whose homotopy, $\pi_*(BP) = BP_*$, is isomorphic to $\mathbb{Z}_2[v_1, v_2, v_3, \ldots]$ where $\mathbb{Z}_2$ denotes the 2-adic integers and $\deg(v_i) = 2(2^i - 1)$. There is a p-adic BP-theory for each prime p which is constructed from the fact that over the p-adics the universal formal group has a canonical summand which is the formal group of BP_* ([225]; see also [9] Part II). We shall be exclusively concerned with the case when $p = 2$.

By virtue of the relation to $MU_*(-;\mathbb{Z}_2)$-theory, much of this section is directly analogous to an alternative treatment of MU-theory results of the previous section.

Then we have $BP^*(\mathbb{CP}^\infty) \cong BP^*[[x]]$ where $BP^* = BP_{-*}$ and $\deg(x) = 2$. The series $[2]x \in BP^*[[x]]$ is defined by $[2]x = f^*(x)$ where $f : \mathbb{CP}^\infty = BS^1 \longrightarrow \mathbb{CP}^\infty$ is induced by the squaring map on the circle, S^1. From ([294] Lemma 3.17 p. 20) we have

$$[2]x \equiv 2x + \sum_{i \geq 1} v_i x^{2^i} \ (\text{modulo } \langle 2, v_1, v_2, v_3, \ldots \rangle^2 BP^*[[x]]).$$

6.2.2. Now consider $BP^*(\mathbb{RP}^{2t}) \cong BP^* \oplus \tilde{BP}^*(\mathbb{RP}^{2t})$. The composition of f with the canonical map, $i : \mathbb{RP}^{2t} \longrightarrow \mathbb{CP}^{2t} \longrightarrow \mathbb{CP}^\infty$, is trivial. Also x^{t+1} is zero in $\tilde{BP}^*(\mathbb{CP}^{2t})$ and there is an induced isomorphism of the form

$$i^* : BP^*[[x]]/\langle x^{t+1}, [2]x \rangle \xrightarrow{\cong} BP^*(\mathbb{RP}^{2t}).$$

This isomorphism is established together with the assertion that every element of $\tilde{BP}^{2m}(\mathbb{RP}^{2t})$ may be written *uniquely* as (the image under i^* of) $\sum_{I,j} \epsilon_I v^I x^j$ where the sum is taken over all sequences of non-negative integers, $I = (i_1, \ldots, i_r)$, $v^I = v_1^{i_1} \ldots v_r^{i_r}$, $2j - \sum_{s=1}^r i_s 2(2^s - 1) = 2m$ and $1 \leq j \leq t$ with each $\epsilon_I = 0$ or 1. To prove both assertions one observes that the Atiyah-Hirzebruch spectral sequence for the reduced group, $\tilde{BP}^*(\mathbb{RP}^{2t})$, collapses because it is concentrated in even total degree. The E_2-term is generated by BP^* and x so that i^* is surjective. Also $E_2^{p,q}$ is zero unless $0 < p \leq 2t$ is even and $q = 2n$ in which case it is isomorphic to $BP^{2n} \otimes \mathbb{Z}/2$. Therefore the order of $\tilde{BP}^{2m}(\mathbb{RP}^{2t})$ is 2^a where $a = a_1 + \cdots + a_t$ and a_j is equal to the number of sequences, I, such that $2j - \sum_{s=1}^r i_s 2(2^s - 1) = 2m$.

This is also the number of expressions of the form $\sum_{I,j} \epsilon_I v^I x^j$ in dimension $2m$. On the other hand, the form of $[2]x$ shows that every element of $\tilde{BP}^{2m}(\mathbb{RP}^{2t})$ may be written in at least one way in the desired form. Hence, by counting group orders, this expression must be unique and i^* must be an isomorphism.

6.2.3. The S-dual of $\mathbb{RP}^{2t}$ is homotopy equivalent to

$$\Sigma^{1-2^i}\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2}$$

for i sufficiently large, by ([116] pp. 205–208), and the previous discussion yields short exact sequences of the form

$$\begin{aligned} 0 \longrightarrow BP^{2^i-2h}&(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2}) \\ &\longrightarrow BP^{2^i-2h}(\mathbb{RP}^{2^i-2}) \longrightarrow BP^{2^i-2h}(\mathbb{RP}^{2^i-2t-2}) \longrightarrow 0. \end{aligned}$$

In addition, we have S-duality isomorphisms [116] of the form

$$\begin{aligned} &BP^{2^i-2h}(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2}) \\ &\cong BP^{1-2h}(\Sigma^{1-2^i}\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2}) \\ &\cong BP_{2h-1}(\mathbb{RP}^{2t}). \end{aligned}$$

For $1 \leq h \leq t$ the element, $x^{2^{i-1}-h} \in BP^{2^i-2h}(\mathbb{RP}^{2^i-2})$ maps to zero in [5] $BP^{2^i-2h}(\mathbb{RP}^{2^i-2t-2})$ and we may define $x_{2h-1} \in BP_{2h-1}(\mathbb{RP}^{2t})$ to be equal to the image of $x^{2^{i-1}-h} \in BP^{2^i-2h}(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2})$ under the S-duality isomorphism.

Every element of $\tilde{BP}^{2^i-2s}(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2t-2})$ is uniquely writeable in the form $\sum_{I,j} \epsilon_I v^I x^j$ with $2^{i-1} - t \leq j \leq 2^{i-1} - 1$ and each $\epsilon_I \in \{0, 1\}$, Hence every element of $\tilde{BP}_{2s-1}(\mathbb{RP}^{2t}) = BP_{2s-1}(\mathbb{RP}^{2t})$ is uniquely writeable in the form $\sum_{I,k} \epsilon_I v^I x_{2k+1}$ with $1 \leq 2k + 1 \leq 2t - 1$ and each $\epsilon_I \in \{0, 1\}$. The relation that $x^{2^{i-1}-h-1} \cdot [2]x = 0$ translates into a congruence of the form

$$\begin{aligned} 2x_{2h-1} + &\textstyle\sum_{j\geq 1} v_j x_{2h-2^{j+1}+1} \\ &\equiv 0 \ (\text{modulo } \langle 2, v_1, v_2, v_3, \ldots\rangle^2 BP_*(\mathbb{RP}^{2t})). \end{aligned}$$

Recall ([9] p. 89) that if X is a commutative ring spectrum with unit, $\iota : S^0 \longrightarrow X$, there are two maps, $\eta_L = 1 \wedge \iota$ and $\eta_R = \iota \wedge 1$, from X to $X \wedge X$ which give $\pi_*(X \wedge X) = (X \wedge X)_*$ the structure of a left or right $\pi_*(X)$-module, respectively. When $X = BP$ there exist canonical elements, $t_i \in (BP \wedge BP)_{2(2^i-1)}$, ([9] Theorem 16.1 p. 112; [294] Theorem 3.11 p. 17) such that

$$(BP \wedge BP)_* \cong BP_*[t_1, t_2, t_3, \ldots]$$

as a left BP_*-module. From the collapsed Atiyah-Hirzebruch spectral sequence for $(BP \wedge BP)_*(\mathbb{RP}^{2t})$ there is an isomorphism of left BP_*-modules of the form

$$\begin{aligned} &(BP \wedge BP)_*(\mathbb{RP}^{2t}) \\ &\cong (BP \wedge BP)_* \otimes_{BP_*} BP_*(\mathbb{RP}^{2t}) \\ &\cong BP_*(\mathbb{RP}^{2t})_*[t_1, t_2, t_3, \ldots]. \end{aligned}$$

Therefore every element of $(BP \wedge BP)_{2s-1}(\mathbb{RP}^{2t})$ is uniquely writeable in the form $\sum_{I,I',k} \epsilon_I v^I t^{I'} x_{2k+1}$ with $1 \leq 2k+1 \leq 2t-1$ and each $\epsilon_I \in \{0,1\}$. Here $t^{I'} = t^{(i'_1, \ldots, i'_r)}$ denotes $t_1^{i'_1} \ldots t_r^{i'_r}$.

6.3 Application to the Whitehead product $[\iota_n, \iota_n]$

6.3.1. Recall the following constructions from Chapter 1 § 1.5.4. Choose a relative homeomorphism $h : (D^n, S^{n-1}) \longrightarrow (S^n, pt)$ then

$$[\iota_n, \iota_n] : S^{2n-1} \longrightarrow S^n$$

is given by the map sending

$$(x, y) \in S^{n-1} \times D^n \bigcup D^n \times S^{n-1} \cong S^{2n-1}$$

to the appropriate one of $h(x)$ or $h(y)$.

Define an involution on $S^{n-1} \times D^n \bigcup D^n \times S^{n-1}$ by $\tau(x,y) = (y,x)$ so that the orbit space satisfies

$$S^{2n-1}/\mathbb{Z}/2 \cong \Sigma^n \mathbb{RP}^{n-1}$$

and the map $[\iota_n, \iota_n]$ factorises through the orbit space to induce a map

$$w_n : \Sigma^n \mathbb{RP}^{n-1} \longrightarrow S^n.$$

Taking adjoints we obtain maps

$$\lambda_n = \mathrm{adj}([\iota_n, \iota_n]) : \Sigma S^{n-1} = S^n \longrightarrow \Omega^{n-1} S^n$$

and

$$k_n = \mathrm{adj}(w_n) : \Sigma \mathbb{RP}^{n-1} \longrightarrow \Omega^{n-1} S^n.$$

The map k_n is the subject of the Kahn-Priddy theorem which was treated in Chapter 1, § 5 (see also [134], [8], [135], [136]).

If $\pi_{n-1} : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ is the standard quotient map then

$$k_n \cdot \Sigma \pi_{n-1} = \lambda_n : \Sigma S^{n-1} \longrightarrow \Omega^{n-1} S^n.$$

The following result, which uses the KU_*-theory formulae of Proposition 6.1.10(b), first appeared in the book review [248] where it was used to give a (then) new proof of the non-existence of elements of Hopf invariant one.

Theorem 6.3.2. The Whitehead product $[\iota_n, \iota_n]$ is nullhomotopic if and only if $n = 1, 3$ or 7.

Proof. The Whitehead product is trivial when S^n is an H-space [293], which happens when $n = 1, 3$ or 7.

Conversely, if $[\iota_n, \iota_n]$ is trivial then, by § 6.3.1 and Chapter 1, Theorem 1.5.6, the canonical map $\Sigma^\infty \pi_{n-1} : \Sigma^\infty S^{n-1} \longrightarrow \Sigma^\infty \mathbb{RP}^{n-1}$ is 2-locally, stably nullhomotopic. This is because composition with the 2-local stable homotopy equivalence

$$\Sigma^\infty \Sigma \mathbb{RP}^{n-1} \stackrel{\Sigma^\infty k_n}{\longrightarrow} \Sigma^\infty \Omega^{n-1} S^n \stackrel{\rho_n}{\longrightarrow} \Sigma^\infty \Sigma \mathbb{RP}^{n-1}$$

is nullhomotopic. Therefore there is a 2-local stable homotopy equivalence

$$\Sigma^\infty \mathbb{RP}^n \simeq \Sigma^\infty \mathrm{Cone}(\pi_{n-1}) \simeq \Sigma^\infty \mathbb{RP}^{n-1} \vee \Sigma^\infty S^n.$$

By Proposition 6.1.7 this is impossible unless n is odd.

Suppose that $n = 2t + 1$. Then, in

$$\begin{aligned} &KU_{2t+1}(\Sigma^\infty \mathbb{RP}^{2t} \vee \Sigma^\infty S^{2t+1}) \\ &\cong KU_{2t+1}(\Sigma^\infty \mathbb{RP}^{2t}) \oplus KU_{2t+1}(\Sigma^\infty S^{2t+1}), \end{aligned}$$

by Corollary 6.1.9 for any odd integer k,

$$\psi^k(\hat{\beta}_{2t+1} + u\hat{\beta}_{2t-1}) = \hat{\beta}_{2t+1} + u\hat{\beta}_{2t-1}$$

for some integer u. However, by Proposition 6.1.10(b) this means that we must have

$$k^{t+1} \equiv 1 \text{ (modulo } 2^{t+1}).$$

Taking $k = 3$ this can only happen if $t = 0, 1$ or 3, by Chapter 5, Proposition 5.2.3 □

6.3.3. 2-divisibility of the Whitehead product. If the Whitehead product $[\iota_n, \iota_n]$ were trivial, as hypothesised in Theorem 6.3.2 then it would of course be divisible by 2 (or any other positive integer for that matter!) in $\pi_{2n-1}(S^n)$. However, Theorem 6.3.2 shows that this vanishing rarely happens. As a weaker question one might ask: "Does there exist $g \in \pi_{2n-1}(S^n)$ such that $2g = [\iota_n, \iota_n]$?" The next result shows that such 2-divisibility is quite rare and that 4-divisibility is impossible. This result was known to Michael Barratt and Mark Mahowald, at least by many sample calculations, in the 1970's and it was proved in one direction in the early 1980's in a collaboration with John Jones which eventually appeared as [30] where they conjecture that the converse is true. Theorem 6.3.4 shows that their conjecture was correct.

This section seems to be the correct place to include this result but in order to prove it we shall need the main result of Chapter 8 (Theorem 8.1.2) , which is the chapter containing my principal applications of the "upper triangular technology" of Chapter 3.

Theorem 6.3.4. Let m be a positive integer. Suppose that $g \in \pi_{16m-3}(S^{8m-1})$ satisfies $2g = [\iota_{8m-1}, \iota_{8m-1}]$. Then

(i) this can only happen when $8m = 2^{j+1}$ in which case

(ii) the composition $f = \pi_{2^{j+1}-2,2} \cdot \Sigma^\infty(\mathrm{adj}(g))$,

$$f : \Sigma^\infty S^{2^{j+1}-1} \longrightarrow \Sigma^\infty \Omega^{2^{j+1}-2} S^{2^{j+1}-1} \longrightarrow \Sigma^\infty \Sigma \mathbb{RP}^{2^{j+1}-2},$$

is detected by Sq^{2^j} on $\mathrm{Cone}(f)$.

(iii) In addition, $[\iota_{8m-1}, \iota_{8m-1}]$ is never divisible by 4.

Here $\pi_{2^{j+1}-2,2}$ is the quadratic part of the Snaith splitting, as in Chapter 1 § 1.5.4.

Proof. If $2g = [\iota_{8m-1}, \iota_{8m-1}]$ then, in the notation of § 6.3.1 and Chapter 1 § 1.5.4, $\mathrm{adj}(2g) = \lambda_{8m-1}$ and

$$2\,\mathrm{adj}(g) = \lambda_{8m-1} = k_{8m-1} \cdot \Sigma\pi_{8m-2} : \Sigma S^{8m-2} \longrightarrow \Omega^{8m-2} S^{8m-1}.$$

By Chapter 1, Theorem 1.5.6 the map

$$\Sigma^\infty \Sigma \mathbb{RP}^{n-1} \xrightarrow{\Sigma^\infty k_n} \Sigma^\infty \Omega^{n-1} S^n \xrightarrow{\pi_{n-1,2}} \Sigma^\infty \Sigma \mathbb{RP}^{n-1}$$

is a 2-local stable homotopy equivalence for $3 \leq n \leq \infty$. This 2-local equivalence induces an isomorphism of the form

$$\begin{aligned} KU_{8m-1}(\Sigma^{-1}\mathrm{Cone}(2f); \mathbb{Z}_{(2)}) &\cong KU_{8m-1}(\mathrm{Cone}(\pi_{8m-1}); \mathbb{Z}_{(2)}) \\ &= KU_{8m-1}(\mathbb{RP}^{8m-1}; \mathbb{Z}_{(2)}). \end{aligned}$$

By Proposition 6.1.10(b) the effect of the Adams operation ψ^3 on

$$KU_{8m-1}(\Sigma^{-1}\mathrm{Cone}(2f); \mathbb{Z}_{(2)})$$

must satisfy

$$\psi^3(\hat{\beta}_{8m-1}) = \hat{\beta}_{8m-1} + \frac{(1 - 3^{4m})}{2 \cdot 3^{4m-1}} \hat{\beta}_{8m-3}.$$

Both $KU_{8m-1}(\Sigma^{-1}\mathrm{Cone}(2f); \mathbb{Z}_{(2)})$ and $KU_{8m-1}(\Sigma^{-1}\mathrm{Cone}(f); \mathbb{Z}_{(2)})$ are isomorphic to $\mathbb{Z}_{(2)}\langle\hat{\beta}_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-1}\langle\hat{\beta}_{8m-3}\rangle$ where $\hat{\beta}_{8m-1}$ is any element mapping to the generator of $KU_{8m-1}(S^{8m-1}; \mathbb{Z}_{(2)})$. Therefore the Adams operation on $KU_{8m-1}(\Sigma^{-1}\mathrm{Cone}(f); \mathbb{Z}_{(2)})$ must satisfy

$$\psi^3(\hat{\beta}_{8m-1}) = \hat{\beta}_{8m-1} + \frac{(3^{4m} - 1)}{4}(2s+1)\hat{\beta}_{8m-3}$$

for some integer s.

However, Chapter 8, Theorem 8.1.2 shows that this can only happen when m is a power of 2, which proves part (i), and that when $m = 2^{j-2}$ it is equivalent to f

being detected by Sq^{2^j} on Cone(f), proving part (ii). Finally, if $[\iota_{8m-1}, \iota_{8m-1}]$ were divisible by 2^{2+e} with $e \geq 0$ then ψ^3 on $KU_{-1}(\Sigma^{-1}\mathrm{Cone}(f); \mathbb{Z}_{(2)})$ must satisfy

$$\psi^3(\hat{\beta}_{8m-1}) = \hat{\beta}_{8m-1} + \frac{(3^{4m}-1)}{2^{3+e}}(2s+1)\hat{\beta}_{8m-3}$$

for some integer s. The proof of Chapter 8, Theorem 8.1.2 shows that this leads to a contradiction similar to that in part (i). □

6.4 bo, bu and bspin e-invariants

6.4.1. This section is concerned with the relation between e-invariants and Hopf invariants. I will address the same situation as Corollary D of [30]. In fact Theorem 6.4.2 implies both Corollary D of [30] and its converse, as conjectured in [30] (see Corollary 6.4.3).

Firstly recall the notation of [30]. For $k \geq 0$, $s(k)$ is given by the formulae

$$s(0) = 0, s(1) = 1, s(2) = 3, s(3) = 7 \text{ and } s(4a+b) = 8a + s(b).$$

The kth non-trivial homotopy group of SO, the infinite special orthogonal group, is $\pi_{s(k)}(SO)$.

Suppose that we are given $\alpha \in \pi_{2^{k+2}-3-s(k)}(S^{2^{k+1}-1-s(k)})$; then we may form the Hopf invariant [293] of α,

$$H(\alpha) \in \pi_{2^{k+2}-3-s(k)}(S^{2^{k+2}-1-2s(k)})$$

whose construction is explained in § 6.4.5.

Consider the image of $H(\alpha)$ in the stable homotopy group

$$H(\alpha) \in \pi_{s(k)}(\Sigma^\infty S^0) \cong (\mathrm{ImJ})_{s(k)} \oplus (?)$$

where the left-hand summand is the image of the classical J-homomorphism [5]. On the other hand we can consider the adjoint of α,

$$\mathrm{adj}(\alpha) : S^{2^{k+1}-1} \longrightarrow \Omega^\infty\Sigma^\infty S^1$$

and compose this with $\pi_{\infty,2}$ of Chapter 1 Theorem 1.5.6 to obtain an S-map

$$\pi_{\infty,2} \cdot \mathrm{adj}(\alpha) : \Sigma^\infty S^{2^{k+1}-2} \longrightarrow \Sigma^\infty\mathbb{RP}^\infty.$$

We may choose a skeletal approximation to $\pi_{\infty,2} \cdot \mathrm{adj}(\alpha)$ denoted by

$$\rho(\alpha) : \Sigma^\infty S^{2^{k+1}-2} \longrightarrow \Sigma^\infty\mathbb{RP}^{2^{k+1}-2}.$$

The KU_*-e-invariant is defined using the Adams operation ψ^3 in the following manner (compare [5]). Form the mapping cone, $\mathrm{Cone}(\rho(\alpha))$, which satisfies

$$KU_{2^{k+1}-1}(\mathrm{Cone}(\rho(\alpha)); \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}\langle \tau \rangle \oplus \mathbb{Z}/2^{2^k-1}\langle \tau' \rangle$$

where τ maps to the generator of $KU_{2^{k+1}-1}(S^{2^{k+1}-1}; \mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}$. Then

$$\psi^3(\tau) = \tau + e(\rho(\alpha))\tau'$$

(compare Proposition 6.1.10(b)) and the KU_*-e-invariant of $\rho(\alpha)$ is

$$e(\rho(\alpha)) \in \mathbb{Z}/2^{2^k-1}.$$

The following result will be proved in § 6.4.6 (see also the related second proof of Chapter 7, Theorem 7.2.2 which is given in Chapter 7 § 7.2.4).

Theorem 6.4.2. The $(\mathrm{ImJ})_{s(k)}$-component of $H(\alpha)$ is a generator if and only if the KU_*-e-invariant of $\rho(\alpha)$ is of the form

$$\frac{(3^{2^k}-1)}{4}(2u+1) \in \mathbb{Z}/2^{2^k-1} \cong KU_{2^{k+1}-1}(\mathbb{RP}^{2^{k+1}-2}).$$

Corollary 6.4.3 (Hopf invariant vs. Arf-Kervaire invariant)**.** In the notation of § 6.4.1 the $(\mathrm{ImJ})_{s(k)}$-component of $H(\alpha)$ is a generator if and only if $\rho(\alpha)$ is detected by Sq^{2^k} on the mod 2 cohomology of $\mathrm{Cone}(\rho(\alpha))$.

Proof. Chapter 8, Theorem 8.1.2, which is my main example of the "upper triangular technology" of Chapter 3 in action, shows that detection by Sq^{2^k} is equivalent to the e-invariant condition of Theorem 6.4.2. □

6.4.4. Theorem 6.4.2 is simple enough to prove. However, for completeness, I shall give a brief sketch of the background concerning bo_*, bu_* and $bspin_*$ which is needed in order to understand from [5] how ImJ is detected.

Throughout this section all spectra will be 2-localised. Also all KU-theory and connective K-theory e-invariants will be calculated by means of the operation $\psi^3 - 1$.

Recall (see Chapter 1, Example 1.3.2(v)) that the bu spectrum is given by

$$\begin{gathered} bu = \{\mathbb{Z} \times BU, U, BU, U, BU, SU, BU\langle 4, \infty\rangle, \\ \Omega BU\langle 4, \infty\rangle, BU\langle 6, \infty\rangle, \ldots\} \end{gathered}$$

where $BU\langle 2n, \infty\rangle$ denotes the $(2n-1)$-connected cover of BU. The structure maps of the spectrum $\epsilon : \Sigma^2 bu_{2j} \longrightarrow bu_{2j+2}$ make the following diagram homotopy commute:

$$\begin{array}{ccc} \Sigma^2 BU\langle 2j, \infty\rangle & \xrightarrow{\epsilon} & BU\langle 2j+2, \infty\rangle \\ \downarrow & & \downarrow \\ \Sigma^2 BU & \xrightarrow{(B\cdot -)} & BU \end{array}$$

where $(B\cdot -)$ is multiplication by the Bott generator in $\pi_2(BU)\cong\mathbb{Z}$. Thus, if X is a CW complex,

$$bu_m(X) = \lim_{\vec{n}} \ \pi_{2n+m}(X\wedge bu_{2n})$$

which is zero if $m<0$. Comparing the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,v} = H_s(\mathbb{RP}^{8t-2}; bu_v(S^0)) \Longrightarrow bu_{s+v}(\mathbb{RP}^{8t-2})$$

with the collapsing Atiyah-Hirzebruch spectral sequence for KU_*-theory shows that

$$bu_{2j}(\mathbb{RP}^{8t-2}) = 0 \ \text{ and}$$
$$bu_{8t-1}(\mathbb{RP}^{8t-2}) \xrightarrow{\cong} KU_{8t-1}(\mathbb{RP}^{8t-2}) \cong \mathbb{Z}/2^{4t-1}.$$

Similarly $bu_{2s+1}(\mathbb{RP}^{8t-2}) \longrightarrow KU_{2s+1}(\mathbb{RP}^{8t-2})$ is injective.

The Adams operation $\psi^3 : BU \longrightarrow BU$ yields a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma^2 BU \arrow[r, "(B\cdot -)"] \arrow[d, "\Sigma^2\psi^3"] & BU \arrow[d, "\frac{1}{3}\psi^3"'] \\
\Sigma^2 BU \arrow[r, "(B\cdot -)"] & BU
\end{tikzcd}$$

since $\psi^3(B) = 3B$. Hence we obtain $\psi^3 : bu \longrightarrow bu$ given by

$$\left\{ \frac{1}{3^n}\psi^3 : BU\langle 2n,\infty\rangle \longrightarrow BU\langle 2n,\infty\rangle \right\}.$$

Hence ψ^3 is the identity on $bu_{8t-1}(S^{8t-1})$.

Suppose that we have a map $f : S^{M+8t-2} \longrightarrow \Sigma^M\mathbb{RP}^{8t-2}$ such that, if $\tau\in KU_{M+8t-1}(\mathrm{Cone}(f))$ maps to a generator of $KU_{M+8t-1}(S^{M+8t-1})$,

$$(\psi^3-1)(\tau) = \lambda \in KU_{M+8t-1}(\Sigma^M\mathbb{RP}^{8t-2}) \cong \mathbb{Z}/2^{4t-1}.$$

Then, in $bu_{M+8t-1}(\mathrm{Cone}(f))$,

$$(\psi^3-1)(\tau) = \lambda \in bu_{M+8t-1}(\Sigma^M\mathbb{RP}^{8t-2}) \cong \mathbb{Z}/2^{4t-1}.$$

The spectrum bo is defined in the same manner as bu but with $bo_0 = \mathbb{Z}\times BO$ and $bo_{8n} = BO\langle 8n,\infty\rangle$ for $n>0$. The complexification map $BO\longrightarrow BU$ induces an isomorphism $bo_{8t-1}(\mathbb{RP}^{8t-2}) \xrightarrow{\cong} BU_{8t-1}(\mathbb{RP}^{8t-2})$ so that in $bo_{M+8t-1}(\mathrm{Cone}(f))$

$$(\psi^3-1)(\tau) = \lambda \in bo_{M+8t-1}(\mathrm{Cone}(f)) \cong \mathbb{Z}/2^{4t-1}.$$

Now consider $bspin$ given by $bspin_{8n} = BSpin\langle 8n,\infty\rangle$. Then ψ^3-1 is trivial on $\pi_1(BO)=\mathbb{Z}/2$ and on $\pi_2(BO)=\mathbb{Z}/2$ so that it lifts to yield the following homotopy commutative diagram.

$$\begin{array}{ccc} BO & \xrightarrow{\Psi} & BSpin = BO\langle 3,\infty\rangle \\ \downarrow 1 & & \downarrow \\ BO & \xrightarrow{\psi^3 - 1} & BO \end{array}$$

A similar calculation, using the Atiyah-Hirzebruch spectral sequence, shows that

$$bspin_{8t-1}(\mathbb{RP}^{8t-2}) \cong \mathbb{Z}/2^{4t-3}$$

which injects into $bo_{8t-1}(\mathbb{RP}^{8t-2})$. Therefore, in $bspin_{M+8t-1}(\mathrm{Cone}(f))$,

$$\begin{aligned} \Psi(\tau) = \lambda/4 \quad & \in bspin_{M+8t-1}(\mathrm{Cone}(f)) \\ & \cong bspin_{M+8t-1}(\Sigma^M \mathbb{RP}^{8t-2}) \\ & \cong \mathbb{Z}/2^{4t-3}. \end{aligned}$$

We need these calculations because the ImJ-component of $\pi_*(\Sigma^\infty S^0)_{(2)}$ is detected faithfully by the Ψ-e-invariant (see [5], [30]).

6.4.5. The Hopf invariant. The Hopf invariant $\alpha \mapsto H(\alpha)$ is induced by composition with a map $H : \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$ whose adjoint is the second component of the splitting ([117], [193]; (see also [243])

$$\rho' : \Sigma\Omega S^{n+1} \longrightarrow \Sigma(S^n \vee S^{2n} \vee S^{3n} \vee \cdots).$$

From the compatibility, as m varies, of the stable splitting of $\Omega^m\Sigma^m S^n$ ([243], see also [60] and [61]) we obtain a commutative diagram in which $N = 2^{k+1} - 2$.

$$\begin{array}{ccc} \pi_{2N-s(k)+1}(S^{N-s(k)+1}) & \xrightarrow{\Sigma} & \pi_{N+1}(\Sigma^\infty S^1) \\ \downarrow H & & \downarrow (\pi_{\infty,2})_* \\ \pi_{2N-s(k)+1}(S^{2N-2s(k)+1}) & & \pi_{N+1}(\Sigma^\infty \Sigma\mathbb{RP}^\infty) \\ \downarrow & & \downarrow c_* \\ \pi_{N+1}(\Sigma^\infty S^{N-s(k)+1}) & \xrightarrow{\delta_*} & \pi_{N+1}(\Sigma^\infty\Sigma\frac{\mathbb{RP}^\infty}{\mathbb{RP}^{N-s(k)-1}}) \end{array}$$

In this diagram $\delta_* : \pi_{s(k)}(\Sigma^\infty S^0) \longrightarrow \pi_{N+1}(\Sigma^\infty\Sigma\frac{\mathbb{RP}^\infty}{\mathbb{RP}^{N-s(k)-1}})$ is induced by the inclusion of the bottom cell and c_* is induced by the canonical collapsing map.

If α is as in § 6.4.1, set $g = \pi_{\infty,2}(\Sigma(\alpha))$; then we have the following homotopy commutative diagram of S-maps.

$$\begin{tikzcd}
\Sigma^\infty \mathbb{RP}^N \arrow[r] \arrow[d, "c"] & \mathrm{Cone}(g) \arrow[r] \arrow[d] & \Sigma^\infty S^{N+1} \arrow[d, "1"'] \\
\Sigma^\infty \frac{\mathbb{RP}^N}{\mathbb{RP}^{N-s(k)-1}} \arrow[r] & \mathrm{Cone}(cg) \arrow[r] & \Sigma^\infty S^{N+1} \\
\Sigma^\infty S^{N-s(k)} \arrow[r] \arrow[u, "\delta"] & \Sigma^\infty \mathrm{Cone}(H(\alpha)) \arrow[r] \arrow[u] & \Sigma^\infty S^{N+1} \arrow[u, "1"']
\end{tikzcd}$$

6.4.6. Proof of Theorem 6.4.2. I shall now prove Theorem 6.4.2 by examining the diagram of § 6.4.4 in the various cases $N = 2^{k+1}-2$ and $k \equiv 1, 2, 3$ or 4 (modulo 4).

Case I: $k = 4u$, $s(k) = 8u$, $N = 2^{4u+1} - 2$

In this case $bspin_{N+1}(\mathbb{RP}^N) \cong \mathbb{Z}/2^{2^{4u}-3}$ and from the Atiyah-Hirzebruch spectral sequence $bspin_{N+1}(\frac{\mathbb{RP}^N}{\mathbb{RP}^{N-8u-1}})$ is the quotient group $\mathbb{Z}/2^{4u-1}$.

Hence, in $bspin_{N+1}(\mathrm{Cone}(g))$, if

$$\Psi(\tau) = \frac{(3^{2^k} - 1)}{16}(2v+1) = (2w+1)2^{k-2} = (2w+1)2^{4u-2}$$

the e-invariant in Cone(cg) is given by $\Psi(\tau)$ being the non-zero element of order two, which is the image of the generator under the injection

$$\mathbb{Z}/2 \cong bspin_{N+1}(S^{N-8u}) \longrightarrow bspin_{N+1}(\frac{\mathbb{RP}^N}{\mathbb{RP}^{N-s(k)-1}}).$$

Hence, in $bspin_{N+1}(\mathrm{Cone}(H(\alpha)))$, $\Psi(\tau)$ generates $bspin_{N+1}(S^{N-s(k)})$, which means that the ImJ-component of $H(\alpha)$ is a generator. Clearly each step of the argument is reversible, which completes the proof in Case I.

In the following cases the argument is similar so I shall merely tabulate the appropriate facts.

Case II: $k = 4u+1$, $s(k) = 8u+1$, $N = 2^{4u+2} - 2$

$$bspin_{N+1}(\mathbb{RP}^N) \cong \mathbb{Z}/2^{2^{4u+1}-3}, \quad bspin_{N+1}(\tfrac{\mathbb{RP}^N}{\mathbb{RP}^{N-8u-2}}) \cong \mathbb{Z}/2^{4u}$$
and $\quad \Psi(\tau) = (2w+1)2^{4u-1} \in \mathbb{Z}/2^{4u}$.

Case III: $k = 4u+2$, $s(k) = 8u+3$, $N = 2^{4u+3} - 2$

$$bspin_{N+1}(\mathbb{RP}^N) \cong \mathbb{Z}/2^{2^{4u+2}-3}, \quad bspin_{N+1}(\tfrac{\mathbb{RP}^N}{\mathbb{RP}^{N-8u-4}}) \cong \mathbb{Z}/2^{4u+1}$$
and $\quad \Psi(\tau) = (2w+1)2^{4u} \in \mathbb{Z}/2^{4u+1}$

which is the image of the generator of $bspin_{N+1}(S^{N-s(k)}) \cong \mathbb{Z}_{(2)}$ under δ_*.

Case IV: $k = 4u+3$, $s(k) = 8u+7$, $N = 2^{4u+4} - 2$

$$\begin{array}{l} bspin_{N+1}(\mathbb{RP}^N) \cong \mathbb{Z}/2^{2^{4u+3}-3}, \quad bspin_{N+1}(\frac{\mathbb{RP}^N}{\mathbb{RP}^{N-8u-8}}) \cong \mathbb{Z}/2^{4u+2} \\ \text{and} \quad \Psi(\tau) = (2w+1)2^{4u+1} \in \mathbb{Z}/2^{4u+2} \end{array}$$

which is the image of the generator of $bspin_{N+1}(S^{N-s(k)}) \cong \mathbb{Z}_{(2)}$ under δ_*. □

6.5 MU_*-e-invariants and Arf-Kervaire invariants

In the previous section I examined various connective K-theory e-invariants associated to stable homotopy classes of maps of the form $f : \Sigma^\infty S^{4m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{4m-2}$. In this section I shall show how to extract a little information from the very complicated formulae which are provided by the MU_*-theory e-invariant. Once again the key ingredient will be Theorem 8.1.2 of Chapter 8, which is the chapter containing my principal applications of the "upper triangular technology" of Chapters 3 and 5.

6.5.1. Consider the group $MU_{2m-1}(\mathbb{RP}^{4m-2})$ which, by § 6.1.5 and Proposition 6.1.6, is generated over $MU_*(pt)$ by $\beta_1, \ldots, \beta_{2m-1}$ subject to the relations of the following form:

$$\begin{array}{ll} 0 & = 2\beta_{2m-1} + a_{1,1}\beta_{2m-3} + \sum_{i,j\geq 1,\ i+j\geq 3} \ a_{i,j}\beta_{2m+1-2i-2j}, \\ 0 & = 2\beta_{2m-3} + a_{1,1}\beta_{2m-5} + \cdots , \\ & \vdots \quad \vdots \quad \vdots \\ & \vdots \quad \vdots \quad \vdots \\ 0 & = 2\beta_1. \end{array}$$

Write $MU_{2s}(pt) = \mathbb{Z}\langle a_{1,1}^s \rangle \oplus A_s$ where

$$A_s = \mathrm{Ker}(\gamma : MU_{2s}(pt) \longrightarrow KU_{2s}(pt) \cong \mathbb{Z}).$$

Then

$$\begin{array}{l} MU_{2m-1}(\mathbb{RP}^{4m-2}) \\ = \mathbb{Z}/2^m\langle \beta_{2m-1} \rangle \oplus (\oplus_{s=2}^{m-1} \ A_s \otimes \mathbb{Z}/2^{m-s}\langle \beta_{2m-2s-1} \rangle) \end{array}$$

since $\gamma(a_{i,j}) = 0$ for $i+j \geq 3$. Similarly

$$\begin{array}{l} MU_{4m-1}(\mathbb{RP}^{4m-2}) \\ = \mathbb{Z}/2^{2m-1}\langle a_{1,1}\beta_{4m-3} \rangle \oplus (\oplus_{s=2}^{2m-1} \ A_s \otimes \mathbb{Z}/2^{2m-s}\langle \beta_{4m-2s-1} \rangle). \end{array}$$

6.5.2. Now suppose we have an S-map

$$f : \Sigma^\infty S^{4m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{4m-2}.$$

Then we may study the following diagrams, in which $\tilde{MU}_*$ denotes reduced MU-homology with $\mathbb{Z}_{(2)}$-coefficients $\tilde{MU}(-;\mathbb{Z}_{(2)})$.

$$\begin{tikzcd}
\widetilde{MU}_{4m-1}(\mathbb{RP}^{4m-2}) \arrow[r] \arrow[d, "S_m"] & \widetilde{MU}_{4m-1}(\mathrm{Cone}(f)) \arrow[r] \arrow[d, "S_m"'] & \widetilde{MU}_{4m-1}(S^{4m-1}) \\
\widetilde{MU}_{2m-1}(\mathbb{RP}^{4m-2}) \arrow[r] & \widetilde{MU}_{2m-1}(\mathrm{Cone}(f)) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widetilde{MU}_{4m-1}(\mathbb{RP}^{4m-2}) \arrow[r] \arrow[d, "\psi^3-1"] & \widetilde{MU}_{4m-1}(\mathrm{Cone}(f)) \arrow[r] \arrow[d, "\psi^3-1"'] & \widetilde{MU}_{4m-1}(S^{4m-1}) \arrow[d, "\psi^3-1"'] \\
\widetilde{MU}_{4m-1}(\mathbb{RP}^{4m-2}) \arrow[r] & \widetilde{MU}_{4m-1}(\mathrm{Cone}(f)) \arrow[r] & \widetilde{MU}_{4m-1}(S^{4m-1})
\end{tikzcd}$$

Choose $T \in \tilde{MU}_{4m-1}(\mathrm{Cone}(f);\mathbb{Z}_{(2)})$ which maps to the fundamental class of S^{4m-1}. Then

$$\psi^3(T) - T = \lambda a_{1,1}\beta_{4m-3} + z$$

where $\lambda \in \mathbb{Z}_{(2)}$ and $z = \sum_{j=2}^{2m-1} \alpha_j\beta_{4m-1-2j}$ with $\alpha_j \in A_j$. In this relation z is well defined modulo $(3^{2m}-1)MU_{4m-1}(\mathbb{RP}^{4m-2};\mathbb{Z}_{(2)})$.

Also we have

$$S_m(T) = \mu\beta_{2m-1} + \sum_{j=2}^{m-1} x_j\beta_{2m-1-2j}$$

where $\mu \in \mathbb{Z}_{(2)}$ and $x_j \in A_j$.

These equations are related by $3^{2m}\psi^3(S_m(T)) = S_m(\psi^3(T))$ from Lemma 6.1.14, which implies

$$(3^{2m}-1)S_m(T) = S_m(\psi^3-1)(T)$$

by Lemma 6.1.8. The right-hand side of this relation is equal to the following:

$$\begin{array}{l}
\lambda S_m(a_{1,1}\beta_{4m-3}) + \sum_{j=2}^{2m-1} S_m(\alpha_j\beta_{4m-1-2j}) \\
= (-1)^{m-1}2\lambda \left(\begin{array}{c} 3m-3 \\ m-1 \end{array} \right) \beta_{2m-1} + (-1)^m a_{1,1} \left(\begin{array}{c} 3m-2 \\ m \end{array} \right) \beta_{2m-3} \\
+ \sum_{j=2}^{2m-1} \sum_{a=0}^{m} (-1)^{m-a} \left(\begin{array}{c} 3m-1-j-a \\ m-a \end{array} \right) S_a(\alpha_j)\beta_{2m-2j+2a-1}.
\end{array}$$

From this calculation and that of § 6.1.5 we obtain the following relation in $MU_{2m-1}(\mathbb{RP}^{4m-2};\mathbb{Z}_{(2)})$.

Proposition 6.5.3.

$$\begin{aligned}
&(3^{2m}-1)(\mu\beta_{2m-1}+\textstyle\sum_{j=2}^{m-1}\ x_j\beta_{2m-1-2j}) \\
&= (-1)^m 2\lambda \begin{pmatrix} 3m-2 \\ m \end{pmatrix} \beta_{2m-1} \\
&\quad +(-1)^{m+1}\begin{pmatrix} 3m-2 \\ m \end{pmatrix}(\textstyle\sum_{i,j\geq 1,\ i+j\geq 3}\ a_{i,j}\beta_{2m-2i-2j+1}) \\
&\quad +\textstyle\sum_{j=2}^{2m-1}\ \sum_{a=0}^{m}\ (-1)^{m-a}\begin{pmatrix} 3m-1-j-a \\ m-a \end{pmatrix} S_a(\alpha_j)\beta_{2m-2j+2a-1} \\
&\quad +(-1)^{m-1}2\lambda\begin{pmatrix} 3m-3 \\ m-1 \end{pmatrix}\beta_{2m-1}.
\end{aligned}$$

Definition 6.5.4 (KU-theory characteristic numbers)**.** Let $\gamma : MU_* \longrightarrow KU_*$ be the Conner-Floyd map of § 6.1.3 – inflicted with coefficients in $\mathbb{Z}_{(2)}$. If $\alpha_j \in MU_{2j}(pt)$ and S_a is the Landweber-Novikov operation of § 6.1.11, then $S_a(\alpha_j) \in MU_{2j-2a}(pt)$ and

$$\gamma(S_a(\alpha_j)) = [S_a, \alpha_j]B^{j-a} \in KU_{2j}(pt;\mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}\langle B^{j-a}\rangle$$

where $[S_a, \alpha_j] \in \mathbb{Z}_{(2)}$ is the normal K-theory characteristic number of α_j associated to S_a (see [260] Chapters III and XI).

Hence

$$S_a(\alpha_j) \equiv [S_a, \alpha_j]a_{1,1}^{j-a} \quad (\text{modulo } A_{j-a})$$

and

$$(-1)^{j-a}S_a(\alpha_j)\beta_{2m-2j+2a-1} \equiv [S_a, \alpha_j]2^{j-a}\beta_{2m-1} \quad (\text{modulo } \mathrm{Ker}(\gamma)).$$

Remark 6.5.5. It is important to observe that Sq^{2m} detects f if and only if μ is odd while λ (modulo $(3^{2m}-1)\cdot 2^{2m-1}$) is the KU_*-e-invariant of f defined by studying the action of ψ^3 on $\tilde{KU}_{4m-1}(\mathrm{Cone}(f);\mathbb{Z}_{(2)})$.

Proposition 6.5.6. Let $m = 2^{q+1}$ for some positive integer q. Then, in the notation of § 6.5.2, Proposition 6.5.3 and Definition 6.5.4, let f be a stable homotopy class of maps

$$f : \Sigma^\infty S^{4m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{4m-2}.$$

Then (modulo 2^{q+5})

$$\sum_{j=2}^{2^{q+1}+q+5}\ \sum_{a=0}^{j} (-1)^{m+j}2^{j-a}\begin{pmatrix} 3m-1-j-a \\ m-a \end{pmatrix}[S_a, \alpha_j] \equiv 0.$$

Proof. By Chapter 5, Proposition 5.2.4,

$$\nu_2(3^{2m}-1) = \nu_2(9^m-1) = \nu_2(m)+3 = q+4.$$

In addition

$$\begin{aligned}
&\begin{pmatrix} 3m-2 \\ m \end{pmatrix} + \begin{pmatrix} 3m-3 \\ m-1 \end{pmatrix} \\
&= \tfrac{(3m-2)!}{m!\cdot(2m-2)!} + \tfrac{(3m-3)!}{(m-1)!\cdot(2m-2)!} \\
&= \tfrac{(3m-3)!}{(2m-2)!(m-1)!}(\tfrac{3m-2}{m} + 1) \\
&= \tfrac{(3m-3)!}{(2m-2)!(m-1)!}\tfrac{3m-2+m}{m} \\
&= \tfrac{(3m-3)!}{(2m-2)!(m-1)!}\tfrac{4m-2}{m}.
\end{aligned}$$

By Chapter 5, Proposition 5.2.6 when $m = 2^{q+1}$ the 2-adic valuation of this integer is

$$\begin{aligned}
&\nu_2(\tfrac{(3m-3)!}{(2m-2)!(m-1)!}\tfrac{4m-2}{m}) \\
&= \alpha(2m-2) + \alpha(m-1) - \alpha(3m-3) + 1 - q - 1 \\
&= q + 1 + q + 1 - (q+1) + 1 - (q+1) \\
&= 1.
\end{aligned}$$

By Remark 6.5.5 and Chapter 8, Theorem 8.1.2 we know that if μ is odd,

$$\begin{aligned}
&\nu_2((3^{2m}-1)\mu) \\
&= \nu_2(m) + 3 \\
&= q + 4 \\
&= \nu_2(4\lambda) \\
&= \nu_2\left\{(-1)^{m-1}2\lambda\begin{pmatrix} 3m-2 \\ m \end{pmatrix} + (-1)^{m-1}2\lambda\begin{pmatrix} 3m-3 \\ m-1 \end{pmatrix}\right\},
\end{aligned}$$

and that if μ is even then

$$\nu_2((3^{2m}-1)\mu) \geq q + 5$$

and

$$\nu_2\left\{(-1)^{m-1}2\lambda\begin{pmatrix} 3m-2 \\ m \end{pmatrix} + (-1)^{m-1}2\lambda\begin{pmatrix} 3m-3 \\ m-1 \end{pmatrix}\right\} \geq q + 5.$$

Since

$$\gamma(\beta_{2m-1}) = 2^{m-1} \in \mathbb{Z}/2^{2m-1} \cong \tilde{KU}_{4m-1}(\mathbb{RP}^{4m-2})$$

applying γ to the relation of Proposition 6.5.3 yields (modulo 2^{2m-1})

$$\begin{aligned}
&2^{m-1}(3^{2m}-1)\mu \\
&\equiv 2^{m-1}\left\{(-1)^{m-1}2\lambda\begin{pmatrix} 3m-2 \\ m \end{pmatrix} + (-1)^{m-1}2\lambda\begin{pmatrix} 3m-3 \\ m-1 \end{pmatrix}\right\} \\
&+2^{m-1}\textstyle\sum_{j=2}^{2m-1}\ \sum_{a=0}^{m}\ (-1)^{m+j}2^{j-a}\begin{pmatrix} 3m-1-j-a \\ m-a \end{pmatrix}[S_a, \alpha_j].
\end{aligned}$$

The previous discussion implies, whatever the parity of μ, that

$$2^{m-1}\sum_{j=2}^{2m-1}\sum_{a=0}^{m}(-1)^{m+j}2^{j-a}\left(\begin{array}{c}3m-1-j-a\\ m-a\end{array}\right)[S_a,\alpha_j]\equiv 0$$

(modulo $2^{m-1+q+5}$) and so

$$\sum_{j=2}^{2m-1}\sum_{a=0}^{m}(-1)^{m+j}2^{j-a}\left(\begin{array}{c}3m-1-j-a\\ m-a\end{array}\right)[S_a,\alpha_j]\equiv 0$$

(modulo 2^{q+5}), which yields the result once we discard some obvious terms. □

We conclude this section with a result which is related to Proposition 6.5.6, being another consequence of Chapter 8, Theorem 8.1.2.

Proposition 6.5.7. Suppose that $n = 2^{k-1}$.

(i) Let $g : \Sigma^\infty S^{4n-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{4n-2}$ be an S-map detected by Sq^{2^k} on Cone(g). Let h be an S-map such that the composition

$$\Sigma^\infty\mathbb{RP}^{4n-2} \longrightarrow \mathrm{Cone}(2g) \xrightarrow{h} \Sigma^\infty\mathbb{RP}^{4n-1}$$

is stably homotopic to the inclusion of $\Sigma^\infty\mathbb{RP}^{4n-2}$ into $\Sigma^\infty\mathbb{RP}^{4n-1}$. Then h is a 2-local stable homotopy equivalence.

(ii) Conversely, if the canonical map $\pi : S^{4n-2} \longrightarrow \mathbb{RP}^{4n-2}$ is 2-locally stably homotopic to $2g$ then $g : \Sigma^\infty S^{4n-2} \longrightarrow \Sigma^\infty\mathbb{RP}^{4n-2}$ is detected by Sq^{2^k} on Cone(g).

Proof. Part (ii) is proved by the argument of Theorem 6.3.4(ii) (proof).

I shall prove part (i) using $KU_*(-;\mathbb{Z}_{(2)})$ e-invariants. It suffices to show that h_* induces an isomorphism on $H_*(-;\mathbb{Z}/2)$. By Lemma 6.1.15 there exists $\tau_{2g} \in MU_{4n-1}(\mathrm{Cone}(2g))$ mapping to a generator of $MU_{4n-1}(S^{4n-1};\mathbb{Z}_{(2)})$ and satisfying

$$h_*(\tau_{2g}) = \lambda\beta_{4n-1} + \sum_{t=2}^{2n-1} x_t\beta_{4n-1-2t}$$

with $x_t \in A_t$, where A_j is as in § 6.5.1. Similarly there exists $\tau_g \in MU_{4n-1}(\mathrm{Cone}(g))$ mapping to a generator of $MU_{4n-1}(S^{4n-1};\mathbb{Z}_{(2)})$.

We must show that $\lambda \in \mathbb{Z}_{(2)}$ is a unit.

Let

$$\gamma : MU_{4n-1}(-;\mathbb{Z}_{(2)}) \longrightarrow KU_{4n-1}(-;\mathbb{Z}_{(2)})$$

denote the homology Conner-Floyd map (see § 6.1.3). From Chapter 8, Theorem 8.1.2 and the fact that bu-e-invariants and KU-e-invariants coincide in this situation (see Chapter 8, Example 8.3.2), detection by Sq^{2^k} implies that

$$(\psi^3-1)(\gamma(\tau_g)) = u\frac{3^{2n}-1}{4} \in KU_{4n-1}(\mathbb{RP}^{4n-2};\mathbb{Z}_{(2)}) \cong \mathbb{Z}/2^{2n-1}$$

where $u \in \mathbb{Z}_{(2)}^*$. The map from Cone$(2g)$ to Cone(g) may be assumed to send τ_{2g} to a unit times $2\tau_g$ so that

$$(\psi^3 - 1)(\gamma(\tau_{2g})) = u' \frac{3^{2n} - 1}{2} \in KU_{4n-1}(\mathbb{RP}^{4n-2}; \mathbb{Z}_{(2)})$$

where $u' \in \mathbb{Z}_{(2)}^*$. However, by Chapter 8, Example 8.3.2, the KU-e-invariant is a unit times $\frac{3^{2n}-1}{2}$ so applying the map h we obtain

$$u'' \lambda \frac{3^{2n} - 1}{2} \equiv (\psi^3 - 1)(h_*(\gamma(\tau_{2g}))) \equiv u' \frac{3^{2n} - 1}{2} \in \mathbb{Z}/2^{2n-1}$$

where u'' is a unit and therefore so is λ. □

Chapter 7

Hurewicz Images, BP-theory and the Arf-Kervaire Invariant

Alone among the sciences, mathematics is the discipline in which something may be proved, often amid great excitement, to be impossible.

Victor Yaraslaw-Paddon

The objective of this chapter is to prove the conjecture of [30] in its original form, as stated in Chapter 1, Theorem 1.8.10. This result is reiterated in this chapter as Theorem 7.2.2. The conjecture states that an element of $\pi_{2^{n+1}-2}(\Sigma^{\infty}\mathbb{RP}^{\infty})$ corresponds under the Kahn-Priddy map to the class of a framed manifold of Arf-Kervaire invariant one if and only if it has a non-zero Hurewicz image in ju-theory. I shall prove this result in three ways (§§ 7.2.3–7.2.5) – one of which uses an excursion into BP-theory.

Material in this chapter is included from [251] because in the BP-theory context the material illustrates the importance of the difference between left and right unit maps η_L and η_R in the construction of operations in a generalised cohomology theory. In the BP-theory context these unit maps give rise to the Quillen operations (see [9] or [294]) which have had important applications in homotopy theory (see [137]). The application of BP-theory to the Arf-Kervaire invariant one problem, as sketched here, is very involved and serves to illustrate how relatively simple is the proof of the same result via the upper triangular technology of Chapters 3 and 5. The reason for this simplification is that, as explained in Chapter 8, the upper triangular technology gives a simple computation of the effect of the 2-local unit map

$$\eta \wedge 1 : bo \longrightarrow bu \wedge bo$$

in connective K-theory, starting from data concerning ψ^3 on $bo_*(X)$. In § 1 I recapitulate the facts and formulae which are needed about BP and $\mathbb{RP}^{\infty}$ and introduce the homology theories, J_* and J'_*, which are to BP and $BP \wedge BP$ what ju is to bu. The crux of the BP approach to Theorem 7.2.2 is to restrict the possibilities for Hurewicz images in these theories by analysing the canonical anti-involution induced by switching the factors in $BP \wedge BP$. This is done in Theorem 7.1.4 and Theorem 7.1.5.

In § 2, three proofs of Theorem 7.2.2 are given. The first proof, a sketch of the argument of [251], uses Theorem 7.1.5. The second proof uses ju rather than J-theory and Chapter 8, Theorem 8.1.2. The third proof uses the calculations of Chapter 6, § 6.4.6 together with the upper triangular technology result of Chapter 8, Theorem 8.1.2.

7.1 J_* and J'_*

7.1.1. In this chapter we shall retain the notation of Chapter 6 § 6.2.1 $\pi_*(BP) = BP_*$ and $BP^* = BP_{-*}$.

Let $\psi^3 : BP \longrightarrow BP$ denote the Adams operation in BP-theory ([9] Part II; [211]; [245] pp. 59/60). Hence ψ^3 is equal to multiplication by 3^k on BP_{2k} and by 3^{j+1} on $BP_{2j+1}(\mathbb{RP}^{2t})$. The last fact follows easily from the formula $\psi^3(x) = 3^{-1}x$ ([245] Corollary 4.3 p. 60) since the S-duality isomorphism is given by slant product with the BP-Thom class of the tangent bundle of $\mathbb{RP}^{2t}$ ([9] p. 264) and since ψ^3 commutes with slant products. It also follows that $\psi^3 \wedge \psi^3 : BP \wedge BP \longrightarrow BP \wedge BP$ is given by multiplication by 3^k on $(BP \wedge BP)_{2k}$ and by 3^{j+1} on $(BP \wedge BP)_{2j+1}(\mathbb{RP}^{2t})$.

Define spectra, J and J', by the cofibration sequences

$$J \xrightarrow{\pi} BP \xrightarrow{\psi^3-1} BP \xrightarrow{\pi_1} \Sigma J$$

and

$$J' \xrightarrow{\pi'} BP \wedge BP \xrightarrow{\psi^3\wedge\psi^3-1} BP \wedge BP \xrightarrow{\pi'_1} \Sigma J'.$$

Since $\psi^3 \cdot \iota = \iota : S^0 \longrightarrow BP$, η_L and η_R induce maps $\tilde{\eta}_L, \tilde{\eta}_R : J \longrightarrow J'$, respectively. Also ι induces a (unique) map, $\tilde{\iota} : S^0 \longrightarrow J$, such that $\pi \cdot \tilde{\iota} = \iota$.

Let $n \geq 1$ be an integer. Since $\psi^3 - 1$ is injective on

$$BP_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}}) \cong BP_{2^{n+1}-2},$$

there is an isomorphism of the form

$$(\pi_1)_* : BP_{2^{n+1}-1}(\mathbb{RP}^{2^{n+1}}) \otimes \mathbb{Z}/2^{n+2} \xrightarrow{\cong} J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}}),$$

since $3^{2^n} - 1 = 2^{n+2}(2s + 1)$. Similarly there is an isomorphism of the form

$$(\pi'_1)_* : (BP \wedge BP)_{2^{n+1}-1}(\mathbb{RP}^{2^{n+1}}) \otimes \mathbb{Z}/2^{n+2} \xrightarrow{\cong} J'_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}}).$$

By means of the isomorphisms, $(\pi_1)_*$ and $(\pi'_1)_*$, we may represent elements of $J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$ and $J'_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$ by sums in degree $2^{n+1} - 1$ of the form

$$\sum_{I,k} \epsilon_I v^I x_{2k+1} \quad \text{and} \quad \sum_{I,I',k} \epsilon_I v^I t^{I'} x_{2k+1},$$

respectively, as in Chapter 6 § 6.2.1.

Now let $T : BP \wedge BP \longrightarrow BP \wedge BP$ be the map which interchanges the factors. Then $T_* = c$, the conjugation, on $(BP \wedge BP)_*(X)$. We shall need the following formulae for $c(v_k) = (\eta_R)_*(v_k)$. Recall that BP_* embeds, via the Hurewicz homomorphism, into

$$H_*(BP; \mathbb{Z}_2) \cong \mathbb{Z}_2[m_1, m_2, \ldots]$$

where $\deg(m_i) = \deg(v_i)$ and $v_i = 2m_i - \sum_{j=1}^{i-1} m_j v_{i-j}^{2^j}$.

Lemma 7.1.2. Let $I = \langle 2, v_1, v_2, \ldots \rangle \lhd BP_* = \mathbb{Z}_2[v_1.v_2, \ldots]$. Then, for $k \geq 1$,

$$(\eta_R)_*(v_k) = 2t_k + \sum_{j\geq 1}^{k} v_j t_{k-j}^{2^j} \text{ (modulo } I^2[t_1, t_2, \ldots])$$

in $(BP \wedge BP)_* \cong \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, t_3, \ldots]$.

Proof. We use the formulae of ([9] Theorem 16.1 p. 112; [294] Theorem 3.11 p. 17) from which we see that $(\eta_R)_*(v_1) = 2t_1 + v_1$ and that in $(BP \wedge BP)_* \otimes \mathbb{Q}_2$ $(\eta_R)_*(m_k) = \sum_{j=0}^{k} m_j t_{k-j}^{2^j}$. The result follows by induction on k. $\square$

7.1.3. Let E be a commutative ring spectrum and let $(\eta)_* : BP^*(\mathbb{CP}^\infty) \longrightarrow (E \wedge BP)^*(\mathbb{CP}^\infty)$ denote the map induced by η, the (left) unit of E. When $E = BP$, $(BP \wedge BP)^*(\mathbb{CP}^\infty) \cong (BP \wedge BP)_{-*}[[x]]$ where $x = (\eta_L)_*(x)$ in dimension two. On the other hand, $\eta = \eta_R$, the right unit, so that the formula of ([9] Lemma 6.3 p. 60; [294] Lemma 1.51 p. 9) becomes

$$c(x) = \sum_{v\geq 0} b_v^{BP} x^{v+1} \in (BP \wedge BP)^2(\mathbb{CP}^\infty).$$

This formula also holds in $(BP \wedge BP)^2(\mathbb{RP}^\infty)$.

Since $(\eta_L)_*(x_{2k+1}) \in (BP \wedge BP)_*(\mathbb{RP}^{2^{n+1}})$ corresponds under S-duality to $x^{2^{i-1}-k-1}$ in

$$\begin{aligned} &(BP \wedge BP)^*(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2^{n+1}-2}) \\ &\cong \left(\frac{x^{2^{i-1}-2^n}\mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]}{x^{2^{i-1}}\mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots][[x]]} \right) /([2]x) \end{aligned}$$

(where this isomorphism follows from the canonical form for elements which was discussed in Chapter 6 § 6.2.1) then

$$c((\eta_L)_*(x_{2k+1})) = (\eta_R)_*(x_{2k+1}) = \sum_{w=0}^{k} b_{k,w}(\eta_L)_*(x_{2w+1})$$

where

$$\sum_{w=0}^{k} b_{k,w} x^{2^{i-1}-w-1} = \Big(\sum_{v\geq 0} b_v^{BP} x^{v+1} \Big)^{2^{i-1}-k-1}$$

in $(BP \wedge BP)^*(\mathbb{RP}^{2^i-2}/\mathbb{RP}^{2^i-2^{n+1}-2})$. The coefficients,

$$b_v^{BP} \in (BP \wedge BP)_* \cong \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$$

satisfy ([294] Theorem 3.11(proof) p. 17 and Theorem 1.48(c) p. 8)

$$\sum_{i \geq 0} (\eta_L)_*(m_i) = \sum_{s \geq 0} (\eta_R)_*(m_s)(\sum_{v \geq 0} b_v^{BP})^{2^s}.$$

This equation holds in $\mathbb{Q}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$ but setting each v_i to zero we obtain the equation

$$0 = \sum_{i \geq 0} t_i (\sum_{v \geq 0} b_v^{BP})^{2^i}$$

and this equation holds in $\mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]/\langle v_1, v_2, \ldots \rangle$. Hence we find that

$$0 = \sum_{i \geq 0} t_i \sum_{v \geq 0} (b_v^{BP})^{2^i} \in \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]/\langle 2, v_1, v_2, \ldots \rangle.$$

Since $t_0 = 1 = b_0$ one sees by induction that

$$b_v^{BP} \in \langle 2, v_1, v_2, \ldots \rangle \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]$$

except when $v = 2^m - 1$ for some m and for each $v \geq 1$,

$$0 \equiv \textstyle\sum_{j=0}^{v} t_j (b_{2^{v-j}-1}^{BP})^{2^j}$$
$$(\text{modulo } \langle 2, v_1, v_2, \ldots \rangle \mathbb{Z}_2[v_1, v_2, \ldots, t_1, t_2, \ldots]).$$

These formulae may be used, as in [251], in a rather complicated induction argument to prove the following two results. In [251] I used these results to give an alternative proof to that of [141] of the main conjecture of [30] (Theorem 7.2.2). Instead of proving these results here, I shall merely use them in § 7.2.3 as the basis for the first of three proofs of Theorem 7.2.2 given in §§ 7.2.3–7.2.5.

Theorem 7.1.4 ([251])**.** In the notation of Chapter 6 § 6.2.1 and § 7.1.1, let

$$u \in J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$$

be represented as $u = \sum_I \epsilon_I v^I x_{2^{n+1}-\deg(v^I)-1}$. If

$$(\tilde{\eta}_R)_*(u) = (\tilde{\eta}_L)_*(u) \in J'_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$$

then, either for some $0 \leq d \leq n+1$ and $\epsilon = 1$ or for some $d \geq n+2$ and $\epsilon = 0$, we have

$$u = \epsilon 2^d x_{2^{n+1}-1} + \sum_{l(I') \geq d+1} \epsilon_{I'} v^{I'} x_{2^{n+1}-\deg(v^{I'})-1}.$$

Here the length of $I = (i_1, \ldots, i_t)$ is defined to be equal to $l(I) = i_1 + \cdots + i_t$.

Theorem 7.1.4 may be improved to the following more difficult result.

Theorem 7.1.5 ([251])**.** Let $u \in J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$, as in the notation of Chapter 6 § 6.2.1 and § 7.1.1, be represented as

$$u = \epsilon 2^{n+1} x_{2^{n+1}-1} + \sum_{l(I') \geq n+2} \epsilon_{I'} v^{I'} x_{2^{n+1}-\deg(v^{I'})-1}$$

and satisfy

$$(\tilde{\eta}_R)_*(u) = (\tilde{\eta}_L)_*(u) \in J'_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$$

where $\epsilon, \epsilon_{I'} \in \{0, 1\}$. Then $\epsilon_{I'} = 0$ if $l(I') = n + 2$.

Remark 7.1.6. Theorem 7.1.4 may be proved directly by mapping to $bu \wedge BP$, which amounts to setting $v_j = 0$ for all $j \geq 2$ and then following a (simpler) version of the induction of [251]. Alternatively one may derive Theorem 7.1.4 from Theorem 7.1.5 by replacing u by $2^{n+1-d}u$.

7.2 J_*-theory, ju_*-theory and the Arf-Kervaire invariant

7.2.1. Suppose that $\Theta : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^\infty$ is an S-map whose mapping cone is denoted by Cone(Θ). In dimension $2^{n+1} - 2$ the Kahn-Priddy map [135] (see also Chapter 1 Theorem 1.5.10) gives a split surjection of stable homotopy groups

$$\pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty) \longrightarrow \pi_{2^{n+1}-2}(\Sigma^\infty S^0) \otimes \mathbb{Z}_2$$

onto the 2-Sylow subgroup of the stable homotopy groups of spheres. The Arf-Kervaire invariant ([30], [47] [138]; see also Chapter 1 § 1.8.3) of a framed manifold yields a homomorphism from $\pi_{2^{n+1}-2}(\Sigma^\infty S^0) \otimes \mathbb{Z}_2$ to the group of order two. Furthermore, it is well known that the image of $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ has non-trivial Kervaire invariant if and only if the Steenrod operation ([247] [259]; see also Chapter 1 § 1.8.5)

$$Sq^{2^n} : H^{2^n-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow H^{2^{n+1}-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$$

is non-trivial.

Now let bu denote 2-adic connective K-theory and define ju-theory by means of the fibration $ju \longrightarrow bu \stackrel{\psi^3-1}{\longrightarrow} bu$ (see Chapter 1, Example 1.3.4(iv)). Hence ju_* is a generalised homology theory for which $ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$. Recall that, if $\iota \in ju_{2^{n+1}-2}(S^{2^{n+1}-2}) \cong \mathbb{Z}_2$ is a choice of generator, the associated ju-theory Hurewicz homomorphism

$$H_{ju} : \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty) \longrightarrow ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$$

is defined by $H_{ju}([\Theta]) = \Theta_*(\iota)$.

We are now ready to state the main result of this section.

Theorem 7.2.2. For $n \geq 1$ the image of $[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$ under the ju-theory Hurewicz homomorphism

$$H_{ju}([\Theta]) \in ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$$

is non-trivial if and only if Sq^{2^n} is non-trivial on $H^{2^n-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$.

In any case, $2H_{ju}([\Theta]) = 0$.

We shall give, by way of comparison, three proofs of this result. The second and third proofs use Chapter 8, Theorem 8.1.2, which is the main application of the "upper triangular technology" of Chapters 3 and 5. The comparison is intended to convince the reader that the upper triangular technology is much simpler.

7.2.3. First Proof of Theorem 7.2.2 – using Theorem 7.1.5. Consider the commutative diagram which is given in Fig. 7.1 below.

$$\begin{tikzcd}
 & J_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[r] \arrow[dr] & J_{2^{n+1}-2}(\mathbb{RP}^\infty) \\
 & BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d, "\psi^3-1"] & BP_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[d, "\psi^3-1"] \\
BP_{2^{n+1}-1}(\mathbb{RP}^\infty) \arrow[r] \arrow[d] & BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d] & BP_{2^{n+1}-2}(S^{2^{n+1}-2}) \\
J_{2^{n+1}-2}(\mathbb{RP}^\infty) \arrow[r] & J_{2^{n+1}-2}(\mathrm{Cone}(\Theta)) &
\end{tikzcd}$$

Figure 7.1.

Let $\tilde{\iota} \in J_{2^{n+1}-2}(S^{2^{n+1}-2})$ be the class given by the J-theory unit as in § 7.1.1. The J-theory Hurewicz image is given by

$$\Theta_*(\tilde{\iota}) \in J_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}}).$$

It is an element satisfying the conditions of Theorems 7.1.4 and 7.1.5. The image of $\tilde{\iota}$ in $BP_{2^{n+1}-2}(S^{2^{n+1}-2})$ is ι of § 7.1.1, which lifts to $\iota'' \in BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta))$. Then $(\psi^3 - 1)(\iota'')$ lifts to

$$\alpha \in BP_{2^{n+1}-1}(\mathbb{RP}^\infty)$$

and, by ([255] Proposition 2 pp. 241–2), the image of α in $J_{2^{n+1}-2}(\mathbb{RP}^{2^{n+1}})$ is equal to $\Theta_*(\tilde{\iota})$. Therefore, by Theorem 7.1.4, we obtain an equation of the form

$$\begin{aligned}\psi^3(\iota'') = \iota'' &+ \epsilon 2^d x_{2^{n+1}-1} \\ &+ \textstyle\sum_{l(I') \geq d+1} \epsilon_{I'} v^{I'} x_{2^{n+1}-deg(v^{I'})-1} + 2^{n+2}\beta\end{aligned}$$

in $BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta))$ for some $0 \leq d \leq n+1$, $\epsilon = 0, 1$ and $\beta \in BP_{2^{n+1}-1}(\mathbb{RP}^\infty)$.

Define ju by the 2-local fibration, $ju \longrightarrow bu \stackrel{\psi^3-1}{\longrightarrow} bu$, so that

$$ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2} \quad (\text{see } \S\, 7.2.4),$$

generated by $\lambda_*(x_{2^{n+1}-1})$ where

$$\lambda_* : J_*(X) \longrightarrow ju_*(X)$$

is induced by the canonical Conner-Floyd map $\lambda : BP \to bu$. Therefore the ju-theory Hurewicz image of Θ is $\epsilon 2^d \lambda_*(x_{2^{n+1}-1})$.

First we must show that $d \geq n+1$ which will imply that the ju-theory Hurewicz image of Θ is trivial unless $d = n+1$ and in that case is non-trivial if and only if $\epsilon = 1$. If $\epsilon = 1$ and $d < n+1$, we replace ι by $2^{n+1-d}\iota$, then the argument which is to follow shows that $2^{n+1-d}\Theta$ is detected by Sq^{2^n} on the mod 2 cohomology of its mapping cone. This is easily seen to be impossible, by comparing the mapping cone sequences for $2^{n+1-d}\Theta$ and $2^{n-d}\Theta$.

The fact that $d \geq n+1$ implies that $2H_{ju}([\Theta]) = 0$. Now write $S_n : BP_*(X) \longrightarrow BP_{*-2n}(X)$ for the Landweber-Novikov operation in BP-homology induced by the MU-operation $S_n = S_{(n,0,0,\ldots)}$ in ([9] p. 12; see also Chapter 6 § 6.1.11). We are going to study the consequences of the relation, $3^m\psi^3 S_m = S_m\psi^3$. This relation is established by observing that the sum of the left and right sides over m correspond to two ring operations in BP-cohomology and therefore are equal if and only if these cohomology operations agree on $x \in BP^2(\mathbb{CP}^\infty)$, which is easily verified (see Chapter 6, Lemma 6.1.14 for the MU analogue). In addition, if $T : BP \longrightarrow H\mathbb{Z}/2$ corresponds to the Thom class then a similar argument (see Chapter 6, Lemma 6.1.15 for the MU analogue) shows that

$$(Sq^{2m})_* T_* = T_* S_m : BP_*(X) \longrightarrow H_{*-2m}(X; \mathbb{Z}/2).$$

Also, if $0 \leq m \leq t$, using the formulae of ([9] Part I § 5 and § 8.1) it is not difficult to show (see Chapter 6, Proposition 6.1.12; we shall only need this formula modulo 2) that

$$S_m(x_{2t+1}) = (-1)^m \binom{m+t}{m} x_{2t-2m+1} \in BP_{2t-2m+1}(\mathbb{RP}^\infty).$$

Bearing in mind the previous discussion about what to do if $d < n+1$, we may suppose that $d = n+1$ and write

$$\psi^3(\iota'') = \iota'' + \epsilon 2^{n+1} x_{2^{n+1}-1} + 2^{n+2}\beta + \gamma \in BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta)).$$

Here $\beta \in BP_{2^{n+1}-1}(\mathbb{RP}^\infty) \subset BP_{2^{n+1}-1}(\mathrm{Cone}(\Theta))$ and, by Theorem 7.1.5,

$$\gamma = \sum_{l(I')>n+2} \epsilon_{I'} v^{I'} x_{2^{n+1}-\deg(v^{I'})-1}.$$

Applying the relation with $m = 2^{n-1}$ we obtain the following equation in $BP_{2^n-1}(\mathrm{Cone}(\Theta)) \cong BP_{2^n-1}(\mathbb{RP}^\infty)$:

$$\begin{array}{l} S_{2^{n-1}}(\iota'') \\ \quad +\epsilon 2^{n+1} \left(\begin{array}{c} 2^{n-1} + 2^n - 1 \\ 2^{n-1} \end{array} \right) x_{2^n-1} + 2^{n+2} S_{2^{n-1}}(\beta) + S_{2^{n-1}}(\gamma) \\ = 3^{2^{n-1}} 3^{2^{n-1}} S_{2^{n-1}}(\iota'') \end{array}$$

because ψ^3 acts like multiplication by $3^{2^{n-1}}$ on $BP_{2^n-1}(\mathbb{RP}^\infty)$.

We are going to apply

$$\lambda_* : BP_{2^{n+1}-1}(\mathbb{RP}^\infty) \longrightarrow bu_{2^{n+1}-1}(\mathbb{RP}^\infty) \quad (\text{modulo } 2^{n+3})$$

to the above equation, bearing in mind that $\lambda_*(v_k) = 0$ for $k \geq 2$ and that

$$0 = v_1 x_{2j-1} + 2x_{2j+1} \in bu_{2j+1}(\mathbb{RP}^\infty).$$

In $bu_*(\mathbb{RP}^\infty)$ consider $\lambda_*(S_m(v_k))x_{2j+1}$. If $m \neq 2^k - 1, 2^k - 2$ then $\lambda_*(S_m(v_k))$ is a multiple of v_1^{2+e} for some $e \geq 0$ and therefore

$$\lambda_*(S_m(v_k))x_{2j+1} \in 4bu_{2j+2^{k+1}-1-2m}(\mathbb{RP}^\infty).$$

Similarly one sees that $\lambda_*(S_{2^k-2}(v_k))x_{2j+1} \in 2bu_{2j+3}(\mathbb{RP}^\infty)$. Also, since S_{2^k-1} cannot decrease Adams filtration,

$$\lambda_*(S_{2^k-1}(v_k)) \in 2bu_0(S^0) \quad \text{and} \quad \lambda_*(S_{2^k-1}(v_k))x_{2j+1} \in 2bu_{2j+1}(\mathbb{RP}^\infty).$$

Now consider $v_{i_1} v_{i_2} \ldots v_{i_t} x_{2j+1} \in BP_{2^{n+1}-1}(\mathbb{RP}^\infty)$ and

$$\begin{array}{l} \lambda_*(S_{2^{n-1}}(v_{i_1} v_{i_2} \ldots v_{i_t} x_{2j+1})) \\ = \sum_{a_1 + \ldots a_{t+1} = 2^{n-1}} \lambda_*(S_{a_1}(v_{i_1})) \ldots \lambda_*(S_{a_t}(v_{i_t}))\lambda_*(S_{a_{t+1}}(x_{2j+1})). \end{array}$$

The above discussion shows that this lies in $2^t bu_{2^n-1}(\mathbb{RP}^\infty)$ unless $t = 0$. Also $\lambda_*(S_{2^{n-1}}(x_{2^{n+1}-1})) \in 2bu_{2^n-1}(\mathbb{RP}^\infty)$ since

$$\left(\begin{array}{c} 2^{n-1} + 2^n - 1 \\ 2^{n-1} \end{array} \right) = 2(2s+1) \quad \text{for some } s.$$

Hence both $2^{n+2}\lambda_*(S_{2^{n-1}}(\beta))$ and $\lambda_*(S_{2^{n-1}}(\gamma))$ lie in $2^{n+3}bu_{2^n-1}(\mathbb{RP}^\infty)$.

From this discussion, in the previous notation, our equation implies the following congruence in $bu_{2^n-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{2^{n-1}}$:

$$(3^{2^n} - 1)\lambda_*(S_{2^{n-1}}(\iota'')) \equiv 2^{n+2}\epsilon \text{ modulo } 2^{n+3}bu_{2^n-1}(\mathbb{RP}^\infty).$$

However, for $n \geq 1$, $(3^{2^n} - 1) = 2^{n+2}(2w+1)$ for some w so that $\epsilon = 1$ if and only if $\lambda_*(S_{2^{n-1}}(\iota''))$ is a generator of $bu_{2^n-1}(\mathbb{RP}^\infty)$. The factorisation, $T : BP \xrightarrow{\lambda}$

$bu \longrightarrow H\mathbb{Z}/2$, implies that $\epsilon = 1$ if and only if the dual Steenrod operation, $Sq_*^{2^n}$, is non-trivial on $H_{2^{n+1}-1}(\mathrm{Cone}(\Theta);\mathbb{Z}/2)$, which is equivalent to Sq^{2^n} being non-trivial on $H^{2^n-1}(\mathrm{Cone}(\Theta);\mathbb{Z}/2)$. This completes the proof. □

The following proof is closely related to the jo-theory results of Chapter 6, Theorem 6.4.2 and Corollary 6.4.3.

7.2.4. Second Proof – using Chapter 8, Theorem 8.1.2. Consider the commutative diagram which is given in Fig. 7.2 below.

$$\begin{tikzcd}
 & ju_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[r] \arrow[dr] & ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \\
 & bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d, "\psi^3-1"] & bu_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[d, "\psi^3-1"] \\
bu_{2^{n+1}-1}(\mathbb{RP}^\infty) \arrow[r] \arrow[d] & bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d] & bu_{2^{n+1}-2}(S^{2^{n+1}-2}) \\
ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \arrow[r] & ju_{2^{n+1}-2}(\mathrm{Cone}(\Theta)) &
\end{tikzcd}$$

Figure 7.2.

This is a diagram which features the S-map

$$\Theta : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^\infty$$

which is detected by Sq^{2^n} on the mod 2 cohomology of the mapping cone $\mathrm{Cone}(\Theta)$. We are going to use Chapter 8, Theorem 8.1.2 to calculate the equivalent behaviour of the bu-e-invariant of Θ, from which the result will follow almost immediately.

There is a slight subtlety to be taken into account, namely the equivalent e-invariant behaviour in Chapter 8, Theorem 8.1.2 is given in terms of a skeletal approximation to Θ; that is, an S-map

$$\tilde{\Theta} : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{2^{n+1}-2}$$

whose composition with the inclusion into $\mathbb{RP}^\infty$ is stably homotopic to Θ. Let $\mathrm{Cone}(\tilde{\Theta})$ denote the mapping cone of $\tilde{\Theta}$. Then Θ is detected by Sq^{2^n} on the mod 2 cohomology of its mapping cone if and only if the same is true for $\tilde{\Theta}$.

We have two canonical maps between 2-local K-theory groups

$$bu_{2t+1}(\mathbb{RP}^{2t}) \cong \mathbb{Z}/2^t \xrightarrow{\cong} KU_{2t+1}(\mathbb{RP}^{2t};\mathbb{Z}_2)$$

and

$$bu_{2t+1}(\mathbb{RP}^{2t}) \cong \mathbb{Z}/2^t \longrightarrow bu_{2t+1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{t+1}$$

the first of which is an isomorphism and the second is injective, as is seen from the collapsed Atiyah-Hirzebruch spectral sequence. Also we have an isomorphism

$$bu_{2^{n+1}-1}(\mathrm{Cone}(\tilde{\Theta})) \cong \mathbb{Z}/2^{2^n-1} \oplus \mathbb{Z}_2 \stackrel{\cong}{\longrightarrow} KU_{2^{n+1}-1}(\mathrm{Cone}(\tilde{\Theta}); \mathbb{Z}_2)$$

which commutes with the Adams operation ψ^3 and in which the first summand is $bu_{2^{n+1}-1}(\mathbb{RP}^{2^{n+1}-2})$. By Chapter 8, Theorem 8.1.2, Sq^{2^n} detects $\tilde{\Theta}$ on $\mathrm{Cone}(\tilde{\Theta})$ if and only if ψ^3 in

$$bu_{2^{n+1}-1}(\mathrm{Cone}(\tilde{\Theta})) \cong \mathbb{Z}/2^{2^n-1} \oplus \mathbb{Z}_2\langle\tilde{\iota}\rangle$$

satisfies

$$(\psi^3 - 1)(\tilde{\iota}) = \frac{(3^{2^n} - 1)}{4}(2u+1) \in \mathbb{Z}/2^{2^n-1}$$

for some integer u.

Therefore Sq^{2^n} detects Θ on $\mathrm{Cone}(\Theta)$ if and only if ψ^3 in

$$bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \cong \mathbb{Z}/2^{2^n} \oplus \mathbb{Z}_2\langle\iota_2\rangle$$

satisfies

$$(\psi^3 - 1)(\iota_2) = \frac{(3^{2^n} - 1)}{2}(2u+1) \in \mathbb{Z}/2^{2^n} \cong bu_{2^{n+1}-1}(\mathbb{RP}^\infty)$$

for some integer u.

By Chapter 5, Proposition 5.2.4 the 2-adic valuation of $3^{4m} - 1$ is equal to $\nu_2(m) + 4$ which equals $n+2$ when $m = 2^{n-2}$. Since $bu_{8m-2}(\mathbb{RP}^\infty)$ is trivial the fibration sequence defining ju shows that $ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$ and that

$$bu_{2^{n+1}-2}(\mathbb{RP}^\infty) \longrightarrow ju_{2^{n+1}-2}(\mathbb{RP}^\infty)$$

is surjective.

$$\begin{tikzcd}
 & ju_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[r] \arrow[dr] & ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \\
 & bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d, "\psi^3-1"] & bu_{2^{n+1}-2}(S^{2^{n+1}-2}) \arrow[d, "\psi^3-1"] \\
bu_{2^{n+1}-1}(\mathbb{RP}^\infty) \arrow[r] \arrow[d] & bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta)) \arrow[r] \arrow[d] & bu_{2^{n+1}-2}(S^{2^{n+1}-2}) \\
ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \arrow[r] & ju_{2^{n+1}-2}(\mathrm{Cone}(\Theta)) &
\end{tikzcd}$$

Now let ι generate $ju_{2^{n+1}-2}(S^{2^{n+1}-2}) \cong \mathbb{Z}_2$, the 2-adic integers. Now, as in the first proof of § 7.2.3, we use the result of ([255] Proposition 2 pp. 241–2), to assert that the image of ι in $ju_{2^{n+1}-2}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$ – the Hurewicz image $H_{ju}(\Theta)$ – is equal to the element obtained by zig-zagging through the diagram of Fig. 2 in the following manner. Map ι to the generator $\iota_1 \in bu_{2^{n+1}-2}(S^{2^{n+1}-2}) \cong \mathbb{Z}_2$, lift ι_1 to $\iota_2 \in bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta))$ and calculate $(\psi^3 - 1)(\iota_2)$ in the torsion subgroup of $bu_{2^{n+1}-1}(\mathrm{Cone}(\Theta))$, which is isomorphic to $bu_{2^{n+1}-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{2^n}$, and map this element vertically down to $ju_{2^{n+1}-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$.

Therefore Sq^{2^n} detects Θ on $\mathrm{Cone}(\Theta)$ if and only if

$$H_{ju}([\Theta]) = (3^{4m} - 1)/2 = 2^{n+1} \in ju_{2^{n+1}-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^{n+2}$$

as required. □

7.2.5. Third Proof – using Chapter 6 § 6.4.6. A minor modification of the proof in Chapter 6 § 6.4.6 of Chapter 6, Theorem 6.4.2 and Corollary 6.4.3 also proves, using Chapter 8, Theorem 8.1.2, that the image of

$$[\Theta] \in \pi_{2^{n+1}-2}(\Sigma^\infty \mathbb{RP}^\infty)$$

under the jo-theory Hurewicz homomorphism is non-trivial if and only if Sq^{2^n} is non-trivial on $H^{2^n-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$. Here jo is defined by the fibration of spectra $jo \longrightarrow bo \longrightarrow bspin$ as in Chapter 1, Example 1.3.4(iv). The result then follows since

$$jo_{2^{n+1}-2}(\mathbb{RP}^\infty) \longrightarrow ju_{2^{n+1}-2}(\mathbb{RP}^\infty)$$

is an injection of cyclic 2-groups. □

Chapter 8

Upper Triangular Technology and the Arf-Kervaire Invariant

"Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life!"

from "Alice in Wonderland" by Lewis Carroll [55]

The objective of this chapter is to give applications of the upper triangular results of Chapter 3 and Chapter 5. § 1 gives some background concerning Adams operations, recapitulates the results and proves the main one (Theorem 8.1.2) assuming the results of § 4. § 2 contains technical computations including some multiplicative relations in the collapsed Adams spectral sequences which will be used later. The most important corollary of this is Proposition 8.2.10 which implies that the maps $\iota_{k,k-1}$ corresponding to the super-diagonal entries in the matrix for $1 \wedge \psi^3$ induce almost injective homomorphisms on the connective K-theory groups which we shall study (see Proposition 8.3.7). § 3 analyses the Adams spectral sequences associated with the summands in the Mahowald decomposition (see Chapter 3) of $(bu \wedge bo)_*(X)$ for the examples which will be used in § 4. § 4 applies the upper triangular technology to the case when X is a projective space and the mapping cone of a homotopy class $\Theta_{8m-2} \in \pi_{8m-2}(\Sigma^\infty \mathbb{RP}^{8m-2})$. The central results in the latter case are Theorem 8.4.6, Theorem 8.4.7 and Corollary 8.4.9.

8.1 Adams operations

8.1.1. This is the main chapter of my book because it contains the main application to date of the "upper triangular technology" of Chapters 3 and 5. Some of the details in this chapter are rather technical – although by the yardstick of classical calculations in stable homotopy the contents of this chapter are the acme of simplicity and conceptual transparency. Nonetheless, before getting into the details I shall try to set the chapter in context by some references to a few historical corner-stones of algebraic topology of the stable homotopy persuasion.

Upper triangular technology is the successor to the famous paper of Michael Atiyah concerning operations in periodic unitary K-theory [25]. The main results of [25] are (i) results about the behaviour of operations in KU-theory with respect to the filtration which comes from the Atiyah-Hirzebruch spectral sequence and (ii) results which relate operations in KU-theory to the Steenrod operations of [259] in mod p singular cohomology in the case of spaces whose integral cohomology is torsion free.

One may ask: "Was [25] important? Did it have any important applications?" Any historical account must begin by addressing these questions. Firstly, in the case of torsion free spaces the results of [25] gave simple proofs of the results of the classic paper of Frank Adams [3] which led on to [14] and [168] (see also [10] and [13]). Secondly, the connection between Adams operations and Steenrod cohomology operations was used by John Hubbuck in [115] to settle the longstanding problem of classifying homotopy commutative finite H-spaces (see also [15]) – they are all homotopy equivalent to the torus! The proof uses a classical paper of Bill Browder [46] to show that such H-spaces have torsion free cohomology and then the relation of Adams operations to Steenrod operations to complete the classification. In fact, Adams operations have been involved in (and were invented in [4] for) solving some classical problems.

The heart of [25] is the observation that it is possible to define operations in K-theory by using the symmetric group, following Steenrod's method for defining operations in singular cohomology [259]. Steenrod's method has a fine pedigree and in a more modern context it is the basis of Alexander Vishik's construction of operations on algebraic cobordism [278] and Vladimir Voevodsky's Steenrod operations on motivic cohomology [282] (see also [45]).

Let E_0 and E_1 be complex vector bundles over X so that $x = [E_0] - [E_1] \in KU^0(X)$ and let Σ_k denote the symmetric group of permutations of $\{1, 2, \ldots, k\}$. Then Σ_k acts on the graded vector bundle $(E_0 \oplus E_1)^{\otimes k}$ so that one obtains an element in the equivariant KU-theory, depending only on x,

$$y \in KU^0_{\Sigma_k}(X) \cong K^0(X) \otimes R(\Sigma_k)$$

where $R(\Sigma_k)$ denotes the complex representation ring of Σ_k. Given any homomorphism $\theta : R(\Sigma_k) \longrightarrow \mathbb{Z}$ one can apply $1 \otimes \theta$ to obtain an element of $KU^0(X)$ and a natural operation from $KU^0(X)$ to itself. Let $\mathrm{Op}(K)$ denote the set of all such natural operations so one has a function

$$j_k : \mathrm{Hom}(R(\Sigma_k), \mathbb{Z}) \longrightarrow \mathrm{Op}(K)$$

where in Atiyah's paper $\mathrm{Hom}(R(\Sigma_k), \mathbb{Z})$ is playing the role that $H_*(\Sigma_k; \mathbb{Z}/p)$ does in Steenrod's situation [259]. $\mathrm{Op}(K)$ can be made into a ring by adding and multiplying the values of operations so that

$$j = \sum_k j_k : R_* = \sum_k \mathrm{Hom}(R(\Sigma_k), \mathbb{Z}) \longrightarrow \mathrm{Op}(K)$$

becomes a homomorphism of rings. Following ideas of Schur concerning the duality between representations of the symmetric groups and the general linear groups, Atiyah shows that R_* is isomorphic to a limit of a ring of symmetric polynomials. Given this structure it is possible to define explicit elements of R_* which coincide with Grothendieck's λ^i's and the Adams operations ψ^k which were introduced in [4].

The KU^0-groups have a skeletal filtration furnished by the Atiyah-Hirzebruch spectral sequence. Atiyah shows that if x lies in filtration greater than or equal to q, then the image of y in $KU^0(X \times B\Sigma_k)$ lies in filtration greater than or equal to kq. Furthermore, if X is a space with torsion-free integral cohomology, and $x \in KU^0(X)$ has filtration greater than or equal to $2q$ and p is a prime, it is shown that there exist elements x_i of filtration greater than or equal to $2q+2i(p-1)$ such that $\psi^p(x) = \sum_i p^{q-i}x_i$ with $x_q = x^p$. If X has no torsion and $KU^*(X)$ is treated as mod 2 graded then

$$\mathrm{Gr}KU^*(X) \otimes \mathbb{Z}/p \cong H^*(X;\mathbb{Z}/p).$$

If x lies in filtration greater than or equal to $2q$ write $\overline{x}$ for the corresponding element of $H^{2q}(X;\mathbb{Z}/p)$. Then Atiyah's main filtration result is that $\overline{x}_i = P^i(\overline{x})$ where P^i is the Steenrod operation (interpreted as Sq^{2i} if $p = 2$).

Generalising Atiyah's results to spaces with torsion in their integral cohomology has remained unsolved since the appearance of [25]. In this chapter I shall apply the upper triangular technology of Chapters 3 and 5 to offer a solution to this problem, although the solution will look at first sight very different from [25].

To begin with, we shall phrase our results in terms of 2-local connective K-theory – bu_* or bo_* – and the Adams operations ψ^3 upon it. There is no loss of generality in choosing ψ^3 since in this context it is 2-adically dense among all the Adams operations [168]. It would be very difficult to phrase the outcome of our method as elegantly as [25] and it would take us far afield from our obsession with the Arf-Kervaire invariant. Therefore I shall content myself with the following result, which is my best example of upper triangular technology in action.

Recall from Chapter 6 § 6.4.5 (see also Chapter 8, Example 8.3.2) that if

$$\Theta_{8m-2} : S^{8m-2} \longrightarrow \mathbb{RP}^{8m-2}$$

is an S-map then

$$bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) \cong \mathbb{Z}_2\langle\iota\rangle \oplus \mathbb{Z}/2^{4m-1}$$

and the bo e-invariant of Θ_{8m-2} [5] is the value of

$$(\psi^3 - 1)(\iota) \in \mathbb{Z}/2^{4m-1}$$

in $bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}))$.

Theorem 8.1.2. Let m be a positive integer and let $\Theta_{8m-2} : S^{8m-2} \longrightarrow \mathbb{RP}^{8m-2}$ be a morphism in the 2-local stable homotopy category. Then the *bo*-e-invariant of Θ_{8m-2} is $\frac{(3^{4m}-1)}{4}(2u+1)$ (modulo 2^{4m-1}) if and only if $m = 2^q$ and Θ_{8m-2} is detected by the Steenrod operation $Sq^{2^{q+2}}$.

Remark 8.1.3. Recall from Chapter 1 § 1.8.5 that the detection by $Sq^{2^{q+2}}$ is equivalent, by Chapter 1, Theorem 1.5.10, to the image of Θ_{8m-2} in $\pi_{8m-2}(\Sigma^\infty S^0) \otimes \mathbb{Z}_2$ under the Kahn-Priddy map having Arf-Kervaire invariant equal to one.

On the other hand, by the proofs of Chapter 7, Theorem 7.2.2 given in either Chapter 7 § 7.2.4 or § 7.2.5, the *bo*-e-invariant condition is equivalent to the ju-theory or jo-theory Hurewicz image of Θ_{8m-2} being non-trivial (of order two); as explained in Chapter 1 § 1.8.9 and Theorem 1.8.10, this result was conjectured in [30].

The upper triangular technology referred to in the title of this chapter (or rather two-thirds of it) consists of the following two results, which were proved in Chapters 3 and 5, and which I shall recapitulate here for the reader's convenience.

Theorem 8.1.4 (Chapter 3 Theorem 3.1.2; [252] § 2.1))**.** There is an isomorphism of the form

$$\psi : \mathrm{Aut}^0_{\text{left-}bu\text{-mod}}(bu \wedge bo) \xrightarrow{\cong} U_\infty \mathbb{Z}_2.$$

Theorem 8.1.5 (Chapter 5 Theorem 5.1.2; [27] § 1.1)**.** Under the isomorphism ψ the automorphism $1 \wedge \psi^3$ corresponds to an element in the conjugacy class of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The remaining third of the upper triangular technology is the material of this chapter which first appeared in [254]. In the stable homotopy category of 2-local spectra I shall describe how to use the knowledge of the crucial Adams operation ψ^3 on connective K-theory $\pi_*(bo \wedge X)$ or $\pi_*(bu \wedge X)$ to evaluate the unit maps

$$(\eta \wedge 1 \wedge 1)_* : \pi_*(bo \wedge X) \longrightarrow \pi_*(bo \wedge bo \wedge X)$$

or

$$(\eta \wedge 1 \wedge 1)_* : \pi_*(bu \wedge X) \longrightarrow \pi_*(bu \wedge bu \wedge X).$$

Shortly I shall describe the general method. However, the extended examples which constitute the most important part of this chapter concern the case when X is a real projective space and in these examples it will be easier and more convenient to use the knowledge of ψ^3 on $\pi_*(bo \wedge X)$ to evaluate the unit map

$$(\eta \wedge 1 \wedge 1)_* : \pi_*(bo \wedge X) \longrightarrow \pi_*(bu \wedge bo \wedge X).$$

The reason why my upper triangular technology may be considered as the extension of Atiyah's result [25] to the case of spaces with torsion in their homology is that the unit map $(\eta \wedge 1 \wedge 1)_*$ is the origin of cohomology operations. Briefly, one splits $E \wedge E$ as a left E-module spectrum into a sum of pieces $E \wedge X_k$. Then, in terms of the splitting, the unit $(1 \wedge \eta \wedge 1)_*$ is a rather boring left E-module map but $(\eta \wedge 1 \wedge 1)_*$ (called η_R in Chapter 7, § 1) leads to very useful operations. This is illustrated in BP-theory where the Quillen operations are constructed using the formula of ([9] Theorem 16.1 p. 112; [294] Theorem 3.11 p. 17; see also Chapter 7, Lemma 7.1.2). Substantial use of these operations was made by Richard Kane in the classification of finite H-spaces [137]. Therefore one may expect the upper triangular technology to be useful in passing from the Adams operation to the calculation of many more operations than just the $Sq^{2^{q+2}}$ of Theorem 8.1.2.

To understand the upper triangular technology method without delving into the technicalities consider the diagram of Fig. 8.1.

In the diagram η is the unit of bu, c is complexification, μ is the bu-multiplication and ψ^3 is the Adams operation. The homomorphism λ_* is equal to $(\mu \wedge 1)_* \cdot (1 \wedge c \wedge 1)_*$. The diagram does *not*[1] commute because the right-hand trapezium does not commute. However, the outer rectangle, the upper and lower triangles and the left-hand trapezium do commute. In addition the right-hand trapezium *would* commute if the right-hand vertical $(1 \wedge \psi^3 \wedge 1)_*$ were replaced by $(\psi^3 \wedge \psi^3 \wedge 1)_*$.

Recall that Theorem 8.1.4 was proved in Chapter 3 as a consequence of a 2-local left-bu-module equivalence of spectra of the form

$$bu \wedge bo \simeq \vee_{k=0}^{\infty}\ bu \wedge (F_{4k}/F_{4k-1})$$

and that the (i,j)th entry of a matrix corresponds to a left-bu-module map from $bu \wedge (F_{4j}/F_{4j-1})$ to $bu \wedge (F_{4i}/F_{4i-1})$ with $j \geq i$. In particular 9^i in the (i,i)th entry corresponds to 9^i times the identity map and a 1 in the $(i-1,i)$th entry corresponds to a specific left-bu-module map

$$\iota_{i,i-1} : bu \wedge (F_{4i}/F_{4i-1}) \longrightarrow bu \wedge (F_{4i-4}/F_{4i-5}).$$

Suppose that we have a good understanding of the maps

$$\pi_*(bu \wedge (F_{4i}/F_{4i-1}) \wedge X) \stackrel{(\iota_{i,i-1} \wedge 1)_*}{\longrightarrow} \pi_*(bu \wedge (F_{4i-4}/F_{4i-5}) \wedge X).$$

For example, when $X = \Sigma^\infty \mathbb{RP}^{8m-2}$ and other closely related spectra $(\iota_{i,i-1} \wedge 1)_*$ is almost injective. Suppose that we have $x \in \pi_*(bo \wedge X)$ for which we know $(\psi^3 \wedge$

[1] Notwithstanding this fact, I did encounter a representative of the Spitalfields Mathematical Society who insisted otherwise!

$$
\begin{tikzcd}
\pi_j(bo \wedge X) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"'] \arrow[dr, "(c \wedge 1)_*"] \arrow[ddd, "(\psi^3 \wedge 1)_*"] & & \pi_j(bu \wedge bo \wedge X) \arrow[dl, "\lambda_*"'] \arrow[ddd, "(1 \wedge \psi^3 \wedge 1)_*"'] \\
& \pi_j(bu \wedge X) \arrow[d, "(\psi^3 \wedge 1)_*"] & \\
& \pi_j(bu \wedge X) & \\
\pi_j(bo \wedge X) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"] \arrow[ur, "(c \wedge 1)_*"'] & & \pi_j(bu \wedge bo \wedge X) \arrow[ul, "\lambda_*"]
\end{tikzcd}
$$

Figure 8.1.

$1)_*(x)$; then chasing the diagram gives us the first entry in each of $(\eta \wedge 1 \wedge 1)_*(x)$ and $(\eta \wedge 1 \wedge 1)_*(\psi^3 \wedge 1)_*(x)$. This information and the fact that we know the matrix for $(1 \wedge \psi^3 \wedge 1)_*$ permits us to compute by induction each of the entries in

$$(\eta \wedge 1 \wedge 1)_*(x) \in \oplus_k \ \pi_*(bu \wedge (F_{4i}/F_{4i-1}) \wedge X).$$

In my main example X will be $\mathrm{Cone}(\Theta_{8m-2})$, the mapping cone of

$$\Theta_{8m-2} : S^{8m-2} \longrightarrow \mathbb{RP}^{8m-2}.$$

This example was chosen because it has lots of torsion in its integral homology and because of its relation to the Arf-Kervaire invariant problem. Assuming the formula for $(\eta \wedge 1 \wedge 1)_*(x)$ when $X = \mathrm{Cone}(\Theta_{8m-2})$, which is proved in Theorem 8.4.6 and Theorem 8.4.7, I shall now prove Theorem 8.1.2.

8.1.6. Proof of Theorem 8.1.2 when $m = 2^q$. Let $m = 2^q$ then we have ι_{8m-1} generating a copy of the 2-adic integers in

$$bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) \cong \mathbb{Z}_2\langle \iota_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-1}$$

giving a stable homotopy class

$$\iota_{8m-1} : S^{2^{q+3}-1} \longrightarrow bo \wedge \mathrm{Cone}(\Theta_{8m-2}).$$

Let $\iota : bu \longrightarrow H\mathbb{Z}/2$ be the canonical cohomology class. Then, if $h_{8m-1} \in H_{2^{q+3}-1}(bo \wedge \mathrm{Cone}(\Theta_{8m-2}); \mathbb{Z}/2)$ is the mod 2 Hurewicz image of ι_{8m-1}, it is represented by either of the compositions

$$\begin{aligned} S^{2^{q+3}-1} &\stackrel{\iota_{8m-1}}{\longrightarrow} bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \stackrel{(\eta\wedge 1\wedge 1)}{\longrightarrow} bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \\ &\stackrel{(\iota\wedge 1\wedge 1)}{\longrightarrow} H\mathbb{Z}/2 \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \end{aligned}$$

or

$$\begin{aligned} S^{2^{q+3}-1} &\stackrel{\iota_{8m-1}}{\longrightarrow} S^0 \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \\ &\stackrel{(\tilde{\eta}\wedge 1\wedge 1)}{\longrightarrow} H\mathbb{Z}/2 \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \end{aligned}$$

where $\tilde{\eta}$ is the unit for $H\mathbb{Z}/2$.

We have an isomorphism of $\mathbb{Z}/2$-vector spaces, from Chapter 3 § 3.1.5,

$$H_*(bo; \mathbb{Z}/2) \cong \oplus_{k \geq 0} H_*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$$

and

$$H_j(\mathrm{Cone}(\Theta_{8m-2}); \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2\langle \iota_{8m-1}\rangle & \text{if } j = 8m-1, \\ \mathbb{Z}/2\langle v_j\rangle & \text{if } 1 \leq j = 8m-2, \\ 0 & \text{otherwise.} \end{cases}$$

Here ι_{8m-1} in mod 2 homology is the image under the canonical map $bo \longrightarrow H\mathbb{Z}/2$ of ι_{8m-1} in connective K-theory while v_j is the image of the generator of $H_j(\mathbb{RP}^{8m-2}; \mathbb{Z}/2)$.

If the bo-e-invariant of § 8.1.1 is given by

$$(\psi^3 - 1)(\iota) = \frac{(3^{4m} - 1)}{4}(2u + 1) \in \mathbb{Z}/2^{4m-1}$$

in $bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}))$ then I shall show in Corollary § 8.4.10 that

$$h_{8m-1} \in H_*(bo; \mathbb{Z}/2) \otimes H_*(\mathrm{Cone}(\Theta_{8m-2}); \mathbb{Z}/2)$$

has the form

$$h_{8m-1} = \tilde{z}_0 \otimes \iota_{2^{q+3}-1} + \tilde{z}_{2^{q+2}} \otimes v_{2^{q+2}-1} + \sum_{j=1}^{2^{q+2}-2} x_{8m-1-j} \otimes v_j$$

with $x_{8m-1-j} \in H_*(F_{2^{q+2}+t}/F_{2^{q+2}-1+t}; \mathbb{Z}/2)$ with $t \geq 0$. The classes denoted by $\tilde{z}_{4k} \in H_{4k}(bo; \mathbb{Z}/2)$ are the elements which correspond to the mod 2 Hurewicz image of the bottom cell in F_{4k}/F_{4k-1} under the homology isomorphism mentioned above so that, in particular, $\tilde{z}_0 = 1 \in H_0(bo; \mathbb{Z}/2)$.

If $X \in \mathcal{A}^{2^{q+2}}$ is an element of the mod 2 Steenrod algebra of degree 2^{q+2} we write X_* for the dual homomorphism on mod 2 homology, which decreases dimensions by 2^{q+2}. Since $H^*(bo; \mathbb{Z}/2)$ is a cyclic $\mathcal{A}$-module generated by 1 in dimension zero, there exists an X such that $X_*(\tilde{z}_{2^{q+2}}) = \tilde{z}_0$. In fact, since the Hurewicz image of the bottom cell in $H_{4k}(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$ equals ξ_1^{4k}, unravelling the homology isomorphism between $H_*(bu; \mathbb{Z}/2) \otimes H_*(bo; \mathbb{Z}/2)$ and $H_*(bu; \mathbb{Z}/2) \otimes H_*(\vee_{l \geq 0} F_{4l}/F_{4l-1}; \mathbb{Z}/2)$ described in [252] one can show that $Sq_*^{4k}(\tilde{z}_{4k}) = \tilde{z}_0$. This will be accomplished in Proposition 8.1.9 and Corollary 8.1.10 below; for the moment we shall assume this fact.

Since h_{8m-1} is stably spherical we have $0 = Sq_*^{2^{q+2}}(h_{8m-1})$ so that

$$\begin{aligned} &\tilde{z}_0 \otimes Sq_*^{2^{q+2}}(\iota_{2^{q+3}-1}) \\ &= \tilde{z}_0 \otimes v_{2^{q+2}-1} + \tilde{z}_{2^{q+2}} \otimes Sq_*^{2^{q+2}}(v_{2^{q+2}-1}) \\ &\quad + \textstyle\sum_{j=1}^{2^{q+2}-1} Sq_*^j(\tilde{z}_{2^{q+2}}) \otimes Sq_*^{2^{q+2}-j}(v_{2^{q+2}-1}) \\ &\quad + \textstyle\sum_{j=1}^{2^{q+2}-2} x_{8m-1-j} \otimes Sq_*^{2^{q+2}}(v_j) \\ &\quad + \textstyle\sum_{a=1}^{2^{q+2}-1} \sum_{j=1}^{2^{q+2}-2} Sq_*^a(x_{8m-1-j}) \otimes Sq_*^{2^{q+2}-a}(v_j) \end{aligned}$$

which implies, comparing coefficients of $\tilde{z}_0$, that

$$Sq_*^{2^{q+2}}(\iota_{2^{q+3}-1}) = v_{2^{q+2}-1}$$

and therefore Θ_{8m-2} is detected by $Sq^{2^{q+2}}$ on its mapping cone, as required. That is:

$$\begin{aligned} &H_{2^{q+3}-1}(\mathrm{Cone}(\Theta_{2^{q+3}-2}); \mathbb{Z}/2) \xrightarrow{\cong} H_{2^{q+3}-1}(S^{2^{q+3}-1}; \mathbb{Z}/2), \\ &Sq_*^{2^{q+2}} : H_{2^{q+3}-1}(\mathrm{Cone}(\Theta_{2^{q+3}-2}); \mathbb{Z}/2) \xrightarrow{\cong} H_{2^{q+2}-1}(\mathrm{Cone}(\theta_{2^{q+3}-2}); \mathbb{Z}/2), \\ &H_{2^{q+2}-1}(\Sigma\mathbb{RP}^{2^{q+3}-2}; \mathbb{Z}/2) \xrightarrow{\cong} H_{2^{q+2}-1}(\mathrm{Cone}(\Theta_{2^{q+3}-2}); \mathbb{Z}/2). \end{aligned}$$

Conversely, if Θ_{8m-2} is detected by $Sq^{2^{q+2}}$ on its mapping cone, reversing the argument shows that the coefficient of $\tilde{z}_{2^{q+2}} \otimes v_{2^{q+2}-1}$ in h_{8m-1} must be non-zero. This implies that the

$$\pi_{8m-1}(bu \wedge (F_{2^{q+2}}/F_{2^{q+2}-1}) \wedge \mathrm{Cone}(\Theta_{8m-2}))$$

component of

$$(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) \in \oplus_k \ \pi_*(bu \wedge (F_{4i}/F_{4i-1}) \wedge \mathrm{Cone}(\Theta_{8m-2}))$$

must be non-zero and detected by mod 2 homology (that is, represented in the classical Adams spectral sequence on the $s = 0$ line). However, in Theorem 8.4.7 and Corollary 8.4.8, I shall show that this can only happen when the bo-e-invariant of Θ_{8m-2} is equal to $\frac{(3^{4m}-1)}{4}(2u+1)$ for some integer u (and, of course, only when m is a power of 2). □

8.1.7. Proof of Theorem 8.1.2 when $m \neq 2^q$. When m is not a power of 2, Theorem 8.4.6 shows that Θ_{8m-2} with this e-invariant cannot exist. For mod 2 cohomology Θ_{8m-2} cannot be detected by a primary operation on $H^*(\mathrm{Cone}(\Theta_{8m-2}); \mathbb{Z}/2)$ because, as explained in Chapter 1 § 1.8.5(ii), this would imply the existence of a non-trivial element in the 2-primary torsion of the stable homotopy groups of spheres in degree $8m - 2$ which is represented in the classical Adams spectral sequence of Chapter 1, Theorem 1.4.3 on the $s = 2$ line. However, the main theorem of [47] shows that this is impossible unless m is a power of 2.

Incidentally, another proof of Bill Browder's theorem [47], using the Kahn-Priddy theorem, was given by John Jones and Elmer Rees in [128] and another alternative proof can be given using the BP-Hurewicz homomorphism calculations of [251] (see also Chapter 7). □

8.1.8. Discussion in preparation for Proposition 8.1.9. In this subsection I shall use the results concerning the mod 2 Steenrod algebra $\mathcal{A}$ and its dual $\mathcal{A}^* = \mathrm{Hom}(\mathcal{A}, \mathbb{Z}/2)$ from ([259] Chapter II § 2; see also Chapter 1 Section 6). The results there concern the right $\mathcal{A}$-module structure on $\mathcal{A}^*$. If $f : \mathcal{A} \longrightarrow \mathbb{Z}/2$ and $\alpha, \beta \in \mathcal{A}$ define $(f \cdot \alpha)$ by the formula

$$(f \cdot \alpha)(\beta) = f(\alpha\beta)$$

for all β. This is, of course, a right module action because

$$((f \cdot \alpha_1) \cdot \alpha_2)(\beta) = (f \cdot \alpha_1)(\alpha_2\beta) = f(\alpha_1\alpha_2\beta) = (f \cdot (\alpha_1\alpha_2))(\beta).$$

Now the left multiplication action of $\mathcal{A}$ on itself corresponds to the usual action of $\mathcal{A}$ on the mod 2 cohomology of the mod 2 Eilenberg-Maclane spectrum $H^*(H\mathbb{Z}/2; \mathbb{Z}/2)$. Therefore right multiplication by Sq^n on $\mathcal{A}^*$ corresponds to the dual homomorphism $(Sq^n)_*$ on the mod 2 homology of the mod 2 Eilenberg-Maclane spectrum $H_*(H\mathbb{Z}/2; \mathbb{Z}/2)$.

Furthermore, since the mod 2 cohomology of bu and bo are both quotients of $\mathcal{A}$ and their mod 2 homologies are right $\mathcal{A}$-submodules of $\mathcal{A}^*$, we may read off formulae and properties of the $(Sq^n)_*$'s acting on $H_*(bu; \mathbb{Z}/2)$ and $H_*(bo; \mathbb{Z}/2)$ from the results of [259] concerning the right $\mathcal{A}$-module structure on $\mathcal{A}^*$.

Let

$$\hat{L} : bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo$$

denote the 2-adic homotopy equivalence of (Chapter 3, Theorem 3.1.6(ii); see also [252] Theorem 2.3). As usual, let η denote the unit of a ring spectrum. The mod 2 homology element $\tilde{z}_{4k} \in H_{4k}(bo; \mathbb{Z}/2)$, which appears in the proof given in § 8.1.6, maps under the injection

$$H_*(bo; \mathbb{Z}/2) \xrightarrow{(\eta \wedge 1)_*} H_*(bu \wedge bo; \mathbb{Z}/2) \cong H_*(bu; \mathbb{Z}/2) \otimes H_*(bo; \mathbb{Z}/2)$$

to the mod 2 Hurewicz image of

$$S^{4k} \subset S^0 \wedge F_{4k}/F_{4k-1} \xrightarrow{\eta \wedge 1} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\hat{L}} bu \wedge bo$$

where the first map is the inclusion of the bottom cell.

Next we observe that, although $\hat{L}$ is only defined up to composition with a left-bu-module homotopy equivalence which corresponds to an upper triangular matrix via (Chapter 3, Theorem 3.1.2; see also [252] Theorem 1.2), the element $\tilde{z}_{4k}$ is uniquely defined. This is because the group of upper triangular matrices corresponds to 2-adic left-bu-module equivalences of $bu \wedge bo$ which induce the identity of mod 2 homology.

The dual Steenrod algebra (see Chapter 1, § 6) is a Hopf algebra whose ring structure is a polynomial ring

$$\mathcal{A}^* \cong \mathbb{Z}/2[\xi_1, \xi_2, \ldots, \xi_n, \ldots]$$

with $\deg(\xi_n) = 2^n - 1$. As explained in Chapter 3, the homology of bu and bo is given by the subalgebras

$$H_*(bu; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^2, \xi_2^2, \ldots, \xi_n, \ldots]$$

and

$$H_*(bo; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^4, \xi_2^2, \ldots, \xi_n, \ldots].$$

Also $H_*(\Omega^2 S^3; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1, \xi_2, \ldots, \xi_n, \ldots]$. There is a stable splitting (see Chapter 1, § 5) $\Omega^2 S^3 \simeq \vee_{k \geq 1} (F_k/F_{k-1})$ and the H-space product on $\Omega^2 S^3$ induces stable homotopy classes of maps of the form

$$(F_k/F_{k-1}) \wedge (F_l/F_{l-1}) \longrightarrow (F_{k+l}/F_{k+l-1}).$$

These maps make $\vee_{k \geq 1} (F_{4k}/F_{4k-1})$ into a submonoid of $\Omega^2 S^3$ whose homology equals the augmentation ideal of the algebra (by convention, $F_0/F_{-1} = S^0$)

$$H_*(\vee_{k \geq 0} (F_{4k}/F_{4k-1}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^4, \xi_2^2, \ldots, \xi_n, \ldots].$$

Furthermore the algebra map which sends each monomial in the elements ξ_1^4, ξ_2^2, ξ_3, ... to itself gives a left B-module isomorphism

$$\Phi : H_*(\vee_{k \geq 0} (F_{4k}/F_{4k-1}); \mathbb{Z}/2) \xrightarrow{\cong} H_*(bo; \mathbb{Z}/2)$$

where $B = E(Sq^1, Sq^{0,1})$, the exterior subalgebra of $\mathcal{A}$ generated by Sq^1 and $Sq^{0,1} = Sq^1Sq^2 + Sq^2Sq^1 = Sq^3 + Sq^2Sq^1$. On the left-hand side the Hurewicz image of the bottom cell in $(F_{2^{q+2}}/F_{2^{q+2}-1})$ is $\xi_1^{2^{q+2}}$ for $q \geq 0$ ([9] p. 341 (proof of 16.4)). When $k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$ with $0 \leq a_1 < a_2 < \cdots < a_r$ the bottom cell of (F_{4k}/F_{4k-1}) is given by the product of bottom cells

$$\wedge_{j=1}^{r}\ S^{2^{a_j+2}} \longrightarrow \wedge_{j=1}^{r}\ (F_{2^{a_j+2}}/F_{2^{a_j+2}-1}) \longrightarrow (F_{4k}/F_{4k-1})$$

whose Hurewicz image is ξ_1^{4k}.

There are isomorphisms of left $\mathcal{A}$-modules

$$\psi_1 : \mathcal{A} \otimes_B H^*(bo;\mathbb{Z}/2) \xrightarrow{\cong} H^*(bu;\mathbb{Z}/2) \otimes H^*(bo;\mathbb{Z}/2)$$

and

$$\begin{array}{l} \mathcal{A} \otimes_B (\oplus_{k\geq 0}\ H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2)) \\ \qquad \stackrel{\psi_2}{\rightarrow} H^*(bu;\mathbb{Z}/2) \otimes (\oplus_{k\geq 0}\ H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2)) \end{array}$$

given by $\psi(v \otimes_B w) = (\sum v' \otimes_B 1) \otimes v'' \cdot w$ where the diagonal of v is $\Delta(v) = \sum\ v' \otimes v''$.

If $\epsilon : H^*(bu;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ denotes the augmentation homomorphism given by $\epsilon = \eta^*$ then, for any left $\mathcal{A}$-module M, the composition

$$(\epsilon \otimes 1) \cdot \psi : \mathcal{A} \otimes_B M \longrightarrow \mathcal{A} \otimes M \longrightarrow M$$

is given by $(\epsilon \otimes 1) \cdot \psi(a \otimes m) = a \cdot m = \mu(a \otimes m)$ where μ denotes the $\mathcal{A}$-module multiplication.

Proposition 8.1.9. The mod 2 Hurewicz image of

$$S^{4k} \subset S^0 \wedge F_{4k}/F_{4k-1} \xrightarrow{\eta\wedge 1} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\hat{L}} bu \wedge bo,$$

where the first map is the inclusion of the bottom cell, is equal to

$$1 \otimes \xi_1^{4k} \in H_*(bu;\mathbb{Z}/2) \otimes H_*(bo;\mathbb{Z}/2).$$

Proof. The mod 2 homology of $F_{2^{q+2}}/F_{2^{q+2}-1}$ is equal to the "lightning flash" B-module described in ([9] p. 341) and having $\xi_1^{2^{q+2}}$ as the generator of its lowest-dimensional homology. Therefore, by the discussion of § 8.1.8, when $k = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_r}$ with $0 \leq a_1 < a_2 < \cdots < a_r$ the Hurewicz image bottom cell of (F_{4k}/F_{4k-1}) is given by the product $\xi_1^{2^{a_1+2}}\xi_1^{2^{a_2+2}} \ldots \xi_1^{2^{a_r+2}} = \xi_1^{4k}$. Therefore the image in

$$\begin{array}{l} H_*(bu \wedge (F_{4k}/F_{4k-1});\mathbb{Z}/2) \\ \qquad \subset H_*(bu;\mathbb{Z}/2) \otimes H_*(\wedge_{l\geq 0}\ (F_{4l}/F_{4l-1});\mathbb{Z}/2) \end{array}$$

is given by $(\eta \wedge 1)_*(\xi_1^{4k}) = 1 \otimes \xi_1^{4k}$.

The homomorphism $\hat{L}_*$ is given by composing the dual of ψ_2 with the dual of $1 \otimes_B \hat{\Phi}$ and the inverse of the dual of ψ_1 where $\hat{\Phi}$ is the dual of Φ.

The composition $\psi_2^{-1} \cdot (\eta \wedge 1)_* = \mu_*$ which by homology-cohomology duality may be identified with the map

$$\begin{array}{c} \oplus_{l \geq 0} \operatorname{Hom}(H^*((F_{4l}/F_{4l-1}); \mathbb{Z}/2), \mathbb{Z}/2) \\ (- \cdot \mu^*) \ \downarrow \\ \oplus_{l \geq 0} \operatorname{Hom}(\mathcal{A} \otimes_B H^*((F_{4l}/F_{4l-1}); \mathbb{Z}/2), \mathbb{Z}/2). \end{array}$$

Thus the homomorphsim $\langle \xi_1^{4k}, - \rangle$, which corresponds to ξ_1^{4k}, is mapped to the homomorphism

$$v \otimes_B z \mapsto \langle \xi_1^{4k}, vz \rangle$$

which is zero unless z has a non-zero component in the summand

$$H^*((F_{4k}/F_{4k-1}); \mathbb{Z}/2)$$

and then only if $\deg(v) = 0$. Hence $\psi_2^{-1} \cdot (\eta \wedge 1)_*(\xi_1^{4k})$ corresponds to the homomorphism

$$v \otimes_B z \mapsto \epsilon(v) \langle \xi_1^{4k}, z \rangle.$$

Note that this homomorphism is well defined because $Sq_*^1(\xi_1^{4k}) = 0 = Sq_*^{0,1}(\xi_1^{4k})$ and so for $b \in B$ of strictly positive degree

$$0 = \epsilon(v) \langle b_*(\xi_1^{4k}), z \rangle = \epsilon(v) \langle \xi_1^{4k}, bz \rangle$$

and $\epsilon(vb) = 0$, too.

By definition of the homomorphism Φ,

$$(1 \otimes_B \hat{\Phi})^*(\psi_2^{-1} \cdot (\eta \wedge 1)_*(\xi_1^{4k}))$$

also corresponds to the homomorphism in

$$\operatorname{Hom}(\mathcal{A} \otimes_B H^*(bo; \mathbb{Z}/2), \mathbb{Z}/2)$$

given by $v \otimes_B z \mapsto \epsilon(v) \langle \xi_1^{4k}, z \rangle$, which is trivial unless $v = 1$.

Finally $H^*(bu; \mathbb{Z}/2) \cong \mathcal{A} \otimes_B \mathbb{Z}/2$ ([9] Proposition 16.6 p. 335) and we have

$$\begin{aligned} \psi_1(v \otimes_B z) &= \sum (v' \otimes_B 1) \otimes v'' z \sum \epsilon(v') \langle \xi_1^{4k}, v'' z \rangle \\ &= \epsilon(v) \langle \xi_1^{4k}, z \rangle \end{aligned}$$

so that the image of ξ_1^{4k} in $H_*(bu; \mathbb{Z}/2) \otimes H_*(bo; \mathbb{Z}/2)$ is equal to $1 \otimes \xi_1^{4k}$, as required. □

Corollary 8.1.10. In the notation of § 8.1.6, in $H_*(bo; \mathbb{Z}/2)$,

$$Sq_*^{4k}(\tilde{z}_{4k}) = \tilde{z}_0.$$

Proof. This follows since, by the Cartan formula and the isomorphism

$$H^*(bo; \mathbb{Z}/2) \cong \mathcal{A}/(\mathcal{A}Sq^1 + \mathcal{A}Sq^2) \quad ([9] \text{ p. } 336),$$

$$Sq_*^{4k}(\tilde{z}_{4k}) = (Sq_*^4(\tilde{z}_4))^k = \tilde{z}_0^k = \tilde{z}_0.$$ □

8.2 K-theory examples

8.2.1. Let $bu_*(X)$ (resp. $KU_*(X)$) denote the reduced, connective (resp. periodic) complex K-theory of a (based) CW complex X. (That is, we shall not need 2-adic coefficients just yet.) When X equals the zero-dimensional sphere we have $bu_*(S^0) \cong \mathbb{Z}[u]$ and $KU_*(S^0) \cong \mathbb{Z}[u^{\pm 1}]$ where $\deg(u) = 2$. Let $\mathbb{RP}^n$ denote n-dimensional real projective space. Let $\mathbb{Z}/t\langle w\rangle$ denote a cyclic group of order t with generator w The following result is well known.

Proposition 8.2.2. For $1 \leq m \leq \infty$,

$$bu_j(\mathbb{RP}^{2m}) = \begin{cases} 0 & \text{if } j \text{ is even,} \\ \mathbb{Z}/2^i\langle v_{2i-1}\rangle & \text{if } 1 \leq j = 2i-1 < 2m, \\ \mathbb{Z}/2^m\langle v_{2m-1}u^{i-m}\rangle & \text{if } 2m < j = 2i-1. \end{cases}$$

In addition, the generators may be chosen to satisfy $uv_{2i-1} = 2v_{2i+1}$ for $1 \leq i \leq m-1$.

Proof. The Atiyah-Hirzebruch spectral sequences for computing $bu_*(\mathbb{RP}^{2m})$ and $KU_*(\mathbb{RP}^{2m})$ collapse for dimensional reasons. This implies that $bu_j(\mathbb{RP}^{2m})$ has the correct order. It also implies the injectivity of the canonical maps

$$h_j(\mathbb{RP}^{2m}) \longrightarrow h_j(\mathbb{RP}^{2m+2}) \ (h = bu, KU)$$

and

$$\pi_* : bu_j(\mathbb{RP}^{2m}) \longrightarrow KU_j(\mathbb{RP}^{2m}).$$

However, by the universal coefficient theorem for KU and the results of ([26] p. 107) we have $KU_{2i-1}(\mathbb{RP}^\infty) \cong \mathbb{Z}/2^\infty$ so each $bu_{2i-1}(\mathbb{RP}^{2m})$ is cyclic. The relation $uv_{2i-1} = 2v_{2i+1}$ follows from Bott periodicity and the fact that the injection π_* commutes with multiplication by u. □

8.2.3. $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2),\mathbb{Z}/2)$**.** Let $B = E(Sq^1, Sq^{0,1})$ denote the exterior subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ [259] generated by Sq^1 and $Sq^{0,1} = [Sq^1, Sq^{0,1}]$. There is an isomorphism of bigraded algebras

$$\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b],$$

the polynomial algebra on a and b with $\mathrm{bideg}(a) = (1,1), \mathrm{bideg}(b) = (1,3)$. Also $\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2) = \langle x, x^2, \ldots\rangle/(x^{2m+1})$ with $Sq^1(x^n) = nx^{n+1}, Sq^{0,1}(x^n) = nx^{n+3}$.

Consider the bigraded $\mathbb{Z}/2[a,b]$-module

$$\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2),\mathbb{Z}/2).$$

Denote the non-zero element of $\mathrm{Ext}_B^{0,2i-1}(\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2),\mathbb{Z}/2)$ by $\tilde{v}_{2i-1}$ for $1 \leq i \leq m$.

Proposition 8.2.4. For $1 \leq m \leq \infty$ the bigraded $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$-module

$$\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$$

is equal to

$$\frac{\mathbb{Z}/2[a,b]\langle \tilde{v}_1, \tilde{v}_3, \ldots, \tilde{v}_{2m-1}\rangle}{\{a^i \tilde{v}_{2i-1},\ b\tilde{v}_{2i-1} - a\tilde{v}_{2i+1}\}}.$$

First Proof – using Proposition 8.2.2. We prove this by induction on m. When $m = 1$ we have

$$\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}/2), \mathbb{Z}/2) \cong \mathrm{Ext}_B^{*,*}(E(Sq^1)[1], \mathbb{Z}/2) \cong \mathbb{Z}/2[b]\langle \tilde{v}_1\rangle$$

where $X[n]$ denotes X with a dimension shift by n so that $X[1] = \Sigma X$ in the notation of [9] and [252]. We have a short exact sequence of B-modules

$$0 \longrightarrow \tilde{H}^*(\mathbb{RP}^{2m-2}; \mathbb{Z}/2)[-2] \stackrel{(x^2 \cdot -)}{\longrightarrow} \tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) \\ \longrightarrow \tilde{H}^*(\mathbb{RP}^2; \mathbb{Z}/2) \longrightarrow 0.$$

By induction, for each non-negative integer r the resulting long exact sequence yields an upper bound for the sums of $\mathbb{F}_2$-dimensions

$$\sum_{s=0}^{\infty} \sum_{t-s=r} \dim_{\mathbb{F}_2}(\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)) \leq d_r$$

where $d_r = 0$ if r is even, $d_{2i-1} = i$ for $1 \leq i \leq m$ and $d_{2i-1} = m$ for $m \leq i$. On the other hand, if $\mathbb{Z}_2$ denotes the 2-adic integers, the Adams spectral sequence ([9]; see also Chapter 1 Theorem 1.4.3) for $\pi_*(bu \wedge \mathbb{RP}^{2m}) \otimes \mathbb{Z}_2 = bu_*(\mathbb{RP}^{2m}) \otimes \mathbb{Z}_2$ has the form ([27]; [252]; see also Chapters 3 and 5)

$$E_2^{s,t} = \mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow bu_{t-s}(\mathbb{RP}^{2m}) \otimes \mathbb{Z}_2$$

and collapses for dimensional reasons, being concentrated where $t - s$ is odd. Therefore Proposition 8.2.2 shows that d_r is also a lower bound. The relations follow from the fact that a and b represent 2 and u respectively in the Adams spectral sequence for $bu_*(S^0) \otimes \mathbb{Z}_2$. □

Second Proof – using a resolution. The following uses an economical free B-resolution familiar to the experts. Map the free B-module $B\langle \sigma_1, \sigma_3, \ldots, \sigma_{2m-1}\rangle$ onto

$$\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) \cong \langle x, x^2, \ldots\rangle/(x^{2m+1})$$

by $\sigma_{2i-1} \mapsto x^{2i-1}$. The kernel of this surjection consists of

$$\langle Sq^{0,1}\sigma_{2i-1} + Sq^1\sigma_{2i+1} \mid i = 1, \ldots, m-1\rangle \\ \oplus \langle Sq^{0,1}\sigma_{2m-1}\rangle \oplus \langle Sq^1 Sq^{0,1}\sigma_{2i-1} \mid i = 1, \ldots, m\rangle,$$

which, as a B-module, is isomorphic to $\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2)[3]$, so that iterating this construction yields a free B-resolution of $\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2)$.

A careful inspection of this resolution shows that

$$\mathrm{Ext}_B^{s,*}(\tilde{H}^*(\mathbb{RP}^{2m};\mathbb{Z}/2),\mathbb{Z}/2) \cong \langle b^s\tilde{v}_{2i-1} \mid i = 1,\ldots,m\rangle$$

subject to the relations $a\tilde{v}_1 = 0$ and $b\tilde{v}_{2i-1} = a\tilde{v}_{2i+1}$ for $i = 1,\ldots,m-1$. $\square$

8.2.5. $\mathrm{Ext}_B^{*,*}(\tilde{H}^*(X \wedge (F_{4k}/F_{4k-1});\mathbb{Z}/2),\mathbb{Z}/2)$**.** For the reader's convenience I shall recapitulate some material from Chapters 3 and 5. Consider the second loopspace of the 3-sphere, Ω^2S^3. There exists a model for Ω^2S^3 which is filtered by finite complexes ([51],[243])

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \cdots \subset \Omega^2S^3 = \bigcup_{k\geq 1} F_k$$

and there is a stable homotopy equivalence, an example of the Snaith splitting (see Chapter 1, § 5), of the form

$$\Omega^2S^3 \simeq \vee_{k\geq 1}F_k/F_{k-1}.$$

Consider the finite complexes F_{4k}/F_{4k-1} with the convention that $F_0/F_{-1} = S^0$, the 0-sphere. Let $\alpha(n)$ denote the number of 1's in the dyadic expansion of the positive integer n. The results of Adams-Margolis ([9], [12]; see also Chapter 3 § 3.2.1, [27] and [252]) yield $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$-module isomorphisms of the form

$$\begin{aligned}&\mathrm{Ext}_B^{s,t}(\tilde{H}^*(X \wedge (F_{4k}/F_{4k-1});\mathbb{Z}/2),\mathbb{Z}/2)\\ &\cong \mathrm{Ext}_B^{s+2k-\alpha(k),t-2k-\alpha(k)}(\tilde{H}^*(X;\mathbb{Z}/2),\mathbb{Z}/2)\end{aligned}$$

for all $s > 0$. The theory of B-modules developed in [12] and giving rise to dimension-shifting isomorphisms of this type is described in detail in Chapter 3 § 3.2.1.

We shall need this isomorphism in the case where X is either a real projective space or a sphere. The case when X is a sphere is described extensively in ([252]; see Chapter 3) in connection with the left bu-module equivalence of 2-local spectra (see also [27] § 2)

$$\hat{L} : \vee_{k\geq 0}\ bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo.$$

The groups $\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1});\mathbb{Z}/2),\mathbb{Z}/2)$, when depicted in the traditional Adams spectral sequence manner with s along the vertical and $t-s$ along the horizontal axis, look as in Figure 8.2 below. The figure is interpreted as follows: the groups are $\mathbb{F}_2$-vector spaces which are possibly non-zero only when $s = 0$ and $t-s \geq 4k+1$ or a copy of $\mathbb{Z}/2$ at each point with $(s,t-s) = (v, 8k-2\alpha(k)-1+2w+2v)$ with $v = 1,2,3,\ldots$ and $1 \leq w \leq m$.

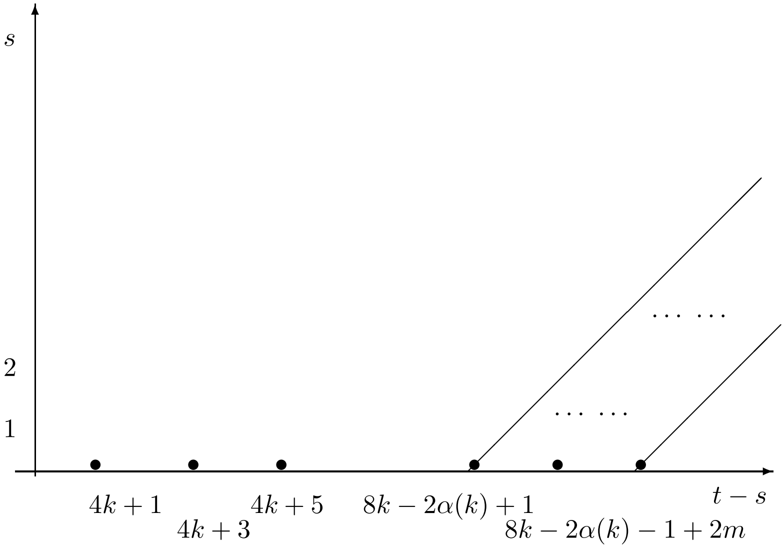

Figure 8.2.

We have the following result which describes the important aspects of the module structure over the bigraded algebra $\mathrm{Ext}_B^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]$. Let $\tilde{v}_{2i-1} \in \mathrm{Ext}_B^{0,2i-1}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$ be as in Proposition 8.2.4 and let

$$\hat{z}_{4k} \in \mathrm{Ext}_B^{0,4k}(\tilde{H}^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

be the element represented as a homomorphism on mod 2 cohomology by the inclusion of the bottom cell of F_{4k}/F_{4k-1} ([27] Theorem 2.12; see also Chapter 5 Theorem 5.2.12). Hence we have a (non-zero) external product

$$\tilde{v}_{2i-1}\hat{z}_{4k} \in \mathrm{Ext}_B^{0,4k+2i-1}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$$

for $1 \leq i \leq m$.

The following result concerning the multiplicative structure of the Ext-groups will be crucial in the applications of § 4.

Theorem 8.2.6. In $\mathrm{Ext}_B^{s,t}(\tilde{H}^*(\mathbb{RP}^{2m} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$ we have:

(i) $\tilde{v}_{2i-1}\hat{z}_{4k} \neq 0$ for each $1 \leq i \leq m$,

(ii) $b\tilde{v}_{2i-1}\hat{z}_{4k} = 0 = a\tilde{v}_i\hat{z}_{4k}$ for $i = 1, \ldots, 2k - \alpha(k)$,

(iii) $b^e\tilde{v}_{2i-1}\hat{z}_{4k} \neq 0$ for $e \geq 1$ and $i = 2k - \alpha(k) + 1, \ldots, m$.

The proof of Theorem 8.2.6 will be given in § 8.2.8 after some preliminaries concerning B-resolutions.

8.2.7. Resolutions. We begin by examining the low dimensions of the free B-resolution used in the second proof of Proposition 8.2.4 (to which I have added a generator called ν_5 for convenience).

For $1 \leq m \leq \infty$, the B-action on

$$\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) = \langle x, x^2, \ldots, x^{2m} \rangle$$

is given by $Sq^1(x^i) = ix^{i+1}$ and $Sq^{0,1}(x^i) = ix^{i+3}$. The beginning of a free B-resolution

$$\cdots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} \tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2) \longrightarrow 0$$

may be given by

$$P_0 = B\langle \sigma_1, \sigma_3, \ldots, \sigma_{2m-1} \rangle$$

and

$$P_1 = B\langle \nu_5, \nu_4, \nu_6, \ldots, \nu_{2m} \rangle$$

where $\deg(\sigma_i) = i$, $\epsilon(\sigma_i) = x^i$, $d(\nu_{2t}) = Sq^1\sigma_{2t-1} + Sq^{0,1}\sigma_{2t-3}$ and $d(\nu_5) = Sq^1 Sq^{0,1}\sigma_1$. By Proposition 8.2.4 $b\tilde{v}_{2i-1}$ is the only non-zero element in the group $\mathrm{Ext}_B^{1,2i+2}(\tilde{H}^*(\mathbb{RP}^{2m}; \mathbb{Z}/2), \mathbb{Z}/2)$ for $1 \leq i \leq m$. Therefore it must be represented by a homomorphism $h_i \in \mathrm{Hom}_B(P_1, \mathbb{Z}/2)$ given by $h_i(\nu_{2i+2}) \equiv 1$ (modulo 2) and $h_i(\nu_j) \equiv 0$ otherwise.

We now examine the beginning of the "lightning flash" B-module, denoted here by $H(k)$, which is the B-module $\Sigma^{2^k} L(2^{k-1} - 1)$ in the notation of [14]. The connection with $\tilde{H}^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2)$, via the theory of [12], was described in Chapter 3 § 3.2.1.

Let $H(k)$ be the graded $\mathbb{F}_2$-vector space with basis

$$y_{k,2^k}, y_{k,2^k+2}, y_{k,2^k+4}, \ldots, y_{k,2^{k+1}-2}, y_{k,2^k+3}, y_{k,2^k+5}, \ldots, y_{k,2^{k+1}-1}$$

where $\deg(y_i) = i$, with the "lightning flash" B-module structure given by

$$\begin{aligned} Sq^{0,1} y_{k,2^k} = y_{k,2^k+3} = Sq^1 y_{k,2^k+2}, \ldots, \\ Sq^{0,1} y_{k,2^{k+1}-4} = y_{k,2^{k+1}-1} = Sq^1 y_{k,2^{k+1}-2}. \end{aligned}$$

We define the start of a free B-resolution

$$\cdots \longrightarrow R_{k,1} \xrightarrow{d(k)} R_{k,0} \xrightarrow{\epsilon(k)} H(k) \longrightarrow 0$$

by

$$\begin{aligned} R_{k,0} &= B\langle \Sigma_{k,0,2^k}, \Sigma_{k,0,2^k+2}, \ldots, \Sigma_{k,0,2^{k+1}-2} \rangle \text{ and} \\ R_{k,1} &= B\langle \Sigma_{k,1,2^k+1}, \Sigma_{k,1,2^k+3}, \ldots, \Sigma_{k,1,2^{k+1}+1} \rangle \end{aligned}$$

where $\deg(\Sigma_{k,0,i}) = i = \deg(\Sigma_{k,1,i})$. Also $\epsilon(k)$ and $d(k)$ are given by $\epsilon(k)(\Sigma_{k,0,2i}) = y_{k,2i}$ and

$$\begin{aligned}
d(k)(\Sigma_{k,1,2^k+1}) &= Sq^1\Sigma_{k,0,2^k},\\
d(k)(\Sigma_{k,1,2^k+3}) &= Sq^1\Sigma_{k,0,2^k+2} + Sq^{0,1}\Sigma_{k,0,2^k},\\
d(k)(\Sigma_{k,1,2^k+5}) &= Sq^1\Sigma_{k,0,2^k+4} + Sq^{0,1}\Sigma_{k,0,2^k+2},\\
\vdots \quad & \quad \vdots \quad \quad \vdots\\
d(k)(\Sigma_{k,1,2^{k+1}+1}) &= Sq^{0,1}\Sigma_{k,0,2^{k+1}-2}.
\end{aligned}$$

We are now ready to embark on the proof of Theorem 8.2.6.

8.2.8. Proof of Theorem 8.2.6. Part (i) follows since the exterior product of two non-zero B-homomorphisms to $\mathbb{Z}/2$ is also non-zero and part (ii) follows because the elements in question lie in groups which are zero, by Proposition 8.2.4 and the discussion of § 8.2.5. Part (iii) is more substantial. By naturality it suffices to work with $m = \infty$.

Let $k = 2^{\epsilon_1} + 2^{\epsilon_2} + \cdots + 2^{\epsilon_t}$ with $0 \leq \epsilon_1 < \epsilon_2 < \cdots < \epsilon_t$ so we are interested in $i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \cdots + 2^{\epsilon_t+1} - t + 1$ and $4k = 2^{\epsilon_1+2} + 2^{\epsilon_2+2} + \cdots + 2^{\epsilon_t+2}$. From ([9] pp. 341–2) or ([252] p. 1267; see also Chapter 3) $\tilde{H}^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2) \cong \otimes_{j=1}^t H(\epsilon_j + 2)$. We have a free B-resolution given by the tensor product

$$\begin{aligned}
\cdots \to \otimes_{j=1,\ \sum a_j=1}^t R_{\epsilon_j+2,a_j} &\xrightarrow{d} \otimes_{j=1}^t R_{\epsilon_j+2,0}\\
&\longrightarrow \tilde{H}^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2) \to 0.
\end{aligned}$$

We introduce the convention that

$$\Sigma_{\epsilon_j+2,1,2s+1} = 0 = \Sigma_{\epsilon_j+2,0,2s} \text{ if } s \leq 2^{\epsilon_j+1} \text{ or } 2^{\epsilon_j+2} \leq s.$$

With this convention the differential has the form

$$d(\epsilon_j + 2)(\Sigma_{\epsilon_j+2,1,2s+1}) = Sq^1\Sigma_{\epsilon_j+2,1,2s} + Sq^{0,1}\Sigma_{\epsilon_j+2,1,2s-2}.$$

The element $\hat{z}_{4k}$ is represented by the B-homomorphism

$$g_k \in \mathrm{Hom}_B(\otimes_{j=1}^t R_{\epsilon_j+2,0}, \mathbb{Z}/2)$$

given by $g_k(\otimes_{j=1}^t \Sigma_{\epsilon_j+2,0,2^{\epsilon_j+2}}) \equiv 1$ (modulo 2) and $g_k(\otimes_{j=1}^t \Sigma_{\epsilon_j+2,0,w_j}) \equiv 0$ otherwise.

We must show that there does not exist a B-homomorphism

$$f \in \mathrm{Hom}_B(P_0 \otimes (\otimes_{j=1}^t R_{\epsilon_j+2,0}), \mathbb{Z}/2)$$

such that $f \cdot d = (0, h_i \otimes g_k)$ in the group of B-homomorphisms

$$\mathrm{Hom}_B(P_0 \otimes (\otimes_{j=1,\ \sum a_j=1}^t R_{\epsilon_j+2,a_j}) \oplus P_1 \otimes (\otimes_{j=1}^t R_{\epsilon_j+2,0}), \mathbb{Z}/2)$$

when $m = \infty$ and i lies in the range $i \geq 2k - \alpha(k)$. This will show that $b\tilde{v}_{2i-1}\hat{z}_{4k} \neq 0$ from which $b^e\tilde{v}_{2i-1}\hat{z}_{4k} \neq 0$ for $e \geq 1$ follows because the isomorphism of § 8.2.5 commutes with multiplication by b when $s > 0$.

I shall first give the argument to prove that f does not exist and finally I shall explain where the $i \geq 2k - \alpha(k)$ is necessary.

In degree $4k + 2i + 2$, suppose that we have the relation $(0, h_i \otimes g_k) = f \cdot d$. Then we shall apply $f \cdot d$ to all the B-basis elements in $P_* \otimes (\otimes_{j=1}^t R_{(\epsilon_j+2,*)})$ in resolution degree 1 and homological degree $4k + 2i + 2$ and add the results in two ways to get a contradiction. The basis elements in question are

$$\left\{ \nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}) \mid 2q + 2 + \sum_j 2s_j = 4k + 2i + 2 \right\}$$

and (where $0 \leq a_j \leq 1$ and $\sum_j a_j = 1$)

$$\left\{ \sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,a_j,2s_j+a_j}) \mid 2q + 2 + \sum_j 2s_j + a_j = 4k + 2i + 2 \right\}$$

disregarding, of course, the ones of this list which are zero by the convention introduced above.

We have

$$\begin{aligned}
&f(d(\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}))) \\
&= f(Sq^1\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})) \\
&\qquad + f(Sq^{0,1}\sigma_{2q-1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}))
\end{aligned}$$

and (where $0 \leq a_j \leq 1$ and $\sum_j a_j = 1$)

$$\begin{aligned}
&f(d(\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,a_j,2s_j+a_j}))) \\
&= f(\sigma_{2q+1} \otimes \cdots \otimes Sq^1\Sigma_{\epsilon_j+2,0,2s_j} \otimes \cdots) \\
&\qquad + f(\sigma_{2q+1} \otimes \cdots \otimes Sq^{0,1}\Sigma_{\epsilon_j+2,0,2s_j-2} \otimes \ldots)
\end{aligned}$$

where in the last expression the Sq's appear precisely in the unique factor for which a_j was equal to 1.

Now fix a $(t+1)$-tuple $(q, s_1, \ldots, s_t)$ such that $4k + 2i + 2 = 2q + 2 + \sum_j \ 2s_j$ and consider the sum

$$\begin{aligned}
&f(Sq^1\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})) \\
&\qquad + \sum_{j=1}^t \ f(\sigma_{2q+1} \otimes \cdots \otimes Sq^1\Sigma_{\epsilon_j+2,0,2s_j} \otimes \cdots) \\
&= Sq^1(f(\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j}))) \\
&= 0
\end{aligned}$$

because Sq^1 acts trivially on $\mathbb{Z}/2$ for dimensional reasons. Similarly

$$\begin{aligned} &f(Sq^{0,1}\sigma_{2q-1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})) \\ &\qquad\qquad + \textstyle\sum_{j=1}^t \ f(\sigma_{2q+1} \otimes \cdots \otimes Sq^{0,1}\Sigma_{\epsilon_j+2,0,2s_j} \otimes \cdots) \\ &= 0. \end{aligned}$$

Therefore applying $f \cdot d$ to each of the basis elements listed above and adding the results yields zero modulo 2.

Now consider what happens if we apply $(0, h_i \otimes g_k)$ to each of the basis elements listed above and add the results. The sum equals 1 because the map is zero on $P_0^* \otimes (\otimes_{j=1,\ \sum\ a_j=1}^t \ R_{\epsilon_j+2,a_j})$ and is also zero on $\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})$ unless $q = i$ and $2s_j = 2^{\epsilon_j+2}$ for $j = 1, \ldots, t$.

This contradiction completes the proof of part (iii) except that it remains to explain why we need the condition that

$$i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \cdots + 2^{\epsilon_t+1} - t + 1.$$

We require that i be large enough so that all the elements $\nu_{2q+2} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,0,2s_j})$ and $\sigma_{2q+1} \otimes (\otimes_j \ \Sigma_{\epsilon_j+2,a_j,2s_j+a_j})$ over which we want to sum are permissible within homological degree $2i + 2 + 4k$. However, if $i \geq 2^{\epsilon_1+1} + 2^{\epsilon_2+1} + \cdots + 2^{\epsilon_t+1} - t + 1$ then

$$2i + 2 + 4k \geq 2^{\epsilon_1+3} - 2 + 2^{\epsilon_2+3} - 2 + \cdots + 2^{\epsilon_t+3} - 2 + 2 \geq \sum_{j=1}^t \ 2s_j$$

for all possible choices of the s_j's involved in the sum. □

8.2.9. The maps $\iota_{k,l}$. As in § 8.2.5 let *bu* and *bo* denote the 2-localised, connective unitary and orthogonal K-theory spectra, respectively. Consider a left-*bu*-module spectrum map

$$\iota : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1}).$$

This map is determined up to homotopy by its restriction, via the unit of *bu*, to (F_{4k}/F_{4k-1}). By S-duality this restriction is equivalent to a map of the form

$$S^0 \longrightarrow D(F_{4k}/F_{4k-1}) \wedge bu \wedge (F_{4l}/F_{4l-1}),$$

where DX denotes the S-dual of X. Maps of this form are studied by means of the (collapsed) Adams spectral sequence (see [252] § 3.1; see also Chapter 3 § 3.2.1)

$$\begin{aligned} E_2^{s,t} &= Ext_B^{s,t}(\tilde{H}^*(D(F_{4k}/F_{4k-1}); \mathbb{Z}/2) \otimes \tilde{H}^*(F_{4l}/F_{4l-1}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2 \end{aligned}$$

where $\mathbb{Z}_2$ denotes the 2-adic integers. It is shown in [252] that such maps ι are trivial when $l < k$ and form a copy of the $\mathbb{Z}_2$ when $l \geq k$. Following [27] and [252] we choose left-*bu*-module spectrum maps

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

to satisfy $\iota_{k,k} = 1$, $\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \ldots \iota_{k,k-1}$ for all $k - l \geq 2$ and each $\iota_{t+1,t}$ is a $\mathbb{Z}_2$-module generator of the group of such left-bu-module maps.

Let $\tilde{z}_{4k} \in \pi_{4k}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2$ denote the element represented by the smash product of the unit η of the bu-spectrum with the inclusion of the bottom cell j_k into F_{4k}/F_{4k-1} (see [27] § 2.12; see also Chapter 5 Theorem 5.2.12)

$$S^0 \wedge S^{4k} \stackrel{\eta \wedge j_k}{\longrightarrow} bu \wedge F_{4k}/F_{4k-1}$$

and let $v_{2i-1} \in \pi_{2i-1}(bu \wedge \mathbb{RP}^\infty) \otimes \mathbb{Z}_2 = bu_{2i-1}(\mathbb{RP}^\infty)$ be as in Proposition 8.2.2. Then we have the exterior product

$$v_{2i-1}\tilde{z}_{4k} \in \pi_{4k+2i-1}(bu \wedge \mathbb{RP}^\infty \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2$$

which is non-zero and is represented by $\tilde{v}_{2i-1}\hat{z}_{4k}$ in the collapsed Adams spectral sequence whose E_2-term is described in § 8.2.5.

The following formula is central to the proof in § 4 of our main results (Theorem 8.4.6 and Theorem 8.4.7) of this chapter.

Proposition 8.2.10. For $l < k$, for some 2-adic unit $\mu_{4k,4l}$,

$$(\iota_{k,l})_*(v_{2i-1}\tilde{z}_{4k}) = \mu_{4k,4l}2^{4k-4l-\alpha(k)+\alpha(l)}v_{2i+4k-4l-1}\tilde{z}_{4l}.$$

Proof. Since $\iota_{k,l}$ is a left-bu-module map we have

$$(\iota_{k,l})_*(v_{2i-1}\tilde{z}_{4k}) = v_{2i-1}(\iota_{k,l})_*(\tilde{z}_{4k})$$

and, by ([27] Proposition 3.2; see also Chapter 5 Proposition 5.3.2),

$$(\iota_{k,l})_*(\tilde{z}_{4k}) = \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}\tilde{z}_{4l}$$

for some 2-adic unit $\mu_{4k,4l}$. The result follows since, by Proposition 8.2.2,

$$\begin{aligned} & v_{2i-1}\mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}\tilde{z}_{4l} \\ &= v_{2i+4k-4l-1}\mu_{4k,4l}2^{4k-4l-\alpha(k)+\alpha(l)}\tilde{z}_{4l}. \end{aligned}$$ □

8.3 The upper triangular technology examples

8.3.1. Deductions from the main diagram. In this section we are going to apply the results of the previous section together with the upper triangular yoga of Chapters 3 and 5 ([252], [27]) to the *partially* commutative diagram of Fig. 8.1 (appearing after Theorem 8.1.5) to prove the remaining tally of results which were used in §§ 8.1.6 and 8.1.7 to prove Theorem 8.1.2. Two types of the space X which we shall consider have the property that their 2-local K-theory in dimension $8m - 1$ is isomorphic *as a group* to $bu_{8m-1}(\mathbb{RP}^{8m-1})$. The third type of example arises from modifying these examples by means of self-maps of $\mathbb{RP}^{8m-2}$ which were constructed by Toda in [276]. All three types of examples have similar connective K-groups. Therefore we begin by reviewing the K-theory of $\mathbb{RP}^{8m-1}$.

Example 8.3.2 (K-theory of $\mathbb{RP}^{8m-1}$)**.** From [26] we have

$$\tilde{KU}^0(\mathbb{RP}^{2t-1}) \cong \tilde{KU}^0(\mathbb{RP}^{2t-2}) \cong \mathbb{Z}/2^{t-1} \quad \text{and} \quad \tilde{KU}^1(\mathbb{RP}^{2t-1}) \cong \mathbb{Z}_2.$$

The KU-theory universal coefficient theorem (proved by the method of [24]) shows that

$$\tilde{KU}_1(\mathbb{RP}^{2t-1}) \cong \mathbb{Z}_2\langle F_2\rangle \oplus \mathbb{Z}/2^{t-1}\langle F_1\rangle, \ \tilde{KU}_0(\mathbb{RP}^{2t-1}) = 0.$$

Shifting dimensions, by Bott periodicity, we have

$$\tilde{KU}_{2n-1}(\mathbb{RP}^{2n-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2\langle\hat{\beta}_{2n-1}\rangle \oplus \mathbb{Z}/2^{n-1}\langle\hat{\beta}_{2n-3}\rangle$$

in the notation of Chapter 6, Corollary 6.1.9 and Proposition 6.1.10 with the Adams operation ψ^3 satisfying the formulae

$$\psi^3(\hat{\beta}_{2n-1}) = \hat{\beta}_{2n-1} + \frac{(1-3^n)}{2\cdot 3^n}\hat{\beta}_{2n-3} \text{ and } \psi^3(\hat{\beta}_{2n-3}) = 3^n\hat{\beta}_{2n-3}.$$

Notice that if we replace $\hat{\beta}_{2n-1}$ by $\hat{\beta}_{2n-1} + a\hat{\beta}_{2n-3}$ these equations imply that

$$(\psi^3 - 1)(\hat{\beta}_{2n-1} + a\hat{\beta}_{2n-3}) = \frac{(1-3^n)}{2\cdot 3^n}\hat{\beta}_{2n-3} + a(3^n-1)\hat{\beta}_{2n-3}$$

so that the KU-e-invariant

$$(\psi^3 - 1)(\hat{\beta}) \in \mathbb{Z}/2^{n-1}\langle\hat{\beta}_{2n-3}\rangle,$$

where $\hat{\beta}$ is any element whose image generates $\tilde{KU}_{2n-1}(\mathbb{RP}^{2n-1}; \mathbb{Z}_2)$ modulo torsion, is well defined modulo $(3^n - 1)$.

In the book review [248] I gave a (then) new, one-line proof of the non-existence of maps of Hopf invariant one based on the above formula for the action of ψ^3.

The canonical map from $bu_{2n-1}(\mathbb{RP}^{2n-1})$ to $\tilde{KU}_{2n-1}(\mathbb{RP}^{2n-1})$ is an isomorphism commuting with ψ^3 so that

$$bu_{2n-1}(\mathbb{RP}^{2n-1}) \cong \mathbb{Z}_2\langle\hat{\beta}_{2n-1}\rangle \oplus \mathbb{Z}/2^{n-1}\langle\hat{\beta}_{2n-3}\rangle$$

with the ψ^3 acting on the generators by the formulae

$$\psi^3(\hat{\beta}_{2n-1}) = \hat{\beta}_{2n-1} + \frac{(1-3^n)}{2\cdot 3^n}\hat{\beta}_{2n-3} \text{ and } \psi^3(\hat{\beta}_{2n-3}) = 3^n\hat{\beta}_{2n-3}.$$

When $n = 4m$ the complexification map is an isomorphism giving, in the notation of Proposition 8.2.2,

$$\begin{aligned} bo_{8m-1}(\mathbb{RP}^{8m-1}) &\cong bu_{8m-1}(\mathbb{RP}^{8m-1}) \\ &\cong \mathbb{Z}_2\langle\iota_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u\rangle \end{aligned}$$

where the second summand is $bo_{8m-1}(\mathbb{RP}^{8m-2}) \cong bu_{8m-1}(\mathbb{RP}^{8m-2})$ and

$$\begin{aligned} &\psi^3(\iota_{8m-1}) = \iota_{8m-1} + \tfrac{(1-3^{4m})}{2\cdot 3^{4m}} v_{8m-3}u, \\ &\psi^3(v_{8m-3}u) = 3^{4m} v_{8m-3}u. \end{aligned}$$

Example 8.3.3 (The maps Θ_{2j}). In this example we shall study homotopy classes of maps in the stable homotopy category of the form

$$\Theta_{2j} : \Sigma^{\infty} S^{2j} \longrightarrow \Sigma^{\infty}\mathbb{RP}^{2j}$$

with mapping cone, Cone(Θ_{2j}), such that, on 2-local connective K-theory

$$bu_{2j+1}(\mathrm{Cone}(\Theta_{2j})) \cong \mathbb{Z}_2\langle \iota_{2j+1}\rangle \oplus \mathbb{Z}/2^j\langle v_{2j-1}u\rangle$$

for some 2-adic unit u_j,

$$\psi^3(\iota_{2j+1}) = \iota_{2j+1} + u_j((3^j - 1)/4)v_{2j-1}u.$$

In other words, the bu_*-e-invariant (see [5]) of Θ_{2j} is half that of the canonical map $\Theta : S^{2j} \longrightarrow \mathbb{RP}^{2j}$ whose mapping cone is homotopy equivalent to $\mathbb{RP}^{2j+1}$ in Example 8.3.2.

For simplicity we shall restrict ourselves to the case when $j = 4m - 1$. In this case the 2-local K-groups which we shall need are very similar (as abstract groups) to those of Example 8.3.2; that is,

$$bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) \cong bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) \cong bu_{8m-1}(\mathbb{RP}^{8m-1}).$$

Example 8.3.4 (The maps $\Theta_{8m-2,r}$). In ([276] Theorem 2.7 p. 311; see also Remark 8.3.13) Toda constructed stable maps of the form

$$\tau_r : \Sigma^{\infty}\mathbb{RP}^{8m-2} \longrightarrow \Sigma^{\infty}\mathbb{RP}^{8m-8r-2}$$

for $0 \leq r \leq m-1$ which induce a surjection from $bu_{8m-1}(\mathbb{RP}^{8m-2})$ to (see Proposition 8.2.2)

$$bu_{8m-1}(\mathbb{RP}^{8m-8r-2}) \cong \mathbb{Z}/2^{4m-4r-1}\langle v_{8m-8r-3}u^{4r+1}\rangle.$$

Given such a τ_r we may form the composition

$$\Theta_{8m-2,r} : \Sigma^{\infty} S^{8m-2} \xrightarrow{\Theta_{8m-2}} \Sigma^{\infty}\mathbb{RP}^{8m-2} \longrightarrow \Sigma^{\infty}\mathbb{RP}^{8m-8r-2}$$

where Θ_{8m-2} is as in Example 8.3.3. Let Cone($\Theta_{8m-2,r}$) denote the mapping cone of $\Theta_{8m-2,r}$.

Therefore

$$bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r})) \cong \mathbb{Z}_2\langle \iota_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-4r-1}\langle v_{8m-8r-3}u^{4r+1}\rangle$$

and for some 2-adic unit μ,

$$\psi^3(\iota_{8m-1}) = \iota_{8m-1} + \mu\frac{(3^{4m} - 1)}{4}v_{8m-8r-3}u^{4r+1}.$$

8.3.5. $bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1}))$ **for** $k \geq 1$**.** In order to make further progress with Example 8.3.2 we shall need to know the structure of the bu_*-homology group $bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1}))$ for a lot of values of $k \geq 1$. This is calculated from the Adams spectral sequence whose E_2-term is the direct sum of that for $\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})$ and that for $S^{8m-1} \wedge (F_{4k}/F_{4k-1})$. The first of these is depicted in Fig. 8.3 below, which was first introduced in § 8.2.5 in order to compute the multiplicative structure in Theorem 8.2.6, and the second consists of some $\mathbb{Z}/2$'s along the $s = 0$ line and columns of $\mathbb{Z}/2$'s connected by multiplication by a (see Chapter 3 § 3.2.1) for each odd value of $t - s \geq 4k + 1$. The rest of the $E_2^{s,t}$'s are zero.

This spectral sequence collapses (Chapter 3 § 3.1.5, Chapter 5, Lemma 5.2.10; see also [9] Lemma 17.12 p. 361 or Proposition 8.2.4, the discussion of § 8.2.5 and Theorem 8.2.6) by the following simple argument. Each differential raises the value of s and changes the parity of $t - s$. Consider the first non-zero differential. Since for all $s > 0$ the only non-zero groups having $t - s$ odd as the only possibility must originate in $E_r^{0,2j}$ and land in a group on which multiplication by b (the bidegree of b is $(1, 3)$) is injective; but this is nonsense since multiplication by b commutes with the differential and multiplication by b sends $E_r^{0,2j}$ to $E_r^{1,2j+3} = 0$.

The collapsed Adams spectral sequence and the multiplicative structure described in Theorem 8.2.6 easily show that for $1 \leq k \leq 2m - 1$ and $4m \geq 4k - \alpha(k) + 1$,

$$\begin{aligned} & bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) \\ & \cong bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \\ & \cong V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle \end{aligned}$$

where V_k is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e., in Adams filtration zero, represented on the $s = 0$ line) in the spectral sequence. If $8m - 1 \leq 8k - 2\alpha(k) + 1$ then the group is a $\mathbb{Z}/2$-vector space of the form

$$bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) \cong V_k \oplus \mathbb{Z}/2\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$$

entirely in Adams filtration zero and if $k \geq 2m$ the group is zero.

8.3.6. $bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1}))$ **for** $k \geq 1$**.** In order to make further progress with Example 8.3.3 we shall need the observation that the Adams spectral sequences for

$$bu_*(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1})) \text{ and } bu_*(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1}))$$

are isomorphic and therefore we have

$$bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \cong bu_{8m-1}(\mathbb{RP}^{8m-1} \wedge (F_{4k}/F_{4k-1}))$$

for $k \geq 1$.

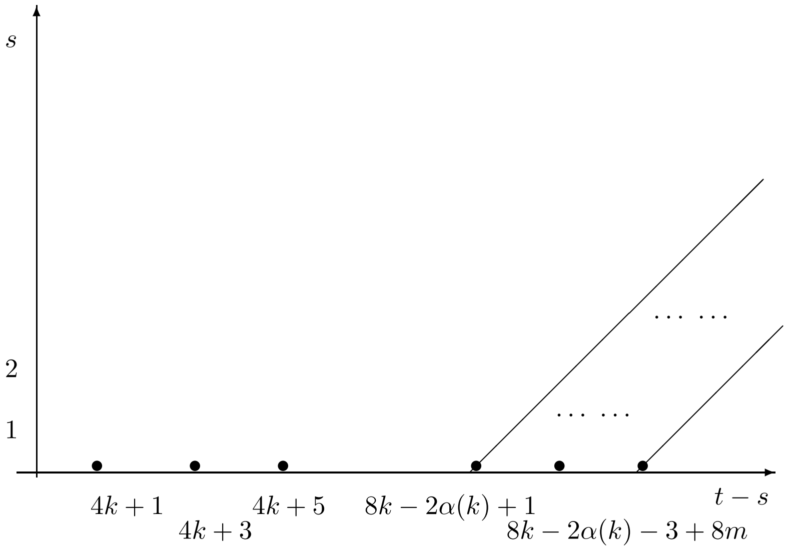

Figure 8.3.

In addition to the group structure in § 8.3.5 and § 8.3.6 we shall need the following result concerning the homomorphism induced by the map $\iota_{k,k-1}$.

$$bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 \longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k-4}/F_{4k-5})) \otimes \mathbb{Z}_2.$$

Proposition 8.3.7. In the notation of Proposition 8.2.10 and Example 8.3.2

(i) $(\iota_{1,0})_*(v_{8m-5}\tilde{z}_4) = \mu_{4,0}2^2 v_{8m-3}u$,

(ii) if $2 \leq k \leq 2m-1$ and $4m \geq 4k - \alpha(k) + 1$ then $(\iota_{k,k-1})_*$

$$V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle \longrightarrow V_{k-1} \oplus \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}\langle v_{8m-4k+3}\tilde{z}_{4k-4}\rangle$$

satisfies

$$(\iota_{k,k-1})_*(v_{8m-4k-1}\tilde{z}_{4k}) = \mu_{4k,4k-4}2^{4-\alpha(k)+\alpha(k-1)}v_{8m-4k+3}\tilde{z}_{4k-4}$$

where $\mu_{4k,4k-4}$ is a 2-adic unit.

In particular, $(\iota_{k,k-1})_*$ is injective on $\mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$ in cases (i) and (ii).

Proof. These formulae follow from those of Proposition 8.2.10, concerning $\mathbb{RP}^\infty$ together with the injectivity of the map from $bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$ to $bu_{8m-1}(\mathbb{RP}^\infty \wedge (F_{4k}/F_{4k-1}))$, which follows from the Adams spectral sequence via Proposition 8.2.4 and § 8.2.5. The formulae make sense because, by Chapter 5 Proposition 5.2.6, $\alpha(k-1) = \alpha(k) - 1 + \nu_2(k)$. □

8.3.8. $bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1}))$**.** From the cofibration sequence for the mapping cone we know that

$$\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}) \longrightarrow \mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1})$$

induces an isomorphism

$$\begin{aligned} &bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \\ &\qquad \stackrel{\cong}{\longrightarrow} bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1})). \end{aligned}$$

Now consider the homomorphism induced by the inclusion of projective spaces

$$\begin{aligned} &bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \\ &\qquad \longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \end{aligned}$$

beginning with the associated homomorphism of Adams spectral sequences. Since

$$\begin{aligned} &\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2) \\ &\qquad \longrightarrow \tilde{H}^*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2) \end{aligned}$$

is surjective the dual map

$$\begin{aligned} &\mathrm{Hom}^*(\tilde{H}^*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \longrightarrow \mathrm{Hom}^*(\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \end{aligned}$$

is injective and therefore the map

$$\begin{aligned} &\mathrm{Ext}_B^{0,*}(\tilde{H}^*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \longrightarrow \mathrm{Ext}_B^{0,*}(\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \end{aligned}$$

is injective also.

The Adams-Margolis theory ([12]; see also Chapter 3 §3.2.1) gives an isomorphism between

$$\mathrm{Ext}_B^{s,*}(\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2)$$

for $s > 0$ and a part of $Ext_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{8m-2}; \mathbb{Z}/2), \mathbb{Z}/2)$. This dimension-shifting isomorphism is natural so the fact that, by Proposition 8.2.4,

$$\begin{aligned} &\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{8m-8r-2}; \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \longrightarrow \mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{8m-2}; \mathbb{Z}/2), \mathbb{Z}/2) \end{aligned}$$

is injective implies that

$$\begin{aligned} &\mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \\ &\qquad \longrightarrow \mathrm{Ext}_B^{*,*}(\tilde{H}^*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}); \mathbb{Z}/2), \mathbb{Z}/2) \end{aligned}$$

is also injective for all $s > 0$.

Therefore, since the map on the E_2-terms of the collapsed Adams spectral sequences is injective in all bidegrees, we see that

$$bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$$

is injective.

Now we are ready to determine the structure of

$$bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}))$$

when $1 \leq r \leq m-1$ and $1 \leq k$. From the preceding discussion we know that the E_2-term of the 2-adic Adams spectral sequence for $bu_*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}))$ given by Fig. 8.3 maps injectively to the E_2-term for $bu_*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$, which is depicted in Fig. 8.4.

Lemma 8.3.9. Suppose that $1 \leq r \leq m-1$, $1 \leq k$ and $4k - \alpha(k) < 4r$ in § 8.3.8. If $w \in bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1}))$ is represented in $E_2^{0,8m-1}$, then $2w = 0$.

Proof. If $2w \neq 0$ it must be represented in $E_2^{s,8m-1+s}$ for some $s \geq 2$ since $E_2^{1,8m} = 0$. The image of w in

$$bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \cong V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}$$

will be a non-zero element of order strictly greater than 2 and represented in $E_2^{0,8m-1}$, by injectivity of E_2-terms. Therefore, from the structure of the $\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})$ spectral sequence, the image of $2w$ will be represented in $E_2^{1,8m}$, which is impossible since the Adams filtration of $2w$ – and hence of its image – is greater than 2. □

Corollary 8.3.10. Suppose that $1 \leq r \leq m-1$, $1 \leq k$ and $4k - \alpha(k) < 4r$ in § 8.3.8. Then

$$\begin{aligned} &bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \\ &\cong W_k \oplus \mathbb{Z}/2^{4m-4r-1}\langle d_{m,r,k}\rangle \end{aligned}$$

where the element $d_{m,r,k}$ is represented by the generator of

$$E_2^{4r+1-4k+\alpha(k),8m+4r-4k+\alpha(k)}$$

and W_k is an $\mathbb{F}_2$-vector space represented entirely in $E_2^{0,8m-1}$.

Lemma 8.3.11. Suppose that $1 \leq r \leq m-1$, $1 \leq k$ and $8r + 2 \leq 4k$ in § 8.3.8. Then

$$\begin{aligned} &bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \\ &\cong W_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle \tilde{z}_{4k} v_{8m-1-4k}\rangle \end{aligned}$$

where W_k is an $\mathbb{F}_2$-vector space represented entirely in $E_2^{0,8m-1}$.

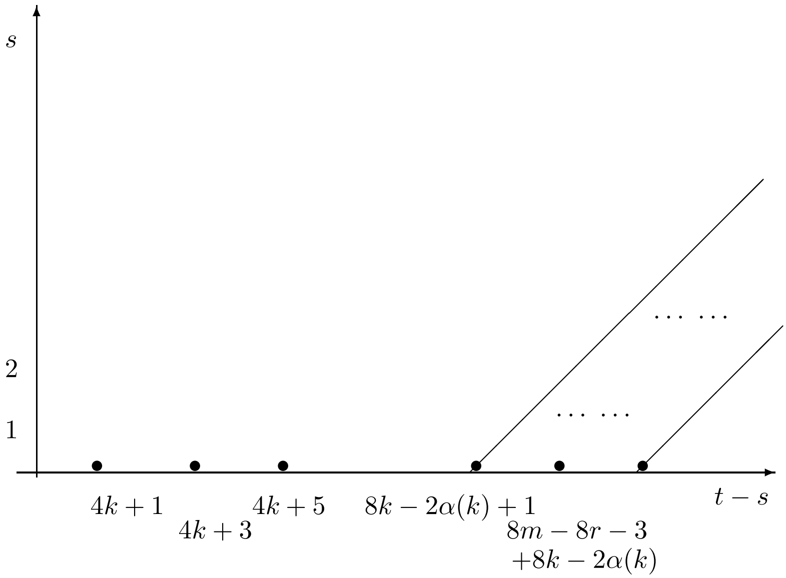

Figure 8.4.

Proof. Immediate from the spectral sequence in § 8.3.8, Proposition 8.2.4 and Theorem 8.2.6. ☐

Lemma 8.3.12. Suppose that $1 \leq r \leq m-1$, $1 \leq k$ and $4k < 8r+2 \leq 8k-2\alpha(k)+2$ in § 8.3.8. Then

$$\begin{aligned} & bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})) \\ & \cong W_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle d_{m,r,k}\rangle \end{aligned}$$

where W_k is an $\mathbb{F}_2$-vector space represented entirely in $E_2^{0,8m-1}$.

Proof. Immediate from the injectivity of the homomorphism of spectral sequences induced by the inclusion of $\mathbb{RP}^{8m-8r-2}$ into $\mathbb{RP}^{8m-2}$ proved in § 8.3.8. ☐

Remark 8.3.13. In Lemma 8.3.12 the element $d_{m,r,k}$ is represented in $E_2^{0,8m-1}$ and therefore its image in

$$\begin{aligned} & bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \\ & \cong V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle \tilde{z}_{4k} v_{8m-4k-1}\rangle \end{aligned}$$

has the form $(?, (2s+1)\tilde{z}_{4k} v_{8m-4k-1})$ for some integer s.

The construction of $\Theta_{8m-2,r}$ in Example 8.3.4 implies that we have a "Toda map"

$$T_r : \mathrm{Cone}(\Theta_{8m-2}) \longrightarrow \mathrm{Cone}(\Theta_{8m-2,r})$$

which restricts on $\Sigma^\infty \mathbb{RP}^{8m-2}$ to give the map

$$\tau_r : \Sigma^\infty \mathbb{RP}^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-8r-2}$$

of Example 8.3.4. In Proposition 8.3.14 we shall partially evaluate

$$\begin{aligned}(T_r \wedge 1)_* : bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \\ \longrightarrow bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1}))\end{aligned}$$

for $k \geq 1$. In preparation we need to recall from ([276] Theorem 2.7 p. 311) how τ_r is constructed.

In ([276] Theorem 2.7 p. 311) it is shown that 2^4 times the identity map of $\Sigma^\infty(\mathbb{RP}^{8t-2}/\mathbb{RP}^{8t-10})$ is nullhomotopic. Hence, for each $t \geq 1$, there exists a map

$$T : \Sigma^\infty \mathbb{RP}^{8t-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8t-10}$$

whose composition with the inclusion of $\mathbb{RP}^{8t-10}$ into $\mathbb{RP}^{8t-2}$ is homotopic to 2^4 times the identity map of $\Sigma^\infty \mathbb{RP}^{8t-2}$. Composing r of these maps gives τ_r such that the composition

$$\Sigma^\infty \mathbb{RP}^{8m-2} \xrightarrow{\tau_r} \Sigma^\infty \mathbb{RP}^{8m-8r-2} \xrightarrow{\mathrm{inc}} \Sigma^\infty \mathbb{RP}^{8m-2}$$

is homotopic to 2^{4r} times the identity map. Therefore we have a homotopy commutative diagram of mapping cone cofibrations.

$$\begin{array}{ccccc}
\Sigma^\infty S^{8m-2} & \xrightarrow{\Theta_{8m-2}} & \Sigma^\infty \mathbb{RP}^{8m-2} & \longrightarrow & \mathrm{Cone}(\Theta_{8m-2}) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tau_r} & & \downarrow{\scriptstyle T_r} \\
\Sigma^\infty S^{8m-2} & \xrightarrow{\Theta_{8m-2,r}} & \Sigma^\infty \mathbb{RP}^{8m-8r-2} & \longrightarrow & \mathrm{Cone}(\Theta_{8m-2,r}) \\
\downarrow{\scriptstyle 2^{4r}} & & \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\
\Sigma^\infty S^{8m-2} & \xrightarrow{\Theta_{8m-2}} & \Sigma^\infty \mathbb{RP}^{8m-2} & \longrightarrow & \mathrm{Cone}(\Theta_{8m-2})
\end{array}$$

Proposition 8.3.14. In Remark 8.3.13 suppose that $1 \leq k$ and $1 \leq r \leq m-1$. Then the map

$$\begin{aligned}(T_r \wedge 1)_* : bu_*(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \\ \longrightarrow bu_*(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1}))\end{aligned}$$

satisfies

$$\begin{aligned}&(T_r \wedge 1)_*(\tilde{z}_{4k} v_{2i-1}) \\ &= \begin{cases} \tilde{z}_{4k} 2^{4r} v_{2i-1} & \text{if } 1 \leq i \leq 4m-4r-1, \\ \tilde{z}_{4k} 2^{4m-i-1} u^{i-4m+4r+1} v_{8m-8r-3} & \text{if } 4m-4r \leq i \leq 4m-2k-1 \end{cases}\end{aligned}$$

for $1 \leq i \leq 4m-2k-1$.

Proof. The homomorphism $(T_r \wedge 1)_*$ may be identified with

$$\begin{aligned}(\tau_r \wedge 1)_* : bu_*(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))\\ \longrightarrow bu_*(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})).\end{aligned}$$

We know from § 8.3.8 that

$$(\mathrm{inc})_* : bu_*(\mathbb{RP}^{8m-8r-2}) \longrightarrow bu_*(\mathbb{RP}^{8m-2})$$

is injective and sends v_{2i-1} to v_{2i-1} for $1 \leq 2i-1 \leq 8m-8r-3$ (i.e., $1 \leq i \leq 4m-4r-1$). Also composition of the Toda map

$$(\tau_r)_* : bu_*(\mathbb{RP}^{8m-2}) \longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-8r-2})$$

with $(inc)_*$ is multiplication by 2^{4r}. The result follows since, in $bu_*(\mathbb{RP}^{8m-2})$,

$$\begin{aligned}&2^{4r}v_{2i-1}\\ &= \begin{cases} 2^{4r}v_{2i-1} & \text{if } 1 \leq i \leq 4m-4r-1, \\ 2^{4m-i-1}u^{i-4m+4r+1}v_{8m-8r-3} & \text{if } 4m-4r \leq i \leq 4m-1 \end{cases}\end{aligned}$$

as required. □

8.4 Applications of upper triangular technology

8.4.1. This section contains my main application of the upper triangular technology in which we shall examine the consequences of combining the *partially* commutative diagram Fig.1 with Chapter 5, Theorem 5.1.2 in the case when $X = \mathrm{Cone}(\Theta_{8m-2})$ of Example 8.3.3. The idea is very simple. By Chapter 3, Theorem 3.1.6(ii) we have a 2-adic equivalence of spectra

$$\hat{L} : \vee_{k \geq 0}\ bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge bo$$

which induces a direct sum splitting of the form

$$(\hat{L} \wedge 1)_* : \vee_{k \geq 0}\ bu_*((F_{4k}/F_{4k-1}) \wedge X) \xrightarrow{\cong} \pi_*(bu \wedge bo \wedge X).$$

Therefore we may apply the unit map

$$(\eta \wedge 1 \wedge 1) : bo_*(X) = \pi_*(bo \wedge X) \longrightarrow \pi_*(bu \wedge bo \wedge X)$$

to $x \in bo_*(X)$ to produce a vector $(x_0, x_1, x_2, \ldots)$ with

$$x_k \in bu_*((F_{4k}/F_{4k-1}) \wedge X).$$

Given a formula for $\psi^3(x) \in bo_*(X)$ the corresponding vector is related to $(x_0, x_1, \ldots)$ by the matrix of Chapter 5, Theorem 5.1.2.

The result of this comparison will be a family of informative equations *provided* that we know the x_0-coordinate. The map $x \mapsto x_0$ is a homomorphism between 2-adic connective K-theories of the form

$$bo_*(X) \longrightarrow bu_*(X)$$

and the diagram of Fig.1 suggests that this map is the canonical complexification map c. The map is probably $\mu \cdot c$ for some 2-adic unit, which can be verified by checking the cases when X is a sphere using the formulae of Chapters 3 and 5 and the formula for the canonical involution on $\pi_*(bu \wedge bu)$ [58]. The difficulty is, of course, that the x_0-coordinate map corresponds to a map of 2-adic spectra

$$\tilde{\pi}_0 : bo \longrightarrow bu$$

which is *not* a left-bu-module map.

Incidentally the map

$$\lambda : \vee_{k \geq 0} \ bu \wedge (F_{4k}/F_{4k-1}) \simeq bu \wedge bo \stackrel{1 \wedge c}{\longrightarrow} bu \wedge bu \longrightarrow bu$$

of Fig. 1 is a left-bu-module map so that

$$\lambda = \sum_{k \geq 0} \ \alpha_k \cdot \iota_{4k,0}$$

for some 2-adic integers α_k.

For our purposes the following result will suffice.

Proposition 8.4.2. There is a 2-adic equivalence $\beta : bu \stackrel{\simeq}{\longrightarrow} bu$ such that the x_0-coordinate map is homotopic to $\beta \cdot c : bo \longrightarrow bu \stackrel{\simeq}{\longrightarrow} bu$.

Proof. We shall begin by evaluating the x_0-coordinate map

$$\begin{aligned} bo \stackrel{\eta \wedge 1}{\longrightarrow} bu \wedge bo \stackrel{\hat{L}^{-1}}{\longrightarrow} \vee_{k \geq 0} \ bu \wedge (F_{4k}/F_{4k-1}) \\ \longrightarrow bu \wedge (F_0/F_{-1}) = bu \end{aligned}$$

in mod 2-cohomology. This map takes the form

$$\begin{aligned} \mathcal{A}/B \otimes \mathbb{Z}/2 = \mathcal{A} \otimes_B \mathbb{Z}/2 \stackrel{1 \otimes \eta}{\longrightarrow} \mathcal{A} \otimes_B H^*(bo; \mathbb{Z}/2) \\ \stackrel{\cong}{\longrightarrow} \mathcal{A}/B \otimes H^*(bo; \mathbb{Z}/2) \stackrel{\epsilon}{\longrightarrow} H^*(bo; \mathbb{Z}/2) = \mathcal{A}/\simeq \end{aligned}$$

where η is the Hopf algebra unit map and ϵ is the augmentation map. As explained in Chapter 3 § 3.1.5, the isomorphism in the centre is given by $a \otimes_B x \mapsto \sum \ a' \otimes a'' x$ where the comultiplication in the Steenrod algebra is given by $\Delta(a) = \sum \ a' \otimes a''$. Hence the x_0-coordinate map is given on mod 2 cohomology by sending $a \in \mathcal{A}/B = H^*(bu; \mathbb{Z}/2)$ to $a \in (\mathcal{A}/ \simeq) = H^*(bo; \mathbb{Z}/2)$. In other words, the map induced on mod 2 cohomology coincides with the homomorphism c^* induced by the complexification map. Therefore we may replace the complexification map by the x_0-coordinate map in the proof of the Anderson-Wood result given in Chapter 5 § 5.5.1 to give a 2-adic equivalence of the form

$$\hat{\beta} : bo \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu \wedge bu \xrightarrow{\mu} bu.$$

By construction the composition $\hat{\beta}$ with the canonical inclusion of $bo = bo \wedge S^0$ into $bo \wedge \Sigma^{-2}\mathbb{CP}^2$ is equal to the x_0-coordinate map. Identifying $bo \wedge \Sigma^{-2}\mathbb{CP}^2$ with bu via the Anderson-Wood map completes the proof. □

8.4.3. Application to Θ_{8m-2}. Consider the diagram of Fig. 8.1 with X replaced by the mapping cone $\mathrm{Cone}(\Theta_{8m-2})$ of Example 8.3.3. This gives Fig. 8.5.

From Examples 8.3.2 and 8.3.3 we have

$$\begin{aligned} bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) &\cong bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})) \\ &\cong \mathbb{Z}_2\langle \iota_{8m-1}\rangle \oplus \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u\rangle \end{aligned}$$

where the second summand is $bo_{8m-1}(\mathbb{RP}^{8m-2}) \cong bu_{8m-1}(\mathbb{RP}^{8m-2})$ and

$$\begin{aligned} &\psi^3(\iota_{8m-1}) = \iota_{8m-1} + (2s+1)\tfrac{(3^{4m}-1)}{4} v_{8m-3}u, \\ &\psi^3(v_{8m-3}u) = 3^{4m} v_{8m-3}u \end{aligned}$$

for some integer s. Therefore, if ι_{8m-1} is replaced by $\iota'_{8m-1} = a\cdot\iota_{8m-1} + b\cdot v_{8m-3}u$ with $a \in \mathbb{Z}_2^*$, then

$$\begin{aligned} &\psi^3(\iota'_{8m-1}) \\ &= a\cdot\iota_{8m-1} + (2s+1)\tfrac{(3^{4m}-1)}{4} a\cdot v_{8m-3}u + b\cdot 3^{4m} v_{8m-3}u \\ &= \iota'_{8m-1} + (2t+1)\tfrac{(3^{4m}-1)}{4}\cdot v_{8m-3}u \end{aligned}$$

for some integer t.

By means of the 2-local equivalence $\hat{L}$ of § 8.4.1 we have a direct sum decomposition

$$\begin{aligned} \hat{L}_* :\ &\oplus_{k\geq 0}\, bu_*(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \\ &\xrightarrow{\cong} \pi_*(bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2})) \end{aligned}$$

and by means of this identification we may write the element

$$(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) \in \pi_{8m-1}(bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}))$$

as a vector $(w_0, w_1, \ldots, w_{2m-1})$ with

$$w_k \in bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})).$$

Similarly we have

$$(\eta \wedge 1 \wedge 1)_*(v_{8m-3}u) = (\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_{2m-1}).$$

$$
\begin{tikzcd}
\pi_j(bo \wedge \mathrm{Cone}(\Theta_{8m-2})) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"'] \arrow[dr, "(c \wedge 1)_*"] \arrow[ddd, "(\psi^3 \wedge 1)_*"] & & \pi_j(bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2})) \arrow[dl, "\lambda_*"'] \arrow[ddd, "(1 \wedge \psi^3 \wedge 1)_*"'] \\
& \pi_j(bu \wedge \mathrm{Cone}(\Theta_{8m-2})) \arrow[d, "(\psi^3 \wedge 1)_*"] & \\
& \pi_j(bu \wedge \mathrm{Cone}(\Theta_{8m-2})) & \\
\pi_j(bo \wedge \mathrm{Cone}(\Theta_{8m-2})) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"] \arrow[ur, "(c \wedge 1)_*"'] & & \pi_j(bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2})) \arrow[ul, "\lambda_*"]
\end{tikzcd}
$$

Figure 8.5.

In addition, by Proposition 8.4.2, the leading coordinate $\tilde{w}_0$ of $(\eta \wedge 1 \wedge 1)_*(v_{8m-3}u)$ must have the form, for some integer a,

$$\tilde{w}_0 = (2a+1)v_{8m-3}u \in bu_{8m-1}(\mathbb{RP}^{8m-2}) \cong \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u \rangle.$$

According to the main theorem of Chapter 3 (see also [252]) a left-bu-module self-equivalence of $bu \wedge bo$ inducing the identity on mod 2 homology determines a unique conjugacy class in the upper triangular group with entries in the 2-adic integers. According to the main theorem of Chapter 5 (see also [27]) the conjugacy class associated to the map $1 \wedge \psi^3$ is equal to

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In practical terms this means that we may choose $\hat{L}$ in § 8.4.1 so that $1 \wedge \psi^3$ maps the wedge summand $bu \wedge (F_{4k}/F_{4k-1})$ to itself by 9^k times the identity map, to $bu \wedge (F_{4k-4}/F_{4k-5})$ by $\iota_{k,k-1}$ and to all other wedge summands $bu \wedge (F_{4t}/F_{4t-1})$ trivially. If we choose $\hat{L}$ in this manner, we have

$$\begin{aligned} &(1 \wedge \psi^3 \wedge 1)_*((\eta \wedge 1 \wedge 1)_*(\iota_{8m-1})) \\ &= (1 \wedge \psi^3 \wedge 1)_*(w_0, w_1, w_2, \ldots, w_{2m-1}) \\ &= (w_0 + (\iota_{1,0})_*(w_1), 9w_1 + (\iota_{2,1})_*(w_2), \\ &\qquad 9^2 w_2 + (\iota_{3,2})_*(w_3), \ldots, 9^{2m-1} w_{2m-1}). \end{aligned}$$

On the other hand this element is equal to

$$\begin{aligned} &(\eta \wedge 1 \wedge 1)_*((\psi^3 \wedge 1)_*(\iota_{8m-1})) \\ &= (\eta \wedge 1 \wedge 1)_*(\iota_{8m-1} + (2s+1)((3^{4m}-1)/4) v_{8m-3} u) \\ &= (w_0, w_1, w_2, \ldots, w_{2m-1}) \\ &\qquad +(2s+1)((3^{4m}-1)/4)(\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{2m-1}) \\ &= (w_0, w_1, w_2, \ldots, w_{2m-1}) \\ &+(2s+1)((3^{4m}-1)/4)((2a+1) v_{8m-3} u, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{2m-1}). \end{aligned}$$

Equating coordinates we obtain a string of equations

$$\begin{aligned} &w_0 + (\iota_{1,0})_*(w_1) \\ &\quad = w_0 + (2s+1)((3^{4m}-1)/4)(2a+1) v_{8m-3} u \\ &\qquad\qquad \text{in } bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2})), \\ &(\iota_{1,0})_*(w_1) = (2s+1)((3^{4m}-1)/4)(2a+1) v_{8m-3} u \\ &\qquad\qquad \text{in } bu_{8m-1}(\mathbb{RP}^{8m-2}), \\ &(9-1) w_1 + (\iota_{2,1})_*(w_2) = (2s+1)((3^{4m}-1)/4) \tilde{w}_1 \\ &\qquad\qquad \text{in } bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_4/F_3)), \\ &(9^2-1) w_2 + (\iota_{3,2})_*(w_3) = (2s+1)((3^{4m}-1)/4) \tilde{w}_2 \\ &\qquad\qquad \text{in } bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_8/F_7)), \\ &\qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \end{aligned}$$

$$(9^k - 1)w_k + (\iota_{k+1,k})_*(w_{k+1}) = (2s+1)((3^{4m}-1)/4)\tilde{w}_k$$
$$\text{in } bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})),$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \ddots$$

This table of relations (i.e., from the 2nd equation downwards) takes place in

$$bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \quad \text{for } k \geq 0.$$

By § 8.3.6 this group may be identified with $bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$. The second equation in the list lies in the torsion subgroup in the case when $k = 0$ and by Example 8.3.3 this subgroup may be identified with $bu_{8m-1}(\mathbb{RP}^{8m-2})$.

Definition 8.4.4. We pause to introduce some convenient notation. Recall from § 8.3.5 and § 8.3.6 that for $1 \leq k \leq 2m-1$ and $4m \geq 4k - \alpha(k) + 1$ there are isomorphisms of the form

$$bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1}))$$
$$\cong V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$$

where V_k is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e., in Adams filtration zero, represented on the $s = 0$ line) in the spectral sequence and from Example 8.3.3,

$$bu_{8m-1}(\mathbb{RP}^{8m-2}) \cong \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u\rangle.$$

Suppose that we have an element

$$w \in V_i \oplus \mathbb{Z}/2^{4m-4i+\alpha(i)}\langle v_{8m-4i-1}\tilde{z}_{4i}\rangle;$$

we shall write $w \simeq 2^N$ if $w = (x, 2^N(2t+1)v_{8m-4i-1}\tilde{z}_{4i})$ for some integers t, $N < 4m - 4i + \alpha(i)$ and some element x.

8.4.5. Application to Θ_{8m-2} continued. By Proposition 8.3.7 we know, in the notation of § 8.4.3 and Definition 8.4.4, that each of the homomorphisms

$$(\iota_{1,0})_* : V_1 \oplus \mathbb{Z}/2^{4m-3}\langle v_{8m-5}\tilde{z}_4\rangle \longrightarrow \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u\rangle$$

and

$$(\iota_{k,k-1})_* : V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle$$
$$\longrightarrow V_{k-1} \oplus \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}\langle v_{8m-4k+3}\tilde{z}_{4k-4}\rangle$$

for $2 \leq k \leq 2m-1$ and $4m \geq 4k - \alpha(k) + 1$ is injective on

$$\mathbb{Z}/2^{4m-4k+\alpha(k)}\langle v_{8m-4k-1}\tilde{z}_{4k}\rangle.$$

Also we observe that in Proposition 8.3.7(i) or (ii) we may choose V_k so that $(\iota_{1,0})_*(V_1) = 0$ and

$$(\iota_{k,k-1})_*(V_k) \bigcap \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}\langle v_{8m-4k+3}\tilde{z}_{4k-4}\rangle = 0.$$

Choosing V_k in this manner will simplify our subsequent calculations, so we shall assume this property of the $V(k)$'s henceforth.

We are now ready to prove our main result.

Theorem 8.4.6. Let $m = (2p+1)2^q$ with $p \geq 1$. In the notation of § 8.4.3, Definition 8.4.4 and § 8.4.5:

(i) for $2 \leq k \leq 2^{q+1}$, $(\iota_{k,k-1})_*(w_k) \simeq 2^{3+q}$,
(ii) $w_{2^{q+1}} \simeq 1$,
(iii) under these hypotheses Θ_{8m-2} does not exist.

Proof. First we observe that $k \leq 2^{q+1}$ implies that $4m = p2^{q+3} + 2^{q+2} \geq 4k - \alpha(k) + 1$ so that we may apply the discussion of § 8.4.5. Therefore the relation

$$(\iota_{1,0})_*(w_1) = (2s+1)((3^{4m}-1)/4)(2a+1)v_{8m-3}u$$

and (Chapter 5, Proposition 5.2.4) $\nu_2((9^{2m}-1)/4) = 3 + \nu_2(2m) - 2 = 2 + q$ implies $w_1 \simeq 2^q$.

Consider the relation

$$(9-1)w_1 + (\iota_{2,1})_*(w_2) = (2s+1)((3^{4m}-1)/4)\tilde{w}_1$$

in $V_1 \oplus \mathbb{Z}/2^{4m-3}\langle v_{8m-5}\tilde{z}_4\rangle$ where $(9-1)w_1 \simeq 2^{q+3}$. In Proposition 8.4.14(iii), as a consequence of the application of upper triangular technology to $\mathbb{RP}^{8m-1}$ of Example 8.3.2, we shall see that $\tilde{w}_1$ is divisible by 4 and therefore $(2s+1)((3^{4m}-1)/4)\tilde{w}_1$ is divisible by 2^{q+4} which implies that $(\iota_{2,1})_*(w_2) \simeq 2^{3+q}$, which starts an induction on k.

Suppose for $2 \leq k < 2^{q+1}$ that

$$(\iota_{k,k-1})_*(w_k) \simeq 2^{3+q} \in V(k-1) \oplus \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}.$$

Therefore $w_k \in V(k) \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}$ satisfies

$$w_k \simeq 2^{\alpha(k)-\alpha(k-1)+q-1} = 2^{q-\nu_2(k)}$$

by Chapter 5, Proposition 5.2.6. Then, since $\tilde{w}_k$ is divisible by 4 in the range $1 \leq k \leq 2^{q+1} - 1$, by Proposition 8.4.14(iii), and $\nu_2(9^k - 1) = 3 + \nu_2(k)$, the relation

$$(9^k-1)w_k + (\iota_{k+1,k})_*(w_{k+1}) = (2s+1)((9^{4m}-1)/4)\tilde{w}_k$$

implies that $\iota_{k+1,k}(w_{k+1}) \simeq 2^{3+q}$, as required. Since

$$\begin{aligned}&(\iota_{2^{q+1},2^{q+1}-1})_*(v_{8m-2^{q+3}-1}\tilde{z}_{2^{q+3}})\\ &= \mu_{2^{q+3},2^{q+3}-4}2^{q+4}v_{8m-2^{q+3}+3}\tilde{z}_{2^{q+3}-4},\end{aligned}$$

by Proposition 8.3.7(ii), we see that $w_{2^{q+1}}$ cannot exist. □

Theorem 8.4.7. Let $m = 2^q$. In the notation of § 8.4.3, Definition 8.4.4 and § 8.4.5:

(i) for $2 \leq k \leq 2^q$, $(\iota_{k,k-1})_*(w_k) \simeq 2^{3+q}$,

(ii) in $\mathbb{Z}/2 \oplus V_{2^q}$, $w_{2^q} \simeq 1$.

Proof. This time we observe that $k \leq 2^q$ implies $4m = 2^{q+2} \geq 4k - \alpha(k) + 1$ so that we may apply the discussion of § 8.4.5 and therefore part (i) follows as in Theorem 8.4.6. For part (ii), by Proposition 8.3.7(ii), we have

$$(\iota_{2^q,2^q-1})_*(v_{8m-2^{q+2}-1}\tilde{z}_{2^{q+2}}) = \mu_{2^{q+2},2^{q+2}-4}2^{3+q}v_{8m-2^{q+2}+3}\tilde{z}_{2^{q+2}-4}.$$

The homomorphism $(\iota_{2^q,2^q-1})_*$ has the form

$$\mathbb{Z}/2 \oplus V_{2^q} \longrightarrow \mathbb{Z}/2^{q+4} \oplus V_{2^q-1}$$

so that the first component of $(\iota_{2^q,2^q-1})_*(w_{2^q})$ is non-zero and therefore so is that of w_{2^q}. □

Corollary 8.4.8 (Converse to Theorem 8.4.7)**.** Suppose that $m = 2^q$ and that

$$\Theta_{8m-2} : \Sigma^\infty S^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-2}$$

is any map. Suppose that elements w_j are produced via the method of § 8.4.3 and that $w_{2^q} \simeq 1$ as in Theorem 8.4.7(ii). Then

$$\psi^3(\iota_{8m-1}) = \iota_{8m-1} + (2s+1)\frac{(3^{4m}-1)}{4}v_{8m-3}u$$

for some integer s.

Proof. Work upwards through the string of equations in § 8.4.3 □

In the course of proving Theorem 8.4.7 we established the following result.

Corollary 8.4.9. If $m = 2^q$ in Theorem 8.4.7 and Definition 8.4.4, then $w_k \simeq 2^{q-\nu_2(k)}$ for $1 \leq k \leq 2^q$.

The following result was important in the proof of Theorem 8.1.2:

Corollary 8.4.10. If $m = 2^q$ in Theorem 8.4.7 denote by

$$h_{8m-1} \in H_*(bo; \mathbb{Z}/2) \otimes H_*(\mathrm{Cone}(\Theta_{8m-2}); \mathbb{Z}/2)$$

the mod 2 homology class represented by the map

$$\begin{aligned} S^{2^{q+3}-1} &\stackrel{\iota_{8m-1}}{\longrightarrow} S^0 \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \\ &\stackrel{(\tilde{\eta}\wedge 1\wedge 1)}{\longrightarrow} H\mathbb{Z}/2 \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2}) \end{aligned}$$

where $\tilde{\eta}$ is the unit for $H\mathbb{Z}/2$ and $\iota_{8m-1} \in bo_{8m-1}(\mathrm{Cone}(\Theta_{8m-2}))$ is as in § 8.4.3. Then

$$h_{8m-1} = \tilde{z}_0 \otimes \iota_{2^{q+3}-1} + \tilde{z}_{2^{q+2}} \otimes v_{2^{q+2}-1} + \sum_{j=1}^{2^{q+2}-2} x_{8m-1-j} \otimes v_j$$

with $x_{8m-1-j} \in H_*(F_{2^{q+2}+t}/F_{2^{q+2}-1+t}; \mathbb{Z}/2)$ with $t \geq 0$. The classes denoted by $\tilde{z}_{4k} \in H_{4k}(bo; \mathbb{Z}/2)$ are the elements which correspond to the mod 2 Hurewicz image of the bottom cell in F_{4k}/F_{4k-1} under the homology isomorphism of Chapter 3 § 3.1.5 so that, in particular, $\tilde{z}_0 = 1 \in H_0(bo; \mathbb{Z}/2)$.

Proof. As explained in the preamble to the proof given in § 8.1.6, h_{8m-1} is the image of $(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) \in \pi_*(bu \wedge bo \wedge C(\Theta_{8m-1}))$ under the map of spectra $bu \longrightarrow H\mathbb{Z}/2$. Therefore h_{8m-1} is equal to the representative of $(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1})$ on the $s = 0$ line of the Adams spectral sequence. By Theorem 8.4.7 this representative is equal to the sum of $\tilde{z}_0 \otimes \iota_{2^{q+3}-1} + \tilde{z}_{2^{q+2}} \otimes v_{2^{q+2}-1}$ with an unknown element lying in the sum of the V_k's with $k \neq 2^q$.

This will establish the required result once we show that the unknown elements must have zero component in any V_k with $k < 2^q$. However, if this were false, let $k_0 > 0$ be minimal such that V_{k_0} is a summand with non-zero coordinate. If $k_0 < 2^q$ we run the argument of § 8.1.6 using $Sq_*^{4k_0}$ instead of $Sq_*^{2^{q+2}}$ and conclude that Θ_{8m-2} is detected by $Sq_*^{4k_0}$ on its mapping cone. However, it is a well-known consequence of Bill Browder's theorem [47] and the Kahn-Priddy theorem of Chapter 1, Theorem 1.5.10 that Θ_{8m-2} cannot be detected by any $Sq_*^{4k_0}$ of degree strictly less than 2^{q+2}. □

8.4.11. Application to Example 8.3.2. In this application we replace X by $\mathbb{RP}^{8m-1}$ of Example 8.3.2 in the diagram of Fig. 1 to give Fig. 5.

I am going to use the diagram of Fig. 5 and the resulting string of equations, similar to those of § 8.4.3, to deduce the 4-divisibility of the $\tilde{w}_k$'s which was essential in the proof of Theorems 8.4.6 and 8.4.7. This is a rather delicate point which hinges on the fact that, unlike Fig. 4, we know that the diagram of Fig. 5 *exists* and furthermore it is related to similar diagrams in which $\mathbb{RP}^{8m-1}$ is replaced by other odd-dimensional real projective spaces.

In the notation of Example 8.3.2 we have

$$v_{8m-3}u \in bu_{8m-1}(\mathbb{RP}^{8m-1}) \cong \mathbb{Z}_2\langle \iota_{8m-1} \rangle \oplus \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u \rangle.$$

This element originates in

$$bu_{8m-1}(\mathbb{RP}^{8m-2}) \cong \mathbb{Z}/2^{4m-1}\langle v_{8m-3}u \rangle.$$

We have a cofibration

$$S^{8m-3} \xrightarrow{\pi} \mathbb{RP}^{8m-3} \xrightarrow{inc} \mathbb{RP}^{8m-2}.$$

$$
\begin{tikzcd}
\pi_j(bo \wedge \mathbb{RP}^{8m-1}) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"'] \arrow[dr, "(c \wedge 1)_*"] \arrow[ddd, "(\psi^3 \wedge 1)_*"] & & \pi_j(bu \wedge bo \wedge \mathbb{RP}^{8m-1}) \arrow[dl, "\lambda_*"'] \arrow[ddd, "(1 \wedge \psi^3 \wedge 1)_*"'] \\
& \pi_j(bu \wedge \mathbb{RP}^{8m-1}) \arrow[d, "(\psi^3 \wedge 1)_*"] & \\
& \pi_j(bu \wedge \mathbb{RP}^{8m-1}) & \\
\pi_j(bo \wedge \mathbb{RP}^{8m-1}) \arrow[rr, "(\eta \wedge 1 \wedge 1)_*"] \arrow[ur, "(c \wedge 1)_*"'] & & \pi_j(bu \wedge bo \wedge \mathbb{RP}^{8m-1}) \arrow[ul, "\lambda_*"]
\end{tikzcd}
$$

Figure 8.6.

If H is the canonical line bundle over $\mathbb{RP}^{8m-3}$ then we may identify S^{8m-3} with the unit sphere bundle $S(H)$. Then π is homotopic to the inclusion of $S(H)$ into the unit disc bundle $D(H)$ with cofibre the Thom space $T(H)$ (see Chapter 1, Definition 1.2.3).

Alternatively π may be identified with the canonical double covering. Therefore we have an epimorphism

$$
\begin{aligned}
bu_{8m-3}(\mathbb{RP}^{8m-3}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}/2^{4m-2} \\
&\qquad \longrightarrow bu_{8m-3}(\mathbb{RP}^{8m-2}) \cong \mathbb{Z}/2^{4m-1}\langle v_{4m-1} \rangle.
\end{aligned}
$$

Next we observe that $\pi : S^{8m-3} \longrightarrow S^{8m-3}/\mathbb{Z}/2 = \mathbb{RP}^{8m-3}$ induces

$$\pi_* : \mathbb{Z}_2 = bu_{8m-3}(S^{8m-3}) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}/2^{4m-2}$$

which hits $2\mathbb{Z}_2$ (modulo torsion) (to see this classical fact we consider the long exact sequence obtained by applying $H_*(-;\mathbb{Z})$ to the cofibration).

The composition

$$\begin{aligned} bu_{8m-3}(S^{8m-3}) &\cong \mathbb{Z}_2\langle\iota\rangle \\ &\xrightarrow{\pi_*} bu_{8m-3}(\mathbb{RP}^{8m-3}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2^{4m-2} \\ &\longrightarrow bu_{8m-3}(\mathbb{RP}^{8m-2}) \cong \mathbb{Z}/2^{4m-1} \end{aligned}$$

is equal to zero. Since the torsion in the central group injects into the right-hand group we see that $2\iota_{8m-3} + (2d+1)v_{8m-5}u \in \Im(\pi_*)$. Similarly

$$2\iota_{8m-1} + (2d+1)v_{8m-3}u \in \pi_*(bu_{8m-1}(S^{8m-1})).$$

for some integer d. Since the left unit and right unit maps coincide on elements originating in $\pi_{8m-1}(S^{8m-1})$ we have, by Proposition 8.4.2,

$$\begin{aligned} &(\eta \wedge 1 \wedge 1)_*(2\iota_{8m-1} + (2d+1)v_{8m-3}u) \\ &\qquad = (2\mu\iota_{8m-1} + (2e+1)v_{8m-3}u, 0, 0, \ldots, 0, \ldots) \end{aligned}$$

for some integer e and some 2-adic unit μ.

From § 8.4.3 we have

$$(\eta \wedge 1 \wedge 1)_*(v_{8m-3}u) = ((2a+1)v_{8m-3}u, \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{2m-1})$$

and for $\iota_{8m-1} \in bo_{8m-1}(\mathbb{RP}^{8m-1})$ we write

$$(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) = (y_0, y_1, \ldots, y_{2m-1}).$$

This discussion proves the following result.

Lemma 8.4.12. In the notation of § 8.4.11 for each $k \geq 1$,

$$2y_k + (2d+1)\tilde{w}_k = 0.$$

Proof. For $1 \leq k$ the kth coordinate of

$$(\eta \wedge 1 \wedge 1)_*(2\iota_{8m-1} + (2d+1)v_{8m-3}u)$$

vanishes so that $2y_k + (2d+1)\tilde{w}_k = 0$. □

8.4.13. Application to Example 8.3.2 **continued.** Imitating the upper triangular technology method of § 8.4.3 and substituting the relation of Lemma 8.4.12 we obtain a string of equations:

$$\begin{aligned}
(\iota_{1,0})_*(y_1) &= (2s+1)((3^{4m}-1)/2)(2a+1)v_{8m-3}u \\
&\qquad \text{in } bu_{8m-1}(\mathbb{RP}^{8m-2}), \\
(9-1)y_1 + (\iota_{2,1})_*(y_2) &= (2s+1)((3^{4m}-1)/2)\tfrac{(-2y_1)}{(2d+1)} \\
&\qquad \text{in } bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_4/F_3)), \\
(9^2-1)y_2 + (\iota_{3,2})_*(y_3) &= (2s+1)((3^{4m}-1)/2)\tfrac{(-2y_2)}{(2d+1)} \\
&\qquad \text{in } bu_{8m-1}(bu \wedge \mathbb{RP}^{8m-2} \wedge (F_8/F_7)), \\
\vdots \qquad \vdots \qquad & \vdots \qquad \vdots \\
(9^k-1)y_k + (\iota_{k+1,k})_*(y_{k+1}) &= (2s+1)((3^{4m}-1)/2)\tfrac{(-2y_k)}{(2d+1)} \\
&\qquad \text{in } bu_{8m-1}(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})), \\
\vdots \qquad \vdots \qquad & \vdots \qquad \ddots
\end{aligned}$$

The next result completes the proof of Theorem 8.4.6 and Theorem 8.4.7 by establishing the 4 divisibility of the $\tilde{w}_k$ for $k \geq 1$. It is a simple consequence of the above equations and a minor modification of the proof of Theorem 8.4.6.

Proposition 8.4.14. Let $m = (2p+1)2^q$ with $p \geq 0$. Suppose that either $p > 0$ and $k \leq 2^{q+1}$ or $p = 0$ and $k \leq 2^q$. Then in the notation of § 8.4.3, Definition 8.4.4 and § 8.4.11:

(i) for $2 \leq k \leq 2^{q+1}-1$, $(\iota_{k,k-1})_*(y_k) \simeq 2^{4+q}$ and when $p > 0$ $(\iota_{2^{q+1},2^{q+1}-1})_*(y_{2^{q+1}})$ is divisible by 2^{q+4},

(ii) for $1 \leq k \leq 2^{q+1}-1$ ($1 \leq k \leq 2^q$ if $p = 0$), $y_k \simeq 2^{q+1-\nu_2(k)}$,

(iii) for $1 \leq k \leq 2^{q+1}-1$ ($1 \leq k \leq 2^q$ if $p = 0$), in § 8.4.3 $\tilde{w}_k \simeq 2^{q+2-\nu_2(k)}$.

Proof. As in the proof of Theorem 8.4.6 and Theorem 8.4.7 we begin with the observation that when $p > 0$ we may apply the discussion of § 8.4.5 in the range $1 \leq k \leq 2^{q+1}$ and when $p = 0$ in the range $1 \leq k \leq 2^q$. Therefore the relation

$$(\iota_{1,0})_*(y_1) = (2s+1)((3^{4m}-1)/2)(2a+1)v_{8m-3}u$$

and (Chapter 5, Proposition 5.2.4) $\nu_2((9^{2m}-1)/2) = 3+q$ implies $y_1 \simeq 2^{q+1}$, which starts an induction on k for parts (ii) and (iii).

Consider the relation

$$(9-1)y_1 + (\iota_{2,1})_*(y_2) = (2s+1)((3^{4m}-1)/2)\frac{(-2y_1)}{(2d+1)}$$

in $V_1 \oplus \mathbb{Z}/2^{4m-3}\langle v_{8m-5}\tilde{z}_4\rangle$ where $(9-1)y_1 \simeq 2^{q+4}$ and $(3^{4m}-1)y_1 \simeq 2^{2q+5}$ which implies that $(\iota_{2,1})_*(y_2) \simeq 2^{4+q}$, which starts an induction on k for part (i).

Suppose for all $2 \leq k$ with $k+1$ in the admissible range that

$$(\iota_{k,k-1})_*(y_k) \simeq 2^{4+q} \in V_{k-1} \oplus \mathbb{Z}/2^{4m-4k+4+\alpha(k-1)}.$$

Therefore $y_k \in V_k \oplus \mathbb{Z}/2^{4m-4k+\alpha(k)}$ satisfies

$$y_k \simeq 2^{\alpha(k)-\alpha(k-1)+q} = 2^{q+1-\nu_2(k)}$$

by Chapter 5, Proposition 5.2.6 which completes the induction for parts (ii) and (iii), by Lemma 8.4.12. Therefore, since $\nu_2(9^k-1) = 3+\nu_2(k)$, in the relation

$$(9^k-1)y_k + (\iota_{k+1,k})_*(y_{k+1}) = (2s+1)((3^{4m}-1)/2)\frac{(-2y_k)}{(2d+1)}$$

we have $(9^k-1)y_k \simeq 2^{q+4}$ and $(3^{4m}-1)y_k \simeq 2q+5-\nu_2(k)$ which completes the induction for part (i). $\square$

Remark 8.4.15. Unlike the results of Theorem 8.4.6 those of Proposition 8.4.14 *are* possible because $\mathbb{RP}^{8m-1}$ exists. This observation implies that when $p > 0$ the second coordinate of

$$(\iota_{2^{q+1},2^{q+1}-1})_*(y_{2^{q+1}}) \in V_{2^{q+1}} \oplus \mathbb{Z}/2^{4m-2^{q+3}+q+5}$$

must either be zero or divisible by 2^{q+5}. We know that it is divisible by 2^{q+4} but if $(\iota_{2^{q+1},2^{q+1}-1})_*(y_{2^{q+1}}) \simeq 2^{q+4}$ then $y_{2^{q+1}} \simeq 1$ and the argument of § 8.1.6 would show that there was a non-trivial primary Steenrod operation hitting the top mod 2 cohomology of $\mathbb{RP}^{8m-1}$, which would contradict Bill Browder's theorem [47].

8.4.16. Application to Example 8.3.4. Let $m = 2^q$ and, in the notation of Example 8.3.4, consider

$$(\eta \wedge 1 \wedge 1)_*(\iota_{8m-1}) = (a_0, a_1, a_2, \ldots, a_{2m-1})$$

in $\pi_{8m-1}(bu \wedge bo \wedge \mathrm{Cone}(\Theta_{8m-2,r}))$, where

$$a_k \in bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1})) \quad \text{satisfies} \quad (T_r \wedge 1)_*(w_k) = a_k.$$

Here w_k is as in § 8.4.3 and Theorem 8.4.7 while T_r is the map of Remark 8.3.13 and Proposition 8.3.14. When $k \geq 1$ the homomorphism

$$\begin{aligned}(T_r \wedge 1)_* : bu_{8m-1}&(\mathrm{Cone}(\Theta_{8m-2}) \wedge (F_{4k}/F_{4k-1})) \\ &\longrightarrow bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}) \wedge (F_{4k}/F_{4k-1}))\end{aligned}$$

may be identified with the homomorphism of Example 8.3.4 and Remark 8.3.13

$$\begin{aligned}(\tau_r \wedge 1)_* : bu_{8m-1}&(\mathbb{RP}^{8m-2} \wedge (F_{4k}/F_{4k-1})) \\ &\longrightarrow bu_{8m-1}(\mathbb{RP}^{8m-8r-2} \wedge (F_{4k}/F_{4k-1})).\end{aligned}$$

I shall conclude this section with a partial evaluation of the a_i's.

Proposition 8.4.17. Suppose that $m = 2^q$ and $r \geq 1$. Then, in the notation of Example 8.3.4 and § 8.4.16,

$$a_0 = \mu \cdot \iota_{8m-1} + e \cdot v_{8m-8r-3} u^{4r+1} \in bu_{8m-1}(\mathrm{Cone}(\Theta_{8m-2,r}))$$

for some integer e and 2-adic unit μ. For $4r - 1 \leq 2k \leq 2^{q+1}$

$$a_k \in (2e_k + 1)2^{4r+q-\nu_2(k)} \tilde{z}_{4k} v_{8m-4k-1} + (T_r \wedge 1)_*(V_k)$$

for some integer e_k.

Proof. The value of a_0 follows from Proposition 8.4.2. When $k \geq 1$, by § 8.4.16, $a_k = (T_r \wedge 1)_*(w_k)$. Therefore, by Corollary 8.4.9, for some integer e_k

$$a_k \in (T_r \wedge 1)_*((2e_k + 1)2^{q-\nu_2(k)} \tilde{z}_{4k} v_{8m-4k-1}) + (T_r \wedge 1)_*(V_k)$$

and the result follows since

$$(T_r \wedge 1)_*(2^{q-\nu_2(k)} \tilde{z}_{4k} v_{8m-4k-1}) = 2^{q-\nu_2(k)} \tilde{z}_{4k} 2^{4r} v_{8m-4k-1}$$

if $4r - 1 \leq 2k$, by Proposition 8.3.14. $\square$

Chapter 9

Futuristic and Contemporary Stable Homotopy

"Live as one of them, Kal-El, to discover where your strength and your power are needed. Always hold in your heart the pride of your special heritage. They can be a great people, Kal-El – they wish to be. They only lack the light to show the way."

from Marlon Brando as Jor-El in "Superman, the Movie"

The objective of this chapter is to give a brief overview of some current themes and developments in stable homotopy theory. In the 1990's a new stable homotopy category was designed by Vladimir Voevodsky in order to construct MGL, a new type of spectrum which Voevodsky used to prove a famous conjecture of John Milnor [191] concerning the norm residue map mod 2 on the Milnor K-groups of a field [180]. In § 1 I shall sketch the construction of this new type of stable homotopy category. In § 2 I shall sketch some spectral constructions which are possible in $\mathbb{A}^1$-stable homotopy theory which imitate the classical construction which I made in [245], [246] and [249] as part of my premature attempt to construct algebraic cobordism. Conjectures due to David Gepner and me, which are given in § 9.2.15, predict that these spectra are equivalent to algebraic K-theory and (the periodic version of) MGL, by analogy with Chapter 1, Theorem 8.3.3. In Example 9.2.15(ii) I have given the proof of my original result which we are convinced goes through *mutatis mutandis* in the $\mathbb{A}^1$-stable homotopy category. § 2 concludes with a paragraph concerning other recent, related stable homotopy theory developments.

9.1 $\mathbb{A}^1$-homotopy theory

In this section I shall outline the foundations of $\mathbb{A}^1$-homotopy theory [281]. This theory is based on the idea that one can define homotopies in the algebro-geometric context using the affine line $\mathbb{A}^1$ as a replacement for the topological unit interval.

Vladimir Voevodsky's construction of the $\mathbb{A}^1$-stable homotopy category consists of a sequence of intermediate categorical constructions which are each similar to the constructions involved in making the classical stable homotopy category of Chapter 1, § 3 but at each stage (of which there are many!) there is a very delicate and difficult choice (of which there are many!) of the next correct definition to prevent the process from stalling. The overall construct has already proved itself to be one of the most fundamental and important new ideas – namely the right place to do motivic cohomological algebraic geometry, which has been a long sought after objective since the advent of the work of Alexandre Grothendieck and his collaborators in the 1960's, recorded in the famous SGA series.

Let $\mathcal{C}$ be a category. Usually $\mathcal{C}$ as it stands will be inadequate for the construction of a (stable) homotopy category and first one has to find a suitable category of "space" $\mathcal{S}pc$ which contains $\mathcal{C}$ but also has useful properties such as the existence of internal Hom-objects and all (small) limits and colimits. For example, in topology $\mathcal{C}$ may consist of CW complexes and $\mathcal{S}pc$ of compactly generated spaces.

Next one defines the class of weak equivalences on $\mathcal{S}pc$ and the localisation [71] of $\mathcal{S}pc$ with respect to this class of morphisms gives the homotopy category H. Experience has shown that the most effective localisations of this type involve the choice of classes of fibrations and cofibrations in such a way as to obtain a closed model structure in the sense of Daniel Quillen [224]. These days one imbibes Quillen's axioms in the formulation given in ([114] Definition 3.2.3).

To get to stable homotopy one needs to stabilise with respect to some kind of suspension functors and typically one obtains a new category SW, called the Spanier-Whitehead category. For suitable suspensions the category SW will be additive and triangulated [208]. In order for this construction to work it is usual for the category H to be pointed (i.e., the initial object and final object coincide). Therefore one applies localisation with respect to suspensions to the homotopy category of pointed "spaces". Usually SW will not have the property of being closed under taking infinite coproducts (i.e., infinite direct sums). For this reason one requires construction of a further triangulated category SH called the stable homotopy category whose objects are "spectra" rather than spaces. Infinite sums are very important because using them one can prove the analogue, due to Amnon Neeman, of Ed Brown's representability theorem ([205], [206], [207]). Therefore the standard sequence of categorical constructions leading to stable homotopy takes the form

$$\mathcal{C} \rightarrow \mathcal{S}pc \rightarrow H \rightarrow SW \rightarrow SH.$$

Voevodsky designs such a procedure starting with the category Sm/S of smooth schemes over a Noetherian base scheme S. The most important applications to date have involved Sm/k, the category of smooth algebraic varieties over a field k (i.e., $S = \mathrm{Spec}(k)$).

At the end of his construction [281] Voevodsky has constructed three cohomology theories on $\mathcal{S}pc(S)$ for any base scheme S. These are algebraic K-theory, motivic cohomology and algebraic cobordism, each defined by making a representing spectrum in the $\mathbb{A}^1$-stable homotopy category. Algebraic K-theory defined in this manner on Sm/S coincides with the homotopy algebraic K-theory constructed by Chuck Weibel in [290]. In turn, particularly with finite coefficients, homotopy K-theory often coincides with Quillen K-theory of ([49], [226]). Voevodsky's motivic cohomology coincides on Sm/k with Bloch's higher Chow groups [36]. Algebraic cobordism is a new cohomology theory for which the correct definition has been sought ever since my paper [245] and which receives a homomorphism (not known to be an isomorphism) from the algebraic cobordism theory defined by generators and relations by Fabien Morel and Marc Levine in ([160], [161]).

There are several antecedents for pieces of the $\mathbb{A}^1$-homotopy picture – for example Rick Jardine's ([118], [119]) and Chuck Weibel's [290].

Incidentally, there are close connections between the motivic cohomology spectral sequence which converges to $K'_*(X)$ [80] and that constructed in Quillen's original paper [226] – these are related to the Gersten conjecture ([67], [82], [83], [88], [89], [139], [199], [213], [232], [241], [271] – for algebraic geometry background see [77], [78], [104], [187]).

9.2 Spaces

The main problem which prevents one from applying the constructions of abstract homotopy theory directly to the category Sm/S of smooth schemes over $\mathrm{Spec}(S)$ is the non-existence of colimits. In classical algebraic geometry this is known as the non-existence of "contractions". Voevodsky solves this problem for particular types of "contractions" by extending the category to include non-smooth varieties and algebraic spaces. For his purposes it is important to have *all* colimits, which is not possible in any of these extended categories.

There is a way, due to Alexandre Grothendieck, to add formally all colimits of small diagrams to a category $\mathcal{C}$. Consider the category of all contravariant functors to the category of sets, which is called the category of presheaves on $\mathcal{C}$ and is denoted by $PreShv(\mathcal{C})$. Any object X represents a presheaf $R_X : Y \mapsto Hom_{\mathcal{C}}(Y, X)$ which, by the Yoneda Lemma, embeds $\mathcal{C}$ as the subcategory of representable presheaves. The category $PreShv(\mathcal{C})$ has all small colimits and limits and any presheaf is canonically colimit of a diagram of representable presheaves.

Now one could take the "spaces" as $PreShv(Sm/S)$ but applying $R_{(-)}$ to a small pushout diagram in Sm/S (or more generally a colimit diagram) does not transform to a pushout diagram in $PreShv(Sm/S)$, although there is a canonical map from $R_{(-)}$ applied to the colimit to the colimit of the diagram of representable presheaves. This problem underlines that one wants to add new colimits while taking into account already existing ones – the technique for doing this is the theory of sheaves on Grothendieck topologies.

Let us specialise to the case when $S = \mathrm{Spec}(k)$ where k is a perfect field.

Definition 9.2.1 (The three principal topologies ([200] § 5.1.1 p. 364))**.** Let $X \in Sm/k$. Let $\{f_\alpha : U_\alpha \longrightarrow X\}_\alpha$ be a finite family of étale morphisms in the Grothendieck topology $\mathcal{V}$. Then:

(i) $\{f_\alpha : U_\alpha \longrightarrow X\}_\alpha$ is a covering family in the étale topology if and only if X is the union of the open sets $f_\alpha(U_\alpha)$,

(ii) $\{f_\alpha : U_\alpha \longrightarrow X\}_\alpha$ is a covering family in the Nisnevich topology if and only if for any point $x \in X$ there exists α and a point $y \in U_\alpha$ such that $f_\alpha(y) = x$ and the residue fields are equal, i.e., $k(x) = k(y)$ (via f_α),

(iii) $\{f_\alpha : U_\alpha \longrightarrow X\}_\alpha$ is a covering family in the Zariski topology if and only if $X = \bigcup_\alpha U_\alpha$.

This list of the τ-topologies and coverings satisfies the chain of inclusions:

$$\text{Zariski} \subset \text{Nisnevich} \subset \text{étale}.$$

If $\mathcal{V}$ is a category then $\mathrm{Fun}(\mathcal{V}^{\mathrm{op}}, \mathrm{Sets}) = \mathrm{Preshv}(\mathcal{V})$ is the category of presheaves of sets on $\mathcal{V}$. For Vladimir Voevodsky's purposes the most important topology is the one discovered by Yevsey Nisnevich and introduced to the West in [210].

Definition 9.2.2 ([200] § 2.1.4, p. 365; [281])**.** A distinguished square in $\mathcal{V}$ is a cartesian square of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

in which p is étale, i is an open immersion and

$$p^{-1}((X-U)_{\mathrm{red}}) \xrightarrow{\cong} (X-U)_{\mathrm{red}}$$

is an isomorphism of schemes (the reduced induced structures).

Remark 9.2.3. For a square in Definition 9.2.2 $\{U \longrightarrow X,\ V \longrightarrow X\}$ is a covering family in the Nisnevich topology. If p is an open immersion the second condition implies that U and V cover X. Squares like that of Definition 9.2.2 do for the Nisnevich topology what squares of the form

$$\begin{tikzcd} U \bigcap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \bigcup V \end{tikzcd}$$

do for the Zariski topology.

Definition 9.2.4 ([200] § 2.12 p. 304)**.** A presheaf of sets $F : \mathcal{V}^{op} \longrightarrow \text{Sets}$ is a sheaf in the τ-topology if F on the empty set is a point and

$$F(X) \longrightarrow \bigcup_{\alpha} F(U_\alpha) \rightrightarrows \bigcup_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$

defines $F(X)$ as the equaliser set of the two right-hand maps for every τ-covering. Hence

$$\text{Shv}(\mathcal{V}_\tau) \subset \text{Preshv}(\mathcal{V})$$

and

$$\mathcal{V} \subseteq \text{Shv}(\mathcal{V}_{\text{et}}) \subseteq \text{Shv}(\mathcal{V}_{\text{Nis}}) \subseteq \text{Shv}(\mathcal{V}_{\text{Zar}}) \subseteq \text{Preshv}(\mathcal{V})$$

where the left-hand inclusion is given by

$$X \in \mathcal{V} \mapsto (Y \longrightarrow \text{Hom}_{\mathcal{V}}(Y, X))$$

sending X to the functor represented by X.

In particular, on the distinguished square of Definition 9.2.2,

$$F(X) = F(U) \times_{F(W)} F(V).$$

9.2.5. The "spaces" in Voevodsky's theory are the Nisnevich sheaves $\text{Shv}_{\text{Nis}}(Sm/S)$ and the functor R embeds Sm/S into $\text{Shv}_{\text{Nis}}(Sm/S)$. This category has all small limits and colimits and, in addition, the forgetful functor from Nisnevich sheaves to presheaves has a left adjoint denoted by

$$a_{\text{Nis}} : \text{PreShv}_{\text{Nis}}(Sm/S) \to \text{Shv}_{\text{Nis}}(Sm/S)$$

– called sheafification.

Along with these spaces goes a subcategory of spaces of finite type (the analogue of compact spaces in topology) which include Sm/S and all suitable push-forward squares of finite type spaces.

In order to form smash products one also needs points and closely related to these are the neighbourhoods of a point and a fibre at a point.

Definition 9.2.6. A τ-point in $\mathcal{V}$ is a morphism of schemes

$$x : \text{Spec}(K) \longrightarrow X$$

such that

(i) K is a separably closed field if $\tau = \text{et}$,
(ii) K is the residue field of the image (also denoted by x) of the point of $\text{Spec}(K)$ in X if τ equals Nis or Zar.

Definition 9.2.7 ([200] § 2.1.9, p. 367)**.** Let $x : \mathrm{Spec}(K) \longrightarrow X$ be a τ-point in $\mathcal{V}$. The category of neighbourhoods of x – denoted by Neib_τ^x – is:

(i) when τ equals either étale or Nisnevich, the category of pairs $(U \xrightarrow{f} X, y : \mathrm{Spec}(K) \longrightarrow U)$ with f étale, U irreducible and y a τ-point of U with the same field K,

(ii) when τ equals Zariski, the category of open sets containing x.

Definition 9.2.8 ([200] § 2.1.10, p. 367)**.** Let $x : \mathrm{Spec}(K) \longrightarrow X$ be a τ-point in $\mathcal{V}$.

(i) For a τ-presheaf F, the fibre at x is the set

$$F_x = \mathrm{colim}_{(U \to X,\ y) \in \mathrm{Neib}_\tau^x}\ F(U).$$

(ii) The fibre functor associated to x is

$$X \mapsto F_x, \qquad \mathrm{Preshv}(\mathcal{V}) \longrightarrow \mathrm{Sets}.$$

Example 9.2.9 ([200] § 2.1.11, p. 367)**.** The fibre of $\mathbb{A}^1$ at x is:

(i) the strict Henselianisation $\mathcal{O}_{X,x}^{sh}$ when $\tau = \mathrm{et}$,

(ii) the Henselianisation $\mathcal{O}_{X,x}^{h}$ when $\tau = \mathrm{Nis}$,

(iii) the local ring $\mathcal{O}_{X,x}$ when $\tau = \mathrm{Zar}$.

Remark 9.2.10. Isomorphisms, monomorphisms and epimorphisms are tested faithfully on fibre functors at τ-points. For example, being a monomorphism on all such fibre functors is necessary and sufficient to be a monomorphism.

9.2.11. Armed with “spaces” and “spaces of finite type” Voevodsky’s construction of the $\mathbb{A}^1$-stable homotopy category proceeds along the lines sketched above in Section 1. Details of this are to be found in [83], [130], [180], [200], [202], [203], [279], [280], [281], [283], [284], [285], [286], [287], [288] and [289] (see also [39], [101], [102], [103], [122], [139], [155], [156], [157], [160], [161], [199], [208], [210], [213], [215], [216], [223], [241], [266], [271], [295]).

A pointed space (X, x) is a space together with a morphism $x : pt \longrightarrow X$ (“space” and “point” in the new exotic sense). For a space X and a subspace $A \subset X$ the space X/A is the pushout of the diagram

$$x \longleftarrow A \longrightarrow X$$

which is a canonically pointed space. Similarly if (Y, y) is another pointed space then the smash product is defined to be

$$X \times Y/(X \times y \bigcup x \times Y).$$

For example, to a vector bundle $E \longrightarrow U$ over a smooth scheme $U \in Sm/S$ we may define a Thom space in the $\mathbb{A}^1$-stable homotopy category as $E/(E - s(U))$ where s is the zero section – this contrasts well with the clumsy étale constructions of ([245] Part IV). This Thom space construct enjoys all the classical properties – for example, the Thom space of a sum of two bundles is canonically equivalent to the smash product of their individual Thom spaces.

Definition 9.2.12 ([200] § 5.1.1 p. 415)**.** Once again let k be a perfect field. Set $\mathcal{V} = Sm/k$ ([200] p. 363). Here is what spectra look like in the $\mathbb{A}^1$-stable homotopy category. Since $\mathbb{P}^1$ lies in Sm/k we may formulate the following definition:

a $\mathbb{P}^1$-spectrum E is a collection $\{E_n, \sigma_n\}_{n \geq 0}$ where

- E_n is a pointed simplicial sheaf,
- $\sigma_n : E_n \wedge \mathbb{P}^1 \longrightarrow E_{n+1}$ is a morphism of pointed simplicial sheaves,
- $f : E \longrightarrow E'$, morphisms of $\mathbb{P}^1$-spectra, are given by the usual definition, analogous to Chapter 1, § 3.

This gives the category $Sp^{\mathbb{P}^1(k)}$ of $\mathbb{P}^1$-spectra over k.

9.2.13. The $\mathbb{A}^1$-homotopy Hopf construction. The $\mathbb{A}^1$-stable homotopy category admits small (filtered) colimits. The topological Hopf construction takes a based map $f : X \times Y \longrightarrow Z$ and constructs $H(f)$ defined by $H(f)(x, t, y) = (t, f(x, y))$,

$$H(f) : X * Y \longrightarrow \Sigma Z.$$

Now we shall make the corresponding construction in the $\mathbb{A}^1$-stable homotopy category ([200] p. 417).

Let (X, x) be a pointed space in the sense of ([200], [203], [281] p. 581) and let $(\mathbb{A}^1, 0)$ be the space given by $\mathbb{A}^1 = \mathrm{Spec}(k[t]$ pointed at 0 (i.e., $k[t] \longrightarrow k$, $h(t) \mapsto h(0)$). Then $\{0\} \times X \longrightarrow \mathbb{A}^1 \times X$ is a pointed morphism with cofibre CX. That is, the following diagram is a cartesian square of based maps:

$$\begin{array}{ccc}
\{0\} \times X & \xrightarrow{i} & \mathbb{A}^1 \times X \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\
\text{pt.} & \xrightarrow{g} & CX.
\end{array}$$

Since i is a weak equivalence in the $\mathbb{A}^1$-stable homotopy category so is g.

Suppose that (Y, y) is another pointed space then

$$\begin{array}{ccc}
\{0\} \times X \times Y & \xrightarrow{i \times 1} & \mathbb{A}^1 \times X \times Y \\
{\scriptstyle j \times 1}\downarrow & & \downarrow{\scriptstyle f \times 1} \\
Y = \text{pt.} \times Y & \xrightarrow{g \times 1} & CX \times Y
\end{array}$$

is also a cartesian square ([203] Lemma 2.2.29 p. 43) of based maps. Similarly we have

$$\begin{array}{ccc}
\{0\} \times Y \times X & \xrightarrow{i' \times 1} & \mathbb{A}^1 \times Y \times X \\
{\scriptstyle j' \times 1}\downarrow & & \downarrow{\scriptstyle f' \times 1} \\
X = \text{pt.} \times X & \xrightarrow{g' \times 1} & CY \times X
\end{array}$$

and the $\mathbb{A}^1$-join $(X *_{\mathbb{A}^1} Y, (0, x, y))$ is the based space defined by the following cocartesian square:

$$\begin{tikzcd} \mathbb{A}^1 \times X \times Y \arrow[r, "f"] \arrow[d, "f'"'] & CX \times Y \arrow[d, "h"] \\ CY \times X \arrow[r, "h'"] & X *_{\mathbb{A}^1} Y. \end{tikzcd}$$

This cocartesian square yields a distinguished triangle in the $\mathbb{A}^1$-stable homotopy category ([200] p. 401)

$$\Sigma^\infty(X \times Y) \xrightarrow{(f,f')} \Sigma^\infty X \vee \Sigma^\infty Y \\ \longrightarrow \Sigma^\infty(X *_{\mathbb{A}^1} Y) \longrightarrow \Sigma^\infty(X \times Y)[1]$$

in which (f, f') is split by $X \vee Y \longrightarrow X \times Y$ so we obtain an equivalence

$$\Sigma^\infty(X *_{\mathbb{A}^1} Y) \simeq \Sigma^\infty(X \wedge Y)[1] = \Sigma^\infty(S^1 \wedge X \wedge Y).$$

Similarly, if (Z, z) is a based space we have a cartesian square

$$\begin{tikzcd} \mathbb{A}^1 \times Z \arrow[r] \arrow[d] & C_+ Z \arrow[d] \\ C_- Z \arrow[r] & \Sigma_{\mathbb{A}^1} Z \end{tikzcd}$$

defining $\Sigma_{\mathbb{A}^1} Z$ and

$$\Sigma^\infty(\Sigma_{\mathbb{A}^1} Z) \simeq \Sigma^\infty(\mathbb{A}^1 \times Z)[1] \simeq \Sigma^\infty Z[1] = \Sigma^\infty S^1 \wedge Z.$$

Finally a based map $F : X \times Y \longrightarrow Z$ yields

$$1 \times F : \mathbb{A}^1 \times X \times Y \longrightarrow \mathbb{A}^1 \times Z$$

and, in the $\mathbb{A}^1$-stable homotopy category,

$$H(F) : X *_{\mathbb{A}^1} Y \longrightarrow S_{\mathbb{A}^1} Z,$$

the Hopf construction on F. Hence, in the $\mathbb{A}^1$-stable homotopy category, we have a canonical morphism

$$H(F) : \Sigma^\infty(X \wedge Y) \longrightarrow \Sigma^\infty Z$$

which is natural and associative.

Example 9.2.14.

(i) Our main example is the construction of the $\mathbb{P}^1$-spectrum which uses the Hopf construction on $\mathbb{P}^1 \times \mathbb{P}^n \longrightarrow \mathbb{P}^{n+1}$ to construct the spectrum

$$\mathbb{P}^\infty[1/\beta] = \{\mathbb{P}^\infty, \mathbb{P}^\infty, \ldots, \mathbb{P}^\infty, \ldots\}.$$

(ii) The Hopf construction on the map which multiplies the reduced line bundle on $\mathbb{P}^1$ by the reduced universal bundle

$$\mathbb{P}^\infty \times BGL \longrightarrow BGL$$

produces the $\mathbb{P}^1$-spectrum of [281] which is Voevodsky's manifestation of homotopy algebraic K-theory,

$$K = \{BGL, BGL, \ldots, BGL, \ldots\}.$$

There is a canonical morphism

$$\mathbb{P}^\infty[1/\beta] \longrightarrow K$$

which is a homotopy equivalence on complex points by Chapter 1, Theorem 1.3.3 ([245], [246] Theorem 2.12)

$$\mathbb{P}^\infty[1/\beta](\mathbb{C}) \longrightarrow K(\mathbb{C}).$$

(iii) The Hopf construction on the *addition* map

$$\mathbb{P}^\infty \times BGL \longrightarrow BGL$$

defines a $\mathbb{P}^1$-spectrum

$$BGL[1/\beta] = \{BGL, BGL, BGL, \ldots\}.$$

On complex points we obtain

$$BGL[1/\beta](\mathbb{C}) = \Sigma^\infty BU[1/B] \simeq PMU,$$

the spectrum of Chapter 1, Theorem 1.3.3.

By the theory of Chern classes and the fact that $BGL[1/\beta]$ is universal for such theories (in the same sense that unitary cobordism is among classical spectra [245]) there is a morphism in the $\mathbb{A}^1$-stable homotopy category

$$BGL[1/\beta] \longrightarrow MGL$$

where MGL is the $\mathbb{P}^1$-spectrum of [281].

9.2.15. Assertions and conjectures. In this subsection I shall present without rigorous proof (but sometimes with a sketch-proof) a number of results in the $\mathbb{A}^1$-stable homotopy category which are analogous to classical stable homotopy results which I have mentioned and used earlier in this book. The results and conjectures here – and the progress towards their verification – are the result of animated discussion with my Sheffield colleague David Gepner[1].

[1] These results are now proved in greater generality. In the end David Gepner and I used a slightly differently method, details of which were given in his talk at the October 2007 Midwest Topology Seminar at the University of Illinois.

(i) The morphism

$$\mathbb{P}^{\infty}[1/\beta] \longrightarrow K$$

is presumably an equivalence.

The proof of this which I have in mind will be sketched below, after I have stated the companion assertion. It is also possible that an alternative proof may be based upon the unique characterisation of K among oriented spectra as proved in [215] together with the techniques of Marc Levine from [159]. This should at least be true for Sm/k when k is a field of characteristic zero – possibly required to be algebraically closed.

(ii) Under similar conditions the morphism

$$BGL[1/\beta] \longrightarrow MGL$$

should also be an equivalence.

Originally the proof of this which I had in mind was based upon the Atiyah-Hirzebruch spectral sequence which is mentioned and used in [295] and also used in [223]. This spectral sequence is due to Mike Hopkins and Fabien Morel which currently remains unpublished.

Instead, I will give a (tentative) proof of my classical result of Chapter 1, Theorem 1.3.3 which carries over to the $\mathbb{A}^1$-stable homotopy category with modest changes which I will list as we go along.

The idea is that $MGL_{*,*}$ often behaves like MU_* tensored over the Lazard ring $L \cong MGL_{2*,*} \cong MU_*$ so that a classical proof based on MU_* calculations carries over to the new setting. The MU-like properties of MGL were explained to me by Mike Hopkins during a lunch-break at the Fields Institute in May 2007. The following sketch proof resembles the proof due to Rob Arthan [23] in a couple of places.

- *An MU-based proof of my two classical localised stable homotopy theorems, apparently modifiable to the $\mathbb{A}^1$-stable homotopy category* (see Chapter 1, Theorem 1.3.3):

Step 1: We begin by proving $\Sigma^{\infty}BU_+[1/B] \simeq PMU$ – periodic MU-theory which we might write as $MU[t, t^{-1}]$. Originally I proved this by geometrical transfer constructions combined with the Snaith splitting for $Q\mathbb{CP}^{\infty}$, which also originates as a geometrical construction. The following argument is more MU-based.

Recall that in the classical stable homotopy category ([9]; see also Chapter 1 § 1.3.1) a spectrum E is a sequence of spaces (or suspension spectra $\Sigma^{\infty}X$ will suffice) $\{E_n\}$ and maps $\epsilon_n : \Sigma E_n \longrightarrow E_{n+1}$. The associated E-homology and cohomology are defined by

$$E^r(X) = [X, E]_{-r}, \qquad E_r(X) = [S^0, X \wedge E]_r$$

where $[E, F]_s$ denotes maps of degree s; that is, $\{f_n : E_n \longrightarrow F_{n-s}\}_n$. The kth suspension of E is $(\Sigma^k E)_m = E_{m+k}$ – that is, we move the E_n's k places to the *left*, $\Sigma^k E = E[-k]$.

For the MU-spectrum we have

$$MU_n = \begin{cases} MU(m) & \text{if } n = 2m,\ m \geq 1, \\ \Sigma MU(m), & \text{if } n = 2m+1,\ m \geq 1 \end{cases}$$

where $MU(m)$ is the Thom space $MU(m) \simeq BU(m) \bigcup \mathrm{Cone}(BU(m-1))$ and in particular $MU(1) = BU(1) = \mathbb{CP}^\infty$.

From ([9] p. 46) we have isomorphisms

$$MU_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots, \beta_j, \ldots] \qquad \deg(\beta_i) = 2i$$

and

$$MU^*(BU) \cong \mathbb{Z}[[c_1, c_2, \ldots, c_k, \ldots]] \qquad \deg(c_i) = 2i.$$

The β_i originate in $MU_{2i}(BU(1))$ and c_i vanishes in $MU^{2i}(BU(r))$ for $i > r$. Also

$$\langle c_i, \beta_{j_1}^{\alpha_1} \beta_{j_2}^{\alpha_2} \ldots \rangle = \begin{cases} 1 & \text{if } \beta_{\underline{j}}^{\underline{\alpha}} = \beta_1^i, \\ 0 & \text{otherwise} \end{cases}$$

and c_1 restricts to the orientation class x^{MU} in $MU^2(S^2) = MU^2(\mathbb{CP}^1)$. The diagonal $\Delta : BU \longrightarrow BU \times BU$ satisfies $\Delta_*(\beta_i) = \sum_j \ \beta_j \otimes \beta_{i-j}$ and the direct sum $\oplus : BU \times BU \longrightarrow BU$ satisfies $\oplus^*(c_i) = \sum_j \ c_j \otimes c_{i-j}$.

The spectrum $\Sigma^\infty BU_+[1/B]$ has

$$\Sigma^\infty BU_+[1/B]_m = \begin{cases} \Sigma^\infty BU_+, & \text{if } m = 2n \geq 0, \\ \Sigma^\infty \Sigma(BU_+), & \text{if } m = 2n+1 \geq 1. \end{cases}$$

The structure maps come from choosing a generator $B : \Sigma^\infty S^2 \longrightarrow \Sigma^\infty(S^2_+) \simeq \Sigma^\infty(S^2 \vee S^0)$, corresponding to the inclusion of the S^2-wedge summand, and forming a map of degree 2,

$$c : S^2 \wedge \Sigma^\infty BU_+ \longrightarrow \Sigma^\infty(S^2_+ \wedge BU_+) = \Sigma^\infty((S^2 \times BU)_+) \stackrel{\Sigma^\infty \oplus_+}{\longrightarrow} \Sigma^\infty BU_+.$$

Hence the reduced MU-theory of $\Sigma^\infty BU_+$ is

$$MU_*(\Sigma^\infty BU_+) \cong MU_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots]$$

and

$$\epsilon_* : MU_*(\Sigma^\infty BU_+) \longrightarrow MU_{*+2}(\Sigma^\infty BU_+)$$

given by $\epsilon_*(\beta_{\underline{j}}^{\underline{\alpha}}) = \beta_1 \beta_{\underline{j}}^{\underline{\alpha}}$ so that

$$MU_*(\Sigma^\infty BU_+[1/B]) = \lim_{\vec{n}} \ MU_{*+2n}(\Sigma^\infty BU_+[1/B]) \cong \mathbb{Z}[\beta_1^{\pm 1}, \beta_2, \beta_3, \ldots].$$

Now $\Sigma^\infty BU_+[1/B]$ has an orientation given by

$$\Sigma^\infty BU(1)_+ \longrightarrow \Sigma^\infty BU_+[1/B]$$

and this orientation restricts to the B on $\Sigma^\infty S^2$. As explained by ([9] p. 46 et seq) MU has a universal mapping property to spectra with orientations (equivalently Chern classes for complex vector bundles) so we obtain a canonical map of spectra $MU \longrightarrow \Sigma^\infty BU_+[1/B]$ which extends to $PMU = \vee_{-\infty}^{\infty} \Sigma^{2k} MU$ as a map of ring spectra.

Now, as I pointed out in [245], $\Sigma^\infty BU_+[1/B]$ also has a universal property for ring spectra with Chern classes. Actually it has an even better property: universality for theories "with genera restricting to B". First let us see what Chern classes will do for us.

Consider the Chern class

$$c_i : \Sigma^\infty BU_+ \longrightarrow \Sigma^{2i} MU,$$

which fits into the following homotopy commutative diagram:

$$\begin{array}{ccc} S^2 \wedge \Sigma^\infty BU_+ & \xrightarrow{\epsilon} & \Sigma^\infty BU_+ \\ \downarrow{\scriptstyle 1\wedge c_{i-1}} & & \downarrow{\scriptstyle c_i} \\ S^2 \wedge \Sigma^{2i-2} MU & \xrightarrow{1} & \Sigma^{2i} MU \end{array}$$

because $\oplus^*(c_i) = \sum_j \ c_j \otimes c_{i-j}$ and c_j restricts to zero on S^2 when $j = 0$ and $j \geq 2$ and c_1 restricts to B, the orientation class. Hence we obtain

$$\Sigma^\infty BU_+ \stackrel{c_1 \vee c_2 \vee \cdots}{\longrightarrow} \Sigma^2 MU \vee \Sigma^4 MU \vee \cdots$$

which induces a map of ring spectra

$$\underline{c} : \Sigma^\infty BU_+[1/B] \longrightarrow PMU.$$

However, the map $\underline{c}$ does *not* induce an isomorphism

$$\underline{c}_* : MU_*(\Sigma^\infty BU_+[1/B]) \longrightarrow MU_*(PMU).$$

This is because $< c_k, \beta_{\underline{j}}^{\underline{\alpha}} >= 0$ except when $\beta_{\underline{j}}^{\underline{\alpha}} = \beta_1^k$. Note that ([9] p. 46 et seq)

$$MU_*(PMU) \cong \mathbb{Z}[b_0^{\pm 1}, b_1, b_2, b_3, \ldots]$$

where b_j is the image of $\beta_{j+1} \in MU_{2j+2}(\mathbb{CP}^\infty)$. However, there is a "genus" C – that is, a nice characteristic class giving a map of ring spectra ([245], [246])

$$C : \Sigma^\infty BU_+[1/B] \longrightarrow PMU$$

inducing an isomorphism

$$C_* : MU_*(\Sigma^\infty BU_+[1/B]) \xrightarrow{\cong} MU_*(PMU).$$

I shall now explain where C comes from and why it generalises to the Morel-Voevodsky situation immediately.

In view of the isomorphism

$$MU^*(BU) \cong \mathbb{Z}[[c_1, c_2, \ldots]]$$

C is induced by

$$\vee_i \ C_i : \Sigma^\infty BU_+ \longrightarrow \vee_{-\infty}^{\infty} \ \Sigma^{2k} MU$$

(the Chern class example was $1 \vee c_1 \vee c_2 \ldots$) where the C_i are power series in the c_i's with $\mathbb{Z}$ coefficients. Currently I do not know their algebraic formulae explicitly, but I imagine that this could be worked out.

As with the Chern classes, the following diagram is homotopy commutative.

$$\begin{tikzcd}
S^2 \wedge \Sigma^\infty BU_+ \arrow[r, "\epsilon"] \arrow[d, "1 \wedge C_{i-1}"] & \Sigma^\infty BU_+ \arrow[d, "C_i"] \\
S^2 \wedge \Sigma^{2i-2} MU \arrow[r, "1"] & \Sigma^{2i} MU
\end{tikzcd}$$

The map C_k is given by the wedge sum of the "transfer" maps

$$\Sigma^\infty BU_+ \longrightarrow \Sigma^\infty BU/BU(n-1) \longrightarrow \Sigma^\infty MU(n)$$

which I constructed in ([245], [246]). The map C induces an isomorphism

$$C_* : MU_*(\Sigma^\infty BU_+[1/B]) \xrightarrow{\cong} MU_*(PMU)$$

and therefore

$$C_* : PMU_*(\Sigma^\infty BU_+[1/B]) \xrightarrow{\cong} PMU_*(PMU).$$

Hence we have two maps of ring spectra

$$PMU \xrightarrow{\gamma} \Sigma^\infty BU_+[1/B] \xrightarrow{C} PMU$$

and $C \cdot \gamma \simeq 1$, by universality of MU and PMU. Therefore

$$\Sigma^\infty BU_+[1/B] \simeq PMU \vee Y$$

– in classical topology we already know that $Y \simeq *$ but if we did not then the splitting would tell us

$$\pi_*(PMU \wedge Y) = 0$$

and Y is a PMU-module so $\pi_*(Y) = 0$.

Replacing ([9] p. 46) by [216] (see also [215]) shows, by the same argument, in the $\mathbb{A}^1$-stable homotopy category,

$$\Sigma^\infty BGL_+[1/B] \simeq PMGL \vee Y$$

and $\pi_{\mathbb{P}^1,*,*}(Y) = 0$.

Step 2: Now we want to show that

$$\Sigma^\infty \mathbb{CP}^\infty_+[1/B] \simeq KU.$$

The determinant map induces a map of ring spectra

$$\det : \Sigma^\infty BU_+[1/B] \longrightarrow \Sigma^\infty \mathbb{CP}^\infty_+[1/B].$$

In [245] I explained why this map may be identified with the Conner-Floyd map. The Hopf line bundle gives a map of ring spectra

$$C' : \Sigma^\infty \mathbb{CP}^\infty_+[1/B] \longrightarrow KU.$$

We have a commutative diagram of maps of ring spectra

$$\begin{tikzcd} \Sigma^\infty BU_+[1/B] \arrow[r, "C"] \arrow[d, "\det"] & PMU \arrow[d, "\text{Conner-Floyd}"] \\ \Sigma^\infty \mathbb{CP}^\infty_+[1/B] \arrow[r, "C'"] & KU. \end{tikzcd}$$

By the Conner-Floyd theorem [68], KU is Landweber exact so

$$PMU_*(X) \otimes_{PMU_*} KU_* \longrightarrow KU_*(X)$$

is an isomorphism. The tensor product over $MU_*[t^{\pm 1}] \cong L[t^{\pm 1}]$ (L is the Lazard ring of the universal formal group) uses the formal group law.

The formal group of $\Sigma^\infty \mathbb{CP}^\infty_+[1/B]$ is the multiplicative formal group, which is the same as that of KU. This means, by the theory of orientations and formal groups explained in ([9] p. 46 et seq), that

$$PMU \xrightarrow{\gamma} \Sigma^\infty BU_+[1/B] \xrightarrow{\det} \Sigma^\infty \mathbb{CP}^\infty_+[1/B]$$

factorises through $PMU \longrightarrow KU$ to give a homotopy commutative diagram of the form

$$\begin{tikzcd} PMU \arrow[r, "\gamma"] \arrow[d, "\text{Conner-Floyd}"] & \Sigma^\infty BU_+[1/B] \arrow[d, "\det"] \\ KU \arrow[r, "\gamma'"] & \Sigma^\infty \mathbb{CP}^\infty_+[1/B]. \end{tikzcd}$$

This splits off KU from $\Sigma^\infty \mathbb{CP}^\infty_+[1/B]$ as a KU-module summand.

In the $\mathbb{A}^1$-stable homotopy category it is true that the Conner-Floyd theorem and Landweber exactness remain true when MU, KU are replaced by MGL, K so the analogous result should still hold.

Step 3: The discussion of Step 2 shows that $C' \cdot \gamma' = 1$. The diagram of Step 2 shows that

$$\pi_*(\Sigma^\infty BU_+[1/B]) \xrightarrow{\det} \pi_*(\Sigma^\infty \mathbb{CP}^\infty_+[1/B])$$

is surjective, because before inverting B it is induced by a split epimorphism of spaces. Therefore

$$\gamma'_* : KU_* \longrightarrow \pi_*(\Sigma^\infty \mathbb{CP}^\infty_+[1/B])$$

is surjective and, being a split monomorphism, it is an isomorphism. This shows that there is a spectrum W with $\pi_*(W) = 0$ such that $\Sigma^\infty \mathbb{CP}^\infty_+[1/B] \simeq KU \vee W$.

In the original situation of [245] is was easy to prove the additional fact that $W \simeq *$. Except for this final fact, the proof applies to

$$\pi_{\mathbb{A}^1,*,*}(B\mathbb{G}_{m+}[1/B]) \longrightarrow \pi_{\mathbb{A}^1,*,*}(BGL) = K_{*,*}$$

using Landweber exactness for MGL and K in the form described in ([215], [216]).

In the classical stable homotopy category one could alternatively show that $\pi_*(W) = 0$ by proving that

$$KU_*(\Sigma^\infty \mathbb{CP}^\infty_+[1/B]) \longrightarrow KU_*(KU)$$

is an isomorphism (c.f. [9] p. 93 et seq) and using the fact that W is a KU-module spectrum – by analogy with the argument of Step 1. Incidentally $KU_*(KU)$ is a $\mathbb{Q}$-vector space ([9] p. 98).

(iii) Both $\mathbb{P}^\infty[1/\beta]$ and $BGL[1/\beta]$, after the $\mathbb{A}^1$-homotopy version of p-adic completion á la Bousfield [43], satisfy Gabber-Suslin rigidity in the sense of ([264], [265]; see also Chapter 4). For $BGL[1/\beta]$ follows from Suslin's theorem (in the classical stable homotopy category) and then rigidity for $\mathbb{P}^\infty[1/\beta]$ should follow from that for $BGL[1/\beta]$ using the (in a sense, "surjective") determinant map $BGL \longrightarrow \mathbb{P}^\infty$ in the same way in which I originally used the determination of $\Sigma^\infty BU[1/B] \simeq PMU$ and the determinant map to show that $\Sigma^\infty \mathbb{CP}^\infty[1/B] \simeq KU$ ([245], [246]; see the sketch-proof of Chapter 1, Theorem 1.3.3).

(iv) The speculative results asserted in (i) and (ii) must surely be true for the étale realisations of these $\mathbb{P}^1$-spectra. For (i) the proof I have in mind uses the results of Bill Dwyer and Eric Friedlander [70] and of Rick Jardine [121]. The case of (ii) should follow from the "conditional theorem" of [223], whose étale homotopy type results may also be phrased and proved, I believe, using the technology of [121].

(v) There should be an analogue of the Snaith splitting of Chapter 1, § 1.5.1 in the $\mathbb{A}^1$-stable homotopy category (see Step 1 in (ii) above). Judging by the many applications of these splittings in classical stable homotopy and other parts of algebraic topology, this would be a really useful result. At the moment even the phrasing of the statement is not completely clear. Since Voevodsky's stable homotopy category has two different basic types of suspension the situation is reminiscent of the stable splittings in equivariant stable homotopy theory used by Gunnar Carlsson [54] to prove that the Segal conjecture implies the Sullivan conjecture, which was first proved by Haynes Miller using the unstable Adams spectral sequence and later by Jean Lannes, using his T-functor.

(vi) Finally, there should be behaviour relating Adams operations in algebraic K-theory to reduced power operations in mod 2 motivic cohomology [282] which is analogous to that illustrated by the main result of this monograph in Chapter 8, Theorem 8.4.6 and Theorem 8.4.7.

That is enough $\mathbb{A}^1$-stable homotopy speculation for now. I included it because of the close relationship with my classical results of Chapter 1, Theorem 1.3.3.

9.2.16. As I come to this the penultimate paragraph in the final chapter of this monograph a large number of further aspects of stable homotopy and spectra are currently being developed. Although no longer completely up to date the proceedings [96] of the short Newton Institute programme during September-December 2002, which John Greenlees and I organised, contains several excellent general surveys (e.g., [71], [91], [124], [182], [261], [277]). Persistent themes in modern stable homotopy include the enriching of structures on categories, triangulated categories and derived categories [113]. The remaining surveys of [96] touch on these topics in one way or another (motivic [201]; oriented spectra [214]; derived categories and motivic arithmetic [253]).

Further descriptions of structured spectra and their applications can be found in (e.g., [236], [34], [97]) and in the notes of Charles Rezk [234]. As I mentioned earlier, one of the applications of symmetric spectra [114] was to give the modern formulation of a Quillen model category; for example, as used by Voevodsky. Other important stable homotopy constructions occur in the work of Ib Madsen and his collaborators – including Lars Hesselholt ([106], [107]), Ulrike Tillmann [170] and Michael Weiss – culminating respectively in the calulation of the K-theory of local fields and in the solution of David Mumford's conjecture on the cohomology of moduli spaces [171] (see also [44] for further topology in physics). In addition there are areas which grew out of Ed Witten's construction of elliptic cohomology in the late 1980's. These include the derived algebraic geometry of Jacob Lurie [163], the search for special spectra such as TMF (topological forms) and TAF (topological automorphic forms) and other spectra connected to topological field theories [112] or to the chromatic stable homotopy which originated with (and still substantially features) Jack Morava's K- and E-theories [186]. Finally, there are connections of K-theories to physics, to the Novikov conjecture, to loop-groups and to cyclic homology inspired by the conjecture of Paul Baum and Alain Connes concerning equivariant K-theory of the reduced C^*-group algebra [198].

Finally, a few words on the subject of the "late-breaking news" mentioned in the Preface. The main result which is claimed in the preprints [16], [17] and [18] is that the stable homotopy classes with Arf-Kervaire invariant one (modulo 2) exist only in a finite, unspecified range of dimensions. Such a result would be excellent evidence in support of the conjecture in the Preface, which predicts that only the first five possibilities actually exist. The method of attack adopted in ([16], [17], [18]) interprets the Arf-Kervaire invariant in terms of counting certain multiple points of immersions and features an invariant given by a formula similar to that of Theorem 2.2.3. The immersion interpretation dates back to work in the 1980's by Peter Eccles ([74], [75], [76]; see also [19] and [20]). By analogy, let us take a look at the immersion approach to the Hopf invariant one problem. The results of Chapter 2 easily yield the fact that there exists a stable homotopy class with Hopf invariant one (modulo 2) if and only if there is a bordism of immersions of an n-manifold $M \longrightarrow \mathbb{R}^{n+1}$ such that $\langle w_1(M)^n, [M]\rangle \equiv 1$ (modulo 2). In ([16], [17], [18]) the Arf-Kervaire invariant is interpreted in a similar manner by means of bordism of codimension 2 immersions. Geometric interpretations of this sort in stable homotopy theory intrinsically require very detailed and complex arguments. Accordingly it should come as no surprise that the expert jury is still in session on this one. In the long run, if the approach succeeds, it would be very interesting to see whether the UTT approach to the Arf-Kervaire invariant can be combined with the bordism of immersions interpretation to effect a simplification to the argument.

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