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ON THE LOCALISATION SEQUENCE IN K-THEORY

VICTOR SNAITH[1]

ABSTRACT. A description of the boundary map in Quillen's localisation sequence is given in terms of classifying spaces of categories. Using this description the low dimensional part of the localisation sequence for a Dedekind domain is shown to coincide with the algebraically defined Bass-Tate sequence.

0. Introduction. Let R be a ring with unit and $S \subset R$ a multiplicative set of central, nonzero divisors. Let R_S be the localised ring. If H is the exact category of finitely generated R-modules, M, having projective dimension $\leqslant 1$ and $M_S = 0$ there is a localisation sequence [G-Q] ($q \geqslant 0$)

$$\dots \to K_{q+1}(R_S) \xrightarrow{\partial} K_q(H) \xrightarrow{a} K_q(R) \xrightarrow{b} K_q(R_S) \to \dots.$$

The proof consists of utilising Quillen's Theorem B [Q, §1] to realise a and b by means of a categorical quasi-fibration. This gives nice categorical descriptions of a and b but not of ∂. Using the techniques of [G-Q], [Q] it is shown (in §1.5) how to realise ∂ and a by means of a categorical quasi-fibration. In §2 we apply this to compute ∂ when R is a Dedekind domain with quotient field R_S.

1. Let R be a ring, $S \subset R$ is a multiplicative set of central nonzero divisors and H is the category of finitely generated R-modules, M, of projective dimension $\leqslant 1$ such that $M_S = 0$.

Let G denote the following category. Its objects are surjections $L \twoheadrightarrow M \oplus B$ with L, $B \in \mathbf{P}(R)$, the category of finitely generated projective R-modules, and $M \in H$. An arrow $(L' \twoheadrightarrow M' \oplus B') \to (L \twoheadrightarrow M \oplus B)$ is represented by an equivalence class of diagrams

$$\begin{array}{ccc} L' & \twoheadrightarrow & M' \oplus B' \\ \| & & \uparrow \\ L' & \twoheadrightarrow & M_1 \oplus B_1 \\ \downarrow & \square & \downarrow \\ L & \twoheadrightarrow & M \oplus B \end{array}$$

in which $\square$ denotes a pullback square and in which the vertical arrows are direct sums of arrows from H and $\mathbf{P}(R)$. Two such diagrams are equivalent if they are isomorphic by an isomorphism which is the identity on $M \oplus B$ and $M' \oplus B'$.

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Composition of two morphisms is accomplished by a pullback construction analogous to the composition in Quillen's Q-category **[Q, §2]**.

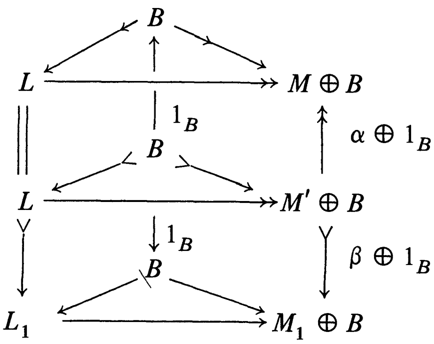
The category, G, was introduced in **[G-Q**, p. 230**]** and there the following is shown.

1.1. Lemma. *The functor* h: $(L \twoheadrightarrow M \oplus B) \mapsto M$ *induces a homotopy equivalence* h: $G \xrightarrow{\simeq} QH$.

Now consider the functor g: $G \to Q\mathbf{P}R$ given by $g(L \twoheadrightarrow M \oplus B) = (B)$. This functor is fibred **[G-Q]**. That is, the functor Γ: $g^{-1}(B) \to B/g$, $\Gamma(L \twoheadrightarrow M \oplus B) = (L \twoheadrightarrow M \oplus B, 1_B)$, has an adjoint, γ. Therefore **[Q**, §1**]** Γ: $g^{-1}(B) \to B/g$ is a homotopy equivalence.

Let us fix $B \in \mathbf{P}(R)$ and consider $g^{-1}(B)$. It has objects $(L \twoheadrightarrow M \oplus B)$, and a morphism $(L' \twoheadrightarrow M' \oplus B) \to (L \twoheadrightarrow M \oplus B)$ is an equivalence class of diagrams as in the category G with the additional restriction that the endomorphisms of B in the diagram should be the identity.

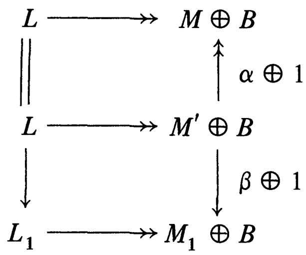
Define a second category $\tilde{g}^{-1}(B)$ whose objects are pairs $(L \twoheadrightarrow M \oplus B, B \rightarrowtail L)$ in which the first entry is an object of G, and the composition $B \rightarrowtail L \twoheadrightarrow M \oplus B$ is the inclusion of the second summand. Morphisms are equivalence classes of commutative diagrams of the following form.



Composition of morphisms is defined as in G and $g^{-1}(B)$.

1.2. Proposition. *The functor* f: $\tilde{g}^{-1}(B) \to g^{-1}(B)$ *which forgets the morphism* $B \rightarrowtail L$ *is a homotopy equivalence.*

Proof. Fix an object $(L_1 \twoheadrightarrow M_1 \oplus B)$ in $g^{-1}B$ and choose a morphism $B \rightarrowtail L_1$ so that $B \rightarrowtail L_1 \twoheadrightarrow M_L \oplus B$ is the inclusion of B. We will now show that $f/(L_1 \twoheadrightarrow M_1 \oplus B)$ is contractible. For an object of this category is represented by a diagram of the following form.



Hence the pullback square gives a natural morphism $B \rightarrowtail L$ from $B \rightarrowtail L_1$ and $B \subset M' \oplus B$. This gives a functor a: $(f/L_1 \to M_1 \oplus B) \to (1/L_1 \to M_1 \oplus B)$ where 1: $g^{-1}(B) \to g^{-1}(B)$ is the identity functor. However forgetting $B \rightarrowtail L$ gives a left inverse to a. Thus $B(f/L_1 \twoheadrightarrow M_1 \oplus B)$ must be contractible since it is a factor in $B(1/L_1 \twoheadrightarrow M_1 \oplus B)$, and $1/L_1 \to M_1 \oplus B$ has a final object.

Let $I = \text{Iso}\,\mathbf{P}(R)$ the category of finitely generated projective R-modules and the isomorphisms. I acts on $g^{-1}(0)$ by

$$A + \left(L \overset{\pi}{\twoheadrightarrow} M\right) = \left(A \oplus L \overset{0+\pi}{\twoheadrightarrow} M\right).$$

The category $\langle I, g^{-1}(0)\rangle$ has the same objects as $g^{-1}(0)$, and a morphism $(L \twoheadrightarrow M) \to (L' \twoheadrightarrow M')$ is an isomorphism class of diagrams of the following form

$$\begin{array}{ccc} L' & \twoheadrightarrow & M' \\ \uparrow & \square & \uparrow \\ A \oplus L & \twoheadrightarrow & M'' \\ \| & & \downarrow \\ A \oplus L & \twoheadrightarrow & M \end{array}$$

Here $A \in \mathbf{P}(R)$ and an isomorphism of diagrams of the above type is required on $A \oplus L$ to be the direct sum of an isomorphism on A and one on L.

1.3. PROPOSITION. *$\langle I, g^{-1}(0)\rangle$ is contractible.*

PROOF. Direct sum makes $\langle I, g^{-1}(0)\rangle$ into an H-space, which is contractible by the proof of [G-Q, p. 227] as follows. The H-space is connected because $(L_i \twoheadrightarrow M_i)$ $(i = 1, 2)$, maps to $(L_1 \oplus L_2 \twoheadrightarrow M_1 \oplus M_2)$ by the inclusion of the ith factor. Therefore we have only to show $\pi_i(B\langle I, g^{-1}(0)\rangle, \mathbf{0}) = 0$ $(i \geqslant 1)$. However multiplication by two on these (abelian) homotopy groups is induced by the functor F: $(L \twoheadrightarrow M) \to (L \oplus L \twoheadrightarrow M \oplus M)$. The inclusion of the first summand gives a natural transformation from α: $1 \to F$ so that $BF \simeq 1$. Hence $2x = x$ for $x \in \pi_i(B\langle I, g^{-1}(0)\rangle, \mathbf{0})$ and the group must be zero.

1.4. THEOREM. *The square*

$$\begin{array}{ccc} I^{-1}g^{-1}(0) & \overset{i}{\to} & I^{-1}G \\ \downarrow & & \downarrow g \\ * & \to & Q\mathbf{P}(R) \end{array}$$

is homotopy cartesian, where $i(A, L \twoheadrightarrow M) = (A, L \twoheadrightarrow M)$.

PROOF. By [Q, §1, Theorem 8] we must show that a morphism $0 \overset{\alpha}{\twoheadleftarrow} B' \overset{\beta}{\to} B$ induces base change homotopy equivalences

$$I^{-1}g^{-1}(0) \overset{I^{-1}\alpha^*}{\to} I^{-1}g^{-1}(B') \overset{I^{-1}\beta^*}{\to} I^{-1}g^{-1}(B)$$

when $I^{-1}\alpha^*$, $I^{-1}\beta^*$ are induced by α^*, β^*.

By §1.2 we may replace $g^{-1}(0)$ etc. by $\tilde{g}^{-1}(0)$. Also there are equivalences

$$\tilde{g}^{-1}(0) \xrightarrow{j_1} \tilde{g}^{-1}(B) \xrightarrow{j_2} \tilde{g}^{-1}(0)$$

given by $j_1(L \twoheadrightarrow M) = (L \oplus B \twoheadrightarrow M \oplus B)$ and

$$j_2\left(\begin{array}{ccc} & & B \\ & j \swarrow & \\ L' & \xrightarrow{\ \pi\ } & M' \oplus B \end{array}\right) = (\operatorname{coker} j \twoheadrightarrow M').$$

With this identification β^*: $\tilde{g}^{-1}(B) \to \tilde{g}^{-1}(B')$ becomes the identity on $\tilde{g}^{-1}(0)$ because β^* is pullback by β.

$$\begin{array}{ccccc} B' & \overset{j'}{\rightarrowtail} & L' & \twoheadrightarrow & M \oplus B' \\ {\scriptstyle\beta}\downarrow & & \downarrow \ \square & & \downarrow {\scriptstyle 1 \oplus \beta} \\ B & \underset{j}{\rightarrowtail} & L & \twoheadrightarrow & M \oplus B \end{array}$$

Furthermore α^* is given by sending $(L \twoheadrightarrow M \oplus B')$ to $(L \twoheadrightarrow M \oplus B' \overset{\pi_2}{\twoheadrightarrow} M)$ which becomes the action of B' on $\tilde{g}^{-1}(0)$. However by Proposition 1.3 and [G-Q, p. 223] multiplication by B' on $I^{-1}\tilde{g}^{-1}(0) \simeq I^{-1}g^{-1}(0)$ is a homotopy equivalence.

1.5. COROLLARY. *There is a commutative diagram* ($q > 0$)

$$\begin{array}{ccccccccc} & & \pi_{q+1}(BI^{-1}g^{-1}(0)) & \xrightarrow{\ i\ } & \pi_{q+1}(BQH) & & & & \\ & & \cong\downarrow & & \| & & & & \\ \cdots & \longrightarrow & K_{q+1}R_S & \longrightarrow & K_q(H) & \longrightarrow & K_q(R) & \longrightarrow & K_q(R_S) \longrightarrow \cdots \end{array}$$

in which the row is the localisation sequence [G-Q, *p.* 233] *and i is induced by the functor* $i(A, L \twoheadrightarrow M) = (M)$.

PROOF. From the derivation of the localisation sequence [G-Q, pp. 229–233] we see that the equivalences $I^{-1}G \simeq I^{-1}QH \simeq QH$ transform the functor g of §1.4 into the map inducing $K_*(H) \to K_*(R)$. Hence Theorem 1.4 extends to the left the (quasi-) fibration sequence $BQH \to BQ\mathbf{P}(R) \to BQ\mathbf{P}(R_S)$ and the result follows from the uniqueness, up to homotopy, of this extension.

1.6. REMARK. Observe that Theorem 1.4 amounts to the assertion that $BI^{-1}g^{-1}(0)$ equals $(K_0R/(\operatorname{im}(K_0H))) \times BGLR_S^+$.

2. Throughout this section set $R = A$, a Dedekind domain, with quotient field $F = R_S$. In this case devissage applied to K_*H transforms the localisation sequence into the form

$$\ldots \to K_iA \xrightarrow{b} K_iF \xrightarrow{\partial} \bigoplus_{P \triangleleft_{\max} A} K_{i-1}(A/P) \xrightarrow{a} K_{i-1}A \to \ldots.$$

Therefore we have $K_1F = F^*$ and $K_0(A/P) = Z(P)$, a copy of the integers. These identifications yield a homomorphism

$$\partial\colon F^* \to \bigoplus_{P \lhd_{\max} A} Z(P). \tag{2.1}$$

We will apply §§1.4–1.5 to prove the following result (asserted by Quillen **[Q**, §7, Remark 5.17**]**). This result did not appear in **[G-Q]** as originally promised in **[Q**, ibid.**]**.

2.2. THEOREM. *The homomorphism of* (2.1) *is given by* $\partial(x) = \Sigma v_P(x)$ *where* $x \in F^*$ *and* v_P *is the P-adic valuation.*

PROOF. Firstly set $J = \text{Iso}\,\mathbf{P}(F)$. Now let us give a functor $f\colon I^{-1}g^{-1}(0) \to J^{-1}J$ which is a homotopy equivalence on base-point components (both equal to $BGLF^+$). Having done this we will be able to use the description given in **[G-Q**, p. 224**]** of the loop in $\pi_1(BGLF^+) = F^*$ corresponding to $x \in A \subset F$ and thereby evaluate $\partial(x)$.

Assigning ($\operatorname{Ker} \pi \leqslant L$) to ($\pi\colon L \twoheadrightarrow M$) gives an equivalence of categories with I-action between $g^{-1}(0)$ and the category, $\mathfrak{L}$, of admissible layers in $\mathbf{P}(A)$. The functor $h(L_0 \leqslant L_1) = L_1 \otimes_A F$ is a homotopy equivalence of categories with I-action, $h\colon \mathfrak{L} \to J$. This is because $(L \otimes_A F)/h$ equals the category of A-lattices which is called $X(L \otimes_A F)$ in **[Q**, §7, Remark 5.17**]** and is shown there to be contractible. Hence $I^{-1}g^{-1}(0) \to I^{-1}J$ is a homotopy equivalence.

Now consider $I^{-1}J \to J^{-1}J$ induced by $I \to J$. This induces a homology isomorphism on base-point components of the classifying spaces, by **[G-Q**, p. 222**]**. However direct sum makes $BI^{-1}J$ and $BJ^{-1}J$ into H-spaces (similarly base-point components) and so the map on base-point components is a homotopy equivalence.

Set f equal to the composite $I^{-1}g^{-1}(0) \to I^{-1}J \to J^{-1}J$.

Now let $x \in A$. In $g^{-1}(0)$ we have two morphisms

$$j\colon \begin{array}{ccc} A & \twoheadrightarrow & 0 \\ \| & & \uparrow \\ A & \twoheadrightarrow & A/xA \\ \| & & \| \\ A & \twoheadrightarrow & A/xA \end{array} \qquad \text{and} \qquad i\colon \begin{array}{ccc} A & \twoheadrightarrow & 0 \\ \| & & \| \\ A & \twoheadrightarrow & 0 \\ x \downarrow & & \downarrow \\ A & \twoheadrightarrow & A/xA \end{array}$$

resulting in a loop $(1_A, i^{-1} \circ j) \in \pi_1(BI^{-1}g^{-1}(0); (A, A \twoheadrightarrow 0))$. The functor f takes this loop to the loop which is shown in **[Q**, §2, Theorem 1**]** to determine $[A/xA] \in K_0H = \bigoplus_{P \lhd_{\max} A} K_0(A/P)$. By definition of the last (devissage) isomorphism **[Q**, §5, Corollary 1**]** A/xA corresponds to $\Sigma v_P(x)$ as required.

2.3. REMARK. The formula of §2.2 agrees with that of **[M**, p. 123**]** for the algebraically defined coboundary in the localisation sequence of Bass and Tate.

Recall that the algebraically defined coboundary **[M**, p. 133**]**

$$\partial\colon K_2F \to \bigoplus_{P \lhd_{\max} A} (A/P)^* \cong \bigoplus_{P \lhd_{\max} A} K_1(A/P) \tag{2.4}$$

is given by the tame symbol. If $x, y \in F^*$ have Steinberg symbol $\{x, y\} \in K_2F$ the P-coordinate of $\partial\{x, y\}$ is given by [**M**, §11.5]

$$\partial\{x, y\}_P = (-1)^{\nu_p(x)\nu_p(y)} x^{\nu_p(y)} y^{-\nu_p(x)}. \tag{2.5}$$

2.6. THEOREM. *The homomorphism of* (2.4) *is equal to the tame symbol whose P-coordinate is given by* (2.5).

PROOF. Recall that K_2F is generated by Steinberg symbols $\{x, y\}$ $(x, y \in F^*)$ and that $\{\_, \_\}$: $K_1F \otimes K_1F \to K_2F$ is equal to the topologically defined K-theory product given by Loday [**L**, §2.2.3]. Furthermore the localisation sequence derived in [**G-Q**, p. 233] is a sequence of left K_*A-modules under Loday's product.

Also by naturality of the localisation sequence we can reduce to the local case. In this case let ν be the valuation associated to the maximal ideal. Let $x, y \in F^*$ satisfy $\nu(x) = n$, $\nu(y) = m$. Choose $z \in F^*$ so that $\nu(z) = 1$. We may write $x = z^n a, y = z^m b$ with $a, b \in A^*$.

We will write K_2F additively and $K_1(A/P)$ multiplicatively.

Since $0 = \{z, 1 - z\}$ the bilinearity and skew-symmetry of the symbol yields $0 = \{z, z^{-1}\} + \{z, 1 - z\} + \{z, -1\}$. Therefore $\{z, z\} = \{z, -1\}$ and $\partial\{z, z\} = \partial\{-1, z\}^{-1} = ((-1)^{\nu(z)})^{-1} = -1$, by the K_1A-module structure and Theorem 2.2.

Similarly we compute in general that

$$\begin{aligned}\partial\{x, y\} &= \partial(mn\{z, z\} + \{a, y\} - \{b, x\}) \\ &= (-1)^{mn} a^{\nu(y)} b^{-\nu(x)}\end{aligned}$$

which is the expression given in (2.5).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO N6A 5B9, CANADA