EXACT MORSE FUNCTIONS IN THE CATEGORY TOP

(IN DIMENSION >5)

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The results in the categories PL and DIFF similar to those studied in this article are well known. The theory of handles developed by Morse, Smale, and Wallace in dimensions >5 cannot be directly extended to the category TOP. Here, one needs a special proof of existence of handle decompositions. In dimension >5 it is given in [1] (III.§3), in dimensions 1, 2, 3 it follows from the fact that a topological manifold of such dimensions admits smoothing. In dimensions 4 and 5 there is the following result (cf. [2]): there exists a closed orientable TOP manifold of dimension 4 or 5 with $w_2 = 0$ which does not admit handle decomposition.

In the category TOP, a new notion of transversality is required (these are two of them) In this article the transversality of microbundles is used; in our case they are simply fittings (a submanifold of V is fitted if it has a neighborhood homeomorphic to $V \times R^n$; the product structure on the neighborhood given by the homeomorphism under which $V \times \{0\}$ is sent to V is called a fitting). In the theorem on TOP microbundle transversality, in fact, dimensional restrictions arise (for details on this matter see [1]).

It is suggested in [1] how one can prove TOP theorems on the h- (s-)cobordism, but this suggestion is relatively easily realized in dimensions >6 (cf. [3]), while in dimension 6 certain difficulties arise. They are expressed by the following question: let $f\colon S^1 \to \partial W^s$ be a locally flat embedding, and f is homotopic to zero in W, does there exist a locally flat disc D^2 in W with $\partial D^2 = f(S^1)$? The proof of this lemma allows to extend all results on exact Morse functions from the categories PL and DIFF (dimension >5) to the category TOP.

Unless stated otherwise, all manifolds are assumed to be topological and all mappings continuous.

1. Statements

Lemma. Let $f\colon S^1 \to \partial W^n$ be a locally flat embedding (henceforth abbreviated to l.f.e.), $n \geqslant 2$, $n \neq 4$. If f is homotopic to zero in W, then there exists an l.f.e. $g\colon (D^2, \partial D^2) \to (W, \partial W)$ with $g|_{\partial D^2} \equiv f$.

In dimension 4 the lemma does not hold as a disc can have self-intersections, even in the smooth case, but under the hypotheses of the lemma it is impossible, at the time of this writing, to say anything on the finiteness of the set of self-intersections.

For a definition of topological Morse functions, see [3].

THEOREM 1. On the triad of manifolds (W^n, V_0, V_1), $n \geqslant 6$, $\pi_1(W) = \pi_1(V_0) = \pi_1(V_1) = 1$ there exists an exact Morse function with the number of critical points of index λ:

$$N_\lambda = \mu(H_\lambda(W^n, V_0; \mathbf{Z})) + \mu(\operatorname{tors} H_{\lambda-1}(W^n, V_0; \mathbf{Z})).$$

THEOREM 2 (s-Cobordism). The triad of manifolds (W^n, V_0, V_1) $n \geqslant 6$ is homeomorphic to $(V_0 \times I, V_0 \times \{0\}, V_0 \times \{1\})$ if and only if $\pi_*(W^n, V_0) = 0$ and the Whitehead torsion $\tau(W, V_0) \in Wh(\pi_1(W))$ is equal to zero.

THEOREM 3. On the triad of manifolds (W^n, V_0, V_1), $n \geqslant 6$, $\pi_1(W) = 1$ there exists an exact Morse function with the number of critical points of index λ: $N_0 = 1$ if $V_0 = \varnothing$; $N_0 = 0$ if

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$V_0 \neq \varnothing$; $N_1 = 0$, $N_{n-1} = 0$, $N_n = 0$, if $V_1 \neq \varnothing$; $N_n = 1$, if $V_1 = \varnothing$, and

$$N_2 = \mu(\pi_2(W^n, V_0)),$$
$$N_3 = \mu(\pi_2(W^n, V_0)) + \mu(H_3(W^n, V_0; \mathbf{Z})) - \mu(H_2(W^n, V_0; \mathbf{Q})),$$
$$N_{n-3} = \mu(\pi_2(W^n, V_1)) + \mu(H_{n-3}(W^n, V_0; \mathbf{Q})) - \mu(H_{n-2}(W^n, V_0; \mathbf{Q})) + \mu(\operatorname{tors} H_{n-4}(W^n, V_0; \mathbf{Z})),$$
$$N_{n-2} = \mu(\pi_2(W^n, V_1)),$$
$$N_\lambda = \mu(H_\lambda(W^n, V_0; \mathbf{Z})) + \mu(\operatorname{tors} H_{\lambda-1}(W^n, V_0; \mathbf{Z}),$$
$$4 \leqslant \lambda \leqslant n-4.$$

To prove Theorems 1, 2, 3, the following facts are necessary: the existence of handle decompositions (cf. [1]), general positioning [ibidem], elimination of handles of index 0 and 1, rearrangement of handles of different indices [3] (Th. 3.4), cancellation of a pair of handles with geometrical index of intersection ±1 [3] (Th. 4.6), the Whitney lemma; addition of handles [4] — the argument is exactly like in the PL case because a locally flat interval has a fitting (cf. [5]), and the middle and co-middle spheres have fittings inherited from the handle; and, for Theorem 2, the theory of the simple homotopy type ([1], III. 4). In the presence of all these facts, the proofs of Theorems 1-3 become repetitions of the corresponding proofs in the smooth or piecewise-linear case.

In the above list, only elimination of handles of index 1 (index 0 is done exactly like in the category PL, cf. [4]) and the Whitney lemma remain unproved, but they easily follow from the lemma and from [1], [6].

2. Proofs. The proof of the lemma for $n = 2$ is obvious, for $n = 3$ it is Den's lemma. So let $n \geqslant 5$, h a homotopy of f to 0, $h\colon D^2 \to W$ and $h|_{\partial D^2} \equiv f$; $C(\partial W) = \operatorname{im}(q)$, where $q\colon \partial W \times [0, 1) \to (W, \partial W)$ is a collaring of ∂W in W. One can assume that h is an l.f.e. in $C(\partial W)$, i.e., there exists a disc $D_0^2 \subset \overset{0}{D^2}$ and h is an l.f.e. on $D^2 \setminus \overset{0}{D_0^2}$ and $h(D_0^2) \cap C(\partial W) = \varnothing$. This can be achieved by the procedure of pushing onto the collaring, the fact that h is an l.f.e. in $C(\partial W)$ follows from f being an l.f.e.

Consider $C_1(\partial W) \overset{\text{def}}{=} q\left(\partial W \times \left[0, \frac{1}{2}\right]\right)$, a finite covering of $h(D_0^2)$ by maps (φ_i, U_i), $i = \overline{1, s}$, such that:

1. $\varphi_i\colon U_i \to \Delta^n$, where Δ^n is the n-dimensional standard simplex.
2. $U_i \cap \partial W = \varnothing$.
3. $\overset{0}{C_1}(\partial W) \cup \left(\bigcup_{i=1}^{s} \varphi_i^{-1}(\overset{0}{\Delta_0^n})\right) \overset{\text{def}}{=} U \supset h(D^2)$, Δ_0^n is a simplex homotetic to the standard one with the homotety center at the baricenter of Δ^n and coefficient $\gamma < 1$; ρ a matrix on W; d the standard metric on Δ^n; μ, $\tilde{\mu}$ are such that

$$\tilde{\mu} d(\varphi_i(x), \varphi_i(y)) < \rho(x, y) < \mu d(\varphi_i(x), \varphi_i(y))$$

for each i ($\tilde{\mu}$ and μ do exist since φ_i are mappings of compacta and there are finitely many of them).

The proof will be conducted by induction, correcting h from one map to another. The base of induction is the neighborhood $C_1(\partial W) - C(\partial W)$.

Suppose that there exists a mapping $h_k\colon D^2 \to (W, \partial W)$ with the properties: $h_k|_{\partial D^2} \equiv f$, h_k is an l.f.e. in a neighborhood V_k of the closed set $\mathscr{D}_k \overset{\text{def}}{=} C_1(\partial W) \cup \left(\bigcup_{i=1}^{k} \varphi_i^{-1}(\Delta_0^n)\right)$, i.e., on $h_k^{-1}(V_k)$ (hereafter, such remarks will be omitted), and $h_k(D^2) \subset U$.

We will show that then there exists h_{k+1} which is an l.f.e. in V_{k+1} and satisfying all inductive hypotheses.

Put $E=\mathcal{D}_k \cap \varphi_{k+1}^{-1}(\partial\Delta_0^n)$, E is closed, $U_\eta(E) \subset \varphi_{k+1}^{-1}(\overset{\circ}{\Delta}{}^n) \cap V_k$, where $U_\eta(E)$ is an η-neighborhood of the set E in W. Since $h_k^{-1}(U_\eta(E))$ is open in D^2, there exists a compact polyhedron N such that $h_k^{-1}(E) \subset \overset{\circ}{N} \subset N \subset h_k^{-1}(U_\eta(E))$.

Then, by the theorem on straightening l.f.e. (see [1], III, App. B and [7]), there exists an isotopy $e_t:\overset{\circ}{\Delta}{}^n \to \overset{\circ}{\Delta}{}^n$ such that $t \in [0, 1]$, $e_0=\mathrm{id}$, $e_1\circ\varphi_{k+1}\circ h_k|$ is piecewise linear, $d(e_{t_1}(x), e_{t_2}(x)) < \varepsilon$ for all $t_1, t_2 \in [0, 1]$ and any given ε, and the support of e_t lies in the ε-neighborhood of $\varphi_{k+1}(N)$.

Choose ε small enough in such a way that:

1. $U_\varepsilon(\varphi_{k+1}(N)) \subset \varphi_{k+1}(U_\eta(E))$.
2. $h_k^{-1}\circ\tilde{e}_1^{-1}(E) \subset \overset{\circ}{N}$.
3. $\mu\cdot\tilde{\mu}\cdot\varepsilon < \dfrac{\rho_k}{3}$,

where $\rho_k = \rho(h_k(D^2), W \setminus U)$ and $\tilde{e}_t$ the isotopy taken from $\overset{\circ}{\Delta}{}^n$ by means of φ_{k+1}^{-1} which is the identity outside U_{k+1}.

Then $\rho(\tilde{e}_1\circ h_k(D^2), W\setminus U) > \dfrac{2}{3}\rho_k$, i.e., $\tilde{e}_1\circ h_k(D^2) \subset U$, $\tilde{e}_t$ is the identity outside some neighborhood $W\setminus V_k$ and $\varphi_{k+1}\circ\tilde{e}_1\circ h_k$ is piecewise-linear in some neighborhood of $\varphi_{k+1}(E)$, say $U_{\eta_1}(\varphi_{k+1}(E)) \subset \Delta^n$.

Since $\varphi_{k+1}\circ\tilde{h}_k$ $(\tilde{h}_k \overset{\text{def}}{=} \tilde{e}_1\circ h_k)$ is PL on N, there exists a set of 2-simplexes T_i, $l=\overline{1, m}$, $N \subseteq \bigcup T_i$, such that $\varphi_{k+1}\circ\tilde{h}_k$ is linear on each simplex.

It is easily seen that $\tilde{h}_k$ satisfies all induction hypotheses for h_k.

Let $\gamma_r \to \gamma$, $\gamma_{r+1} < \gamma_r$ $(\Delta_0^n = \gamma\Delta^n)$ and $\Delta_r^n = \gamma_r\Delta^n$; $S_r \overset{\text{def}}{=} \mathcal{D}_k \cap \varphi_{k+1}^{-1}(\Delta_r^n)(S_{r+1} \subset S_r)$. It is clear that $\bigcap_{r=1}^{\infty} S_r = E$, then there exists r such that $S_r \subset U_{\eta_2}(E)$, where $U_{\eta_2}(E) \subset \varphi_{k+1}^{-1}(U_{\eta_1}(\varphi_{k+1}(E)))$.

We will construct a mapping $h^*:\tilde{h}_k^{-1}\circ\varphi_{k+1}^{-1}(\overset{\circ}{\Delta}{}_r^n) \to \overset{\circ}{\Delta}{}_r^n$; to this end, we triangulate $\tilde{h}_k^{-1}\circ\varphi_{k+1}^{-1}(\overset{\circ}{\Delta}{}_r^n) \overset{\text{def}}{=} G$ in such a way that each simplex σ in the triangulation of the open set G would satisfy the following conditions:

1. $\operatorname{diam}(\sigma) < \dfrac{\rho_k}{6\mu}$.
2. Either $\sigma \cap \left(\bigcup_i T_i\right) = \varnothing$ or $\sigma \subset T_i$ for some i.
3. There exist $\lambda_1, \lambda_2 \in \mathbf{R}$ such that $\lambda_1\cdot\rho(\sigma, \partial G) < \operatorname{diam}(\sigma) < \lambda_2\cdot\rho(\sigma, \partial G)$.

We define the mapping $h^*: G \to \overset{\circ}{\Delta}{}_r^n$ by the formula:

$$h^*(x) = \begin{cases} \varphi_{k+1}\tilde{h}_k(x), & \text{x is a vertex of the triangulation of G} \\ \sum_i \lambda_i \tilde{h}(x_i), & x = \sum_i \lambda_i x_i,\ x_i \text{ are vertices of the simplex } \sigma \supset x. \end{cases}$$

By construction $h^*|_N \equiv \varphi_{k+1}\circ\tilde{h}_k|_N$.

Bring h^* to a general position on $\overset{\circ}{\Delta}{}_r^n$ without changing it on N in such a way that $d(h^*(x), h_1^*(x)) < \dfrac{\rho_k}{9\mu}$ (where h_1^* is h^* brought to a general position).

Put

$$h_{k+1}(x)=\begin{cases}\tilde{h}_k(x), & x\in D^2\setminus G,\\ \varphi_{k+1}^{-1}h_1^*(x), & x\in G,\end{cases}$$

then $h_{k+1}|_{\partial D^2}\equiv f$ since $\varphi_{k+1}^{-1}(\Delta^n)\cap\partial W=\varnothing$; h_{k+1} is an l.f.e. on $(V_k\setminus\varphi_{k+1}^{-1}(\Delta_r^n))\cup U_{\eta_2}(E)\cup\varphi_{k+1}^{-1}(\mathring{\Delta}_{r+1}^n)$ – on the first component by virtue of the fact that h_k was changed there by means of the isotropy $\tilde{e}_t$ and an isotropy preserves the property of being an l.f.e., on the second component for the same reasons, and on the third by construction (an embedding of a two-dimensional PL-manifold in a five-dimensional one is an l.f.e. if it is piecewise-linear, cf. [4]). It is easy to verify the continuity of h_{k+1} which can be violated only on ∂G, but there h_{k+1} is continuous since diam $\sigma\to 0$ as $\rho(\sigma,\partial G)\to 0$. Furthermore,

$$\rho(h_k(x),h_{k+1}(x))\leqslant\rho(h_k(x),\tilde{h}_k(x))+\rho(\tilde{h}_k(x),h_{k+1}(x))\leqslant\frac{\rho k}{3}+$$

$$+\mu d(\varphi_{k+1}\circ\tilde{h}_k(x),h_1^*(x))\leqslant\frac{\rho k}{3}+\mu(d(\varphi_{k+1}\circ\tilde{h}_k(x),h^*(x))+d(h^*(x),h_1^*(x)))\leqslant\frac{\rho_k}{3}+\frac{\rho_k}{3}+\frac{\rho_k}{9}=\frac{7}{9}\rho_k.$$

Therefore, $\rho_{k+1}>\frac{2}{9}\rho_k$, i.e., $h_{k+1}(D^2)\subset U$. Thus, one can take the above union as V_{k+1} and complete the induction.

Thus, h_s is an l.f.e. in U and coincides with f on the boundary and, since $h_s(D^2)\subset U$ by induction, h_s is the required embedding.

We will show that the lemma implies the theorem on elimination of handles of index 1 in dimension$\geqslant$5 and the Whitney lemma, in the same dimensions, for topological manifolds.

The proof of the first fact is similar to the PL-case (cf. [4]), only instead of polygons α and β one should take fitted embeddings of intervals whose existence follows from [5]; instead of the theorem on the general position, for finding a locally flat disc with boundary $\alpha\cup\beta$, one has to apply the lemma; by virtue of theorems in [6], on the locally flat disc there exists a fitting coinciding with the fitting on the boundary. The rest of the proof is done like in the piecewise-linear case (cf. [4]) with the appropriate replacement of the PL general position by the TOP general position (cf. [1]). To prove the Whitney lemma, one should use the argument in [1] (III. 3), where, instead of referring to the techniques of Newman, Gluck, and Homma, one should make use of the lemma. In view of the remarks made in the beginning of the article, this completes the proof of Theorems 1 and 2. The proof of Theorem 3 is a repetition of the corresponding smooth version (cf. [8]), except that for the choice of fitted generators of $\pi_2(W, V_0)$ and $\pi_2(W, V_1)$ we apply the lemma and then a theorem from [6].

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REPRESENTATION OF AN INFINITELY DIFFERENTIABLE FUNCTION AS A SUM OF TWO FUNCTIONS BELONGING TO QUASIANALYTIC CLASSES

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In this article a theorem of Mandelbrojt [1] on the representation of an infinitely differentiable function of one independent variable as a sum of two functions belonging to quasianalytic classes is carried over to a function $f(x)$, $x = (x_1, x_2, \ldots, x_n)$ of several real variables which is infinitely differentiable in an arbitrary closed bounded oriented n-dimensional cube $D(x_0, 2\delta)$ with center at $x_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ and with edge at length 2δ. We prove the following theorem, which improves a result in [2].

THEOREM. A function $f(x)$ which is infinitely differentiable in a closed bounded n-dimensional cube is the sum of two functions which belong to elementary quasianalytic classes of functions.

A class $C_n\{M_i(q)\}$ of functions $f(x)$ which are infinitely differentiable in $D(x_0, 2\delta)$ and which satisfy the inequalities

$$|D_x^k f(x)| \leqslant C^k M_1(k_1) M_2(k_2) \ldots M_n(k_n), \tag{1}$$

where $k = k_1 + k_2 + \ldots + k_n$, $D_x^k = (\partial/\partial x_1)^{k_1} (\partial/\partial x_2)^{k_2} \ldots (\partial/\partial x_n)^{k_n}$, the constant C depends on $f(x)$, and the sequences $M_i(q)$, $i = 1, 2, \ldots, n$, $q = 0, 1, 2, \ldots$, of positive numbers are the same for all functions in the class, is called an elementary quasianalytic class if the sequences $M_i(q)$ are logarithmically convex with respect to q and satisfy the conditions

$$\sum_{q=1}^{\infty} \frac{M_i(q-1)}{M_i(q)} = +\infty, \qquad i = 1, 2, \ldots, n. \tag{2}$$

The *Proof* splits into two parts: 1) we expand $f(x)$ into a series of Chebyshev polynomials and obtain bounds for the partial derivatives of the terms of the series; 2) with the help of auxiliary functions and sequences constructed in [2] (pp. 295-296), we break up the series representing $f(x)$ into a sum of two series, $f_1(x)$ and $f_2(x)$, and show that $f_1(x)$ and $f_2(x)$ belong to elementary quasianalytic classes of functions.

1. Series of Chebyshev Polynomials. Let e_j be the unit coordinate vectors in n-dimensional space, let l_i be an edge of the cube $D(x_0, 2\delta)$ issuing from one of the vertices, let $u_{j,i}$ be the direction cosines of l_i, let $x = (x_1, x_2, \ldots, x_n)$ and $t = (t_1, t_2, \ldots, t_n)$. The mapping $t = v(x) \equiv (v_1(x), v_2(x), \ldots, v_n(x))$, which maps the cube $D(x_0, 2\delta)$ onto the cube $D(0, 2) = \{t, -1 \leqslant t_i \leqslant 1,\ i = 1, 2, \ldots, n\}$ is defined by $t_i = v_i(x) \equiv [(x_1 - x_1^0) u_{i,1} + \ldots + (x_n - x_n^0) u_{i,n}]/\delta$, $i = 1, 2, \ldots, n$, and the inverse of v, $x = v^{-1}(t) \equiv (v_1^{-1}(t), v_2^{-1}(t), \ldots, v_n^{-1}(t))$, which maps $D(0, 2)$ onto $D(x_0, 2\delta)$, is given by $x_i = v_i^{-1}(t) \equiv x_i^0 + \delta(t_1 u_{1,i} + \ldots + t_n u_{n,i})$, $i = 1, 2, \ldots, n$. Let $T_k(y) = \cos(k \arccos y)$ by Chebyshev polynomials, let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\lambda_\nu = 2^{-q}$, where q is the number

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