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# A formula of Atiyah and Hirzebruch

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## Introduction

In a recent paper [1] ATIYAH and HIRZEBRUCH have proved a formula (3.2 of [1]) for differentiable manifolds which can be formulated for topological manifolds, and they ask the question whether the formula is valid for such manifolds. It is the point of the present note to establish the formula for topological manifolds and to show how this implies their result. Actually we generalize their formula somewhat to obtain one for manifolds with boundary.

We work over a fixed prime field $Z_q$ and all homology and cohomology groups which occur in the note have $Z_q$ as coefficient group. $\lambda$ will denote a cohomology automorphism over $Z_q$ in the sense of [1]. If $X$ is a topological manifold (or a space in which there is a Poincaré duality given by cup product), then such a cohomology automorphism $\lambda$ corresponds to a class $\mathrm{Wu}(\lambda, X) \in H^*(X)$ (see (1.1) below). The main result, Theorem I, which is stated and proved in §1, asserts that in a certain sense $\mathrm{Wu}(\lambda^{-1}, X)$ gives a correction term for the non-commutativity of $\lambda$ with the Gysin homomorphism $f_*$: $H^*(Y) \to H^*(X)$ corresponding to a continuous map $f$ of one such space $Y$ to another $X$.

In §2 we use the diagonal map $X \to X \times X$ in order to express $\mathrm{Wu}(\lambda, X)$ in terms of cohomology properties of $X \times X$ (see Theorem II). This was the technique originally introduced by WU [4] to study characteristic classes, and our treatment follows closely that of MILNOR [2]. These results are used in §3, when $X$ is assumed to have a differentiable structure, to prove Theorem III which asserts that $\mathrm{Wu}(\lambda, X)$ is essentially a characteristic class of the tangent bundle of $X$. This result was also proved by ATIYAH-HIRZEBRUCH (see (17) of §3 of [1]) by a different method and shows that their definition of $\mathrm{Wu}(\lambda, X)$ based on the tangent bundle is the same as our definition based on Poincaré duality. Theorem I and Theorem III together then imply 3.2 of [1].

## 1. The main formula

Let $X$ be a compact topological manifold with boundary $\dot{X}$ (which may be empty) and assume $X$ is orientable over $Z_q$. If $X$ is oriented and $x \in H^*(X, \dot{X})$, we let $x[X]$ denote the value of $x$ on the homology class of $X$ modulo $\dot{X}$ which is the orientation class of $X$. If $x \in H^*(X, \dot{X})$ and $x' \in H^*(X)$,

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their cup product $x \cup x'$ is an element of $H^*(X, \dot{X})$, and the map of $H^*(X, \dot{X}) \times H^*(X)$ to $Z_q$ defined by $x \times x' \to (x \cup x')[X]$ is a dual pairing.

If $\lambda$ is a cohomology automorphism, then because of the above duality properties of $X$ there exists an element $\mathrm{Wu}(\lambda, X) \in H^*(X)$ defined by the condition that for any $x \in H^*(X, \dot{X})$

$$(1.1) \qquad \big(x \cup \mathrm{Wu}(\lambda, X)\big)[X] = \big(\lambda(x)\big)[X].$$

Let $Y$ be an oriented compact topological manifold with boundary $\dot{Y}$ and let $f: (Y, \dot{Y}) \to (X, \dot{X})$ be continuous. Then $f$ induces ring homomorphisms $f^*$ from $H^*(X)$ (or $H^*(X, \dot{X})$) to $H^*(Y)$ (or $H^*(Y, \dot{Y})$). The Gysin homomorphism (see [3]) $f_*$ from $H^*(Y)$ (or $H^*(Y, \dot{Y})$) to $H^*(X)$ (or $H^*(X, \dot{X})$) is defined using the dual pairings in $X$ and $Y$ by the equation

$$(1.2) \qquad (x \cup f_* y)[X] = (f^* x \cup y)[Y]$$

for $y \in H^*(Y)$ and $x \in H^*(X, \dot{X})$ (or $y \in H^*(Y, \dot{Y})$ and $x \in H^*(X)$). Then $f_*$ is not a ring homomorphism but does have the following easily verified property:

$$(1.3) \qquad f_*(f^* x \cup y) = x \cup f_* y$$

for $x \in H^*(X)$ or $H^*(X, \dot{X})$, $y \in H^*(Y)$ or $H^*(Y, \dot{Y})$ and the equation holds in $H^*(X, \dot{X})$ unless $x \in H^*(X)$ and $y \in H^*(Y)$ in which case it holds in $H^*(X)$.

THEOREM I. *Let $f: (Y, \dot{Y}) \to (X, \dot{X})$ be a continuous map between compact oriented topological manifolds with boundary. If $\lambda$ is an arbitrary cohomology automorphism, then*

$$f_*\big(\lambda y \cup \mathrm{Wu}(\lambda^{-1}, Y)\big) = \lambda f_* y \cup \mathrm{Wu}(\lambda^{-1}, X)$$

*where $y \in H^*(Y)$, in which case the equation holds in $H^*(X)$, or $y \in H^*(Y, \dot{Y})$, in which case the equation holds in $H^*(X, \dot{X})$.*

PROOF. We prove both statements, the absolute and relative, at the same time. Let $y \in H^*(Y)$ (or $H^*(Y, \dot{Y})$) and let $x \in H^*(X, \dot{X})$ (or $H^*(X)$) be arbitrary. Then because $\lambda^{-1}$ is also a cohomology automorphism,

$$\begin{aligned}
\big(x \cup f_*(\lambda y \cup \mathrm{Wu}(\lambda^{-1}, Y))\big)[X] &= \big((f^* x \cup \lambda y) \cup \mathrm{Wu}(\lambda^{-1}, Y)\big)[Y] && \text{by (1.2)}\\
&= \big(\lambda^{-1}(f^* x \cup \lambda y)\big)[Y] && \text{by (1.1)}\\
&= (\lambda^{-1} f^* x \cup y)[Y].
\end{aligned}$$

We also have

$$\begin{aligned}
\big(x \cup (\lambda f_* y \cup \mathrm{Wu}(\lambda^{-1}, X))\big)[X] &= \big((x \cup \lambda f_* y) \cup \mathrm{Wu}(\lambda^{-1}, X)\big)[X]\\
&= \big(\lambda^{-1}(x \cup \lambda f_* y)\big)[X] && \text{by (1.1)}\\
&= (\lambda^{-1} x \cup f_* y)[X]\\
&= (f^* \lambda^{-1} x \cup y)[Y] && \text{by (1.2).}
\end{aligned}$$

Since the cohomology automorphism $\lambda^{-1}$ commutes with the induced homomorphism $f^*$, we see that

$$\left(x \cup f_*(\lambda y \cup \mathrm{Wu}(\lambda^{-1}, Y))\right)[X] = \left(x \cup (\lambda f_* y \cup \mathrm{Wu}(\lambda^{-1}, X))\right)[X].$$

Since $x$ is arbitrary, it follows from the duality properties of $X$ that

$$f_*\left(\lambda y \cup \mathrm{Wu}(\lambda^{-1}, Y)\right) = \lambda f_* y \cup \mathrm{Wu}(\lambda^{-1}, X),$$

which completes the proof.

Formula 3.2 of [1] is Theorem I for the case where $X$ and $Y$ are differentiable manifolds without boundary and $\mathrm{Wu}(\lambda, X)$ is defined using the differentiable structure of $X$. In [1] ((17) of §3) it is shown that $\mathrm{Wu}(\lambda, X)$ defined in this way by the differentiable structure satisfies our (1.1) so, using this fact, Theorem I implies 3.2 of [1]. However, the proof in [1] that the two definitions of $\mathrm{Wu}(\lambda, X)$ in the differentiable case are equal uses 3.2 of [1], and although only a special case of 3.2 need be applied to deduce this equality, we prefer to give a different proof of the equality which also is somewhat more general than the case treated in [1] because it is valid for manifolds with boundary. The method of proof in [1] uses the imbeddability of a manifold in a sphere. Our proof of the equality of the two definitions of $\mathrm{Wu}(\lambda, X)$ uses the original diagonal technique of WU [2, 4]. This seems to be natural because the existence of Poincaré duality implies the existence of a cohomology class $U \in H^*(X \times X, X \times \dot{X})$ with rather special properties (see (2.4), (2.5) below).

## 2. Cohomology properties of $X \times X$

If $X, \dot{X}$ are as in §1 so that $H^*(X, \dot{X})$, $H^*(X)$ are dually paired by cup product, it follows that $H^*(X \times X, \dot{X} \times X)$, which is isomorphic to $H^*(X, \dot{X}) \otimes H^*(X)$ by the Künneth formula, and $H^*(X \times X, X \times \dot{X})$, which is isomorphic to $H^*(X) \otimes H^*(X, \dot{X})$, are dually paired by the map $u \times v \to (u \cup v)[X \times X]$ where if $u = \sum \alpha_i \otimes \beta_i$, $v = \sum \beta'_j \otimes \alpha'_j$ with $\alpha_i, \alpha'_j \in H^*(X, \dot{X})$ and $\beta_i, \beta'_j \in H^*(X)$ all homogeneous then

$$u \cup v = \sum (-1)^{\dim \beta_i \dim \beta'_j} (\alpha_i \cup \beta'_j) \otimes (\beta_i \cup \alpha'_j) \in H^*(X, \dot{X}) \otimes H^*(X, \dot{X})$$

and

$$(u \cup v)[X \times X] = \sum (-1)^{\dim \beta_i \dim \beta'_j} (\alpha_i \cup \beta'_j)[X] (\beta_i \cup \alpha'_j)[X] \in Z_q.$$

We define a homomorphism

$$h: H^*(X \times X, X \times \dot{X}) \to H^*(X),$$

as in [2], by the equation

$$h(\alpha \otimes \beta) = \beta[X] \alpha \in H^*(X) \quad \text{for} \quad \alpha \in H^*(X), \qquad \beta \in H^*(X, \dot{X}).$$

Letting $1 \in H^0(X)$ denote the unit class then for $x \in H^*(X, \dot{X})$, $v \in H^*(X \times X, \dot{X} \times X)$ we have the formula

$$((x \otimes 1) \cup v)[X \times X] = (x \cup h(v))[X] \tag{2.1}$$

which is proved for $v = \alpha \otimes \beta$ by direct verification from the definitions, whence it follows for arbitrary $v$ because both sides are linear in $v$.

Let $d: (X, \dot{X}) \to (X \times X, \dot{X} \times X)$ be the diagonal map (so $d(t) = (t, t)$ for $t \in X$). It follows from the dual pairings in $X$ and $X \times X$ that there is a unique $U \in H^*(X \times X, X \times \dot{X})$ such that for $y \in H^*(X \times X, \dot{X} \times X)$

$$(y \cup U)[X \times X] = (d^* y)[X]. \tag{2.2}$$

The class $U$ will be called the *basic class* of $X \times X$. Some of its properties are summarized in the following.

LEMMA (2.3). *If $U$ is the basic class of $X \times X$, then*

$$h(U) = 1, \tag{2.4}$$

$$(1 \otimes x) \cup U = (x \otimes 1) \cup U \quad \textit{for} \quad x \in H^*(X). \tag{2.5}$$

PROOF. To prove (2.4) let $x \in H^*(X, \dot{X})$ be arbitrary. Then

$$\begin{aligned} (x \cup h(U))[X] &= ((x \otimes 1) \cup U)[X \times X] && \text{by (2.1)} \\ &= d^*(x \otimes 1)[X] && \text{by (2.2)} \\ &= x[X] \quad \text{because } d^*(x \otimes 1) = x. \end{aligned}$$

Since this is true for all $x \in H^*(X, \dot{X})$, it follows by the duality property of $X$ that $h(U) = 1$.

To prove (2.5) let $y \in H^*(X \times X, \dot{X} \times X)$. Then

$$\begin{aligned} (y \cup (1 \otimes x) \cup U)[X \times X] &= d^*(y \cup (1 \otimes x))[X] \quad \text{by (2.2)} \\ &= (d^* y \cup x)[X] \quad \text{because } d^*(1 \otimes x) = x. \end{aligned}$$

Similarly

$$(y \cup (x \otimes 1) \cup U)[X \times X] = (d^* y \cup x)[X].$$

Hence

$$(y \cup (1 \otimes x) \cup U)[X \times X] = (y \cup (x \otimes 1) \cup U)[X \times X]$$

for every $y \in H^*(X \times X, \dot{X} \times X)$. By the duality property of $X \times X$, this implies

$$(1 \otimes x) \cup U = (x \otimes 1) \cup U,$$

which completes the proof.

LEMMA (2.6). *If $U \in H^*(X \times X, X \times \dot{X})$ satisfies* (2.4) *and* (2.5), *then $U$ is the basic class of $X \times X$.*

PROOF. Let $\{\alpha_i\}$ be a homogeneous base for $H^*(X)$, $\{\beta_j\}$ a homogeneous base for $H^*(X, \dot{X})$, and define $d_{ji} = (\beta_j \cup \alpha_i)[X] \in Z_q$. In terms of the above bases we can write $U = \sum c_{ij} \alpha_i \otimes \beta_j$ where $c_{ij} \in Z_q$. If $n = \dim X$ we see that

$$\begin{aligned}
\alpha_k &= \alpha_k \cup h(U) \quad \text{because } U \text{ satisfies (2.4)}\\
&= h\big((\alpha_k \otimes 1) \cup U\big) \quad \text{by definition of } h\\
&= h\big((1 \otimes \alpha_k) \cup U\big) \quad \text{because } U \text{ satisfies (2.5)}\\
&= (-1)^{n \dim \alpha_k} h\big(U \cup (1 \otimes \alpha_k)\big) \quad \text{by standard cup product property}\\
&= (-1)^{n \dim \alpha_k} \sum c_{ij} (\beta_j \cup \alpha_k)[X] \alpha_i \quad \text{by definition of } h\\
&= (-1)^{n \dim \alpha_k} \sum c_{ij} d_{jk} \alpha_i \quad \text{by definition of } d_{jk}.
\end{aligned}$$

Therefore, $\sum c_{ij} d_{jk} = (-1)^{n \dim \alpha_k} \delta_{ik}$, so the product of the matrix $(c_{ij})$ by the matrix $(d_{jk})$ is the matrix $\big((-1)^{n \dim \alpha_k} \delta_{ik}\big)$, and since $(d_{jk})$ is non-singular because $X$ has Poincaré duality, the matrix $(c_{ij})$ is uniquely characterized by this property. Since (2.3) shows that the basic class satisfies (2.4) and (2.5), it follows from the uniqueness property above that if $U$ satisfies (2.4) and (2.5) then $U$ is the basic class, and this completes the proof.

Our next result expresses $\mathrm{Wu}(\lambda, X)$ in terms of the basic class of $X \times X$.

THEOREM II. *Let $U \in H^*(X \times X, X \times \dot{X})$ be the basic class of $X \times X$. For any cohomology automorphism $\lambda$ we have*

$$\mathrm{Wu}(\lambda, X) = \lambda^{-1} h \lambda(U).$$

PROOF. As in the proof of (2.6) let $\{\alpha_i\}$ be a homogeneous base for $H^*(X)$, $\{\beta_j\}$ be a homogeneous base for $H^*(X, \dot{X})$, and $U = \sum c_{ij} \alpha_i \otimes \beta_j$. We define $s_j = (\lambda \beta_j)[X]$. Then

$$h\lambda(U) = h\big(\sum c_{ij} \lambda\alpha_i \otimes \lambda\beta_j\big) = \sum c_{ij} s_j \lambda(\alpha_i).$$

Therefore, $\lambda^{-1} h \lambda(U) = \sum c_{ij} s_j \alpha_i$. Then

$$(2.7) \qquad \big(\beta_k \cup \lambda^{-1} h \lambda(U)\big)[X] = \sum c_{ij} s_j (\beta_k \cup \alpha_i)[X] = \sum d_{ki} c_{ij} s_j.$$

Let $C$ be the matrix $(c_{ij})$ and $D$ the matrix $(d_{ji})$. If $n$ is even or the characteristic $q$ of the coefficient field is 2, then it was shown in the proof of (2.6) that $CD = I$. Therefore, $DC = I$ so (if $n$ is even or $q = 2$)

$$\big(\beta_k \cup \lambda^{-1} h \lambda(U)\big)[X] = s_k = (\lambda \beta_k)[X].$$

Since this is true for every element $\beta_k$ of the base $\{\beta_j\}$ for $H^*(X, \dot{X})$, it follows that for any $x \in H^*(X, \dot{X})$

$$\big(x \cup \lambda^{-1} h \lambda(U)\big)[X] = (\lambda x)[X].$$

Hence, $\lambda^{-1} h \lambda(U)$ satisfies (1.1), which characterizes $\mathrm{Wu}(\lambda, X)$, so we have proved the theorem if $n$ is even or $q = 2$.

If $n$ is odd and we enumerate $\{\alpha_i\}$ (and $\{\beta_j\}$) so that all even dimensional $\alpha$'s (and $\beta$'s) have smaller subscript than all odd dimensional $\alpha$'s (and $\beta$'s), then $C = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix}$ for some square matrices $C_1, C_2, D_1, D_2$ where $C_1$ (or $D_2$) corresponds to those $c_{ij}$ (or $d_{ji}$) for which $\dim \alpha_i$ is even and

$\dim\beta_j$ is odd and $C_2$ (or $D_2$) corresponds to those $c_{ij}$ (or $d_{ji}$) for which $\dim\alpha_i$ is odd and $\dim\beta_j$ is even. Then the proof of (2.6) shows that $C_1D_2=I$ and $C_2D_1=-I$ so $D_2C_1=I$ and $D_1C_2=-I$, and $DC=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}$. Therefore $\sum d_{ki}c_{ij}=(-1)^{1+\dim\beta_j}\delta_{kj}$ so (2.7) becomes

$$(2.8)\qquad (\beta_k\cup\lambda^{-1}h\lambda(U))[X]=(-1)^{1+\dim\beta_k}s_k=(-1)^{1+\dim\beta_k}(\lambda\beta_k)[X].$$

If $q\neq 2$, $\lambda$ is a sum of cohomology operations of even degree (see 1.7 of [1]) so $(\lambda\beta_k)[X]=0$ unless $\dim\beta_k$ has the same parity as $n$. Since $n$ is odd, we see that the only time the right hand side of (2.8) is nonzero is when $\dim\beta_k$ is odd so (2.8) implies

$$(\beta_k\cup\lambda^{-1}h\lambda(U))[X]=(\lambda\beta_k)[X],$$

and, as before, this implies that $\lambda^{-1}h\lambda(U)=\mathrm{Wu}(\lambda, X)$, which proves the result in general.

## 3. Differentiable manifolds

Throughout this section we shall assume $X$ is a compact differentiable manifold with boundary $\dot{X}$ and that $X$ is oriented over $Z_q$. We shall use Theorem II to show that $\mathrm{Wu}(\lambda, X)$ is closely related to cohomology properties of the tangent bundle of $X$.

Let $X$ be provided with a Riemannian metric and let $E$ be the disk bundle over $X$ of tangent vectors of length $\leqq 1$ and let $E'$ be the sphere bundle contained in $E$ of tangent vectors of length 1. Then $E$ is oriented over $Z_q$ by the orientation of $X$, and there is a corresponding Thom-Gysin isomorphism [2, 3]

$$\Phi: H^*(X)\approx H^*(E, E')$$

which has the following properties (see [2] for details). For $t\in X$ let $E_t, E'_t$ be the fibers of $E, E'$ over $t$. Then we have an inclusion map $i_t: (E_t, E'_t)\subset(E, E')$. The orientation class of $X$ modulo $\dot{X}$ determines a homology class $X_t$ of $E_t$ modulo $E'_t$. Then for any $t\in X$ we have

$$(3.1)\qquad (i_t^*\Phi(1))[X_t]=1.$$

Furthermore, if $\pi: E\to X$ is the bundle projection, then for any $x\in H^*(X)$ we have

$$(3.2)\qquad \Phi(x)=\pi^*(x)\cup\Phi(1).$$

The class $\Phi(1)$ determines $\mathrm{Wu}(\lambda, X)$ by the following result.

THEOREM III. *If $\lambda$ is a cohomology automorphism, then*

$$\mathrm{Wu}(\lambda, X)=\lambda^{-1}\Phi^{-1}\lambda\Phi(1).$$

PROOF. Theorem III is quite similar to Theorem II, and we shall deduce it from Theorem II. In order to do so it suffices to construct a homomorphism

$$\mu: H^*(E, E')\to H^*(X\times X, X\times\dot{X})$$

such that:

(a) $\mu$ commutes with arbitrary cohomology automorphisms.

(b) $h\mu\Phi(1)=1$.

(c) $(1\otimes x)\cup\mu\Phi(1)=(x\otimes 1)\cup\mu\Phi(1)=\mu\Phi(x)$ for $x\in H^*(X)$.

Certainly if such a homomorphism $\mu$ exists then (b) and the first part of (c) show that $\mu\Phi(1)$ satisfy (2.4) and (2.5) so, by (2.6), $\mu\Phi(1)$ is the basic class of $X\times X$. Hence, by Theorem II and (a)

$$\mathrm{Wu}(\lambda, X)=\lambda^{-1}h\lambda\big(\mu\Phi(1)\big)=\lambda^{-1}h\mu\lambda\Phi(1).$$

On the other hand, from the second part of (c), the definition of $h$, and (b) we have

$$h\mu\Phi(x)=h\big((x\otimes 1)\cup\mu\Phi(1)\big)=x\cup h\mu\Phi(1)=x$$

so $h\mu=\Phi^{-1}$, and $\lambda^{-1}h\mu\lambda\Phi(1)=\lambda^{-1}\Phi^{-1}\lambda\Phi(1)$, which shows that it suffices to construct $\mu$ satisfying (a), (b), (c).

To construct $\mu$ we consider separately the two cases when $\dot{X}$ is empty and when $\dot{X}$ is non-empty. If $\dot{X}$ is empty, $E$ can be identified with a tubular neighborhood $N$ of the diagonal $d(X)\subset X\times X$ in such a way that $E'$ corresponds to the boundary of $N$ and $\pi$ corresponds to $p_1|N$ where $p_1:X\times X\to X$ is projection to the first factor. If $N^0$ is the interior of $N$, then

$$H^*(E, E')\approx H^*(X\times X, X\times X-N^0).$$

There is an injection

$$H^*(X\times X, X\times X-N^0)\to H^*(X\times X),$$

and the composite of these is a homomorphism

$$\mu: H^*(E, E')\to H^*(X\times X)$$

which has property (a) because it is induced by continuous maps. We shall omit the proof that $\mu$ has properties (b) and (c). It is similar to the proof for the case when $\dot{X}$ is not empty, which we shall give, and details for the case when $\dot{X}$ is empty can be found in [2].

If $\dot{X}$ is not empty, we shrink $X$ to $\overline{X}\subset X-\dot{X}$. That is, there is a homeomorphism $g:X\to\overline{X}$ such that if $j:\overline{X}\subset X$, then $jg\simeq$identity map of $X$ and $gj=$ identity map of $\overline{X}$. Let $\overline{E}, \overline{E}'$ be the bundles over $\overline{X}$ corresponding to $E, E'$ (i.e. $\overline{E}, \overline{E}'$ are induced from $E, E'$ by $j:\overline{X}\subset X$) and let $\bar{j}:(\overline{E}, \overline{E}')\to(E, E')$ be the bundle map covering $j$. If $\overline{\Phi}:H^*(\overline{X})\approx H^*(\overline{E}, \overline{E}')$ is the Thom-Gysin isomorphism of the induced bundle, it follows from standard naturality properties that

$$\overline{\Phi}j^*=\bar{j}^*\Phi.$$

Let $\bar{h}:H^*(\overline{X}\times X, \overline{X}\times\dot{X})\to H^*(\overline{X})$ be the homomorphism defined on $\bar{\alpha}\otimes\beta$ for $\bar{\alpha}\in H^*(\overline{X})$, $\beta\in H^*(X, \dot{X})$ by $\bar{h}(\bar{\alpha}\otimes\beta)=\beta[X]\bar{\alpha}$. Then we have the isomorphism

$$(g\times 1)^*: H^*(\bar{X}\times X, \bar{X}\times\dot{X}) \to H^*(X\times X, X\times\dot{X}),$$

and the equation

$$g^*\bar{h} = h(g\times 1)^*.$$

Furthermore, if $x\in H^*(X)$, then

$$(g\times 1)^*(j^*x\otimes 1) = g^*j^*x\otimes 1 = x\otimes 1.$$

It follows that if we have a homomorphism

$$\bar{\mu}: H^*(\bar{E}, \bar{E}') \to H^*(\bar{X}\times X, \bar{X}\times\dot{X})$$

such that:

($\bar{\text{a}}$) $\bar{\mu}$ commutes with arbitrary cohomology automorphisms,

($\bar{\text{b}}$) $\bar{h}\bar{\mu}\bar{\Phi}(1) = 1$,

($\bar{\text{c}}$) $(1\otimes g^*\bar{x})\cup\bar{\mu}\bar{\Phi}(1) = (\bar{x}\otimes 1)\cup\bar{\mu}\bar{\Phi}(1) = \bar{\mu}\bar{\Phi}(\bar{x})$ for $\bar{x}\in H^*(\bar{X})$,

then the composite $(g\times 1)^*\bar{\mu}\bar{j}^*$ is a homomorphism $\mu$ from $H^*(E, E')$ to $H^*(X\times X, X\times\dot{X})$ which satisfies (a), (b), (c). Therefore, it suffices to construct $\bar{\mu}$ satisfying ($\bar{\text{a}}$), ($\bar{\text{b}}$), ($\bar{\text{c}}$).

Now $\bar{E}$ can be imbedded in $\bar{X}\times(X-\dot{X})$ in such a way that the diagonal $d(\bar{X})$ is a deformation retract of $\bar{E}$ and the projection map of the bundle $\bar{E}\to\bar{X}$ is equal to $\bar{p}_1|\bar{E}$ where $\bar{p}_1: \bar{X}\times X\to\bar{X}$ is projection to the first factor. Then the inclusion map $i: (\bar{E}, \bar{E}')\subset(\bar{X}\times X, \bar{X}\times X-(\bar{E}-\bar{E}'))$ induces isomorphisms

$$i^*: H^*(\bar{X}\times X, \bar{X}\times X-(\bar{E}-\bar{E}'))\approx H^*(\bar{E}, \bar{E}'),$$

and we have also an inclusion map

$$\dot{i}: (\bar{X}\times X, \bar{X}\times\dot{X})\subset(\bar{X}\times X, \bar{X}\times X-(\bar{E}-\bar{E}')).$$

We define $\bar{\mu}: H^*(\bar{E}, \bar{E}')\to H^*(\bar{X}\times X, \bar{X}\times\dot{X})$ to be the composite

$$H^*(\bar{E}, \bar{E}') \xrightarrow{i^{*-1}} H^*(\bar{X}\times X, \bar{X}\times X-(\bar{E}-\bar{E}')) \xrightarrow{\dot{i}^*} H^*(\bar{X}\times X, \bar{X}\times\dot{X}).$$

Since $i^*$, $\dot{i}^*$ both commute with cohomology automorphisms, so does $\bar{\mu}$ so (a) is satisfied.

To prove ($\bar{\text{b}}$) is satisfied, note that $\bar{h}\bar{\mu}\bar{\Phi}(1)$ is a 0-dimensional cohomology class of $X$ and that for $\bar{t}\in\bar{X}$ if we define $f_{\bar{t}}: (X, \dot{X})\to(\bar{X}\times X, \bar{X}\times\dot{X})$ by $f_{\bar{t}}(t) = (\bar{t}, t)$ for $t\in X$, then $\bar{h}\bar{\mu}\bar{\Phi}(1)$ is the cohomology class of the cocycle which assigns to $\bar{t}$ the element $(f_{\bar{t}}^*\bar{\mu}\bar{\Phi}(1))[X]\in Z_q$. Let $F_{\bar{t}} = f_{\bar{t}}^{-1}(\bar{E})$, $F'_{\bar{t}} = f_{\bar{t}}^{-1}(\bar{E}')$ be the fiber of $\bar{E}$, $\bar{E}'$ over $\bar{t}$ and let $i_{\bar{t}}: (F_{\bar{t}}, F'_{\bar{t}})\subset(\bar{E}, \bar{E}')$. Then we have a commutative diagram

$$\begin{array}{ccccc}
H^*(\bar{E}, \bar{E}') & \xleftarrow[\approx]{i^*} & H^*(\bar{X}\times X, \bar{X}\times X-(\bar{E}-\bar{E}')) & \xrightarrow{\dot{i}^*} & H^*(\bar{X}\times X, \bar{X}\times\dot{X}) \\
{\scriptstyle i_{\bar{t}}^*}\downarrow & & \downarrow{\scriptstyle f_{\bar{t}}^*} & & \downarrow{\scriptstyle f_{\bar{t}}^*} \\
H^*(F_{\bar{t}}, F'_{\bar{t}}) & \xleftarrow{\approx} & H^*(X, \dot{X}-(F_{\bar{t}}-F'_{\bar{t}})) & \to & H^*(X, \dot{X})
\end{array}$$

where the unlabeled maps are injections and, in homology, map the homology class of $X$ modulo $\dot{X}$ into $X_{\bar{t}}$. It follows from the commutativity of this diagram and the definition of $\bar{\mu}$ that

$$\left(f_{\bar{t}}^{*}\bar{\mu}\,\bar{\Phi}(1)\right)[X]=\left(i_{\bar{t}}^{*}\,\bar{\Phi}(1)\right)[X_{\bar{t}}].$$

By (3.1) this $=1$ for every $\bar{t}\in\bar{X}$ so ($\bar{\text{b}}$) is satisfied.

To prove ($\bar{\text{c}}$) is satisfied note that by (3.2) and the definition of $\bar{\mu}$, for $\bar{x}\in H^{*}(\bar{X})$ we have

$$\begin{aligned}\bar{\mu}\,\bar{\Phi}(\bar{x})&=\bar{\mu}\left((\bar{p}_1|\bar{E})^{*}\,\bar{x}\cup\bar{\Phi}(1)\right)=\bar{i}^{*}\,i^{*-1}\left(i^{*}\,p_1^{*}\,\bar{x}\cup\bar{\Phi}(1)\right)\\&=p_1^{*}\,\bar{x}\cup\bar{i}^{*}\,i^{*-1}\,\bar{\Phi}(1)=(\bar{x}\otimes 1)\cup\bar{\mu}\,\bar{\Phi}(1),\end{aligned}$$

which proves the second half of ($\bar{\text{c}}$). For the first half of ($\bar{\text{c}}$) we have for $\bar{x}\in H^{*}(\bar{X})$

$$\begin{aligned}(\bar{x}\otimes 1)\cup\bar{\mu}\,\bar{\Phi}(1)&=\bar{i}^{*}\left((\bar{x}\otimes 1)\cup i^{*-1}\,\bar{\Phi}(1)\right)=\bar{i}^{*}\,i^{*-1}\left(i^{*}(\bar{x}\otimes 1)\cup\bar{\Phi}(1)\right)\\(1\otimes g^{*}\,\bar{x})\cup\bar{\mu}\,\bar{\Phi}(1)&=\bar{i}^{*}\left((1\otimes g^{*}\,\bar{x})\cup i^{*-1}\,\bar{\Phi}(1)\right)=\bar{i}^{*}\,i^{*-1}\left(i^{*}(1\otimes g^{*}\,\bar{x})\cup\bar{\Phi}(1)\right).\end{aligned}$$

To complete the proof we need only verify that $i^{*}(\bar{x}\otimes 1)=i^{*}(1\otimes g^{*}\,\bar{x})$ in $H^{*}(\bar{E})$. Since $d:\bar{X}\to\bar{E}$ is a homotopy equivalence, we need only check that $d^{*}\,i^{*}(1\otimes g^{*}\,\bar{x})$ and $d^{*}\,i^{*}(\bar{x}\otimes 1)$ are equal in $H^{*}(\bar{X})$. But $d^{*}\,i^{*}(1\otimes g^{*}\,\bar{x})=j^{*}\,g^{*}\,\bar{x}=\bar{x}$ and $d^{*}\,i^{*}(\bar{x}\otimes 1)=\bar{x}$, so they are equal, and the proof is complete.

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