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SPACES SATISFYING POINCARÉ DUALITY

MICHAEL SPIVAK†

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INTRODUCTION

A P-SPACE (Poincaré duality space) of formal dimension n is, roughly speaking, a finite complex X such that for some $\mu \in H_n(X)$ the map

$$\cap\mu : H^*(X) \to H_*(X)$$

is an isomorphism. A more complicated definition is required when X is not simply connected.

According to Browder [3] and Novikov [19] a simply connected P-space X has the homotopy type of a compact C^∞ manifold of dimension n if n is odd and there is a vector bundle $\pi : E \to X$ of fibre dimension k, say, such that the generator of $H^{n+k}(T(E))$ is spherical, where $T(E)$ is the Thom space of the bundle. This paper is concerned with the possibility of deciding whether or not such a vector bundle exists. We consider spherical fibre spaces over X, that is, fibre spaces whose fibres have the homotopy type of a sphere S^{k-1}; the integer k is called the fibre dimension. The Thom space $T(\pi)$ of such a fibre space π, and stable fibre homotopy equivalence of two such fibre spaces, can be defined.

THEOREM A. *If X is a P-space, then there is one and, up to stable fibre homotopy equivalence, only one spherical fibre space π over X such that the generator of $H^{n+k}(T(\pi))$ is spherical.*

The construction of such a fibre space was suggested by Milnor. The proof of uniqueness is a generalization of a theorem of Atiyah [2].

Theorem A provides an obstruction theory for the existence of a vector bundle over X with the desired property. The p^{th} obstruction $\mathcal{O}^p(X)$ lies in $H^p(X; \pi_{p-1}(F))$, where F is the fibre of the fibring map $B_0 \to B_H$ (here $B_H = \varinjlim B_{H(k)}$, where $B_{H(k)}$ is Stasheff's classifying space for spherical fibre spaces of fibre dimension k). If X has formal dimension N and is $(n-1)$-connected the primary obstruction $\mathcal{O}^n(X) \in H^n(X; \pi_{n-1}(F))$ is described in terms of the topology of X as follows.

Let Π_{n-1} denote the stable $(n-1)$-stem and let $\psi : H^{N-n}(X; Z) \to H^N(X; \Pi_{n-1}) \approx \Pi_{n-1}$ be the secondary obstruction defined in [11]. Then $(-1)^{n(N+1)+1}\ \psi$ is $\cup\ \psi^n(X)$ for a unique $\psi^n(X) \in H^n(X; \Pi_{n-1})$.

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THEOREM B. *There is a homomorphism $\Pi_{n-1} \to \pi_{n-1}(F)$ such that the induced coefficient homomorphism $H^n(X; \Pi_{n-1}) \to H^n(X; \pi_{n-1}(F))$ takes $\psi^n(X)$ into $\mathcal{O}^n(X)$, and the sequence*

$$\pi_{n-1}(SO) \xrightarrow{J_{n-1}} \Pi_{n-1} \longrightarrow \pi_{n-1}(F)$$

is exact.

Sections 1 and 2 contain elementary properties of fibre spaces, §3 elementary properties of P-spaces, and §§4 and 5 the material necessary for Theorem A. The obstruction theory is formulated in §6, and Theorem B is proved in §§7 and 8.

If A and B are groups [spaces] then $A \approx B$ means: A is isomorphic to B [A has the same homotopy type as B]. If $f, g: X \to Y$ are maps then $f \simeq g$ means: f is homotopic to g. Singular homology and cohomology are used throughout.

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§1. SPHERICAL FIBRE SPACES

The term "fibre space" always means a map $\pi: E \to X$ with the covering homotopy property CHP for all spaces or, equivalently, the path lifting property, PLP. All base spaces of fibre spaces are assumed to be paracompact and locally contractible. The fibre $\pi^{-1}(x)$ will often be denoted E_x. If π and η are fibre spaces then $\pi \sim \eta$ means "π is fibre homotopy equivalent to η".

If $\pi: E \to X$ is a fibre space let C_π be the mapping cylinder of π and let $r: C_\pi \to X$ be the retraction. Then $r: C_\pi \to X$ is a fibre space with sub-fibre space π; in fact r is (see below) $\pi \oplus 1$, for the identity map $1: X \to X$. The fibre of r over x will be denoted simply C_x, if no confusion is possible. The inclusions of the fibres C_x and E_x in the total spaces will be denoted i_x.

If A and B are subsets of X, then $\mathcal{P}(A, X, B)$ is the set of all paths $p: [0,1] \to X$ such that $p(0) \in A$ and $p(1) \in B$, topologized by the compact-open topology. The *endpoint map* $\omega: \mathcal{P}(A, X, B) \to B$ is defined by $\omega(p) = p(1)$.

1.1 PROPOSITION. *If $\pi: E \to X$ is a fibre space, then π is fibre homotopy equivalent to the endpoint map $\omega: \mathcal{P}(E, C_\pi, X) \to X$.* (Compare with [7], §5, and [8]).

Proof. Let $r: C_\pi \to X$ be the retraction. Let $\mathcal{P}'(E, C_\pi, X)$ be the space of paths in $\mathcal{P}(E, C_\pi, X)$ which remain in a single fibre of r, let $\omega' = \omega | \mathcal{P}'(E, C_\pi, X)$, and let $i: \mathcal{P}'(E, C_\pi, X) \to \mathcal{P}(E, C_\pi, X)$ be the inclusion.

For $p: [0,1] \to X$ and $e \in E$ with $\pi(e) = p(0)$, let $l(p, e): [0,1] \to E$ be a path, depending continuously on p and e, such that $l(p, e)(0) = e$ and $\pi \circ l(p, e) = p$. Any point $x \in C_\pi$ can be written as $(e(x), s(x))$ for $e(x) \in E$ and $s(x) \in [0, 1]$. Define $H: \mathcal{P}(E, C_\pi, X) \times I \to \mathcal{P}(E, C_\pi, X)$ by

$$H(p, u)(t) = (l(r \circ p \,|\, [t, 1], e(p(t)))(u), s(p(t))),$$

so that $H(p, 1) \in \mathscr{P}'(E, C_\pi, X)$. The composition $i \circ H(\ ,1)$ is fibre preserving homotopic to the identity by the homotopy H, and $H(\ ,1) \circ i$ is easily seen to be fibre preserving homotopic to the identity. It therefore suffices to show that $\pi \sim \omega'$.

Define $\alpha : \mathscr{P}'(E, C_\pi, X) \to E$ by $\alpha(p) = p(0)$. If $i : E \to \mathscr{P}(E, C_\pi, X)$ is the obvious map then $\alpha \circ i$ is the identity. If $K : \mathscr{P}'(E, C_\pi, X) \times I \to \mathscr{P}(E, C_\pi, X)$ is defined by

$$K(p, u) = p \mid [0, u) \quad \text{followed by the radial path from } p(u) \text{ to } p(1),$$

then K is a fibre preserving homotopy of $i \circ \alpha$ and the identity.

Proposition 1.1, and the following considerations, allow E to be replaced by a space of the same homotopy type.

If $\pi_i : E_i \to X (i = 1, 2)$ are maps (not necessarily fibre spaces) and $f : E_1 \to E_2$ is a homotopy equivalence such that $\pi_2 f \simeq \pi_1$, it is not hard to show that there are maps $\alpha_i : C_{\pi_i} \to C_{\pi_{3-i}}$ such that

(1) $\alpha_i(E_i) \subset E_{3-i}$

(2) $\alpha_i | X =$ identity map of X

(3) $\alpha_{3-i}\alpha_i \simeq$ identity map of C_{π_i}, where the homotopy keeps X pointwise fixed and E_i fixed as a set.

It follows that the endpoint maps $\omega_i : \mathscr{P}(E_i, C_{\pi_i}, X) \to X$ are fibre homotopy equivalent.

A spherical fibre space of fibre dimension $k \geq 1$, is a fibre space in which every fibre has the homotopy type of S^{k-1}. If $\pi : E \to X$ is a vector bundle of fibre dimension k, and E_0 is the set of non-zero vectors, then $\pi | E_0 : E_0 \to X$ is a spherical fibre space of fibre dimension k, denoted $[\pi]$. (One can similarly define $[\mathfrak{A}]$ for any microbundle $\mathfrak{A}$, by [12].)

The Thom isomorphism theorem holds for spherical fibre spaces: there is a class $U(\pi) \in H^k(C_\pi, E)$, natural with respect to induced fibre spaces, such that the maps

$$\cup U(\pi) : H^p(C_\pi; G) \longrightarrow H^{p+k}(C_\pi, E; G)$$

$$U(\pi) \cap : H_{p+k}(C_\pi, E; G) \to H_p(C_\pi; G)$$

are isomorphisms for $p \geq 0$, where $G = Z_2$ if π is not orientable and G is arbitrary if π is orientable. This can be proved by using the Leray–Serre spectral sequence for (r,π) (c.f. remarks after Theorem 2.1 in [18]).

The isomorphisms

$$H^p(X; G) \xrightarrow{r^*} H^p(C_\pi; G) \xrightarrow{\cup U(\pi)} H^{p+k}(C_\pi, E; G)$$

$$H_{p+k}(C_\pi, E; G) \xrightarrow{U(\pi) \cap} H_p(C_\pi; G) \xrightarrow{r_*} H_p(X; G)$$

are the *Thom isomorphisms* φ and ψ, respectively.

Let $*$ be fixed point outside of all spaces under consideration. If A and B are two spaces their *join* $A * B$ is the set of all (a, t, b) with $t \in [0,1]$ and

$$a \in A \ \text{ if } \ t \neq 1, \qquad a = * \ \text{ if } \ t = 1$$

$$b \in B \ \text{ if } \ t \neq 0, \qquad b = * \ \text{ if } \ t = 0.$$

$A * B$ is given the small topology as in [14]. As a special case, the cone CA is $\{\infty_A\} * A$ where ∞_A is the vertex. There is an obvious homeomorphism $\varphi : A * B \to (A \times CB \cup CA \times B) \subset CA \times CB$ and $CA \times CB$ may be regarded as $C(A \times CB \cup CA \times B) = C(A * B)$.

If $\pi_i : E_1 \to X$ are fibre spaces, their *Whitney join* $\pi = \pi_1 \oplus \pi_2$ is the map $\pi : E_1 \oplus E_2 \to X$ defined as follows.

(1) $E_1 \oplus E_2$ is the subset of $E_1 * E_2$ consisting of all (e_1, t, e_2) such that $\pi_1(e_1) = \pi_2(e_2)$ if $t \in (0,1)$

(2) $\pi(e_1, 0, *) = \pi_1(e_1)$
$\pi(*, 1, e_2) = \pi_2(e_2)$
$\pi(e_1, t, e_2) = \pi_1(e_1) = \pi_2(e_2)$ for $t \in (0,1)$.

Using the PLP it is easy to see that $\pi_1 \oplus \pi_2$ is a fibre space. For any integer $n \geq 1$, the fibre space $X \times S^{n-1} \to X$ will be denoted n_X, or simply n, if no confusion is possible. Then π and η *are stably fibre homotopy equivalent* ($\pi \underset{s}{\sim} \eta$) if and only if there are integers m and n such that $\pi \oplus m \sim \eta \oplus n$. The spherical fibre space π of fibre dimension k is *trivial* if and only if $\pi \sim k$ and *stably trivial* if and only if $\pi \underset{s}{\sim} 1$.

Let $H(n)$ be the space of all homotopy equivalences of S^{n-1}, with the compact-open topology. A "classifying space" $B_{H(n)}$ and a map $i_n : B_{0(n)} \to B_{H(n)}$ are defined in [5]. A fibre space $u : UE \to B_{H(n)}$ is defined in [22], which is universal for spherical fibre spaces of fibre dimension n over CW-complexes. We shall denote this fibre space by $\pi_{H(n)} : E_{H(n)} \to B_{H(n)}$. If $\pi_{0(n)} : E_{0(n)} \to B_{0(n)}$ is the universal vector bundle of fibre dimension n, then $i_n^*(\pi_{H(n)}) \sim [\pi_{0(n)}]$. If $B_0 = \varinjlim B_{0(n)}$ and $B_H = \varinjlim B_{H(n)}$ we obtain the *natural map* $i : B_0 \to B_H$, which is well defined up to homotopy. The classifying space for orientable spherical fibre spaces of dimension will be denoted $B_{SH(n)}$.

Clearly $\oplus$ induces the structure of an abelian semi-group on $\underset{s}{\sim}$ equivalence classes of spherical fibre spaces over X.

1.2 PROPOSITION. *If X is a finite complex this semi-group is a group.*

Proof. Only the existence of inverses is non-trivial. (The stable inverse of π will be denoted π^{-1}). The proof is a replica of [17], Theorem 3, with the following changes.

(1) To define the wedge of π and η, two orientable spherical fibre spaces over X of fibre dimension k, let $\pi = f^*(\pi_{SH(k)})$ and $\eta = g^*(\pi_{SH(k)})$ and define $\pi \vee \eta$ over $X \vee X$ as $(f \vee g)^*(\pi_{SH(k)})$.

(2) If X^n is the n-skeleton of an $(n+1)$-dimensional space X, and $\pi | X^n$ has an inverse η, it is not clear that η can be extended to X, but it is easy to show, using the universal spherical fibre spaces, that there is a spherical fibre space η' over X^n such that $\eta \underset{s}{\sim} \eta'$ and η' can be extended over X.

§2. THOM SPACES

If $\pi : E \to X$ is a fibre space, the *Thom space* $T(\pi)$ is defined as $CE \underset{\pi}{\cup} X$. The vertex ∞ of CE is the natural base point for $T(\pi)$. Note that $(T(\pi), \infty) \approx (C_\pi \cup CE, \infty)$ so that $H^*(T(\pi), \infty)) \approx H^*(C_\pi, E)$.

If $\pi_i : E_i \to X (i = 1,2,3)$ are fibre spaces and $f: E_1 \to E_2$ is a fibre preserving map, a continuous map $f_T : (T(\pi_1), \infty_1) \to (T(\pi_2), \infty_2)$ is defined by

$$f_T((\infty_1, t, e_1)) = (\infty_2, t, f(e_1)) \quad \text{for} \quad t \in [0, 1), \text{ where } f(*) \text{ means } *.$$

$$f_T(\pi_1(e_1)) = \pi_2 f(e_1) = \pi_1(e_1).$$

It is easy to show that if $f: E(\pi_1) \to E(\pi_2)$ is a fibre homotopy equivalence, then $f_T : (T(\pi_1), \infty_1) \to (T(\pi_2), \infty_2)$ is a homotopy equivalence.

2.1 Proposition. *If π is a fibre space over Y and $f: X \to Y$ is a homotopy equivalence, then* $(T(f^*(\pi)), \infty) \approx (T(\pi), \infty)$.

Proof. Let $g : Y \to X$ be a homotopy inverse for f. There are fibre preserving maps $\tilde{f}: E(f^*(\pi)) \to E(\pi)$ and $\tilde{g} : E(g^*(f^*(\pi))) \to E(f^*(\pi))$ and clearly also a fibre homotopy equivalence $\alpha : E(g^*(f^*(\pi))) \to E((fg)^*(\pi))$ such that $\tilde{f}\tilde{g} = (fg)^{\sim}\alpha$. Since $fg \simeq 1$ there is a fibre homotopy equivalence $h: E(\pi) \to E((fg)^*(\pi))$ such that $(fg)^{\sim}h$ is fibre preserving homotopic to 1. If β is a fibre homotopy inverse for α, then $\tilde{f}\tilde{g}\beta h$ is fibre preserving homotopic to 1. Therefore $\tilde{f}_T\tilde{g}_T(\beta h)_T \simeq 1$.

Similarly, there is a fibre homotopy equivalence $k : E(f^*(\pi)) \to E(f^*(g^*(f^*(\pi))))$ such that $\tilde{g}\tilde{f}k$ is fibre preserving homotopic to 1, where $\tilde{f}: E(f^*(g^*(f^*(\pi)))) \to E(g^*(f^*(\pi)))$. Hence $\tilde{g}_T(\tilde{f}k)_T \simeq 1$.

Since $\tilde{f}_T\tilde{g}_T(\beta h)_T \simeq 1$ and $(\beta h)_T$ is a homotopy equivalence, it follows that $(\beta h)_T(\tilde{f}_T\tilde{g}_T) \simeq 1$, or $((\beta h)_T\tilde{f}_T)\tilde{g}_T \simeq 1$.

Thus $\tilde{g}_T$ has the left homotopy inverse $(\beta h)_T\tilde{f}_T$ and the right homotopy inverse $(\tilde{f}k)_T$; consequently $\tilde{g}_T$ is a homotopy equivalence and its left homotopy inverse is a right homotopy inverse. Hence $1 \simeq \tilde{g}_T((\beta h)_T f_T) = (\tilde{g}_T(\beta h)_T)f_T$.

Thus $\tilde{f}_T$ has the left and right homotopy inverse $\tilde{g}_T(\beta h)_T$, and $\tilde{f}_T$ is a homotopy equivalence.

Let Y be a space with base point y_0. Following [9] we say that Y is *reducible* [*S-reducible*] if and only if there is a map [S-map] $f: (S^n,a) \to (Y,y_0)$ inducing isomorphisms of $\tilde{H}_q$ for $q \geq n$. Dually, Y is *coreducible* [*S-coreducible*] if and only if there is a map [S-map] $f: (Y,y_0) \to (S^n,a)$ inducing isomorphisms of $\tilde{H}^q$ for $q \leq n$. Then Y is S-reducible if and only if its S-dual ([21]) is S-coreducible (Y must have the homotopy type of a finite CW-complex and hence ([23], Theorem 13) of a finite complex for this to be meaningful). A spherical fibre space π is called *reducible*, etc., if and only if $(T(\pi), \infty)$ is reducible, etc. (Notice that if $\pi: E \to X$ is a spherical fibre space over a finite complex X and F is a fibre then the pair (E,F) has the homotopy type of a pair of finite complexes (c.f. proof of Proposition (0) of [22]).)

If $\pi: E \to X$ is a fibre space over X, it is easy to see that $T(1 \oplus \pi)$ is homeomorphic to the suspension $\Sigma(T(\pi))$, so $T(n \oplus \pi)$ is homeomorphic to $\Sigma^n(T(\pi))$. Therefore a simple generalization of the argument in [2], Proposition 2.8 proves the following.

2.2 PROPOSITION. *If $\pi : E \to X$ is a spherical fibre space over a connected finite complex X, then π is S-coreducible if and only if π is stably trivial.*

§3. P-SPACES

A pair (X, Y) satisfies *Poincaré duality for dimension n* if and only if for some $\mu \in H_n(X, Y)$ the maps

$$(1) \quad \cap\, \mu : H^*(X) \to H_*(X, Y)$$

$$(2) \quad \cap\, \mu : H^*(X, Y) \to H_*(X)$$

$$(3) \quad \cap\, \partial\mu : H^*(Y) \to H_*(Y)$$

are isomorphisms. Such a class μ is called an *orientation*, and $\partial\mu$ is the *induced orientation* of Y.

Note that $H_*(X)$ is finitely generated: in fact if μ is represented by a finite sum $\sigma = \sigma_1 + \cdots + \sigma_k \in C_n(X, Y)$ of singular n-simplices then every element of $H_*(X)$ is represented by $f \cap \sigma$ for some $f \in C^*(X, Y)$; but each f $\cap\, \sigma$ is in the free group generated by the faces of $\sigma_1, \ldots, \sigma_k$. Similarly, $H_*(Y)$ and $H_*(X, Y)$ are finitely generated.

From the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{p-1}(X, Y), Z) & \to & H^p(X, Y) & \to & Hom(H_p(X, Y), Z) & \to 0 \\ & \big\downarrow {\scriptstyle \mathrm{Ext}(\cap\mu, 1)} & & \big\downarrow {\scriptstyle \cap\mu} & & \big\downarrow {\scriptstyle \mathrm{Hom}(\cap\mu, 1)} & \\ 0 \to & \mathrm{Ext}(H^{n-p+1}(X), Z) & \to & H_{n-p}(X) & \to & Hom(H^{n-p}(X), Z) & \to 0 \end{array}$$

it follows that (1) is an isomorphism if and only if (2) is an isomorphism. Similar diagrams show that (1–3) are isomorphisms for any coefficient group. Moreover, (3) is an isomorphism if (1) or (2) is.

If, in the definition we have given, H_* is replaced by H_*^{LF} (homology based on infinite, locally finite chains) then (X, Y) *satisfies open Poincaré duality for dimension n.*

If (X, Y) is a pair of complexes, $\tilde{X}$ will represent the universal covering space of X. If $\rho : \tilde{X} \to X$ is the covering map, then $(\tilde{X}, \tilde{Y})$ will represent $(\tilde{X}, \rho^{-1}(Y))$. A *P-pair of formal dimension n* is a finite complex (X, Y) such that $(\tilde{X}, \tilde{Y})$ satisfies open Poincaré duality for dimension n. If (X, Y) itself satisfies (open) Poincaré duality for dimension n, then (X, Y) is called *orientable.* X is called a *P-space of formal dimension n* if $(X, \varnothing)$ is a P-pair of formal dimension n.

The most obvious examples of P-pairs are triangulated manifolds with boundaries. The following examples will be used in §§7 and 8.

Let S_1^n and S_2^n be two copies of S^n, for $n \leq 2$, and let $\iota_i : S^n \to S_1^n \vee S_2^n$ $(i = 1, 2)$, be the inclusions. Then $\pi_{2n-1}(S_1^n \vee S_2^n) \approx \pi_{2n-1}(S_1^n) \oplus \pi_{2n-1}(S_2^n) \oplus Z$; any element $\alpha \in \pi_{2n-1}(S_1^n \vee S_2^n)$ can be written as $\iota_1 \circ \alpha_1 \oplus \iota_2 \circ \alpha_2 \oplus m[\iota_1, \iota_2]$ for $\alpha_i \in \pi_{2n-1}(S_i^n)$ and m an integer. Let $g : S^{2n-1} \to S_1^n \vee S_2^n$ and consider the space $X = e^{2n} \cup_g (S_1^n \vee S_2^n)$. The homotopy type of X is determined by $[g] \in \pi_{2n-1}(S_1^n \vee S_2^n)$. Let $\kappa_i : S^n \to X$ be the composition $S^n \xrightarrow{\iota_i} S_1^n \vee S_2^n \subset X$ and let $c_i \in H^n(X)$ be the elements such that $\kappa_i^*(c_i) = 1$ and $\kappa_{3-i}^*(c_i) = 0$. If $[g] = \iota_1 \circ \alpha_1 \oplus \iota_2 \circ \alpha_2 \oplus m[\iota_1, \iota_2]$ then, with respect to the basis (c_1, c_2) for $H^n(X)$, the cup-product pairing $\cup : H^n(X) \otimes H^n(X) \to H^{2n}(X) \approx Z$ has the matrix

$$M = \begin{pmatrix} H(\alpha_1) & m \\ (-1)^n m & H(\alpha_2) \end{pmatrix}$$

where $H(\alpha_i)$ is the Hopf invariant of α_i. Therefore X is a P-space if and only if $\det M = \pm 1$. In particular, let $[g] = \iota_1 \circ \alpha \oplus 0 \oplus [\iota_1, \iota_2]$. Then $X_\alpha = e^{2n} \cup_g (S_1^n \vee S_2^n)$ is a P-space.

If $\pi : E \to X$ is a fibre space and $r : C_\pi \to X$ is the retraction, then H^{rLF} will denote singular homology based on chains $c \in C_*(C_\pi)$ such that $r_\# c$ is a locally finite chain in X.

3.1 Proposition. *Let $\pi: E \to X$ be a spherical fibre space of fibre dimension d over a space X which satisfies Poincaré duality [open Poincaré duality] for dimension n. Then (C_π, E) satisfies Poincaré duality for dimension $n + d$ [with H_* replaced by H_*^{rLF}].*

Proof. The proof will be given when X satisfies open Poincaré duality. A proof is obtained for the other case by deleting LF and rLF whenever they occur.

Let $\mu \in H_n^{LF}(X)$ be an orientation. Let φ amd ψ be the Thom isomorphisms for π. Putting together the Gysin sequences in cohomology and homology we obtain the diagram

$$\begin{array}{ccccccc} H^i(X) & \xrightarrow{\cup\chi} & H^{i+d}(X) & \xrightarrow{\pi^*} & H^{i+d}(E) & \xrightarrow{\varphi^{-1}\delta} & H^{i+1}(X) \\ \downarrow{\scriptstyle \cap\mu} & & \downarrow{\scriptstyle \cap\mu} & & \downarrow{\scriptstyle \cap\tilde\mu} & & \downarrow{\scriptstyle \cap\mu} \\ H_{n-i}^{LF}(X) & \xrightarrow{\chi\cap} & H_{n-i-d}^{LF}(X) & \xrightarrow{\partial\psi^{-1}} & H_{n-i-1}^{rLF}(E) & \xrightarrow{\pi_*} & H_{n-i-1}^{LF}(X) \end{array}$$

where

$\delta : H^{i+d}(E) \to H^{i+d+1}(C_\pi, E)$

$\partial : H_{n-i}^{rLF}(C_\pi, E) \to H_{n-i-1}^{rLF}(E)$

$i : C_\pi \to (C_\pi, E)$ is the inclusion

$\chi = \varphi^{-1}(U \cup U)$

$\tilde\mu = \partial\bar\mu$, where $\bar\mu \in H_{n+d}^{rLF}(C_\pi, E)$ satisfies $\psi(\bar\mu) = \mu$, that is, $r_*(U \cap \bar\mu) = \mu$.

The first square commutes up to sign: $(\alpha \cup \chi) \cap \mu = (-1)^{id}(\chi \cup \alpha) \cap \mu = (-1)^{id}\chi \cap (\alpha \cap \mu)$. *The second square commutes up to sign:* We must show that if $r_*(U \cap \alpha) = \beta \cap \mu$, then $\partial\alpha = \pm \pi^*\beta \cap \mu$. Now

$$\begin{aligned} r_*(U \cap \alpha) &= \beta \cap \mu \\ &= \beta \cap r_*(U \cap \bar\mu) \\ &= r_*(r^*\beta \cap (U \cap \bar\mu)) \\ &= r_*((r^*\beta \cup U) \cap \bar\mu) \\ &= (-1)^{d(1+i)}((U \cup r^*\beta) \cap \bar\mu) \\ &= (-1)^{d(1+i)}(U \cap (r^*\beta \cap \bar\mu)). \end{aligned}$$

Therefore $U \cap \alpha = (-1)^{d(1+i)} U \cap (r^*\beta \cap \bar{\mu})$; hence $\alpha = (-1)^{d(1+i)} r^*\beta \cap \bar{\mu}$, so

$$\begin{aligned}\partial\alpha &= (-1)^{d(1+i)}\, \partial(r^*\beta \cap \bar{\mu}) \\ &= (-1)^{i(d+1)} \pi^*\beta \cap \partial\bar{\mu} \\ &= (-1)^{i(d+1)} \pi^*\beta \cap \tilde{\mu}.\end{aligned}$$

The third square commutes up to sign: We must prove that if $r^*\alpha \cup U = \delta\beta$, then $\alpha \cap \mu = \pm\, \pi_*(\beta \cap \tilde{\mu})$. Now, if $j: E \to C_\pi$ is the inclusion, then

$$\begin{aligned}\pi_*(\beta \cap \tilde{\mu}) &= \pi_*(\beta \cap \partial\bar{\mu}) \\ &= r_* j_*(\beta \cap \partial\bar{\mu}) \\ &= (-1)^{i+d} r_*(\delta\beta \cap \bar{\mu}) \\ &= (-1)^{i+d} r_*((r^*\alpha \cup U) \cap \bar{\mu}) \\ &= (-1)^{i+d} r_*(r^*\alpha \cap (U \cap \bar{\mu})) \\ &= (-1)^{i+d} \alpha \cap r_*(U \cap \bar{\mu}) \\ &= (-1)^{i+d} \alpha \cap \mu.\end{aligned}$$

It follows from the 5-lemma that $\cap\, \tilde{\mu}$ is an isomorphism, and we need only prove that $\cap\, \bar{\mu}$ is an isomorphism. This follows from the diagram

$$\begin{array}{ccc} H^i(C_\pi, E) & \xrightarrow{\ \cap\, \bar{\mu}\ } & H^{rLF}_{n-i+d}(C_\pi) \\ \approx \Big\uparrow \varphi & & \approx \Big\downarrow r_* \\ H^{i-d}(X) & \underset{\approx}{\xrightarrow{\ \cap\, \mu\ }} & H^{LF}_{n-i+d}(X) \end{array}$$

which is commutative, since

$$\begin{aligned}r_*((r^*\alpha \cup U) \cap \bar{\mu}) &= r_*(r^*\alpha \cap (U \cap \bar{\mu})) \\ &= \alpha \cap r_*(U \cap \bar{\mu}) \\ &= \alpha \cap \mu.\end{aligned}$$

§4. NORMAL FIBRE SPACES

In this section we construct certain spherical fibre spaces over a P-pair (X, Y). In the next section we prove that these are reducible if $Y = \varnothing$ and that all S-reducible fibre spaces over a P-space X are stably fibre homotopy equivalent. We will require a lemma on cap products.

The usual definition of the cap product $\cap : H^p(X, A) \otimes H_q(X, A \cup B) \to H_{q-p}(X, B)$ for A and B open in $A \cup B$, uses the complex $\hat{C}_*(A, B)$ generated by singular simplices lying in A or in B. This definition also provides a cap product

$$\cap : H^p(A \cup B, A) \otimes H_q(A \cup B) \to H_{q-p}(B),$$

since $f \cap c \in C_{q-p}(B)$ if $f \in C^p(A \cup B, A)$ and $c \in \hat{C}_q(A, B)$. These maps can also be defined using H_*^{LF}. The following lemma applies for both H_* and H_*^{LF}.

4.1 LEMMA. (1) *Let* $k : (A \cup B, A) \to (X, A)$ *be the inclusion and let* ∂_1 *and* ∂_2 *be the boundary maps of the homology sequences of* $(X, A \cup B)$ *and* (X, B) *respectively. Let* $\alpha \in H^p(X, A)$ *and* $\beta \in H_q(X, A \cup B)$. *Then*

$$k^*\alpha \cap \partial_1\beta = (-1)^p\, \partial_2(\alpha \cap \beta).$$

(2) *Let* $Y \subset A \cap B$ *and let* $i : (B, Y) \to (A \cup B, A)$ *and* $j : A \cup B \to (A \cup B, A)$ *be inclusions. Let* $\alpha \in H^p(A \cup B, A)$, $\beta \in H_q(A \cup B)$, $\gamma \in H_q(B, Y)$. *If* $j_*\beta = i_*\gamma$ *then* $\alpha \cap \beta = i^*\alpha \cap \gamma$.

$$\begin{array}{ccccc}
H^p(A \cup B, A) & \otimes & H_q(A \cup B) & \to & H_{q-p}(B) \\
\Big\downarrow i^* & & \Big\downarrow j_* & & \\
 & & H_q(A \cup B, A) & & \\
 & & \Big\uparrow i_* & & \\
H^p(B, Y) & \otimes & H_q(B, Y) & \to & H_{q-p}(B)
\end{array}$$

Proof. Only the proof of (2) will be given. Let $f \in C^p(A \cup B, A)$ represent α. Let $c \in \hat{C}^q(A, B)$ represent β, so that $f \cap c$ represents $\alpha \cap \beta$. Let $d \in C_q(B)$ represent γ. Since $j_*\beta = i_*\gamma$ we have

$$d = c + c' + \partial c''$$

where $c' \in C_q(A)$ and $c'' \in C_{q+1}(A \cup B)$. Moreover $c'' = c''' + \partial c''''$, for $c''' \in \hat{C}_{q+1}(A, B)$ and $c'''' \in \hat{C}_{q+2}(A \cup B)$, so that

$$d = c + c' + \partial c'''.$$

Now $i^*\alpha \cap \gamma$ is represented by $f \cap d = f \cap c + f \cap c' + f \cap \partial c'''$. But $f \cap c' = 0$ and

$$f \cap \partial c''' = (-1)^p(\delta f \cap c''' + \partial(f \cap c''')) = (-1)^p\, \partial(f \cap c'''),$$

so $f \cap d = f \cap c + (-1)^p\partial(f \cap c''')$, where $f \cap c''' \in C_*(B)$. Hence $i^*\alpha \cap \gamma = \alpha \cap \beta$.

Let H^{n+k} be a closed half-space of R^{n+k}, bounded by R^{n+k-1}. A complex $X \subset R^{n+k}$ is always assumed to be closed. A pair of complexes $(X, Y) \subset R^{n+k}$ is a *subcomplex of* (H^{n+k}, R^{n+k-1}) if $X \subset H^{n+k}$ and $Y = X \cap R^{n+k-1}$. If (X, Y) is a subcomplex of (H^{n+k}, R^{n+k-1}), a *regular neighborhood of* (X, Y) is a triple $(N; N_1, N_2)$ such that

(1) N is a regular neighborhood of X in H^{n+k} (with boundary ∂N),
(2) $N_2 = N \cap R^{n+k-1}$ is a regular neighborhoodof Y in R^{n+k-1},
(3) $N_1 = \text{Closure}\,(\partial N - N_2)$,
(4) There is a deformation retraction $(N, N_2) \to (X, Y)$.

Note that N is a submanifold of R^{n+k} with $\partial N = N_1 \cup N_2$. The case $Y = \varnothing$ is not excluded; N is then a regular neighborhood of X in R^{n+k} with $\partial N = N_1$.

The *cohomology bound* of X, denoted $cb(X)$, is the largest n such that $H^n(X) \neq 0$.

4.2 Proposition. *Let (X, Y) be a subcomplex of (H^{n+k}, R^{n+k-1}) with X connected, where $k > cb(X) + 1$. Let $(N; N_1, N_2)$ be a regular neighborhood of (X, Y). Then (X, Y) satisfies open Poincaré duality for dimension n if and only if the following three conditions hold.*

(1) $H^*(N_1) \approx H^*(N) \oplus H^{*+(k-1)}(N)$,
(2) $H^d(N) \to H^d(N_1)$ *is an isomorphism,* $0 \le d \le n$,
(3) $\cup g \colon H^d(N_1) \to H^{d+(k-1)}(N_1)$ *is an isomorphism,*
$0 \le d \le n$, *where* $g \in H^{(k-1)}(N_1)$ *is a generator.*

Proof. Let $\nu_1 \in H^{LF}_{n+k}(N, \partial N)$ be an orientation. If $Y = \varnothing$ then $\cap\, \nu_1 : H^d(N, N_1) \to H_{n+k-d}(N, N_2)$ is clearly an isomorphism. If $Y \neq \varnothing$ this can be proved as follows.

Let $\nu_2 \in H^{LF}_{n+k-1}(N_2, \partial N_2)$ be an orientation. Let $a : \partial N \to (\partial N, N_1)$, $b : (N_2, \partial N_2) \to (\partial N, N_1)$ and $c : (\partial N, N_1) \to (N, N_1)$ be inclusions. Then $b_* : H^{LF}_*(N_2, \partial N_2) \to H^{LF}_*(\partial N, N_1)$ is an isomorphism and $b_* \nu_2$ is a generator of $H^{LF}_{n+k-1}(\partial N, N_1)$. From the diagram (for $p \in N_2$)

$$\begin{array}{ccccc} & & H^{LF}_{n+k-1}(\partial N, N_1) & & \\ & \overset{a_*}{\nearrow} & & \searrow & \\ H^{LF}_{n+k-1}(\partial N) & & \longrightarrow & & H^{LF}_{n+k-1}(\partial N, \partial N - p) \end{array}$$

it is clear that $a_* \partial \nu_1$ is also a generator of $H^{LF}_{n+k-1}(\partial N, N_1)$. Hence $a_* \partial \nu_1 = \pm\, b_* \nu_2$. It follows from Lemma 4.1 (choosing $A = N_1$, $B = N_2$ and $X = N$) that $b^* c^* \alpha \cap \nu_2 = \pm\, c^* \alpha \cap \partial \nu_1 = \pm\, \partial(\alpha \cap \nu_1)$ for $\alpha \in H^*(\partial N, N_1)$. Therefore the following diagram commutes up to sign.

$$\begin{array}{ccccccc} \longrightarrow & H^d(N, \partial N) & \longrightarrow & H^d(N, N_1) & \xrightarrow{c^*} & H^d(\partial N, N_1) & \longrightarrow \\ & & & & & \approx \downarrow b^* & \\ & \approx \downarrow \cap \nu_1 & & \downarrow \cap \nu_1 & & H^d(N_2, \partial N_2) & \\ & & & & & \approx \downarrow \cap \nu_2 & \\ \longrightarrow & H^{LF}_{n+k-d}(N) & \longrightarrow & H^{LF}_{n+k-d}(N, N_2) & \xrightarrow{\partial} & H_{n+k-d-1}(N_2) & \longrightarrow \end{array}$$

It follows from the 5-lemma that $\cap\, \nu_1 : H^d(N, N_1) \to H_{n+k-d}(N, N_2)$ is an isomorphism.

Suppose now that (X, Y) satisfies open Poincaré duality for dimension n. Let $\mu \in H^{LF}_n(N, N_2)$ be an orientation. We have the exact sequence

$$\begin{array}{ccccccc} \xrightarrow{\delta} & H^d(N, N_1) & \xrightarrow{k^*} & H^d(N) & \xrightarrow{i^*} & H^d(N_1) & \xrightarrow{\delta} \\ & \approx \uparrow & & \uparrow & & & \\ & H^d(H^{n+k}, H^{n+k} - \operatorname{Int} N_1) & \longrightarrow & H^d(H^{n+k}) = 0. & & & \end{array}$$

Hence $k^* = 0$ and we have short exact sequences†

$$\begin{array}{ccccccc} 0 \longrightarrow & H^d(N) & \xrightarrow{i^*} & H^d(N_1) & \xrightarrow{\delta} & H^{d+1}(N, N_1) & \longrightarrow 0 \\ & & & & & \Big\downarrow \cap \nu_1 \ \wr\wr & \\ & & & & & H^{LF}_{n+k-d-1}(N, N_2) & \\ & & & & & \Big\uparrow \cap \mu \ \wr\wr & \\ & & & & & H^{d-(k-1)}(N). & \end{array}$$

If $0 \le d \le n$, then $H^{d-(k-1)}(N) = 0$ and $H^d(N) \to H^d(N_1)$ is an isomorphism. If $n < d \le n + k - 1$, then $H^d(N) = 0$ and we have an isomorphism

$$\varphi : H^d(N_1) \xrightarrow{\delta} H^{d+1}(N, N_1) \xrightarrow{\cap \nu_1} H^{LF}_{n+k-d-1}(N, N_2) \xleftarrow{\cap \mu} H^{d-(k-1)}(N) \xrightarrow{i^*} H^{d-(k-1)}(N_1).$$

We will show that $\pm \varphi^{-1} = \cup g$, for a generator $g \in H^{k-1}(N_1)$. We can take $g = \varphi^{-1}(1)$, where $1 \in H^0(N_1)$; in other words $g = \delta^{-1}(\cap \nu_1)^{-1}(\cap \mu)(i^*)^{-1}(1)$, or

$$\delta g \cap \nu_1 = (i^*)^{-1}(1) \cap \mu = \mu.$$

Thus for $\alpha \in H^d(N_1)$ we have

$$\alpha \cap \mu = \alpha \cap (\delta g \cap \nu_1) = (\alpha \cup \delta g) \cap \nu_1 = (-1)^d \delta(i^*\alpha \cup g) \cap \nu_1$$

so $(-1)^d \varphi^{-1}(\alpha) = \alpha \cup g$. This completes the proof that (1), (2) and (3) hold.

Suppose conversely that (1), (2) and (3) hold. We have the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^{n+k-d-1}(N) & \xrightarrow{i^*} & H^{n+k-d-1}(N_1) & \xrightarrow{\delta} & H^{n+k-d}(N, N_1) & \longrightarrow 0 \\ & & & \Big\uparrow \cup g & & \Big\downarrow \cap \nu_1 \ \wr\wr & \\ & & & H^{n-d}(N_1) & & H^{LF}_d(N, N_2) & \\ & & & \Big\uparrow i^* & & & \\ & & & H^{n-d}(N) & & & \end{array}$$

If $d > n$ then $H^{n-d}(N) = 0$; moreover $i^* : H^{n+k-d}(N) \to H^{n+k-d-1}(N_1)$ is an isomorphism, hence $H^{LF}_d(N, N_2) = 0$.

If $0 \le d \le n$, then $H^{n+k-d-1}(N) = 0$ and we obtain an isomorphism $\theta = (\cap \nu_1) \delta(\cup g)i^* : H^{n-d}(N) \to H^{LF}_d(N, N_2)$. Therefore it suffices to show that $\theta = \pm \cap \mu$, for $\mu \in H^{LF}_n(N, N_2)$ a generator. We can let $\mu = \varphi(1)$ for $1 \in H^0(N)$; in other words

$$\delta g \cap \nu_1 = \mu.$$

Then for $\alpha \in H^{n-d}(N)$ we have

$$\begin{aligned} \theta(\alpha) = \delta(i^*\alpha \cup g) \cap \nu_1 &= (-1)^{n-d}(i^*\alpha \cup \delta g) \cap \nu_1 \\ &= (-1)^{n-d}\alpha \cap (\delta g \cap \nu_1) \\ &= (-1)^{n-d}\alpha \cap \mu. \end{aligned}$$

This completes the proof.

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† (Added in proof) Actually, the following argument is valid even if the long exact sequence is not split into short sequences; the regular neighborhood N may then be replaced by any thickening.

The proof of the following Lemma, which requires the construction of explicit fibre homotopies, is left to the reader.

4.3 LEMMA. *Let (X,Y) be a subcomplex of (H^n,R^{n-1}) and let $(N; N_1, N_2)$ be a regular neighborhood. Let $(M; M_1, M_2)$ be the regular neighborhood $(N\times[-1, 1]; N_1\times[-1, 1] \cup N\times\{-1, 1\}, N_2\times[-1, 1])$ of $(X\times\{0\}, Y\times\{0\})$ in $(H^n\times R, R^{n-1}\times R)$. If $\omega : \mathscr{P}(M_1, M, X)\to X$ and $\eta : \mathscr{P}(N_1, N, X)\to X$ are the endpoint maps, then $\omega\sim\eta \oplus 1$.*

4.4 PROPOSITION. *Let (X,Y) be a subcomplex of (H^{n+k}, R^{n+k-1}) of codimension ≥ 3. Suppose X is simply connected. Let $(N; N_1, N_2)$ be a regular neighborhood of (X,Y). The fibres of the endpoint map $\omega : \mathscr{P}(N_1, N, N)\to N$ (and hence the fibre of $\mathscr{P}(N_1, N, X) \to X$ have the homotopy type of S^{k-1} if and only if X satisfies open Poincaré duality for dimension n.*

Proof. Assume first that $k > cb(X) + 1$. Let $i : N_1 \to N$ be the inclusion. Let $\alpha : N_1 \to \mathscr{P}(N_1, N, N)$ be defined by $\alpha(x)$ = constant path x. Then $\omega\alpha = i$ and α is clearly a homotopy equivalence. Let $\mathscr{P} = \mathscr{P}(N_1, N, N)$. Suppose X satisfies open Poincaré duality for dimension n. It follows from 4.2 that

(1) $H^*(\mathscr{P}) \approx H^*(N) \oplus H^{*+k-1}(N)$

(2) $\omega^* : H^p(N)\to H^p(\mathscr{P})$ is an isomorphism $0 \leq p \leq n$

(3) $\cup g : H^p(\mathscr{P}) \to H^{p+(k-1)}(\mathscr{P})$ is an isomorphism where $g \in H^{k-1}(\mathscr{P})$ is a generator, $0 \leq p \leq n$.

Let $\{E_r^{p,q}\}$ be the Leray–Serre spectral sequence for ω. Since the isomorphism ω^* is the composition $H^p(N) \approx E_2^{p,0} \to E_\infty^{p,0} \to H^p(\mathscr{P})$ all maps $d_r : E_r^{p-r,r-1} \to E_r^{q,0}$ are 0 and $E_2^{q,0} \approx E_\infty^{p,0}$.

Consider $E_2^{0,1}$. The maps d_r all vanish on $E_2^{0,1}$; hence $E_2^{0,1}\approx E_\infty^{0,1}$. But $0 = H^1(\mathscr{P})/E_\infty^{1,0} \approx E_\infty^{0,1} \approx E_2^{0,1}$; hence $H^1(F) = 0$ and all $E_2^{p,1} = 0$. It follows that all maps d_r vanish on $E_2^{0,2}$ so that $E_2^{0,2} \approx E_\infty^{0,2}$. But $0 = H^2(\mathscr{P})/E_\infty^{2,0} \approx E_\infty^{0,2} \approx E_2^{0,2}$; hence $H^2(F) = 0$ and all $E_2^{p,2} = 0$. Continuing in this way we have $H^s(F) = 0$ for $0 < s < k - 1$ and $H^{k-1}(F) \approx Z$. For $r \leq n$ we have the commutative diagram

$$\begin{array}{ccc} E_2^{r,0} \otimes E_2^{0,k-1} & \xrightarrow{\ \cup\ } & E_2^{r,k-1} \\ \approx \downarrow \varphi_{r,0}\otimes\varphi_{0,k-1} & & \downarrow \varphi_{r,k-1} \\ H^r(\mathscr{P}) \otimes H^{k-1}(\mathscr{P}) & \xrightarrow{\ \cup\ } & H^{r+k-1}(\mathscr{P}) \end{array}$$

where the maps φ are inclusions as subgroups. Let $g = \varphi_{0,k-1}(g')$. Since $\cup g : H^r(\mathscr{P}) \to H^{r+(k-1)}(\mathscr{P})$ is an isomorphism, the map $\varphi_{r,k-1}$ is also. It now follows, as before, that $H^s(F) = 0$ for $s > k - 1$. Therefore a fibre F of ω has the cohomology of S^{k-1}. Now $\pi_1(F) \approx \pi_2(N, N_1)$. If $f : (e^2, S^1) \to (N, \partial N)$ is a simplicial map, then, since the co-dimension of X in N is ≥ 3, we can push f off X, and therefore retract f into N_1. Thus $\pi_1(F) = 0$ and F has the homotopy type of S^{k-1}. We can now use 4.3 to eliminate the assumption $k > cb(X) + 1$.

Suppose, conversely, that F has the homotopy type of S^{k-1}. Then it is easy to see that (1), (2) and (3) hold, and therefore conditions (1), (2) and (3) of 4.2 hold, so that X satisfies open Poincaré duality for dimension n.

4.5 *Remarks.* (1) Let $\theta : \mathscr{P}(N_1, N, N) \to N$ be the *initial point map* $\theta(p) = p(0)$. Then $\theta : \omega^{-1}(x) \to N_1$ corresponds to the inclusion of the fibre into the total space $\mathscr{P}(N_1, N, X)$; if (X, Y) satisfies open Poincaré duality for dimension n and $k > n$ then θ induces isomorphisms of H^{k-1}.

(2) It is clear that X need not be simply connected provided that $\pi_1(X, x)$ operates trivially on $H^*(\omega^{-1}(x))$.

4.6 PROPOSITION. *Let (X, Y) be a finite subcomplex of (H^{n+k}, R^{n+k-1}) of codimension ≥ 3. Let $(N; N_1, N_2)$ be a regular neighborhood of (X, Y). Then the fibres of the endpoint map ω: $\mathscr{P}(N_1, N, X) \to X$) have the homotopy of type S^{k-1} if and only if (X, Y) is a P-pair of formal dimension n.*

Proof. Because of 4.3 we can assume without loss of generality that $k > cb(X) + 1$ and $n + k \geq 2\cdot$(*topological dimension* X) $+ 1$. Let $\tilde{N}$ be the universal covering space of N and let $\rho : \tilde{N} \to N$ be the covering map. Let $V_1, \dots, V_p$ be the vertices of N. By [13], Lemma 1, there is an $\varepsilon > 0$ such that, for $W_1, \dots, W_p \in R^{n+k}$ satisfying $|V_i - W_i| < \varepsilon$, the simplicial map $f : N \to R^{n+k}$ with $f(V_i) = W_i$ is a homeomorphism. For each vertex V of $\tilde{N}$ let V' be a point of R^{n+k} such that $|V' - \rho(V)| < \varepsilon$ and such that the points V' are in general position. Then the simplicial map $f : \tilde{N} \to R^{n+k}$ with $f(V) = V'$ is a local homeomorphism which is a homeomorphism on $\tilde{X} = \rho^{-1}(X)$. Hence ([24], Lemma 4.1) if $\tilde{N}$ is shrunk sufficiently the map f is a homeomorphism, whose image $f(\tilde{N})$ may not be closed. Let $g : \tilde{N} \to R$ be a piecewise linear map which goes to ∞ at ∞. Define $h : \tilde{N} \to R^{n+k+1}$ by $h(x) = (f(x), g(x))$. Let $(M; M_1, M_2)$ be the regular neighborhood $(N \times [-1, 1]; N_1 \times [-1, 1] \cup N \times \{-1, 1\}, N_2 \times [-1, 1])$ of $(X \times \{0\}, Y \times \{0\})$ in R^{n+k+1}. Then $(\tilde{M}; \tilde{M}_1, \tilde{M}_2)$ is homeomorphic to $(\tilde{N} \times [-1, 1]; \tilde{N}_1 \times [-1, 1] \cup \tilde{N} \times \{-1, 1\}, \tilde{N}_2 \times [-1, 1])$, which is a regular neighborhood of $(h(\tilde{X}), h(\tilde{Y}))$. By 4.4 the fibres of $\mathscr{P}(\tilde{M}_1, \tilde{M}, \tilde{M}) \to \tilde{M}$ have the homotopy type of S^k if and only if $(\tilde{X}, \tilde{Y})$ satisfies open Poincaré duality in dimension n, that is, if and only if (X, Y) is a P-pair of formal dimension n. But the fibre of $\mathscr{P}(\tilde{M}_1, \tilde{M}, \tilde{M}) \to \tilde{M}$ at $x \in X$ is clearly homeomorphic to the fibre of $\mathscr{P}(M_1, M, M) \to M$ at $\rho(x)$. The proposition now follows using 4.3.

From now until the end of §5 we shall regard the particular triangulation of a P-pair (X, Y) as part of its structure. The fibre spaces given by 4.6, (when (X, Y) is embedded in (H^{n+k}, R^{n+k-1}) piecewise linearly with respect to this triangulation), are called *normal fibre spaces* of (X, Y). They are all stably fibre homotopy equivalent by 4.3, since all regular neighborhoods are homeomorphic for k sufficiently large.

4.7 PROPOSITION. *Let (X, Y) be a subcomplex of (H^{n+k}, R^{n+k-1}) of codimension ≥ 3, which is a P-pair of formal dimension n. Let $(N; N_1, N_2)$ be a regular neighborhood of (X, Y), and let $\omega : \mathscr{P}(N_1, N, N) \to N$ be the endpoint map. Then (X, Y) is orientable if and only if ω is orientable.*

Proof. Again we may assume that $k > cb(X) + 1$. If ω is orientable then (X, Y) satisfies (open) Poincaré duality by 4.5 (2). If ω is not orientable and $\mathscr{S}(\omega)$ is the orientation sheaf of ω then in the spectral sequence of ω the term $E_2^{0,k-1} \approx H^0(X; \mathscr{S}(\omega))$ is 0; it follows that $H^{k-1}(N_1) = 0$, and (X, Y) does not satisfy (open) Poincaré duality for dimension n by 4.2.

§5. REDUCIBLE SPHERICAL FIBRE SPACES

5.1 PROPOSITION. *Let (X, Y) be a subcomplex of (H^N, R^{N-1}). Let $(N; N_1, N_2)$ be a regular neighborhood of (X, Y) and let $\omega : \mathscr{P}(N_1, N, N) \to N$ be the endpoint map. Then $(N \cup C(N_1), \infty) \approx (T(\omega), \infty)$.*

Proof. Let $\mathscr{P} = \mathscr{P}(N_1, N, N)$. Then $T(\omega) = N \underset{\omega}{\cup} C\mathscr{P}$. Let $f : N \cup C(N_1) \to N \underset{\omega}{\cup} C\mathscr{P}$ be defined by

$$f(\infty, 0, *) = (\infty, 0, *)$$

$$f(\infty, t, x) = (\infty, t, \text{constant path } x), \text{ for } x \in N, \text{ and } 0 \le t \le 1$$

$$f(y) = y, \text{ for } y \in N.$$

Let $g : N \underset{\omega}{\cup} C\mathscr{P} \to N \cup C(N_1)$ be defined by

$$g(\infty, 0, *) = (\infty, 0, *)$$

$$g(\infty, t, p) = \begin{cases} (\infty, 2t, p(0)) & 0 < t \le 1/2 \\ p(2t-1) & 1/2 \le t \le 1 \end{cases}$$

$$g(y) = y, \text{ for } y \in N.$$

Then $gf \simeq 1$ by the homotopy H defined by

$$H((\infty, 0, *), u) = (\infty, 0, *)$$

$$H((\infty, t, x), u) = \begin{cases} (\infty, t + ut, x), \text{ for } x \in \partial N, 0 < t \le 1/2 \\ (\infty, t + u(1-t), x), \text{ for } x \in \partial N, 1/2 \le t \le 1 \end{cases}$$

$$H(y, u) = y, \text{ for } y \in N,$$

and $fg \simeq 1$ by the homotopy K defined by

$$K((\infty, 0, *), u) = (\infty, 0, *)$$

$$K((\infty, t, p), u) = \begin{cases} ((\infty, t + ut, p \mid [0, 1-u] & 0 < t \le 1/2 \\ (\infty, t + u/2, p \mid [0, 1-u]) & 1/2 \le t \le 1 - u/2 \\ p(2t-1) & 1 - u/2 \le t \le 1 \end{cases}$$

$$K(y, u) = y, \text{ for } y \in N.$$

5.2 COROLLARY. *If $\omega' = \omega | \omega^{-1}(X)$ then $(T(\omega'), \infty) \approx (N/N_1, *)$.*

Proof. $(N/N_1, *) \approx (N \cup C(N_1), \infty) \approx (T(\omega), \infty) \approx (T(\omega'), \infty)$, by 2.2.

5.3 COROLLARY. *Let X be a P-space which is a subcomplex of R^n of codimension ≥ 3. Let N be a regular neighborhood of X, and $\omega : \mathscr{P}(\partial N, N, X) \to X$ the endpoint map. Then ω is reducible.*

The remainder of this section is devoted to proving that all reducible spherical fibre spaces over a P-space X are stably fibre homotopy equivalent; in particular the normal fibre spaces of X for different triangulations are all stably fibre homotopy equivalent.

Suppose X is a P-space of formal dimension n and $\pi : E \to X$ is a spherical fibre space of fibre dimension d. If $\tilde{X}$ is the universal covering space of X with covering map $\rho : \tilde{X} \to X$, let $\tilde{\pi} : \tilde{E} \to \tilde{X}$ be $\rho^*(\pi)$, and $\tilde{r} : C_{\tilde{\pi}} \to \tilde{X}$ the retraction. Let $f : E \to A$ be a homotopy equivalence, where A is a locally finite complex with cells of bounded dimension (c.f. proof of Proposition (0) in [22]). Let $\omega : A \to X$ be a simplicial map such that $\omega f \simeq \pi$. Then X is a subcomplex of C_ω and the pair (C_ω, A) has the homotopy type of (C_π, E). On the other hand, the pair $(\tilde{C}_\omega, \tilde{A})$ has the homotopy type of $(C_{\tilde{\pi}}, \tilde{E})$. Since $\tilde{X}$ satisfies open Poincaré duality for dimension n, by 3.1 the pair $(C_{\tilde{\pi}}, \tilde{E})$ satisfies Poincaré duality for dimension $n + d$, with H_* replaced by $H_*^{\tilde{r}LF}$. Therefore $(\tilde{C}_\omega, \tilde{A})$ satisfies open Poincaré duality and consequently (C_ω, A) is a P-pair of formal dimension $n + d$.

Let (C_ω, A) be embedded as a subcomplex of $(H^{n+d+k}, R^{n+d+k+1})$, with regular neighborhood $(N; N_1, N_2)$. Let

$$\nu \text{ be the endpoint map } \mathscr{P}(\partial N, N, X) \to X$$

$$\mu \text{ be the endpoint map } \mathscr{P}(N_1, N, X) \to X$$

$$\pi' \text{ be the endpoint map } \mathscr{P}(A, C_\omega, X) \to X.$$

Clearly ν is a normal fibre space of X and μ is the restriction to X of a normal fibre space of (C_ω, A), while $\pi' \sim \pi$, by 1.1.

5.4 Proposition. *If $d, k > n + 1$ then $\nu \sim \mu \oplus \pi'$.*

Proof. It is clear from the proof of 4.6 that it suffices to prove the theorem for simply connected X.

For $x \in X$ a map $g_x : \nu_x \to \mu_x * \pi'_x$ will be constructed in three steps. Each map g_x will be a homotopy equivalence, and the union of all g_x will be a continuous map g, which is a fibre homotopy equivalence by [4], Theorem 6.3.

We will use the following abbreviations:

$$\mathscr{P}(\partial N) = \mathscr{P}(\partial N, N, x)$$

$$\mathscr{P}(N_1) = \mathscr{P}(N_1, N, x)$$

$$\mathscr{P}(N) = \mathscr{P}(N, N, x)$$

$$\mathscr{P}(A) = \mathscr{P}(A, C_\omega, x)$$

$$\mathscr{P}(C_\omega) = \mathscr{P}(C_\omega, C_\omega, x).$$

Step (1). A homeomorphism $\varphi_x : \mathscr{P}(N_1) * \mathscr{P}(A) \to \mathscr{P}(N_1) \times C(\mathscr{P}(A)) \cup C(\mathscr{P}(N_1)) \times \mathscr{P}(A)$ has been defined in §2.

Step (2). Let $U \subset N$ be a neighborhood of N_1 such that

(1) $U \cap X = \varnothing$

(2) $U \cap E^{n+d+k-1}$ is a neighborhood of ∂N_2 in N_2

(3) U is homeomorphic to $N_1 \times [0, 1]$.

Let $\psi : N_1 \times [0, 1] \to U$ be a homeomorphism with $\psi(y, 0) = y$, for $y \in N_1$. Define $i_x : \mathscr{P}(N) \to C(\mathscr{P}(N_1))$ as follows.

If $p(0) \notin U$, then $i_x(p) = (\infty, 0, x)$; if $p(0) = \psi(y, t)$, for $y \in N_1$ and $t \in (0, 1)$, then $i_x(p) = (\infty, t, p')$, where

$$p'(u) = \begin{cases} \psi(y, u) & 0 \le u \le t \\ p((u-t)/(1-t)) & t \le u \le 1; \end{cases}$$

if $p(0) \in N_1$, then $i_x(p) = (*, 1, p)$.

Then i_x is a homotopy equivalence with homotopy inverse $i_x' : C(\mathscr{P}(N_1)) \to \mathscr{P}(N)$ defined by

$$i_x'(\infty, 0, *) = \text{constant path x}$$

$$i_x'(\infty, t, p) = p|[1-t, 1] \quad 0 < t \le 1.$$

Note that $\mathscr{P}(N_1)$ is always left fixed.

Define $j_x : \mathscr{P}(C_\omega) \to C(\mathscr{P}(A))$ as follows.

If $p(0) \in X$, then $j_x(p) = (\infty, 0, *)$; if $p(0) = (a, t)$, for $a \in A$ and $t \in [0, 1)$, then $j_x(p) = (\infty, t, p')$, where

$$p'(u) = \begin{cases} (a, u) & 0 \le u \le t \\ p((u-t)/(1-t)) & t \le u \le 1. \end{cases}$$

Then j_x is a homotopy equivalence with homotopy inverse $j_x' : C(\mathscr{P}(A)) \to \mathscr{P}(C_\omega)$ defined by

$$j_x'(\infty, 0, *) = \text{constant path } x$$

$$j_x'(\infty, t, p) = p\,|\,[1-t, 1] \qquad p < t \le 1.$$

Note that $\mathscr{P}(A)$ is always left fixed.

Using i_x and j_x we obtain a homotopy equivalence

$$\begin{array}{c} \mathscr{P}(N_1) \rtimes\!\!\!\ltimes \mathscr{P}(C_\omega) \cup \mathscr{P}(N) \rtimes\!\!\!\ltimes \mathscr{P}(A) \\ \Big\downarrow \alpha_x \\ \mathscr{P}(N_1) \rtimes\!\!\!\ltimes C(\mathscr{P}(A)) \cup C(\mathscr{P}(N_1)) \rtimes\!\!\!\ltimes \mathscr{P}(A). \end{array}$$

Step (3). Define $f_x : \mathscr{P}(\partial N) \to \mathscr{P}(N_1) \rtimes\!\!\!\ltimes \mathscr{P}(C_\omega) \cup \mathscr{P}(N) \rtimes\!\!\!\ltimes \mathscr{P}(A)$ by $f_x(p) = (p, r \circ p)$, where $r : (N, N_2) \to (C_\omega, A)$ is the retraction.

Now define $g_x : \mathscr{P}(\partial N) \to \mathscr{P}(N_1) * \mathscr{P}(A)$ by $g_x = \varphi_x^{-1} \alpha_x f_x$.

To complete the proof it is sufficient to show that f_x is a homotopy equivalence. Since $\mathscr{P}(N)$ and $\mathscr{P}(C_\omega)$ are contractible it suffices to show that

$$f_x : (\mathscr{P}(N), \mathscr{P}(\partial N)) \longrightarrow (\mathscr{P}(N), \mathscr{P}(N_1)) \rtimes\!\!\!\ltimes (\mathscr{P}(C_\omega), \mathscr{P}(A))$$

induces isomorphisms of H^{k+d}. We have the commutative diagram

$$\begin{array}{ccccc} (\mathscr{P}(N), \mathscr{P}(\partial N)) & \xrightarrow{f_x} & & & (\mathscr{P}(N), \mathscr{P}(N_1)) \rtimes\!\!\!\ltimes (\mathscr{P}(C_\omega), \mathscr{P}(A)) \\ \Big\downarrow \theta_1 & & & & \Big\downarrow \theta_2 = \theta_2' \times \theta_2'' \\ (N, \partial N) & \xrightarrow{\Delta} & (N, N_1) \rtimes\!\!\!\ltimes (N, N_2) & \xrightarrow{1 \times r} & (N, N_1) \rtimes\!\!\!\ltimes (C_\omega, A) \end{array}$$

where θ_1, θ_1' and θ_2'' are initial point maps and Δ is the diagonal.

To see that θ_1 induces an isomorphism of H^{d+k}, consider the diagram

$$\begin{array}{ccccccc} 0 = H^{k+d-1}(\mathscr{P}(N)) & \longrightarrow & H^{k+d-1}(\mathscr{P}(\partial N)) & \longrightarrow & H^{k+d}(\mathscr{P}(N), \mathscr{P}(\partial N)) & \longrightarrow & H^{k+d}(\mathscr{P}(N)) = 0 \\ & & \uparrow{\scriptstyle \theta^*} & & \uparrow{\scriptstyle \theta_1^*} & & \\ 0 = H^{k+d-1}(N) & \longrightarrow & H^{k+d-1}(\partial N) & \longrightarrow & H^{k+d}(N, \partial N) & \longrightarrow & H^{k+d}(N) = 0. \end{array}$$

Since (4.6) the map θ^* is an isomorphism, $\theta_1{}^*$ is also. Similarly $(\theta_2')^*$ is an isomorphism of H^k (using $k > n + 1$) and $(\theta_2'')^*$ is an isomorphism of H^d (using $d > n + 1$). Hence $\theta_2{}^*$ is an isomorphism of H^{k+d}.

Let α and β be generators of $H^k(N, N_1)$ and $H^d(C_\omega, A)$, respectively. Let $\mu \in H_{n+d}(N, N_2)$ and $\nu_1 \in H_{n+k+d}(N, \partial N)$ be generators, and let $\mu_1 = r_*\mu$, where $r : (N, N_2) \to (C_\omega, A)$ is the deformation retraction. Then (c.f. proof of 4.2) we have $\alpha \cap \nu_1 = \mu$.

Now β corresponds to $U(\pi)$; hence $\beta \cap \mu_1$ is a generator of $H_n(C_\omega)$ by 1.6. Therefore

$$(r^*\beta \cup \alpha) \cap \nu_1 = r^*\beta \cap (\alpha \cap \nu_1) = r^*\beta \cap \mu$$

is a generator of $H_n(N)$. This proves (c.f. proof of 4.2) that $r^*\beta \cup \alpha$ is a generator of $H^{k+d}(N, \partial N)$.

If γ is a generator of $H^{k+d}((\mathscr{P}(N), \mathscr{P}(N_1)) \times (\mathscr{P}(C_\omega), \mathscr{P}(A)))$, then

$$\begin{aligned} (f_x)^*(\gamma) &= (f_x)^*(\theta_2{}^*(\alpha \times \beta)) \\ &= \theta_1{}^*\Delta^*(1 \times r^*)(\alpha \times \beta) \\ &= \theta_1{}^*\Delta^*(\alpha \times r^*\beta) \\ &= \theta_1{}^*(\alpha \cup r^*\beta), \end{aligned}$$

so $(f_x)^*(\gamma)$ is a generator of $H^{k+d}(\mathscr{P}(N), \mathscr{P}(\partial N))$. This completes the proof of 5.4.

5.5 Proposition. *Let (X, Y) be a P-pair and ν a normal fibre space. Then $T(\nu)$ is the S-dual of X/Y.*

Proof. Let (X, Y) be embedded as a subcomplex of (H^{n+k}, R^{n+k-1}) and let $(N; N_1, N_2)$ be a regular neighborhood. We can assume that we actually have $N \subset (I^{n+k}, I^{n+k-1})$. If $\infty \notin R^{n+k}$, then $\{\infty\} * I^{n+k}$ is S^{n+k}. Now $X \cup (S^{n+k} - \text{Interior } N)$ is an S-dual for N_1 and $S^{n+k} - \text{Interior } N$ is an S-dual for N. Hence $X \cup (S^{n+k} - \text{Interior } N)/(S^{n+k} - \text{Interior } N)$ is an S-dual of N/N_1. But $X \cup (S^{n+k} - \text{Interior } N)/(S^{n+k} - \text{Interior } N) \approx X/Y$ and $N/N_1 \approx T(\nu_X)$ by 5.2.

5.6 Proposition. *Let X be a P-space with normal fibre space ν and let π be an S-reducible spherical fibre space over X. Then $\pi \underset{s}{\sim} \nu$.*

Proof. We can assume that fibre dimension $\pi > 1 +$ formal dimension X. We have, using the notation introduced before 5.4,

$$\nu \sim i^*\mu \oplus \pi$$

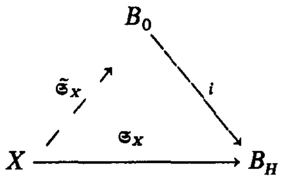
where $i: X \to C_\omega$ is the inclusion. If $r: C_\omega \to X$ is the retraction then $r^*\nu \sim \mu \oplus r^*\pi$ or $\mu \sim r^*\nu \oplus (r^*\pi)^{-1}$.

Since $T(\pi)$ is S-reducible, its S-dual is S-coreducible. Since $T(\pi) \approx C_\omega/A$, by 5.5 the S-dual of $T(\pi)$ is $T(\mu)$ and by 2.1 we have $T(\mu) \approx T(r^*\nu \oplus (r^*\pi)^{-1})$. Thus $T(r^*\nu \oplus (r^*\pi)^{-1})$ is S-coreducible and it follows from 2.5 that $r^*\nu \oplus (r^*\pi)^{-1} \underset{s}{\sim} 1$, or $r^*\nu \underset{s}{\sim} r^*\pi$. Since r is a homotopy equivalence, $\nu \underset{s}{\sim} \pi$.

§6. OBSTRUCTION THEORY

Let X be a P-space and let ν be a normal fibre space of X, of fibre dimension k. Let $f: X \to B_{H(k)}$ be a map such that $f^*(\pi_{H(k)}) \sim \nu$. The composite maps $X \xrightarrow{f} B_{H(k)} \to B_H$ are all in one homotopy class; we denote (maps in) this homotopy class by $\mathfrak{S}_X$. The class $\mathfrak{S}_X$ is called the *normal* or *Gauss map of* X. The condition that there is a reducible vector bundle over X can be expressed very succinctly.

6.1 PROPOSITION. *There is a reducible vector bundle over a P-space X if and only if there is a map $\tilde{\mathfrak{S}}_X: X \to B_0$ such that the following diagram commutes up to homotopy.*



Proof. Suppose $i\tilde{\mathfrak{S}}_X \simeq \mathfrak{S}_X$. For some k we have $\mathfrak{S}_X: X \to B_{H(k)}$ and $\tilde{\mathfrak{S}}_X: X \to B_{0(k)}$. Then $\mathfrak{S}_X^*(\pi_{H(k)}) \sim \tilde{\mathfrak{S}}_X^* i^*(\pi_{H(k)}) \sim \tilde{\mathfrak{S}}_X^*[\pi_{0(k)}] \sim [\tilde{\mathfrak{S}}_X^*\pi_{0(k)}]$. Since $\mathfrak{S}_X^*(\pi_{H(k)})$ is reducible, so is $\tilde{\mathfrak{S}}_X^*(\pi_{0(k)})$. Conversely, if for some $f: X \to B_{0(k)}$ the bundle $f^*(\pi_{0(k)})$ is reducible, then $(if)^*(\pi_{H(k)}) \sim f^*(i^*\pi_{H(k)}) \sim f^*[\pi_{0(k)}]$ and $(if)^*(\pi_{H(k)})$ is reducible. Hence $if \sim \mathfrak{S}_X$.

It follows from 6.1 that if X is a P-space there is an obstruction theory for the existence of a reducible vector bundle over X, namely the obstruction theory for the existence of $\tilde{\mathfrak{S}}_X$. More generally, given a spherical fibre space π over any complex X, suppose $\pi \sim f_\pi^*(\pi_{H(k)})$ for $f_\pi: X \to B_{H(k)}$. Let f_π also denote the composition $X \xrightarrow{f_\pi} B_{H(k)} \to B_H$. Then there is an obstruction theory for the existence of $\tilde{f}_\pi: X \to B_0$ with $i\tilde{f}_\pi \simeq f_\pi$. If $\bar{\imath}: B_0' \to B_H$ is the fibring associated with $i: B_0 \to B_H$, then the existence of $\tilde{f}_\pi$ is equivalent to the existence of a map $\bar{f}_\pi: X \to B_0'$ such that $\bar{\imath}\bar{f}_\pi = f_\pi$, hence to the existence of a cross-section of the induced fibre space $f_\pi^*(\bar{\imath})$. The p^{th} obstruction $\mathcal{O}^p(\pi)$ to finding a cross-section depends only on the $\underset{s}{\sim}$ equivalence class of π and is an element of $H^p(X; \pi_{p-1}(F))$, where F is a fibre of $\bar{\imath}$. As usual, the higher dimensional obstructions are not in general well-defined. If X is a P-space we define $\mathcal{O}^p(X)$ as $\mathcal{O}^p(\nu)$ where ν is a normal fibre space.

The groups $\pi_n(F)$ can be identified as follows. We have $B_0' \approx B_0$ and, for $n \leq k-2$, the following diagram commutes, where J_{n-1} is the J-homomorphism (c.f. [2], pp. 293–295).

$$\begin{array}{ccc} \pi_n(B_{O(k)}) & \xrightarrow{(i_k)_*} & \pi_n(B_{H(k)}) \\ \downarrow \wr\wr & & \wr\wr \downarrow \\ \pi_{n-1}(O(k)) & \xrightarrow{J_{n-1}} & \Pi_{n-1} \end{array}$$

Hence the exact homotopy sequence of $\bar{i}$ becomes

$$\pi_n(0) \xrightarrow{J_n} \Pi_n \longrightarrow \pi_n(F) \longrightarrow \pi_{n-1}(0) \xrightarrow{J_{n-1}} \Pi_{n-1}.$$

Since ([1]) J_{n-1} is a monomorphism for $n - 1 \equiv 0$ or 1 (mod 8), we have

$$\begin{array}{ll} \ker J_{n-1} \approx Z & n - 1 \equiv 3, 7 \pmod 8 \\ \ker J_{n-1} = 0 & \text{otherwise.} \end{array}$$

Hence the sequence always splits and

$$\begin{array}{ll} \pi_n(F) \approx \Pi_n/\text{image } J_n \oplus Z & n \equiv 0, 4 \pmod 8 \\ \pi_n(F) \approx \Pi_n/\text{image } J_n & \text{otherwise.} \end{array}$$

The first few groups are given below.

n	0	1	2	3	4	5	6	7
$\pi_n(F)$	0	0	Z_2	0	Z	0	Z_2	0

Examples. (1) Every P-space of dimension 4 has a reducible vector bundle over it.

(2) There are ([6], p. 44), 3-connected compact PL-8-manifolds M^8, which do not have the homotopy type of a compact C^∞ manifold. But there is a reducible vector bundle over M^8.

If X is an $(n - 1)$-connected P-space the primary obstruction $\mathcal{O}^n(X)$ is well-defined. In the next two sections $\mathcal{O}^n(X)$ will be characterized in terms of a cohomology operation on X, as indicated in Theorem B of the Introduction. The content of this Theorem is contained in 8.3 and 8.4.

§7. THE COHOMOLOGY OPERATION ψ

Let $2 \le k \le n - 2$. In [11], §8, a cohomology operation $\psi : H^k(K, L; Z) \longrightarrow H^n(K, L; \pi_{n-1}(S^k))$ is defined when (K, L) has the homotopy type of a CW-pair and satisfies the condition: $H^p(K, L; G) = 0$ for $k < p < n$ and for all coefficient groups G.

If $n < 2k$ then ψ is a homomorphism and $\pi_{n-1}(S^k) \approx \Pi_{n-k-1}$. If $n = 2k$, then ([11], Lemma 8.2)

$$\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + [i^k, i^k](\alpha \cup \beta)$$

where $[i^k, i^k]$ stands for the coefficient homomorphism $Z \longrightarrow \pi_{2k-1}(S^k)$ which carries 1 into the Whitehead product $[i^k, i^k]$, where $i^k \in \pi_k(S^k)$ is the generator. Since the kernel of the suspension homomorphism $\Sigma : \pi_{2k-1}(S^k) \longrightarrow \pi_{2k}(S^{k+1}) \approx \Pi_{k-1}$ is generated by $[i^k, i^k]$, the composition of ψ and the coefficient homomorphism Σ is also a homomorphism, which will also be denoted ψ. These are the only two cases when ψ will be considered.

7.1 Proposition. *If*

$$\psi : H^m(X_1, Y_1; Z) \to H^{m+q}(X_1, Y_1; \Pi_{q-1})$$
$$\psi : H^n(X_2, Y_2; Z) \to H^{n+q}(X_2, Y_2; \Pi_{q-1})$$
$$\psi : H^{m+n}((X_1, Y_1) \times (X_2, Y_2); Z) \to$$
$$\to H^{m+n+q}((X_1, Y_1) \times (X_2, Y_2); \Pi_{q-1})$$

are all defined and $\alpha \in H^m(X_1, Y_1; Z)$, $\beta \in H^n(X_2, Y_2; Z)$, *then*

$$\psi(\alpha \times \beta) = \psi(\alpha) \times \beta + (-1)^{mq} \alpha \times \psi(\beta).$$

Proof. (Unpublished proof of Milnor). Given spaces A and B with base point define $A \times B$ as $A \times B / A \vee B$. The suspension of a map $f: A \to B$ is the map $f \times i^1 : A \times S^1 \to B \times S^1$. The k-fold suspension is $f \times i^k$. If $f: S^{n+q-1} \to S^n$ is a map then $i^k \times f: S^{n+q-1} \to S^{n+k}$ is also defined and $i^k \times f \simeq (-1)^{k(q-1)} f \times i^k$. Let D^n be the standard n-cell with orientation $\mu_n \in H_n(D^n, S^{n-1})$. Let

$$f : S^{m+q-1} \to S^m \text{ and } g : (D^n, S^{n-1}) \to (S^n, *)$$

be given maps. Then a map

$$h : \partial(D^{m+q} \times D^n) \to S^m \times S^n$$

is defined by

$$h(x, y) = f(x) \times g(y) \quad \text{for} \quad x \in S^{m+q-1}$$
$$h(x, y) = * \quad \text{for} \quad y \in S^{n-1}.$$

Let $\partial(D^{m+q} \times D^n)$ be oriented by $\partial(\mu_{m+q} \times \mu_n)$. Then h corresponds to an element of $\pi_{m+n+q-1}(S^{m+n})$.

Assertion 1. If g has degree d then h corresponds to d times the n-fold suspension of f. For h can be factored as

$$\partial(D^{m+q} \times D^n) \xrightarrow{g'} S^{m+q-1} \times S^n \xrightarrow{f \times i^n} S^m \times S^n$$

where

$$g'(x, y) = x \times g(y) \quad \text{for} \quad x \in S^{m+q-1}$$
$$g'(x, y) = * \quad \text{for} \quad x \in S^{n-1},$$

and g' has degree d.

Similarly given $f: (D^m, S^{m-1}) \to (S^m, *)$ with degree d and $g : S^{n+q-1} \to S^n$ a map

$$h : \partial(D^m \times D^{n+q}) \to S^m \times S^n$$

is defined by

$$h(x, y) = * \quad \text{for} \quad x \in S^{m-1}$$
$$h(x, y) = f(x) \times g(y) \quad \text{for} \quad y \in S^{n+q-1}.$$

Assertion 2. h corresponds to $(-1)^{mq} d$ times the m-fold suspension of g.

The proof is similar to the above except that a sign $(-1)^m$ is introduced from the formula for $\partial(\mu_m \times \mu_{n+q})$, and a sign $(-1)^{m(q-1)}$ is introduced to relate $i^m \times g$ to the m-fold suspension of g.

Now let α and β be represented by cocycles $a \in C^m(X_1, Y_1)$ and $b \in C^n(X_2, Y_2)$, respectively. Let

$$f : (X_1^{m+q-1} \cup Y_1, X_1^{m-1} \cup Y_1) \to (S^m, *)$$
$$g : (X_2^{n+q-1} \cup Y_2, X_2^{n-1} \cup Y_2) \to (S^n, *)$$

be maps representing a and b, respectively. Then the obstruction $o(f) \in C^{m+q}(X_1, Y_1; \pi_{m+q-1}(S^m))$ to extending f represents $\psi(\alpha)$ and $o(g)$ represents $\psi(b)$. Let $(M, N) = (X_1, Y_1) \times (X_2, Y_2)$. A map

$$h : (M^{m+n+q-1} \cup N, M^{m+n-1} \cup N) \to (S^m \times S^n, *)$$

corresponding to $a \times b$ is defined by

$$h(x, y) = f(x) \times g(y) \quad \text{if} \quad x \in X_1^{m+q-1}, y \in X_2^{n+q-1}$$
$$h(x, y) = * \quad \text{if} \quad x \in X_1^{m-1} \cup Y_1, y \in X_2^{n-1} \cup Y_2.$$

This same formula can be used to extend h throughout the interior of a cell $e_1{}^c \times e_2^{m+n+q-c}$, unless $c = m+q$ or m. Using the two assertions, the desired formula is obtained.

7.2 COROLLARY. *If, in the hypothesis of* 7.1, *we have* $X_1 = X_2 = X$, *then* $\psi(\alpha \cup \beta) = \psi(\alpha) \cup \beta + (-1)^{mq}\, \alpha \cup \psi(\beta)$.

Proof. Let $Z_1 = Y_1 \cup X_1^{m-1}$ and $Z_2 = Y_2 \cup X_2^{n-1}$. It follows from the Kunneth theorem that $\psi : H^{m+n}((X, Z_1) \times (X, Z_2); Z) \to H^{m+n+q}((X, \psi_1) \times (X, Z_2); \Pi_{q-1})$ is defined. Since the natural homomorphism $H^m(X, Z_1) \to H^m(X, Y_1)$ is onto, there exists an element $\alpha' \in H^m(X, Z_1)$ representing α. Similarly choose $\beta' \in H^n(X, Z_2)$. Let $\Delta : (X, Y_1 \cup Y_2) \to (X, Z_1) \times (X, Z_2)$ be the diagonal map. The required identity is obtained by applying Δ^* to

$$\psi(\alpha' \times \beta') = \psi(\alpha') \times \beta' + (-1)^{mq} \alpha' \times \psi(\beta').$$

Let X be an $(n-1)$-connected space $(n \geq 2)$ and let $\pi : E \to X$ be a spherical fibre space of fibre dimension $k \geq n$. Then (C_π, E) has the homotopy type of a CW-pair (c.f. discussion preceeding 4.5) and $H^p(C_\pi, E; G) = 0$ for $k < p < n + k$ and for all coefficient groups G. Hence $\psi(U(\pi)) \in H^{n+k}(C_\pi, E; \Pi_{n-1})$ is defined. Let $\psi(U(\pi)) = \varphi(\psi^n(\pi)) = r^*\psi^n(\pi) \cup U(\pi)$ for $\psi^n(\pi) \in H^n(X; \Pi_{n-1})$, where $r : C_\pi \to X$ is the retraction. It is clear that $\psi^n(\pi)$ does not depend on the orientation of π and that $\psi^n(1) = 0$.

7.3 PROPOSITION. $\psi^n(\pi_1 \oplus \pi_2) = \psi^n(\pi_1) + \psi^n(\pi_2)$.

Proof. Let $\pi_1 \oplus \pi_2 = \pi : E \to X$. Let $r_i : C_{\pi_i} \to X$ and $r : C_\pi \to X$ be the retractions. Let

$$C_\pi{}^1 = \bigcup_{x \in X} (E_1)_x \times (C_{\pi_2})_x$$
$$C_\pi{}^2 = \bigcup_{x \in X} (C_{\pi_1})_x \times (E_2)_x.$$

Then (c.f. discussion of the join in §1) we can regard $C_\pi^1 \cup C_\pi^2$ as E and $\bigcup_{x \in X} (C_{\pi_1})_x \times (C_{\pi_2})_x$ as C_π. If $p_i : C_\pi \to C_{\pi_i} (i = 1, 2)$ is projection on the i^{th} factor we have the maps

$$\begin{array}{ccccc} C_\pi^i & \subset & C_\pi & \subset & (C_\pi, C_\pi^i) \\ \downarrow n_i & & \downarrow p_i & & \downarrow q_i \\ E_i & \subset & C_{\pi_i} & \subset & (C_{\pi_i}, E_i) \end{array}$$

where n_i is the restriction of p_i, which is the restriction of q_i.

As in [15], Theorem 11, it follows that $q_1{}^*U(\pi_1) \cup q_2{}^*U(\pi_2) = U(\pi)$. Let $U_i = U(\pi_i)$ and $U = U(\pi)$. Then by (7.2)

$$\begin{aligned} \psi(U) &= \psi(q_1{}^*U_1) \cup q_2{}^*U_2 + (-1)^{nk_1} q_1{}^* U_1 \cup \psi(q_2{}^*U_2) \\ &= q_1{}^*\psi(U_1) \cup q_2{}^*U_2 + (-1)^{nk_1} q_1{}^*U_1 \cup q_2{}^*\psi(U_2) \\ &= q_1{}^*(r_1{}^*\psi^n(\pi_1) \cup U_1) \cup q_2{}^*U_2 \\ &\qquad + (-1)^{nk_1} q_1{}^*U_1 \cup q_2{}^*(r_2{}^*\psi^n(\pi_2) \cup U_2) \\ &= n_1{}^*r_1{}^*\psi^n(\pi_1) \cup q_1{}^*U_1 \cup q_2{}^*U_2 \\ &\qquad + n_2{}^*r_2{}^*\psi^n(\pi_2) \cup q_1{}^*U_1 \cup q_2{}^*U_2 \\ &= [r^*\psi^n(\pi_1) + r^*\psi^n(\pi_2)] \cup U. \end{aligned}$$

Since $\psi(U) = r^*\psi^n(\pi) \cup U$ the proof is complete.

7.4 Corollary. *$\psi^n(\pi)$ depends only on the $\underset{s}{\sim}$ equivalence class of π.*

If X is an $(n-1)$-connected P-space, then $\psi^n(X)$ is defined as $\psi^n(\nu)$ for any normal fibre space ν of X. Let N be the formal dimension of X. If $N - n < p < N$ then we have $H^p(X; G) \approx H_{N-p}(X; G) = 0$ so the homomorphism $\psi : H^{N-n}(X; Z) \to H^N(X; \Pi_{n-1})$ is defined.

7.5 Proposition. *$\cup\, \psi^n(X) : H^{N-n}(X; Z) \to H^N(X; \Pi_{n-1})$ is $(-1)^{n(N+1)+1}$ times the map $\psi : H^{N-n}(X; Z) \to H^N(X; \Pi_{n-1})$.*

(*Remark.* It is clear that $\psi^n(X)$ is the unique class with this property. Hence $\psi^n(X)$ can be computed, knowing only the topology of X).

Proof. Let $\pi : E \to X$ be a normal fibre space of fibre dimension k, and let $r : C_\pi \to X$ be the retraction. Since π is S-reducible we may assume, without loss of generality, that π is reducible. Let $f : (S^{N+k}, a) \to (T(\pi), \infty)$ be a map inducing isomorphisms of $\tilde{H}_q$ for $q \geq N$. Considering $T(\pi)$ as $C_\pi \cup CE$ we have the following commutative diagram.

$$\begin{array}{ccc} H^{N+k-n}(C_\pi, E; Z) & \xrightarrow{\psi} & H^{N+k}(C_\pi, E; \Pi_{n-1}) \\ \uparrow \wr\wr & & \uparrow \wr\wr \\ H^{N+k-n}(T(\pi), \infty; Z) & \xrightarrow{\psi} & H^{N+k}(T(\pi), \infty; \Pi_{n-1}) \\ \downarrow f^* & & \wr\wr \downarrow f^* \\ 0 = H^{N+k-n}(S^{n+k}\ a; Z) & \xrightarrow{\psi} & H^{N+k}(S^{N+k}, a; \Pi_{n-1}) \end{array}$$

Therefore $\psi : H^{N+k-n}(C_\pi, E; Z) \to H^{N+k}(C_\pi, E; \Pi_{n-1})$ is 0. The maps $\psi : H^{N-n}(X; Z) \to H^N(X; \Pi_{n-1})$ and $\psi : H^k(C_\pi, E; Z) \to H^{k+n}(C_\pi, E; \Pi_{n-1})$ are defined. Let $\alpha \in H^{N-n}(X; Z)$. Then by (7.2)

$$0 = \psi(r^*\alpha \cup U(\pi)) = \psi(r^*\alpha) \cup U(\pi) + (-1)^{n(N+1)} r^*\alpha \cup \psi(U(\pi)).$$

Hence $\psi(r^*\alpha) \cup U(\pi) = (-1)^{n(N+1)+1}\, r^*\alpha \cup r^*\psi^n(\pi) \cup U(\pi)$, and therefore

$$\psi(\alpha) = (-1)^{n(N+1)+1}\, \alpha \cup \psi^n(\pi).$$

Let X_α be the P-space defined in §3, where $\alpha \in \pi_{2n-1}(S^n)$, and let $c_1, c_2 \in H^n(X_\alpha)$ be as defined in §3.

7.6 COROLLARY. *$\psi^n(X_\alpha) \in H^n(X_\alpha; \Pi_{n-1}) \approx H^n(X_\alpha; Z) \otimes \Pi_{n-1}$ is $(-1)^{n+1} c_2 \otimes \Sigma\,\alpha$.*

Proof. Consider $\psi : H^n(X_\alpha; Z) \to H^{2n}(X_\alpha; \Pi_{n-1}) \approx \Pi_{n-1}$. A cocycle representing c_i is 1 on S_i^n and 0 on S_{3-i}^n, so a map $f : S_1^n \vee S_2^n \to S^n$ representing c_i is the identity on S_i^n and trivial on S_{3-i}^n. Therefore $\psi(c_1) = \Sigma\,\alpha$ and $\psi(c_2) = 0$, and the result follows from 7.5, and the matrix for the cup-product pairing for X_α, given in §3.

§8. THE PRIMARY OBSTRUCTION

Let $B_{SH(k)}[0, n-1]$ be the space obtained from $B_{SH(k)}$ by attaching cells of dimension $\geq n+1$ to kill off all homotopy groups of dimension $\geq n$. The inclusion $j : B_{SH(k)} \to B_{SH(k)}[0, n-1]$ induces isomorphisms of homotopy groups in dimension $\leq n-1$. Let $\pi : B'_{SH(k)} \to B_{SH(k)}[0, n-1]$ be the fibring associated to j; the fibre will be denoted $B_{SH(k)}[\text{n}, \infty)$. Then the inclusion $B_{SH(k)}[n, \infty) \subset B'_{SH(k)} \approx B_{SH(k)}$ induces isomorphisms of homotopy groups in dimensions $\geq n$, and $B_{SH(k)}[n, \infty)$ is the base space of a spherical fibre space $\pi_{SH(k),n}$ of fibre dimension k, which is universal for spherical fibre spaces of fibre dimension k over $(n-1)$-connected CW-complexes. The stable $B_{SH}[n, \infty)$ is defined in the obvious way.

For $k \geq n+2$ the obstruction $\mathcal{O}^n(\pi_{SH(k),n})$ is an element of $H^n(B_{SH(k)}[n, \infty); \pi_{n-1}(F)) \approx H^n(B_{SH}[n, \infty); \pi_{n-1}(F))$. Identifying these elements for all $k \geq n+2$ we obtain an element

$$\mathcal{O}^n \in H^n(B_{SH}[n, \infty); \pi_{n-1}(F)) \approx \mathrm{Hom}(\pi_n(B_{SH}), \pi_{n-1}(F)).$$

8.1 PROPOSITION. *$\mathcal{O}^n$, considered as an element of $Hom(\pi_n(B_{SH}), \pi_{n-1}(F))$, is the boundary homomorphism $\partial : \pi_n(B_{SH}) \to \pi_{n-1}(F)$ of the fibre space $\tilde{\imath} : B'_{SO} \to B_{SH}$.*

Proof. Let $f : S^n \to B_{SH(k)}$ be a map, where $k \geq n+2$. Under the identification of the following diagram f^* is evaluation of a homomorphism γ on $[f]$, where $[f]$ is the homotopy class of the composition $S^n \xrightarrow{f} B_{SH(k)} \to B_{SH}$.

$$\begin{array}{ccc} H^n(S^n; \pi_{n-1}(F)) & \xleftarrow{f^*} & H^n(B_{SH}(k)[n, \infty); \pi_{n-1}(F)) \\ \wr\wr & & \wr\wr \\ \pi_{n-1}(F) & \xleftarrow{f^*} & \mathrm{Hom}(\pi_n(B_{SH}), \pi_{n-1}(F)) \end{array}$$

Hence $\mathcal{O}^n(f^*(\pi_{SH(k),n})) = f^*\mathcal{O}^n(\pi_{SH(k),n}) = \mathcal{O}^n([f])$.

On the other hand, consider the regular cell complex structure for S^n consisting of two k-cells, σ_1^k and σ_2^k, for $0 \le k \le n$. We can assume that $f|\sigma_2^n$ is trivial. The obstruction $\mathcal{O}^n(f^*(\pi_{SH(k),n})) \in H^n(S^n; \pi_{n-1}(F)) \approx \pi_{n-1}(F)$ is defined by covering the reverse of the radial contraction of σ_1^n into the origin of σ_1^n, obtaining an element of $\pi_{n-1}(F)$. But the element so obtained is just $\partial([f])$.

For $k \ge n+2$ the class $\psi^n(\pi_{SH(k),n})$ is an element of $H^n(B_{SH(k)}[n, \infty); \Pi_{n-1}) \approx H^n(B_{SH}[n, \infty); \Pi_{n-1})$. Identifying these elements for all $k \ge n+2$ we obtain an element.

$$\psi^n \in \mathrm{H}^n(B_{SH}[n, \infty); \Pi_{n-1}) \approx \mathrm{Hom}(\pi_n(B_{SH}), \Pi_{n-1}).$$

8.2 PROPOSITION. ψ^n, *considered as an element of* $Hom(\pi_n(B_{SH}), \Pi_{n-1})$, *is an isomorphism.*

Proof. Let $f: S^{2n-1} \to S^n$ and let ν be a normal fibre space of $X_{[f]}$, of fibre dimension $k \le n+2$. Then $\nu|S_i^n \sim g_i^*(\pi_{SH(k),n})$ for some $g_i: S^n \to B_{SH(k)}[n, \infty)$. Let $\varphi_i([f]) \in \pi_n(B_{SH})$ be $[g_i]$, considered as an element of $\pi_n(B_{SH})$. Then $\psi^n(X_{[f]}) = c_i \otimes \psi^n(\varphi_1[f]) \oplus c_2 \otimes \psi^n(\varphi_2[f])$. But (7.6) $\psi^n(X_{[f]}) = (-1)^{n+1}\ c_2 \otimes \Sigma\,[f]$. Hence $\psi^n(\varphi_2[f]) = (-1)^{n+1}\,\Sigma\,[f]$ (and $\psi^n(\varphi_1[f]) = 0$. Since Σ is onto Π_{n-1}, the homomorphism ψ^n is also onto. Since $\pi_n(B_{SH})$ and Π_{n-1} are isomorphic finite groups, ψ^n is an isomorphism.

8.3 COROLLARY. *Let X be an $(n-1)$-connected P-space of dimension $N \ge 2n$. Under the coefficient homomorphism*

$$\Pi_{n-1} \xrightarrow{(\psi^n)^{-1}} \pi_n(B_{SH}) \xrightarrow{\partial} \pi_{n-1}(F)$$

the class $\psi^n(X)$ goes into $\mathcal{O}^n(X)$.

Proof. The coefficient homomorphism $(\psi^n)^{-1}: \Pi_{n-1} \to \pi_n(B_{SH})$ takes $\psi^n \in Hom(\pi_n(B_{SH}), \Pi_{n-1})$ into $1 \in Hom(\pi_n(B_{SH}), \pi_n(B_{SH}))$. Therefore the coefficient homomorphism $\partial(\psi^n)^{-1}$ takes ψ^n into $\mathcal{O}^n = \partial \in Hom(\pi_n(B_{SH}), \pi_{n-1}(F))$. Therefore this coefficient homomorphism takes $\psi^n(X)$ into $\mathcal{O}^n(X)$.

The composite isomorphism $\pi_{n-1}(SH) \xrightarrow{\approx} \pi_n(B_{SH}) \xrightarrow{\psi^n} \Pi_{n-1}$ multiplied by $(-1)^{n+1}$ will be denoted j_{n-1}.

8.4 PROPOSITION. *The following diagram commutes.*

$$\begin{array}{ccc} \pi_{n-1}(SO) & \longrightarrow & \pi_{n-1}(SH) \\ & {\scriptstyle J_{n-1}} \searrow & \downarrow {\scriptstyle j_{n-1}} \\ & & \Pi_{n-1} \end{array}$$

Therefore the sequence $\pi_{n-1}(SO) \xrightarrow{J_{n-1}} \Pi_{n-1} \xrightarrow{\partial(\psi^n)^{-1}} \pi_{n-1}(F)$ *is exact.*

Proof. Let $J_{n-1}(\alpha) = \Sigma\,[f]$ for $f: S^{2n-1} \to S^n$. Let ν be a normal fibre space of $X_{[f]}$. It suffices (c.f. proof of 8.2) to prove that $\nu|S_2^n$ has characteristic map ig where $[g] = \alpha$.

Consider the S^n-bundle $\pi : E \longrightarrow S^n$ whose characteristic map is $-\alpha$, where α is considered as a member of $\pi_{n-1}(R_{n+1})$. Let $\alpha = k_* \beta$ for $\beta \in \pi_{n-1}(R_n)$, where $k : R_n \longrightarrow R_{n+1}$ is the inclusion. Then $J_{n-1}(\alpha) = J_{n-1}(k_* \beta) = -\Sigma J_{n-1}(\beta)$. Choosing such a β corresponds to choosing a cross-section s of the bundle $\pi : E \longrightarrow S^n$. If $a \in S^n$, the space E is $\pi^{-1}(a) \cup s(S^n) \cup [\pi^{-1}(S^n - a) - s(S_n)] = S_1{}^n \cup S_2{}^n \cup e^{2n}$, where (c.f. [10], p. 206) the attaching map is $-i_1 \circ J_{n-1}(\beta) \oplus 0 \oplus [i_1, i_2]$. Thus the C^∞ manifold E may be chosen for $X_{[f]}$ (with $[f] = -J_{n-1}(\beta)$). Therefore it suffices to prove that the restriction to $S_2{}^n$ of the normal bundle of E in R^{n+k} is stably equivalent to a bundle with classifying map α; this assertion is equivalent to the obvious fact that $\tau_E | S_2{}^n$ is stably equivalent to a bundle with classifying map $-\alpha$.

REFERENCES

1. J. F. Adams: On the groups $J(X)$, Part I, *Topology* **2** (1963), 181–196.
2. M. Atiyah: Thom complexes, *Proc. Lond. Math. Soc.* **11** (1961), 291–310.
3. W. Browder: Homotopy type of differentiable manifolds, in *Colloquium on Algebraic Topology*, pp. 44–46. Matematisk Institut, Aarhus Universitet, 1962.
4. A. Dold: Partitions of unity in the theory of fibrations, *Ann. Math.* **78** (1963), 223–255.
5. A. Dold and R. Lashof: Principal quasi-fibrations and fibre homotopy equivalence of bundles, *Illinois J. Math.* **3** (1959), 285–305.
6. J. Eels and N. Kuiper: Manifolds which are like projective planes, *Publs. Math. Inst. Ht. Etud. Scient.* **14** (1962), 1–44.
7. E. Fadell: Oh fibre homotopy equivalence, *Duke Math. J.* **26** (1959), 699–706.
8. E. Fadell: The equivalence of fibre spaces and bundles, *Bull. Am. Math. Soc.* **66** (1960), 50–53.
9. I. M. James: Spaces associated with Steifel manifolds, *Proc. Lond. Math. Soc.* **9** (1959), 115–140.
10. I. M. James and J. H. C. Whitehead: The homotopy theory of sphere bundles over spheres, *Proc. Lond. Math. Soc.* **4** (1954), 196–218.
11. M. Kervaire and J. Milnor: Groups of homotopy spheres: I, *Ann. Math.* **77** (1963), 504–537.
12. J. M. Kister, Microbundles are fibre bundles, *Ann. Math.* **80** (1964), 190–199.
13. B. Mazur: The definition of equivalence of combinatorial embeddings, *Publs. Math. Inst. Ht. Etud. Scient.* **3** (1959), 5–17.
14. J. Milnor: Construction of universal bundles, II, *Ann. Math.* **63** (1956), 430–436.
15. J. Milnor: Lectures on Characteristic Classes, (Mimeo. notes) Princeton Univ., Spring, 1957.
16. J. Milnor: Microbundles and Differentiable Structures, (Mimeo. notes) Princeton Univ., 1961.
17. J. Milnor: Microbundles, Part I, *Topology*, **3** Suppl. 1 (1964), 53–80.
18. J. C. Moore, Some applications of homology theory to homotopy problems, *Ann. Math.* **58** (1953), 325–350.
19. S. P. Novikov: Some problems in the topology of manifolds connected with the theory of Thom spaces, *Dokl. Akad. Nauk SSSR* **32** (1960), 1031–1034.
20. J.-P. Serre: Homologie singuliere des espaces fibres, *Ann. Math.* **54** (1951), 425–505.
21. E. H. Spanier and J. H. C. Whitehead: Duality in homotopy theory, *Mathematika* **2** (1955), 56–80.
22. J. Stasheff: A classification theorem for fibre spaces, *Topology* **2** (1963), 239–246.
23. J. H. C. Whitehead: Combinatorial homotopy, I, *Bull. Am. Math. Soc.* **55** (1949), 213–245.
24. J. H. C. Whitehead: Manifolds with transverse fields in Euclidean space, *Ann. Math.* 73 (1961), 154–212.

Brandeis University