ALGEBRAIC AND GEOMETRIC SPLITTINGS OF THE K- AND L-GROUPS OF POLYNOMIAL EXTENSIONS

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Introduction

This paper is an account of assorted results concerning the algebraic and geometric splittings of the Whitehead group of a polynomial extension as a direct sum

$$Wh(\pi\times\mathbb{Z}) = Wh(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])$$

and the analogous splittings of the Wall surgery obstruction groups

$$\begin{cases} L^s_*(\pi\times\mathbb{Z}) = L^s_*(\pi)\oplus L^h_{*-1}(\pi) \\ L^h_*(\pi\times\mathbb{Z}) = L^h_*(\pi)\oplus L^p_{*-1}(\pi) \end{cases} .$$

Such a splitting of $Wh(\pi\times\mathbb{Z})$ was first obtained by Bass, Heller and Swan [2]. $\begin{cases}\text{Shaneson [29]}\\ \text{Pedersen and Ranicki [18]}\end{cases}$ obtained such a splitting of $\begin{cases}L^s_*(\pi\times\mathbb{Z})\\ L^h_*(\pi\times\mathbb{Z})\end{cases}$ geometrically. Novikov [17] and Ranicki [20] obtained such L-theory splittings algebraically.

The main object of this paper is to point out that the geometric L-theory splittings of [29] and [18] are not in fact the same as the algebraic L-theory splittings of [17] and [20] (contrary to the claims put forward in [18],[20],[23] and [24] that they coincided), and to express the difference between them in terms of algebra. The splitting maps $\begin{cases}L^s_*(\pi)\rightarrowtail L^s_*(\pi\times\mathbb{Z})\\ L^h_*(\pi)\rightarrowtail L^h_*(\pi\times\mathbb{Z})\end{cases}$, $\begin{cases}L^s_*(\pi\times\mathbb{Z})\twoheadrightarrow L^h_{*-1}(\pi)\\ L^h_*(\pi\times\mathbb{Z})\twoheadrightarrow L^p_{*-1}(\pi)\end{cases}$ are the same in algebra and geometry, the split injections being the ones induced functorially from the split injection of groups $\bar{\varepsilon}:\pi\rightarrowtail\pi\times\mathbb{Z}$. However, the splitting maps $\begin{cases}L^s_*(\pi\times\mathbb{Z})\twoheadrightarrow L^s_*(\pi)\\ L^h_*(\pi\times\mathbb{Z})\twoheadrightarrow L^h_*(\pi)\end{cases}$, $\begin{cases}L^h_{*-1}(\pi)\rightarrowtail L^s_*(\pi\times\mathbb{Z})\\ L^p_{*-1}(\pi)\rightarrowtail L^h_*(\pi\times\mathbb{Z})\end{cases}$ are in general different in algebra and geometry. In particular, the geometric split split surjections are not the algebraic split surjections induced functorially from the split surjection of groups $\varepsilon:\pi\times\mathbb{Z}\twoheadrightarrow\pi$! This may be seen by considering the composite $\varepsilon\bar{B}'$ of the geometric split injection

$$\begin{cases} \bar{B}' : L^h_{n-1}(\pi) \rightarrowtail L^s_n(\pi\times\mathbb{Z}) ; \\ \qquad \sigma^h_*((f,b):M\longrightarrow X) \longmapsto \sigma^s_*((f,b)\times 1:M\times S^1 \longrightarrow X\times S^1) \\ \bar{B}' : L^p_{n-1}(\pi) \rightarrowtail L^h_n(\pi\times\mathbb{Z}) , \\ \qquad \sigma^p_*((f,b):M\longrightarrow X) \longmapsto \sigma^h_*((f,b)\times 1:M\times S^1 \longrightarrow X\times S^1) \end{cases}$$

(denoted $\bar{B}'$ to distinguish from the algebraic split injection $\bar{B}$ of [20]) and the algebraic split surjection

$$\begin{cases} \varepsilon : L^s_n(\pi\times\mathbb{Z}) \twoheadrightarrow L^s_n(\pi) ; \\ \qquad \sigma^s_*((g,c):N\longrightarrow Y) \longmapsto \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi\times\mathbb{Z}]}\sigma^s_*(g,c) \\ \varepsilon : L^h_n(\pi\times\mathbb{Z}) \twoheadrightarrow L^h_n(\pi) ; \\ \qquad \sigma^h_*((g,c):N\longrightarrow Y) \longmapsto \mathbb{Z}[\pi]\otimes_{\mathbb{Z}[\pi\times\mathbb{Z}]}\sigma^h_*(g,c) \end{cases} .$$

Now $\varepsilon\bar{B}'$ need not be zero: if X is a $\begin{cases}\text{finite}\\ \text{finitely dominated}\end{cases}$ $(n-1)$-dimensional geometric Poincaré complex then $X\times S^1$ is a $\begin{cases}\text{simple}\\ \text{homotopy finite}\end{cases}$ n-dimensional geometric Poincaré complex, the boundary of the $\begin{cases}\text{finite}\\ \text{finitely dominated}\end{cases}$ $(n+1)$-dimensional geometric Poincaré pair $(X\times D^2, X\times S^1)$, but not in general the boundary of a $\begin{cases}\text{simple}\\ \text{homotopy finite}\end{cases}$ pair $(W, X\times S^1)$ with $\pi_1(W) = \pi_1(X)$, so that ε and $\bar{B}'$ do not belong to the same direct sum system.

The geometrically significant splittings of $L_*(\pi\times\mathbb{Z})$ obtained in §6 are compatible with the geometrically significant variant in §3 of the splitting of $Wh(\pi\times\mathbb{Z})$ due to Bass, Heller and Swan [2]. In both K- and L-theory the algebraic and geometric splitting maps differ in 2-torsion only, there being no difference if $Wh(\pi) = 0$.

I am grateful to Hans Munkholm for our collaboration on [16]. It is the considerations of the appendix of [16] which led to the discovery that the algebraic and geometric L-theory splittings are not the same.

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Detailed proofs of the results announced here will be found in Ranicki [26], [27], [28].

§1. Absolute K-theory invariants

The definitions of the Wall finiteness obstruction $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of a finitely dominated CW complex X and the Whitehead torsion $\tau(f) \in Wh(\pi_1(X))$ of a homotopy equivalence $f:X \longrightarrow Y$ of finite CW complexes are too well known to bear repeating here. The reduced algebraic K-groups $\tilde{K}_0$, Wh are not as well-behaved with respect to products as the absolute K-groups K_0, K_1. Accordingly it is necessary to deal with absolute versions of the invariants. The projective class of a finitely dominated CW complex X

$$[X] = (\chi(X),[X]) \in K_0(\mathbb{Z}[\pi_1(X)]) = K_0(\mathbb{Z}) \oplus \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

is well-known, with $\chi(X) \in K_0(\mathbb{Z}) = \mathbb{Z}$ the Euler characteristic. It is harder to come by an absolute torsion invariant.

Let A be an associative ring with 1 such that the rank of f.g. free A-modules is well-defined, e.g. a group ring $A = \mathbb{Z}[\pi]$. An A-module chain complex C is finite if it is a bounded positive complex of based f.g. free A-modules

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 \longrightarrow 0 \longrightarrow \ldots,$$

in which case the Euler characteristic of C is defined in the usual manner by

$$\chi(C) = \sum_{r=0}^{n} (-)^r \mathrm{rank}_A(C_r) \in \mathbb{Z} .$$

A finite A-module chain complex C is round if

$$\chi(C) = 0 \in \mathbb{Z} .$$

The absolute torsion of a chain equivalence $f:C \longrightarrow D$ of round finite A-module chain complexes is defined in Ranicki [25] to be an element

$$\tau(f) \in K_1(A)$$

which is a chain homotopy invariant of f such that

i) if f is an isomorphism $\tau(f) = \sum_{r=0}^{\infty} (-)^r \tau(f:C_r \longrightarrow D_r)$.

ii) $\tau(gf) = \tau(f) + \tau(g)$ for $f:C \longrightarrow D$, $g:D \longrightarrow E$.

iii) The reduction of $\tau(f)$ in $\tilde{K}_1(A) = K_1(A)/\{\tau(-1:A \longrightarrow A)\}$ is the usual reduced torsion invariant of f, defined for a chain equivalence $f:C \longrightarrow D$ of finite A-module chain complexes to be the reduction of the torsion $\tau(C(f)) \in K_1(A)$ of the algebraic mapping cone C(f). Thus for $A = \mathbb{Z}[\pi]$ the reduction of $\tau(f) \in K_1(\mathbb{Z}[\pi])$ in the Whitehead group $Wh(\pi) = \tilde{K}_1(\mathbb{Z}[\pi])/\{\pi\}$ is the usual Whitehead torsion of f.

iv) $\tau(f) = \tau(D) - \tau(C) \in K_1(A)$ for contractible finite C,D.

v) In general $\tau(f) \neq \tau(C(f)) \in K_1(A)$, and $\tau(f \oplus f') \neq \tau(f) + \tau(f')$ (although the differences are at most $\tau(-1:A \longrightarrow A) \in K_1(A)$).

vi) The absolute torsion $\tau(f) \in K_1(A)$ of a self chain equivalence $f:C \longrightarrow D = C$ agrees with the absolute torsion invariant $\tau(f) \in K_1(A)$ defined by Gersten [10] for a self chain equivalence $f: C \longrightarrow C$ of a finitely dominated A-module chain complex C.

A $\begin{cases}\text{round}\\ -\end{cases}$ <u>finite structure</u> on an A-module chain complex C is an equivalence class of pairs (F,ϕ) with F a $\begin{cases}\text{round}\\ -\end{cases}$ finite A-module chain complex and $\phi:F \longrightarrow C$ a chain equivalence, subject to the equivalence relation

$$(F,\phi) \sim (F',\phi') \text{ if } \tau(\phi'^{-1}\phi:F \longrightarrow C \longrightarrow F') = 0 \in \begin{cases}K_1(A)\\ \widetilde{K}_1(A)\end{cases} .$$

In the topological applications $A = \mathbb{Z}[\pi]$, and $\widetilde{K}_1(A)$ is replaced by $Wh(\pi)$.

<u>Proposition 1.1</u> A finitely dominated A-module chain complex C admits a $\begin{cases}\text{round}\\ -\end{cases}$ finite structure if and only if it has $\begin{cases}\text{absolute}\\ \text{reduced}\end{cases}$ projective class $[C] = 0 \in \begin{cases}K_0(A)\\ \widetilde{K}_0(A)\end{cases}$, in which case the set of such structures on C carries an affine $\begin{cases}K_1(A)\text{-}\\ \widetilde{K}_1(A)\text{-}\end{cases}$ structure.

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Let X be a (connected) CW complex with universal cover $\widetilde{X}$ and fundamental group $\pi_1(X) = \pi$. The cellular chain complex $C(\widetilde{X})$ is defined as usual, with $C(\widetilde{X})_r = H_r(\widetilde{X}^{(r)},\widetilde{X}^{(r-1)})$ $(r \geqslant 0)$ the free $\mathbb{Z}[\pi]$-module generated by the r-cells of X. The cell structure of X determines for each $C(\widetilde{X})_r$ a $\mathbb{Z}[\pi]$-module base up to the multiplication of each element by $\pm g$ $(g \in \pi)$. Thus for a finite CW complex X the cellular $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{X})$ has a canonical finite structure.

A CW complex X is <u>round finite</u> if it is finite, $\chi(X) = 0 \in \mathbb{Z}$, and there is given a choice of actual base for each $C(\widetilde{X})_r$ $(r \geqslant 0)$ in the class of bases determined by the cell structure of X.

The $\begin{cases}\underline{\text{absolute}}\\ \underline{\text{Whitehead}}\end{cases}$ <u>torsion</u> of a homotopy equivalence $f:X \longrightarrow Y$ of $\begin{cases}\text{round}\\ -\end{cases}$ finite CW complexes is defined by

$$\tau(f) = \tau(\widetilde{f}:C(\widetilde{X}) \longrightarrow C(\widetilde{Y})) \in \begin{cases}K_1(\mathbb{Z}[\pi_1(X)])\\ Wh(\pi_1(X))\end{cases}$$

A $\left\{\begin{array}{l}\text{round}\\ -\end{array}\right.$ finite structure on a CW complex X is an equivalence class of pairs (F,ϕ) with F a $\left\{\begin{array}{l}\text{round}\\ -\end{array}\right.$ finite CW complex and $\phi:F\longrightarrow X$ a homotopy equivalence, subject to the equivalence relation

$$(F,\phi) \sim (F',\phi') \text{ if } \tau(\phi'^{-1}\phi:F\longrightarrow X\longrightarrow F') = 0 \in \left\{\begin{array}{l}K_1(\mathbb{Z}[\pi_1(X)])\\ Wh(\pi_1(X))\end{array}\right. .$$

The finiteness obstruction theory of Wall [34] gives:

Proposition 1.2 The $\left\{\begin{array}{l}\text{round}\\ -\end{array}\right.$ finite structures on a finitely dominated CW complex X are in a natural one-one correspondence with the $\left\{\begin{array}{l}\text{round}\\ -\end{array}\right.$ finite structures on the $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$.

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The mapping torus of a self map $f:X\longrightarrow X$ is defined as usual by

$$T(f) = X\times[0,1]/\{(x,0) = (f(x),1)\,|\,x\in X\} .$$

Proposition 1.3 (Ranicki [26]) The mapping torus T(f) of a self map $f:X\longrightarrow X$ of a finitely dominated CW complex X has a canonical round finite structure.

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The circle $S^1 = [0,1]/(0=1)$ has universal cover $\widetilde{S}^1 = \mathbb{R}$ and fundamental group $\pi_1(S^1) = \mathbb{Z}$. Let $z\in\pi_1(S^1) = \mathbb{Z}$ denote the generator such that

$$z : \mathbb{R}\longrightarrow\mathbb{R} ;\ x\longmapsto x+1 .$$

The canonical round finite structure on the circle $S^1 = e^0\cup e^1 = T(id.:\{pt.\}\longrightarrow\{pt.\})$ is represented by the bases $\widetilde{e}^r\in C(\widetilde{S}^1)_r = \mathbb{Z}[z,z^{-1}]$ $(r = 0,1)$ with

$$d = 1-z : C(\widetilde{S}^1)_1 = \mathbb{Z}[z,z^{-1}] \longrightarrow C(\widetilde{S}^1)_0 = \mathbb{Z}[z,z^{-1}] ;\ \widetilde{e}^1\longmapsto\widetilde{e}^0 - z\widetilde{e}^0 ,$$

corresponding to the lifts $\widetilde{e}^0 = \{0\}$, $\widetilde{e}^1 = [0,1]\subset\mathbb{R}$ of e^0,e^1.

In particular, Proposition 1.3 applies to the product $X\times S^1 = T(id.:X\longrightarrow X)$, in which case the canonical round finite structure is a refinement of the finite structure defined geometrically by Mather [14] and Ferry [8], using the homotopy equivalent finite CW complex $T(fg:Y\longrightarrow Y)$ for any domination of X

$$(Y,\ f:X\longrightarrow Y,\ g:Y\longrightarrow X,\ h:gf\simeq 1:X\longrightarrow X)$$

by a finite CW complex Y.

Given a ring morphism $\alpha:A \longrightarrow B$ let

$$\alpha_! : (\text{A-modules}) \longrightarrow (\text{B-modules}) \; ; \; M \longmapsto B\otimes_A M$$

be the functor inducing morphisms in the algebraic K-groups

$$\alpha_! : K_i(A) \longrightarrow K_i(B) \quad (i=0,1) \quad ,$$

which we shall usually abbreviate to α. Given a ring automorphism $\alpha:A \longrightarrow A$ let $K_1(A,\alpha)$ be the relative K-group in the exact sequence

$$K_1(A) \xrightarrow{1-\alpha} K_1(A) \xrightarrow{j} K_1(A,\alpha) \xrightarrow{\partial} K_0(A) \xrightarrow{1-\alpha} K_0(A) \quad ,$$

as originally defined by Siebenmann [33] in connection with the splitting theorem for $K_1(A_\alpha[z,z^{-1}])$ recalled in §3 below. By definition $K_1(A,\alpha)$ is the exotic group of pairs (P,f) with P a f.g. projective A-module and $f \in \mathrm{Hom}_A(\alpha_! P,P)$ an isomorphism. The <u>mixed invariant</u> of a finitely dominated A-module chain complex C and a chain equivalence $f:\alpha_! C \longrightarrow C$ was defined in Ranicki [26] to be an element

$$[C,f] \in K_1(A,\alpha)$$

such that $\partial([C,f]) = [C] \in K_0(A)$, and such that $[C,f] = 0 \in K_1(A,\alpha)$ if and only if C admits a round finite structure $(F,\phi:F \longrightarrow C)$ with

$$\tau(\phi^{-1}f(\alpha_!\phi) : \alpha_! F \longrightarrow \alpha_! C \longrightarrow C \longrightarrow F) = 0 \in K_1(A) \ .$$

The invariant is a mixture of projective class and torsion, and indeed for $\alpha = 1 : A \longrightarrow A$

$$[C,f] = (\tau(f),[C]) \in K_1(A,1) = K_1(A)\oplus K_0(A) \ .$$

The absolute torsion invariant defined by Gersten [10] for a self homotopy equivalence $f:X \longrightarrow X$ of a finitely dominated CW complex X inducing $f_* = 1 : \pi_1(X) = \pi \longrightarrow \pi$

$$\tau(f) = \tau(\tilde{f}:C(\tilde{X}) \longrightarrow C(\tilde{X})) \in K_1(\mathbb{Z}[\pi])$$

was generalized in Ranicki [26]: the <u>mixed invariant</u> of a self homotopy equivalence $f:X \longrightarrow X$ of a finitely dominated CW complex X inducing any automorphism $f_* = \alpha : \pi_1(X) = \pi \longrightarrow \pi$ is defined by

$$[X,f] = [C(\tilde{X}),\tilde{f}:\alpha_! C(\tilde{X}) \longrightarrow C(\tilde{X})] \in K_1(\mathbb{Z}[\pi],\alpha) \ .$$

This has image $\partial([X,f]) = [X] \in K_0(\mathbb{Z}[\pi])$, and is such that $[X,f] = 0$ if and only if X admits a round finite structure $(F,\phi:F \longrightarrow X)$ such that

$$\tau(\phi^{-1}f\phi : F \longrightarrow X \longrightarrow X \longrightarrow F) = 0 \in K_1(\mathbb{Z}[\pi]) \ .$$

If X admits a round finite structure (F,ϕ) then $[X,f] = j(\tau(\phi^{-1}f\phi))$ is the image of $\tau(\phi^{-1}f\phi:F \longrightarrow F) \in K_1(\mathbb{Z}[\pi])$.

§2. Products in K-theory

For any rings A,B and automorphism $\beta:B\longrightarrow B$ there is defined a product of algebraic K-groups

$$\otimes : K_0(A)\otimes K_1(B,\beta)\longrightarrow K_1(A\otimes B,1\otimes\beta) ;$$

$$[P]\otimes[Q,f:\beta_!Q\longrightarrow Q]\longmapsto[P\otimes Q,1\otimes f:(1\otimes\beta)_!(P\otimes Q) = P\otimes\beta_!Q\longrightarrow P\otimes Q] ,$$

which in the case $\beta = 1$ is made up of the products

$$\otimes : K_0(A)\otimes K_0(B)\longrightarrow K_0(A\otimes B) ; [P]\otimes[Q]\longmapsto[P\otimes Q]$$

$$\otimes : K_0(A)\otimes K_1(B)\longrightarrow K_1(A\otimes B) ; [P]\otimes\tau(f:Q\longrightarrow Q)\longmapsto\tau(1\otimes f:P\otimes Q\longrightarrow P\otimes Q) .$$

The product of a finitely dominated A-module chain complex C and a finitely dominated B-module chain complex D is a finitely dominated $A\otimes B$-module chain complex $C\otimes D$ with projective class

$$[C\otimes D] = [C]\otimes[D] \in K_0(A\otimes B) ,$$

and if $f:\beta_!D\longrightarrow D$ is a chain equivalence then the product chain equivalence $1\otimes f : C\otimes\beta_!D\longrightarrow C\otimes D$ has mixed invariant

$$[C\otimes D,1\otimes f] = [C]\otimes[D,f] \in K_1(A\otimes B,1\otimes\beta)$$

The following product formula is an immediate consequence.

Proposition 2.1 Let X,F be finitely dominated CW complexes with $\pi_1(X) = \pi$, $\pi_1(F) = \rho$, and let $f : F\longrightarrow F$ be a self homotopy equivalence inducing the automorphism $f_* = \beta : \rho\longrightarrow\rho$. The mixed invariant of the product self homotopy equivalence $1\times f : X\times F\longrightarrow X\times F$ is given by

$$[X\times F,1\times f] = [X]\otimes[F,f] \in K_1(\mathbb{Z}[\pi\times\rho],1\otimes\beta) ,$$

identifying $\mathbb{Z}[\pi\times\rho] = \mathbb{Z}[\pi]\otimes\mathbb{Z}[\rho]$.

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In the case $\beta = 1 : \rho\longrightarrow\rho$ the result of Proposition 2.1 is made up of the product formula of Gersten [9] and Siebenmann [30] for the projective class

$$[X\times F] = [X]\otimes[F] \in K_0(\mathbb{Z}[\pi\times\rho])$$

and the product formula of Gersten [10] for torsion

$$\tau(1\times f:X\times F\longrightarrow X\times F) = [X]\otimes\tau(f:F\longrightarrow F) \in K_1(\mathbb{Z}[\pi\times\rho]) .$$

If also X is finite the product formula $\tau(1\times f) = [X]\otimes\tau(f)$ is an absolute version of the special case $e = 1 : X\longrightarrow X' = X$, $f_* = 1$ of the formula of Kwun and Szczarba [12] for the Whitehead torsion of the product $e\times f : X\times F\longrightarrow X'\times F'$ of homotopy equivalences $e : X\longrightarrow X'$, $f : F\longrightarrow F'$ of finite CW complexes

$$\tau(e\times f) = \chi(X)\otimes\tau(f) + \tau(e)\otimes\chi(F) \in Wh(\pi\times\rho)$$

The product $A\otimes B$-module chain complex $C\otimes D$ of a finitely dominated A-module chain complex C and a round finite B-module chain complex D was shown in Ranicki [26] to have a canonical round finite structure, with

$$\tau(e\otimes f:C\otimes D \longrightarrow C'\otimes D') = [C]\otimes\tau(f:D\longrightarrow D') \in K_1(A\otimes B)$$

for any chain equivalences $e:C\longrightarrow C', f:D\longrightarrow D'$ of such complexes. The following product structure theorem of [26] was an immediate consequence.

Proposition 2.2 The product $X\times F$ of a finitely dominated CW complex X and a round finite CW complex F has a canonical round finite structure, with

$$\tau(e\times f:X\times F\longrightarrow X'\times F') = [X]\otimes\tau(f:F\longrightarrow F') \in K_1(\mathbb{Z}[\pi_1(X)\times\pi_1(F)])$$

for any homotopy equivalences $e:X\longrightarrow X', f:F\longrightarrow F'$ of such complexes.

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The canonical round finite structure on $X\times S^1 = T(\mathrm{id.}:X\longrightarrow X)$ given by Proposition 1.3 coincides with the canonical round finite structure given by Proposition 2.2.

The product

$$K_0(\mathbb{Z}[\pi])\otimes K_1(\mathbb{Z}[\rho])\longrightarrow K_1(\mathbb{Z}[\pi\times\rho])$$

has a reduced version

$$\tilde{K}_0(\mathbb{Z}[\pi])\otimes\{\pm\rho\}\longrightarrow Wh(\pi\times\rho)\ ;$$

$$[P]\otimes\tau(\pm g:\mathbb{Z}[\rho]\longrightarrow\mathbb{Z}[\rho])\longmapsto\tau(1\otimes\pm g:P[\rho]\longrightarrow P[\rho])$$

with $\{\pm\rho\} = \{\pm 1\}\times\rho^{ab} = \ker(K_1(\mathbb{Z}[\rho])\twoheadrightarrow Wh(\rho))$. We shall make much use of this reduced version with $\rho = \mathbb{Z}$, for which $\{\pm\mathbb{Z}\} = K_1(\mathbb{Z}[\mathbb{Z}])$.

§3. The Whitehead group of a polynomial extension

In the first instance we recall some of the details of the direct sum decomposition

$$Wh(\pi\times\mathbb{Z}) = Wh(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])$$

obtained by Bass, Heller and Swan [2] and Bass [1 ,XII] for any group We shall call this the algebraically significant splitting of $Wh(\pi\times\mathbb{Z})$. The relevant isomorphism

$$\beta_K = \begin{pmatrix}\varepsilon\\ B\\ \Delta_+\\ \Delta_-\end{pmatrix} : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])$$

and its inverse

$$\beta_K^{-1} = (\bar{\varepsilon}\ \bar{B}\ \bar{\Delta}_+\ \bar{\Delta}_-) : Wh(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi\times\mathbb{Z})$$

involve the split $\left\{\begin{matrix}\text{surjection}\\ \text{injection}\end{matrix}\right.$ of group rings

$$\left\{\begin{array}{l}\varepsilon : \mathbb{Z}[\pi\times\mathbb{Z}] = \mathbb{Z}[\pi][z,z^{-1}] \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}[\pi] \ ;\ \sum_{j=-\infty}^{\infty} a_j z^j \longmapsto \sum_{j=-\infty}^{\infty} a_j \\ \\ \bar{\varepsilon} : \mathbb{Z}[\pi] \rightarrowtail \mathbb{Z}[\pi][z,z^{-1}] \ ;\ a \longmapsto a \qquad (a,a_j\in\mathbb{Z}[\pi]) \ .\end{array}\right.$$

The split injection $\bar{B}:\tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})$ is the evaluation of the product $\tilde{K}_0(\mathbb{Z}[\pi])\otimes K_1(\mathbb{Z}[\mathbb{Z}]) \longrightarrow Wh(\pi\times\mathbb{Z})$ (the reduction of $K_0(\mathbb{Z}[\pi])\otimes K_1(\mathbb{Z}[\mathbb{Z}]) \longrightarrow K_1(\mathbb{Z}[\pi\times\mathbb{Z}])$) on the element $\tau(z)\in K_1(\mathbb{Z}[\mathbb{Z}])$

$$\bar{B} = -\otimes\tau(z) : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \ ;$$
$$[P] \longmapsto \tau(z:P[z,z^{-1}] \longrightarrow P[z,z^{-1}]) \ .$$

If $P = im(p)$ is the image of the projection $p = p^2 : \mathbb{Z}[\pi]^r \longrightarrow \mathbb{Z}[\pi]^r$ then

$$\bar{B}([P]) = \tau(pz+1-p:\mathbb{Z}[\pi\times\mathbb{Z}]^r \longrightarrow \mathbb{Z}[\pi\times\mathbb{Z}]^r) \in Wh(\pi\times\mathbb{Z}) \ .$$

By definition, $\widetilde{Nil}(\mathbb{Z}[\pi])$ is the exotic K-group of pairs (F,ν) with F a f.g. free $\mathbb{Z}[\pi]$-module and $\nu\in Hom_{\mathbb{Z}[\pi]}(F,F)$ a nilpotent endomorphism. The split injections $\bar{\Delta}_+$, $\bar{\Delta}_-$ are defined by

$$\bar{\Delta}_\pm : \widetilde{Nil}(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \ ;$$
$$(F,\nu) \longmapsto \tau(1+z^{\pm 1}\nu:F[z,z^{-1}] \longrightarrow F[z,z^{-1}]) \ .$$

The precise definitions of the split surjections $B, \Delta_{\pm}$ need not detain us here, especially as they are the same for the algebraically and geometrically significant direct sum decompositions of $Wh(\pi\times\mathbb{Z})$.

The exact sequence

$$0 \longrightarrow Wh(\pi) \xrightarrow{\bar{\varepsilon}} Wh(\pi\times\mathbb{Z}) \xrightarrow{\begin{pmatrix} B \\ \Delta_+ \\ \Delta_- \end{pmatrix}} \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi]) \longrightarrow 0$$

was interpreted geometrically by Farrell and Hsiang [5], [7]: if X is a finite n-dimensional geometric Poincaré complex with $\pi_1(X) = \pi$ and $f : M \longrightarrow X \times S^1$ is a homotopy equivalence with M^{n+1} a compact (n+1)-dimensional manifold then the Whitehead torsion $\tau(f) \in Wh(\pi\times\mathbb{Z})$ is such that

$$\begin{aligned} \tau(f) &\in \operatorname{im}(\bar{\varepsilon}: Wh(\pi) \rightarrowtail Wh(\pi\times\mathbb{Z})) \\ &= \ker\left(\begin{pmatrix} B \\ \Delta_+ \\ \Delta_- \end{pmatrix} : Wh(\pi\times\mathbb{Z}) \twoheadrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi])\right) \end{aligned}$$

if (and for $n \geqslant 5$ only if) f is homotopic to a map transverse regular at $X \times \{pt.\} \subset X \times S^1$ with the restriction

$$g = f| : N^n = f^{-1}(X \times \{pt.\}) \longrightarrow X$$

also a homotopy equivalence. Thus $\tau(f) \in \operatorname{coker}(\bar{\varepsilon}: Wh(\pi) \rightarrowtail Wh(\pi\times\mathbb{Z}))$ is the codimension 1 splitting obstruction of f along $X \times \{pt.\} \subset X \times S^1$. For a finitely presented group π every element of $Wh(\pi\times\mathbb{Z})$ is the Whitehead torsion $\tau(f)$ for a homotopy equivalence of pairs $(f, \partial f) : (M, \partial M) \longrightarrow (X, \partial X) \times S^1$ with $(M, \partial M)$ a compact (n+1)-dimensional manifold with boundary, and $(X, \partial X)$ a finite n-dimensional geometric Poincaré pair with $\pi_1(X) = \pi$, for some $n \geqslant 5$. In this case $\tau(f) \in \operatorname{coker}(\bar{\varepsilon}: Wh(\pi) \rightarrowtail Wh(\pi\times\mathbb{Z}))$ is the relative codimension 1 splitting obstruction.

The <u>geometrically significant</u> splitting

$$Wh(\pi\times\mathbb{Z}) = Wh(\pi) \oplus \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi])$$

is defined by the isomorphism

$$\beta_K' = \begin{pmatrix} \varepsilon' \\ \bar{B}' \\ \Delta_+ \\ \Delta_- \end{pmatrix} : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi) \oplus \tilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}(\mathbb{Z}[\pi])$$

with inverse

$$\beta_K'^{-1} = (\bar\epsilon\ \bar B'\ \bar\Delta_+\ \bar\Delta_-) : Wh(\pi)\oplus\tilde K_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\longrightarrow Wh(\pi\times\mathbb{Z}) ,$$

where

$$\bar B' = -\otimes\tau(-z) : \tilde K_0(\mathbb{Z}[\pi])\rightarrowtail Wh(\pi\times\mathbb{Z}) ;\ [P]\longmapsto\tau(-z:P[z,z^{-1}]\longrightarrow P[z,z^{-1}])$$
$$(=\tau(-pz+1-p)\ \text{if}\ P = \mathrm{im}(p=p^2)) ,$$

$$\epsilon' = \epsilon(1-\bar B'B) : Wh(\pi\times\mathbb{Z})\twoheadrightarrow Wh(\pi) ;$$
$$\tau(f:P[z,z^{-1}]\longrightarrow P[z,z^{-1}])\longrightarrow\tau(\epsilon f:P\longrightarrow P) + \tau(-1:Q\longrightarrow Q)$$

with f an automorphism of the f.g. projective $\mathbb{Z}[\pi\times\mathbb{Z}]$-module $P[z,z^{-1}]$ induced from a f.g. projective $\mathbb{Z}[\pi]$-module P, and Q a f.g. projective $\mathbb{Z}[\pi]$-module such that $B(\tau(f)) = [Q]\in\tilde K_0(\mathbb{Z}[\pi])$.

Ferry [8] defined a geometric injection for any finitely presented group

$$\bar B'' : \tilde K_0(\mathbb{Z}[\pi])\rightarrowtail Wh(\pi\times\mathbb{Z}) ;$$
$$[X]\longmapsto\tau(f=\phi^{-1}(1\times-1)\phi : Y\xrightarrow{\phi}X\times S^1\xrightarrow{1\times-1}X\times S^1\xrightarrow{\phi^{-1}}Y) ,$$

with $[X]\in\tilde K_0(\mathbb{Z}[\pi])$ the Wall finiteness obstruction of a finitely dominated CW complex X with $\pi_1(X)=\pi$ and $\tau(f)\in Wh(\pi\times\mathbb{Z})$ the Whitehead torsion of the homotopy equivalence $f=\phi^{-1}(1\times-1)\phi:Y\longrightarrow Y$ defined using the map $-1:S^1\longrightarrow S^1$ reflecting the circle in a diameter and any homotopy equivalence $\phi:Y\longrightarrow X\times S^1$ from a finite CW complex Y in the finite structure on $X\times S^1$ given by the mapping torus construction of Mather [14].

Proposition 3.1 The geometrically significant injection $\bar B'$ agrees with the geometric injection $\bar B''$

$$\bar B' = \bar B'' : \tilde K_0(\mathbb{Z}[\pi])\rightarrowtail Wh(\pi\times\mathbb{Z}) .$$

Proof: By Proposition 2.2

$$\bar B''([X]) = [X]\otimes\tau(-1:S^1\longrightarrow S^1)\in Wh(\pi\times\mathbb{Z}) ,$$

with $\tau(-1:S^1\longrightarrow S^1)\in K_1(\mathbb{Z}[z,z^{-1}])$ the absolute torsion. Now $-1:S^1\longrightarrow S^1$ induces the non-trivial automorphism $z\longmapsto z^{-1}$ of $\pi_1(S^1) = \langle z\rangle$, and the induced chain equivalence of based f.g. free $\mathbb{Z}[z,z^{-1}]$-module chain complexes is given by

$$\begin{array}{ccccc}
(-1)_!C(\tilde S^1) & : & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z^{-1}} & \mathbb{Z}[z,z^{-1}] \\
\big\downarrow\widetilde{(-1)} & & \big\downarrow 1 & & \big\downarrow -z \\
C(\tilde S^1) & : & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z} & \mathbb{Z}[z,z^{-1}]
\end{array} ,$$

so that

$$\tau(-1:S^1 \longrightarrow S^1) = \tau(-z:\mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z}[z,z^{-1}]) \in K_1(\mathbb{Z}[z,z^{-1}]) .$$

Thus

$$\bar{B}'' = -\otimes\tau(-z) = \bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) .$$

[]

Ferry [8] characterized $im(\bar{B}'') \subseteq Wh(\pi\times\mathbb{Z})$ as the subgroup of the elements $\tau \in Wh(\pi\times\mathbb{Z})$ such that $(p_n)^!(\tau) = \tau$ for some $n \geqslant 2$, with $(p_n)^! : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi\times\mathbb{Z})$ the transfer map associated to the n-fold covering of the circle by itself

$$p_n : S^1 \longrightarrow S^1 \; ; \; z \longmapsto z^n .$$

See Ranicki [27] for an explicit algebraic verification that $im(\bar{B}') \subseteq Wh(\pi\times\mathbb{Z})$ is the subgroup of transfer invariant elements.

The algebraically significant decomposition of $Wh(\pi\times\mathbb{Z})$ also has a certain measure of geometric significance, in that it is related to the Bott periodicity theorem in topological K-theory - cf. Bass [1 ,XIV]. More recently, Munkholm [15] identified the infinite structure set $\mathcal{S}(X\times\mathbb{R}^2) = ker(\varepsilon:\tilde{K}_0(\mathbb{Z}[\pi\times\mathbb{Z}]) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]))$ (X compact, $\pi_1(X) = \pi$) of Siebenmann [32] with the lower algebraic K-groups derived from the algebraically significant splitting of $Wh(\pi\times\mathbb{Z})$ by Bass [1 ,XII] - to be precise $\mathcal{S}(X\times\mathbb{R}^2) = (K_{-1}\oplus NK_0\oplus NK_0)(\mathbb{Z}[\pi])$.

Both the injections $\bar{B},\bar{B}':\tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})$ can be realized geometrically for a finitely presented group π, as follows. Given a f.g. projective $\mathbb{Z}[\pi]$-module P let $p = p^2 \in Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi]^r,\mathbb{Z}[\pi]^r)$ be a projection such that $P = im(p)$. Let K be a finite CW complex such that $\pi_1(K) = \pi$. For any integer $N \geqslant 2$ define the finite CW complexes

$$X = (K\times S^1 \vee \bigvee_r S^N) \cup_{pz+1-p} (\bigcup_r e^{N+1})$$

$$X' = (K\times S^1 \vee \bigvee_r S^N) \cup_{-pz+1-p} (\bigcup_r e^{N+1}) \quad ,$$

such that the inclusions define homotopy equivalences

$$K\times S^1 \longrightarrow X \quad , \quad K\times S^1 \longrightarrow X' .$$

<u>Proposition 3.2</u> The injections $\bar{B},\bar{B}'$ are realized geometrically by

$$\bar{B} : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \; ; \; [P] \longmapsto (-)^N\tau(K\times S^1 \longrightarrow X)$$

$$\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \; ; \; [P] \longmapsto (-)^N\tau(K\times S^1 \longrightarrow X') .$$

[]

Nevertheless, $\bar{B}'$ is more geometrically significant than $\bar{B}$.

(Following Siebenmann [31] define a <u>band</u> to be a finite CW complex X equipped with a map $p:X\longrightarrow S^1$ such that the pullback infinite cyclic cover $\overline{X} = p^*(\mathbb{R})$ of X is finitely dominated. For a connected band X the infinite complex $\overline{X}$ has two ends ε^+, ε^- which are contained in finitely dominated subcomplexes $\overline{X}^+$, $\overline{X}^- \subset \overline{X}$ such that $\overline{X}^+ \cap \overline{X}^-$ is finite and $\overline{X}^+ \cup \overline{X}^- = \overline{X}$. The finiteness obstructions are such that

$$[\overline{X}] = [\overline{X}^+] + [\overline{X}^-] \in \widetilde{K}_0(\mathbb{Z}[\pi]) \quad (\pi = \pi_1(\overline{X})) .$$

For a manifold band X the finiteness obstructions $[\overline{X}^\pm] \in \widetilde{K}_0(\mathbb{Z}[\pi])$ are images of the end obstructions $[\varepsilon^\pm] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon^\pm)])$ of Siebenmann [30]. For any finitely presented group π the surjection $B:Wh(\pi\times\mathbb{Z})\twoheadrightarrow\widetilde{K}_0(\mathbb{Z}[\pi])$ is realized geometrically by

$$B(\tau(f:X\longrightarrow Y)) = [\overline{Y}^+] - [\overline{X}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi]) ,$$

with $\tau(f) \in Wh(\pi\times\mathbb{Z})$ the Whitehead torsion of a homotopy equivalence of bands $f:X\longrightarrow Y$ with $\pi_1(X) = \pi\times\mathbb{Z}$, $\pi_1(\overline{X}) = \pi$. For the bands used in Proposition 3.2

$$[\overline{X}^+] = -[\overline{X}^-] = [\overline{X}'^+] = -[\overline{X}'^-] = (-)^N[P] ,$$

$$[\overline{(K\times S^1)}^+] = [\overline{(K\times S^1)}^-] = [K\times\mathbb{R}^+] = [K] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])).$$

We shall now express the difference between the algebraically and geometrically significant splittings of $Wh(\pi\times\mathbb{Z})$ using the generator $\tau(-1:\mathbb{Z}\longrightarrow\mathbb{Z}) \in K_1(\mathbb{Z})$ $(=\mathbb{Z}_2)$ and the product map

$$\omega = -\otimes\tau(-1) : \widetilde{K}_0(\mathbb{Z}[\pi])\longrightarrow Wh(\pi) ; [P]\longmapsto \tau(-1:P\longrightarrow P) .$$

If $P = im(p)$ for a projection $p = p^2 : F\longrightarrow F$ of a f.g. free $\mathbb{Z}[\pi]$-module F then the automorphism $1-2p:F\longrightarrow F$ is such that

$$\omega([P]) = \tau(1-2p:F\longrightarrow F) \in Wh(\pi).$$

<u>Proposition 3.3</u> The algebraically and geometrically significant

$$\begin{cases} \text{surjections } \varepsilon,\varepsilon':Wh(\pi\times\mathbb{Z})\twoheadrightarrow Wh(\pi) \\ \text{injections } \overline{B},\overline{B}':\widetilde{K}_0(\mathbb{Z}[\pi])\rightarrowtail Wh(\pi\times\mathbb{Z}) \end{cases} \text{differ by}$$

$$\begin{cases} \varepsilon' - \varepsilon = \omega B : Wh(\pi\times\mathbb{Z})\xrightarrow{B}\widetilde{K}_0(\mathbb{Z}[\pi])\xrightarrow{\omega}Wh(\pi) \\ \overline{B}' - \overline{B} = \overline{\varepsilon}\omega : \widetilde{K}_0(\mathbb{Z}[\pi])\xrightarrow{\omega}Wh(\pi)\xrightarrow{\overline{\varepsilon}}Wh(\pi\times\mathbb{Z}) \end{cases} .$$

[]

In particular, the difference between the algebraic and geometric splittings is 2-torsion only, since $2\omega = 0$.

It is tempting to identify the geometrically significant surjection $\varepsilon' : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi)$ with the surjection induced functorially by the split surjection of rings defined by $z \longmapsto -1$

$$\eta \ : \ \mathbb{Z}[\pi\times\mathbb{Z}] = \mathbb{Z}[\pi][z,z^{-1}] \longrightarrow \mathbb{Z}[\pi] \ ; \ \sum_{j=-\infty}^{\infty} a_j z^j \longmapsto \sum_{j=-\infty}^{\infty} a_j(-1)^j \ ,$$

and indeed

$$\begin{aligned}\varepsilon'| = \eta| \ : \ & \mathrm{im}((\bar\varepsilon \ \bar B) : Wh(\pi)\oplus\widetilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})) \\ & = \mathrm{im}((\bar\varepsilon \ \bar B') : Wh(\pi)\oplus\widetilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})) \longrightarrow Wh(\pi) \ .\end{aligned}$$

However, in general

$$\begin{aligned}\varepsilon'| \neq \eta| \ : \ & \mathrm{im}((\bar\Delta_+ \ \bar\Delta_-) : \widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})) \\ & \longrightarrow Wh(\pi)\end{aligned}$$

so that $\varepsilon' \neq \eta : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi)$.

For an automorphism $\alpha:\pi\longrightarrow\pi$ of a group π Farrell and Hsiang [6] and Siebenmann [33] expressed the Whitehead group of the α-twisted extension $\pi\times_\alpha\mathbb{Z}$ of π by $\mathbb{Z} = \langle z\rangle$ ($gz = z\alpha(g) \in \pi\times_\alpha\mathbb{Z}$ for $g\in\pi$) as a natural direct sum

$$Wh(\pi\times_\alpha\mathbb{Z}) = Wh(\pi,\alpha)\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha)\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha^{-1})$$

with $Wh(\pi,\alpha)$ the relative group in the exact sequence

$$Wh(\pi) \xrightarrow{1-\alpha} Wh(\pi) \xrightarrow{j} Wh(\pi,\alpha) \xrightarrow{\partial} \widetilde{K}_0(\mathbb{Z}[\pi]) \xrightarrow{1-\alpha} \widetilde{K}_0(\mathbb{Z}[\pi])$$

(the reduced version of the group $K_1(\mathbb{Z}[\pi],\alpha)$ discussed at the end of §1) and $\widetilde{Nil}(\mathbb{Z}[\pi],\alpha^{\pm 1})$ the exotic K-group of pairs (F,ν) with F a f.g. free $\mathbb{Z}[\pi]$-module and $\nu \in \mathrm{Hom}_{\mathbb{Z}[\pi]}((\alpha^{\pm 1})_! F, F)$ nilpotent. Given a f.g. projective $\mathbb{Z}[\pi]$-module P and an isomorphism $f\in \mathrm{Hom}_{\mathbb{Z}[\pi]}(\alpha_! P,P)$ there is defined a mixed invariant $[P,f]\in Wh(\pi,\alpha)$ with $\partial([P,f]) = [P] \in \widetilde{K}_0(\mathbb{Z}[\pi])$.
As in the untwisted case $\alpha = 1$ there are defined an <u>algebraically significant</u> splitting of $Wh(\pi\times_\alpha\mathbb{Z})$, with inverse isomorphisms

$$Wh(\pi\times_\alpha\mathbb{Z}) \underset{(\bar B \ \ \bar\Delta_+ \ \ \bar\Delta_-)}{\overset{\begin{pmatrix} B \\ \Delta_+ \\ \Delta_- \end{pmatrix}}{\rightleftarrows}} Wh(\pi,\alpha)\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha)\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha^{-1}) \ ,$$

and a <u>geometrically significant</u> splitting of $Wh(\pi\times_\alpha\mathbb{Z})$ with inverse isomorphisms

$$\mathrm{Wh}(\pi\times_\alpha\mathbb{Z}) \underset{(\bar{B}'\ \bar{\Delta}_+\ \bar{\Delta}_-)}{\overset{\begin{pmatrix} B' \\ \Delta_+ \\ \Delta_- \end{pmatrix}}{\rightleftarrows}} \mathrm{Wh}(\pi,\alpha)\oplus\widetilde{\mathrm{Nil}}(\mathbb{Z}[\pi],\alpha)\oplus\widetilde{\mathrm{Nil}}(\mathbb{Z}[\pi],\alpha^{-1}) \quad ,$$

with

$$\bar{B} : \mathrm{Wh}(\pi,\alpha) \rightarrowtail \mathrm{Wh}(\pi\times_\alpha\mathbb{Z}) \ ; \ [P,f] \longmapsto \tau(zf:P_\alpha[z,z^{-1}] \longrightarrow P_\alpha[z,z^{-1}])$$

$$\bar{B}' : \mathrm{Wh}(\pi,\alpha) \rightarrowtail \mathrm{Wh}(\pi\times_\alpha\mathbb{Z}) \ ; \ [P,f] \longmapsto \tau(-zf:P_\alpha[z,z^{-1}] \longrightarrow P_\alpha[z,z^{-1}])$$

$$\bar{\Delta}_\pm : \widetilde{\mathrm{Nil}}(\mathbb{Z}[\pi],\alpha^{\pm 1}) \rightarrowtail \mathrm{Wh}(\pi\times_\alpha\mathbb{Z}) \ ;$$
$$(P,\nu) \longmapsto \tau(1+z^{\pm 1}\nu:P_\alpha[z,z^{-1}] \longrightarrow P_\alpha[z,z^{-1}]) \quad ,$$

identifying $\mathbb{Z}[\pi\times_\alpha\mathbb{Z}] = \mathbb{Z}[\pi]_\alpha[z,z^{-1}]$. The automorphism

$$\Omega : \mathrm{Wh}(\pi,\alpha) \longrightarrow \mathrm{Wh}(\pi,\alpha) \ ; \ [P,f] \longmapsto [P,-f]$$

is such that $\Omega^2 = 1$ and

$$\bar{B}' = \bar{B}\Omega : \mathrm{Wh}(\pi,\alpha) \rightarrowtail \mathrm{Wh}(\pi\times_\alpha\mathbb{Z})$$

$$B' = \Omega B : \mathrm{Wh}(\pi\times_\alpha\mathbb{Z}) \twoheadrightarrow \mathrm{Wh}(\pi,\alpha) \quad .$$

In the untwisted case $\alpha = 1$ $\pi\times_\alpha\mathbb{Z}$ is just the product $\pi\times\mathbb{Z}$, and there is defined an isomorphism

$$\mathrm{Wh}(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow \mathrm{Wh}(\pi,1) \ ;$$
$$(\tau(f:P\longrightarrow P),[Q]) \longmapsto [P,f] - [P,1] + [Q,1]$$

with respect to which

$$\Omega = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : \mathrm{Wh}(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow \mathrm{Wh}(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi]) \quad .$$

The algebraically (resp. geometrically) significant splitting of $\mathrm{Wh}(\pi\times_\alpha\mathbb{Z})$ for $\alpha = 1$ corresponds under this isomorphism to the algebraically (resp. geometrically) significant splitting of $\mathrm{Wh}(\pi\times\mathbb{Z})$ defined previously.

A self homotopy equivalence $f:X\longrightarrow X$ of a finitely dominated CW complex X has a <u>mixed invariant</u>

$$[X,f] \in \mathrm{Wh}(\pi,\alpha)$$

with $\alpha = f_* : \pi = \pi_1(X) \longrightarrow \pi$, such that $\partial([X,f]) = [X] \in \tilde{K}_0(\mathbb{Z}[\pi])$, a reduction of the mixed invariant $[X,f] \in K_1(\mathbb{Z}[\pi],\alpha)$ described at the end of §1. Let $f^{-1}:X\longrightarrow X$ be a homotopy inverse, with homotopy $e:f^{-1}f \simeq 1:X\longrightarrow X$. The mapping tori of f and f^{-1} are related by the homotopy equivalence

$$U : T(f^{-1}) \longrightarrow T(f) \ ; \ (x,t) \longmapsto (e(x,t),1-t)$$

inducing the isomorphism of fundamental groups

$$U_* : \pi_1(T(f^{-1})) = \pi \times_{\alpha^{-1}} \mathbb{Z} \longrightarrow \pi_1(T(f)) = \pi \times_\alpha \mathbb{Z} \ ;$$
$$g\ (\in \pi) \longmapsto g \ , \ z \longmapsto z^{-1} \ .$$

The torsion of U with respect to the canonical round finite structures given by Proposition 1.3 is

$$\tau(U) = \tau(-z\tilde{f}:C(\tilde{X})_\alpha[z,z^{-1}] \longrightarrow C(\tilde{X})_\alpha[z,z^{-1}]) \in K_1(\mathbb{Z}[\pi]_\alpha[z,z^{-1}]) \quad ,$$

so that:

<u>Proposition 3.4</u> The geometrically defined split injection is given geometrically by

$$\bar{B}' : Wh(\pi,\alpha) \rightarrowtail Wh(\pi\times_\alpha\mathbb{Z}) \ ; \ [X,f] \longmapsto \tau(U:T(f^{-1}) \longrightarrow T(f)) \ .$$

[]

Proposition 3.3 is just the untwisted case $\alpha = 1$ of Proposition 3.4, with $f = 1 : X \longrightarrow X$ and

$$U = 1 \times -1 : T(1:X \longrightarrow X) = X \times S^1 \longrightarrow T(1) = X \times S^1 \ ,$$
$$-1 : S^1 = \mathbb{R}/\mathbb{Z} \longrightarrow S^1 \ ; \ t \longmapsto 1-t \ .$$

The exact sequence

$$Wh(\pi) \xrightarrow{1-\alpha} Wh(\pi) \xrightarrow{\bar{\varepsilon}} Wh(\pi\times_\alpha\mathbb{Z})$$
$$\xrightarrow{\begin{pmatrix}\partial B\\ \Delta_+\\ \Delta_-\end{pmatrix}} \tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha)\oplus\widetilde{Nil}(\mathbb{Z}[\pi],\alpha^{-1})$$
$$\xrightarrow{(1-\alpha\ \ 0\ \ 0)} \tilde{K}_0(\mathbb{Z}[\pi]) \xrightarrow{\bar{\varepsilon}} \tilde{K}_0(\mathbb{Z}[\pi\times_\alpha\mathbb{Z}])$$
$$(\bar{\varepsilon} = \bar{B}j = \bar{B}'j \ , \ \partial B = \partial B')$$

has a geometric interpretation in terms of codimension 1 splitting obstructions for homotopy equivalences $f:M^n \longrightarrow X$ with $\pi_1(X) = \pi\times_\alpha\mathbb{Z}$ (Farrell and Hsiang [5],[7]), as in the untwisted case $\alpha = 1$.

The obstruction theory of Farrell [4] and Siebenmann [33] for fibering manifolds over S^1 can be used to give the injection $\bar{B}':\tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})$ a further degree of geometric significance, as follows.

Let $p:\overline{X} \longrightarrow X$ be the covering projection of a regular infinite cyclic cover of a connected space X, with $\overline{X}$ connected also. Let $\zeta:\overline{X} \longrightarrow \overline{X}$ be a generating covering translation, inducing the automorphism $\zeta_* = \alpha : \pi_1(\overline{X}) = \pi \longrightarrow \pi$. The map

$$T(\zeta) \longrightarrow X \ ; \ (x,t) \longmapsto p(x)$$

is a homotopy equivalence, inducing an isomorphism of fundamental groups $\pi_1(T(\zeta)) = \pi \times_\alpha \mathbb{Z} \longrightarrow \pi_1(X)$. If X is a finite CW complex and $\overline{X}$ is finitely dominated the canonical (round) finite structure on $T(\zeta)$ given by Proposition 1.3 can be used to define the <u>fibering obstruction</u>

$$\Phi(X) = \tau(T(\zeta) \longrightarrow X) \in Wh(\pi \times_\alpha \mathbb{Z}) \ .$$

This is the invariant described (but not defined) by Siebenmann [31]. If X is a compact n-manifold with the finite structure determined by a handlebody decomposition then $\Phi(X) = 0$ if (and for $n \geqslant 6$ only if) X fibres over S^1 in a manner compatible with p, by the theory of Farrell [4] and Siebenmann [33].

Given a finitely dominated CW complex X with $\pi_1(X) = \pi$ let $Y \longrightarrow X \times S^1$ be a homotopy equivalence from a finite CW complex Y in the canonical finite structure. Embed $Y \subset S^N$ (N large) with closed regular neighbourhood an N-dimensional manifold with boundary $(Z,\partial Z)$, and let $(\overline{Z},\overline{\partial Z})$ be the infinite cyclic cover of $(Z,\partial Z)$ classified by the projection

$$\pi_1(Z) = \pi_1(\partial Z) = \pi_1(X \times S^1) = \pi \times \mathbb{Z} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z} \ .$$

Thicken up the self homotopy equivalence transposing the S^1-factors

$$1 \times T : X \times S^1 \times S^1 \longrightarrow X \times S^1 \times S^1 \ ; \ (x,s,t) \longmapsto (x,t,s)$$

to a self homotopy equivalence of a pair

$$(f,\partial f) \ : \ (Z,\partial Z) \times S^1 \longrightarrow (Z,\partial Z) \times S^1$$

inducing on the fundamental group the automorphism

$$\pi \times \mathbb{Z} \times \mathbb{Z} \longrightarrow \pi \times \mathbb{Z} \times \mathbb{Z} \ ; \ (x,s,t) \longmapsto (x,t,s)$$

transposing the $\mathbb{Z}$-factors. Thus $(f,\partial f)$ lifts to a $\mathbb{Z}$-equivariant homotopy equivalence

$$(\overline{f},\overline{\partial f}) \ : \ (\overline{Z},\overline{\partial Z}) \times S^1 \longrightarrow (Z,\partial Z) \times \mathbb{R} \ .$$

In particular, this shows that ∂Z is a finite CW complex with a finitely dominated infinite cyclic cover $\overline{\partial Z}$.

<u>Proposition 3.5</u> The geometrically significant injection is such that

$$\overline{B}' \ : \ \tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi \times \mathbb{Z}) \ ; \ [X] \longmapsto \Phi(\partial Z) \ .$$

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§4. Absolute L-theory invariants

The duality involutions on the algebraic K-groups of a ring A with involution $^{-}:A\longrightarrow A;a\longmapsto \bar{a}$ are defined as usual by

$$* : K_0(A)\longrightarrow K_0(A)\ ;\ [P]\longmapsto [P^*]\ ,\ P^* = \mathrm{Hom}_A(P,A)$$

$$* : K_1(A)\longrightarrow K_1(A)\ ;\ \tau(f:P\longrightarrow P)\longmapsto \tau(f^*:P^*\longrightarrow P^*)\ ,$$

with reduced versions for $\tilde{K}_0(A)$, $\tilde{K}_1(A)$. We shall only be concerned with group rings $A = \mathbb{Z}[\pi]$ and the involution $\bar{g} = w(g)g^{-1}$ $(g\in\pi)$ determined by a group morphism $w:\pi\longrightarrow\mathbb{Z}_2 = \{\pm 1\}$, so that there is also defined a duality involution $*:Wh(\pi)\longrightarrow Wh(\pi)$.

The $\left\{\begin{array}{l}\text{projective class}\\ \text{Whitehead torsion}\end{array}\right.$ of a $\left\{\begin{array}{l}\text{finitely dominated}\\ \text{finite}\end{array}\right.$ n-dimensional geometric Poincaré complex X with $\pi_1(X) = \pi$

$$\left\{\begin{array}{l}[X] = [C(\tilde{X})]\in K_0(\mathbb{Z}[\pi])\\ \tau(X) = \tau(C(\tilde{X})^{n-*}\longrightarrow C(\tilde{X}))\in Wh(\pi)\end{array}\right.$$

satisfies the usual duality formula

$$\left\{\begin{array}{l}[X]^* = (-)^n[X]\in K_0(\mathbb{Z}[\pi])\\ \tau(X)^* = (-)^n\tau(X)\in Wh(\pi)\ .\end{array}\right.$$

The <u>torsion</u> of a round finite n-dimensional geometric Poincaré complex X

$$\tau(X) = \tau(C(\tilde{X})^{n-*}\longrightarrow C(\tilde{X}))\in K_1(\mathbb{Z}[\pi])$$

is such that

$$\tau(X)^* = (-)^n\tau(X)\in K_1(\mathbb{Z}[\pi])\ .$$

The Poincaré duality chain equivalence for the universal cover $\tilde{S}^1 = \mathbb{R}$ of the circle S^1 is given by

$$\begin{array}{rcccc} C(\tilde{S}^1)^{1-*} & : & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z^{-1}} & \mathbb{Z}[z,z^{-1}]\\ {\scriptstyle [S^1]\cap -}\big\downarrow & & {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle -z}\\ C(\tilde{S}^1) & : & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z} & \mathbb{Z}[z,z^{-1}] \end{array}\ ,$$

so that S^1 has torsion

$$\begin{aligned}\tau(S^1) &= \tau([S^1]\cap -:C(\tilde{S}^1)^{1-*}\longrightarrow C(\tilde{S}^1))\\ &= \tau(-z:\mathbb{Z}[z,z^{-1}]\longrightarrow\mathbb{Z}[z,z^{-1}])\\ &\qquad\qquad \in K_1(\mathbb{Z}[z,z^{-1}])\ .\end{aligned}$$

This is the special case $f = 1 : X = \{pt.\}\longrightarrow\{pt.\}$ of the following formula, which is the Poincaré complex version of Propositions 1.3,3.4.

Proposition 4.1 Let $f:X \longrightarrow X$ be a self homotopy equivalence of a finitely dominated n-dimensional geometric Poincaré complex X inducing the automorphism $f_* = \alpha : \pi_1(X) = \pi \longrightarrow \pi$ and the $\mathbb{Z}[\pi]$-module chain equivalence $\tilde{f} : \alpha_! C(\tilde{X}) \longrightarrow C(\tilde{X})$. The mapping torus T(f) is an (n+1)-dimensional geometric Poincaré complex with canonical round finite structure, with torsion

$$\tau(T(f)) = \tau(-z\tilde{f}:C(\tilde{X})_\alpha[z,z^{-1}] \longrightarrow C(\tilde{X})_\alpha[z,z^{-1}]) \in K_1(\mathbb{Z}[\pi]_\alpha[z,z^{-1}]) .$$

[]

For $f = 1 : X \longrightarrow X$ the formula of Proposition 4.1 gives

$$\tau(X \times S^1) = \tau(-z:C(\tilde{X})[z,z^{-1}] \longrightarrow C(\tilde{X})[z,z^{-1}])$$
$$= [X]\otimes\tau(S^1) = \overline{B}'([X]) \in K_1(\mathbb{Z}[\pi][z,z^{-1}])$$

with $[X] \in K_0(\mathbb{Z}[\pi])$ the projective class and $\overline{B}'$ the absolute version

$$\overline{B}' : K_0(\mathbb{Z}[\pi]) \rightarrowtail K_1(\mathbb{Z}[\pi][z,z^{-1}]) ;$$
$$[P] \longmapsto \tau(-z:P[z,z^{-1}] \longrightarrow P[z,z^{-1}])$$

(also a split injection) of $\overline{B}':\tilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z})$.

For a finitely presented group π every element $x \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction $x = [X]$ of a finitely dominated CW complex X with $\pi_1(X) = \pi$, by the realization theorem of Wall [34]. We need the version for Poincaré complexes:

Proposition 4.2 (Pedersen and Ranicki [18]) For a finitely presented group π every element $x \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction $x = [X]$ for a finitely dominated geometric Poincaré pair $(X,\partial X)$ with $\pi_1(X) = \pi$.

[]

The method of [18] used the obstruction theory of Siebenmann [30]. The construction of Proposition 3.5 gives a more direct method, since $(\overline{Z},\overline{\partial Z})$ is a finitely dominated (N-1)-dimensional geometric Poincaré pair with prescribed $[\overline{Z}] \in \tilde{K}_0(\mathbb{Z}[\pi])$. (Moreover, if the evident map of pairs $(e,\partial e):(Z,\partial Z) \longrightarrow S^1$ is made transverse regular at pt. $\in S^1$ the inclusion

$$(M,\partial M) = (e,\partial e)^{-1}(\{pt.\}) \longrightarrow (Z,\partial Z)$$

lifts to a normal map

$$(f,b) : (M,\partial M) \longrightarrow (\overline{Z},\overline{\partial Z})$$

from a compact (N-1)-dimensional manifold with boundary. This gives a more direct proof of the realization theorem of [18] for the projective surgery groups $L^p_*(\pi)$, except possibly in the low dimensions).

By the relative version of Proposition 4.1 the product of a finitely dominated n-dimensional geometric Poincaré pair $(X,\partial X)$ and the circle S^1 is an $(n+1)$-dimensional geometric Poincaré pair

$$(X,\partial X)\times S^1 = (X\times S^1,\partial X\times S^1)$$

with canonical round finite structure, and torsion

$$\begin{aligned}\tau(X\times S^1,\partial X\times S^1) &= \tau(-z:C(\widetilde{X})[z,z^{-1}]\longrightarrow C(\widetilde{X})[z,z^{-1}])\\ &= [X]\otimes\tau(S^1) = \overline{B}'([X]) \in K_1(\mathbb{Z}[\pi][z,z^{-1}]) \quad .\end{aligned}$$

Combined with Proposition 4.2 this gives:

<u>Proposition 4.3</u> The geometrically significant injection is such that

$$\overline{B}' : \widetilde{K}_0(\mathbb{Z}[\pi]) \rightarrowtail Wh(\pi\times\mathbb{Z}) \ ; \ [X] \longmapsto \tau(X\times S^1,\partial X\times S^1) \ ,$$

for any finitely dominated geometric Poincaré pair $(X,\partial X)$ with $\pi_1(X)=\pi$
[]

In §5 this will be seen to be a special case of the product formula for the torsion of (finitely dominated) × (round finite) Poincaré complexes.

Given a $*$-invariant subgroup $S\subseteq \widetilde{K}_0(\mathbb{Z}[\pi])$ (resp. $S\subseteq Wh(\pi)$) let $\begin{cases} L^n_S(\pi) \\ L^S_n(\pi)\end{cases}$ $(n\geqslant 0)$ be the cobordism group of finitely dominated (resp. finite) n-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complexes over $\mathbb{Z}[\pi]$ $\begin{cases}(C,\phi\in Q^n(C))\\ (C,\psi\in Q_n(C))\end{cases}$ with finiteness obstruction $[C]\in S\subseteq\widetilde{K}_0(\mathbb{Z}[\pi])$ (resp. Whitehead torsion $\begin{cases}\tau(C,\phi) = \tau(\phi_0:C^{n-*}\longrightarrow C)\\ \tau(C,\psi) = \tau((1+T)\psi_0:C^{n-*}\longrightarrow C)\end{cases}$ $\in S\subseteq Wh(\pi)$).

A finitely dominated (resp. finite) n-dimensional geometric Poincaré complex X with $\pi_1(X)=\pi$ and $[X]\in S$ (resp. $\tau(X)\in S$) has a <u>symmetric signature</u> invariant

$$\sigma^*_S(X) = (C(\widetilde{X}),\phi) \in L^n_S(\pi)$$

with $\phi_0 = [X]\cap - : C(\widetilde{X})^{n-*}\longrightarrow C(\widetilde{X})$, and a normal map $(f,b):M\longrightarrow X$ of such complexes has a <u>quadratic signature</u> invariant

$$\sigma^S_*(f,b) \in L^S_n(\pi)$$

such that $(1+T)\sigma^S_*(f,b) = \sigma^*_S(M) - \sigma^*_S(X)$. See Ranicki [22],[23] for the details. In the extreme cases $S = \{0\},\widetilde{K}_0(\mathbb{Z}[\pi])$ (resp. $\{0\},Wh(\pi)$) the notation is abbreviated in the usual fashion

$$\begin{cases} L^n_{\tilde{K}_0(\mathbb{Z}[\pi])}(\pi) = L^n_p(\pi) \\ L_n^{\tilde{K}_0(\mathbb{Z}[\pi])}(\pi) = L^p_n(\pi) \end{cases} , \begin{cases} L^n_{\{0\}\subseteq Wh(\pi)}(\pi) = L^n_s(\pi) \\ L_n^{\{0\}\subseteq Wh(\pi)}(\pi) = L^s_n(\pi) \end{cases}$$

$$\begin{cases} L^n_{\{0\}\subseteq \tilde{K}_0(\mathbb{Z}[\pi])}(\pi) = L^n_{Wh(\pi)}(\pi) = L^n_h(\pi) \\ L_n^{\{0\}\subseteq \tilde{K}_0(\mathbb{Z}[\pi])}(\pi) = L_n^{Wh(\pi)}(\pi) = L^h_n(\pi) \end{cases} .$$

In particular, the simple quadratic L-groups $L^s_*(\pi)$ are the original surgery obstruction groups of Wall [35], with $\sigma^s_*(f,b)$ the surgery obstruction.

The <u>torsion</u> of a round finite n-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complex over $\mathbb{Z}[\pi]$ $\begin{cases}(C,\phi)\\ (C,\psi)\end{cases}$ is defined by

$$\begin{cases} \tau(C,\phi) = \tau(\phi_0 : C^{n-*} \longrightarrow C) \in K_1(\mathbb{Z}[\pi]) \\ \tau(C,\psi) = \tau((1+T)\psi_0 : C^{n-*} \longrightarrow C) \in K_1(\mathbb{Z}[\pi]) \end{cases}$$

and is such that

$$\begin{cases} \tau(C,\phi)^* = (-)^n \tau(C,\phi) \\ \tau(C,\psi)^* = (-)^n \tau(C,\psi) \end{cases} \in K_1(\mathbb{Z}[\pi]) .$$

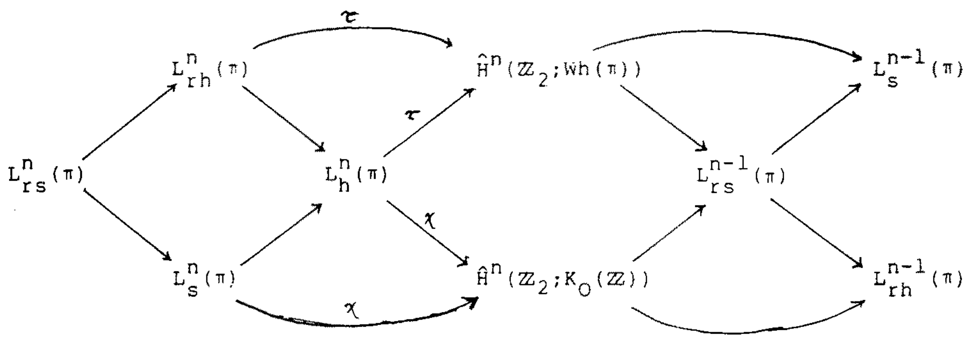
Given a $*$-invariant subgroup $S \subseteq K_1(\mathbb{Z}[\pi])$ define the <u>round</u> $\begin{cases}\underline{\text{symmetric}}\\ \underline{\text{quadratic}}\end{cases}$ <u>L-group</u> $\begin{cases} L^n_{rS}(\pi) \\ L_n^{rS}(\pi)\end{cases}$ $(n \geqslant 0)$ to be the cobordism group of round finite n-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ Poincaré complexes over $\mathbb{Z}[\pi]$ $\begin{cases}(C,\phi)\\ (C,\psi)\end{cases}$ with torsion $\begin{cases}\tau(C,\phi)\\ \tau(C,\psi)\end{cases} \in S \subseteq K_1(\mathbb{Z}[\pi])$. See Hambleton, Ranicki and Taylor [11] for an exposition of round L-theory. We shall only be concerned with the round symmetric L-groups L^*_{rS} here, adopting the terminology

$$L^n_{rh}(\pi) = L^n_{rK_1(\mathbb{Z}[\pi])}(\pi) \quad , \quad L^n_{rs}(\pi) = L^n_{r\{\pm\pi\}}(\pi) .$$

The Rothenberg exact sequence for the quadratic L-groups

$$\ldots \longrightarrow L^s_n(\pi) \longrightarrow L^h_n(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2; Wh(\pi)) \longrightarrow L^s_{n-1}(\pi) \longrightarrow \ldots$$

has versions for the symmetric and round symmetric L-groups which fit together in a commutative braid of exact sequences



with the maps τ (resp. χ) defined by the Whitehead torsion (resp. Euler characteristic). In the case $Wh(\pi) = 0$ the L-groups $\begin{cases} L^*_{rh}(\pi) = L^*_{rs}(\pi) \\ L^*_h(\pi) = L^*_s(\pi) \end{cases}$ are abbreviated to $\begin{cases} L^*_r(\pi) \\ L^*(\pi) \end{cases}$. The L-groups of the trivial group $\pi = \{1\}$ are given by

$$L^n(\{1\}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \\ 0 \\ 0 \end{cases} , \quad L^n_r(\{1\}) = \begin{cases} \mathbb{Z} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ 0 \\ 0 \end{cases} \quad \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 ,$$

with isomorphisms

$$L^{4k}(\{1\}) \longrightarrow \mathbb{Z} \; ; \; (C,\phi) \longmapsto \text{signature}(C,\phi)$$
$$L^{4k+1}(\{1\}) \longrightarrow \mathbb{Z}_2 \; ; \; (C,\phi) \longmapsto \text{deRham}(C,\phi) = \chi_{\frac{1}{2}}(C;\mathbb{Z}_2) + \chi_{\frac{1}{2}}(C;\mathbb{Q})$$
$$L^{4k}_r(\{1\}) \longrightarrow \mathbb{Z} \; ; \; (C,\phi) \longmapsto \tfrac{1}{2}(\text{signature}(C,\phi))$$
$$L^{4k+1}_r(\{1\}) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \; ; \; (C,\phi) \longmapsto (\chi_{\frac{1}{2}}(C;\mathbb{Z}_2), \chi_{\frac{1}{2}}(C;\mathbb{Q})) \quad .$$

(See [11] for details. The F-coefficient semicharacteristic of a $(2i+1)$-dimensional $\mathbb{Z}$-module chain complex C is defined by

$$\chi_{\frac{1}{2}}(C;F) = \sum_{r=0}^{i} (-)^r \text{rank}_F H_r(C) \in \mathbb{Z} \quad ,$$

for any field F).

The torsion of a round finite n-dimensional geometric Poincaré complex X with $\pi_1(X) = \pi$ is the torsion of the associated round finite n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\pi]$ $(C(\tilde{X}),\phi)$

$$\tau(X) = \tau(C(\tilde{X}),\phi) = \tau(\phi_0 = [X] \cap - : C(\tilde{X})^{n-*} \longrightarrow C(\tilde{X})) \in K_1(\mathbb{Z}[\pi]) \quad .$$

If $S \subseteq K_1(\mathbb{Z}[\pi])$ is a $*$-invariant subgroup such that $\tau(X) \in S$ the <u>round</u>

symmetric signature of X is defined by

$$\sigma^*_{rS}(X) = (C(\widetilde{X}),\phi) \in L^n_{rS}(\pi) .$$

In the case $S = K_1(\mathbb{Z}[\pi])$ (resp. $\{\pm\pi\}$) this is denoted $\sigma^*_{rh}(X) \in L^n_{rh}(\pi)$ (resp. $\sigma^*_{rs}(X) \in L^n_{rs}(\pi)$), and if also $Wh(\pi) = 0$ by $\sigma^*_r(X) \in L^n_r(\pi)$.

We shall be particularly concerned with the round symmetric signature of the circle S^1

$$\sigma^*_r(S^1) = (C(\widetilde{S}^1),\phi) \in L^1_r(\mathbb{Z}) .$$

The image of the $\mathbb{Z}[z,z^{-1}]$-module chain complex

$$C(\widetilde{S}^1) : \mathbb{Z}[z,z^{-1}] \xrightarrow{1-z} \mathbb{Z}[z,z^{-1}]$$

under the morphism of rings with involution

$$\begin{cases} \varepsilon : \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z} ; \ z \longmapsto 1 \\ \eta : \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z,z^{-1}] \longrightarrow \mathbb{Z} ; \ z \longmapsto -1 \end{cases} \qquad (\bar{z} = z^{-1})$$

is the $\mathbb{Z}$-module chain complex

$$\begin{cases} \varepsilon_! C(\widetilde{S}^1) : \mathbb{Z} \xrightarrow{0} \mathbb{Z} \\ \eta_! C(\widetilde{S}^1) : \mathbb{Z} \xrightarrow{2} \mathbb{Z} \end{cases}$$

with mod2 and rational semicharacteristics $\begin{cases} (\chi_{\frac{1}{2}}(C;\mathbb{Z}_2),\chi_{\frac{1}{2}}(C;\mathbb{Q})) = (1,1) \\ (\chi_{\frac{1}{2}}(D;\mathbb{Z}_2),\chi_{\frac{1}{2}}(D;\mathbb{Q})) = (1,0) \end{cases}$

so that $\sigma^*_r(S^1) \in L^1_r(\mathbb{Z})$ has images

$$\begin{cases} \varepsilon_!\sigma^*_r(S^1) = (1,1) \\ \eta_!\sigma^*_r(S^1) = (1,0) \end{cases} \in L^1_r(\{1\}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 .$$

The algebraic proof of the splitting theorem for the quadratic L-groups $L^s_n(\pi\times\mathbb{Z}) = L^s_n(\pi)\oplus L^h_{n-1}(\pi)$ discussed in §6 below can be extended to prove analogous splitting theorems for the symmetric and round symmetric L-groups

$$L^n_s(\pi\times\mathbb{Z}) = L^n_s(\pi)\oplus L^{n-1}_h(\pi) \quad , \quad L^n_{rs}(\pi\times\mathbb{Z}) = L^n_{rs}(\pi)\oplus L^{n-1}_h(\pi) .$$

Thus $L^1_r(\mathbb{Z}) = L^1_r(\{1\})\oplus L^0(\{1\}) = \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}$, although we do not actually need this computation here.

§5. Products in L-theory

The product of an m-dimensional $\left\{\begin{array}{l}\text{symmetric}\\\text{quadratic}\end{array}\right.$ Poincaré complex over A

(C,ϕ) and an n-dimensional symmetric Poincaré complex over B (D,θ) is an $(m+n)$-dimensional $\left\{\begin{array}{l}\text{symmetric}\\\text{quadratic}\end{array}\right.$ Poincaré complex over $A\otimes B$

$$(C,\phi)\otimes(D,\theta) = (C\otimes D,\phi\otimes\theta) \quad ,$$

allowing the definition (in Ranicki [22]) of products in L-theory of the type

$$\left\{\begin{array}{l} L^m(A)\otimes L^n(B) \longrightarrow L^{m+n}(A\otimes B) \\ L_m(A)\otimes L^n(B) \longrightarrow L_{m+n}(A\otimes B) \ . \end{array}\right.$$

We shall only be concerned with the product $L_m\otimes L^n \longrightarrow L_{m+n}$ here, with $A = \mathbb{Z}[\pi]$, $B = \mathbb{Z}[\rho]$ group rings, so that $A\otimes B = \mathbb{Z}[\pi\times\rho]$.

The product of a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ m-dimensional symmetric (resp. quadratic) Poincaré complex over $\mathbb{Z}[\pi]$ (C,ϕ) and a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\rho]$ (D,θ) is a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ $(m+n)$-dimensional symmetric (resp. quadratic) Poincaré complex over $\mathbb{Z}[\pi\times\rho]$ $(C\otimes D,\phi\otimes\theta)$ with $\left\{\begin{array}{l}\text{projective class}\\\text{Whitehead torsion}\end{array}\right.$

$$\left\{\begin{array}{l} [C\otimes D] = [C]\otimes[D] \in K_0(\mathbb{Z}[\pi\times\rho]) \\ \tau(C\otimes D,\phi\otimes\theta) = \tau(C,\phi)\otimes\chi(D) + \chi(C)\otimes\tau(D,\theta) \in Wh(\pi\times\rho) \ . \end{array}\right.$$

The following product formulae for geometric Poincaré complexes are immediate consequences.

<u>Proposition 5.1</u> The product of a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ m-dimensional geometric Poincaré complex X with $\pi_1(X) = \pi$ and a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ n-dimensional geometric Poincaré complex F with $\pi_1(F) = \rho$ is a $\left\{\begin{array}{l}\text{finitely dominated}\\\text{finite}\end{array}\right.$ $(m+n)$-dimensional geometric Poincaré complex $X\times F$

with $\begin{cases}\text{projective class}\\ \text{Whitehead torsion}\end{cases}$

$$\begin{cases}[X\times F] = [X]\otimes[F] \in K_0(\mathbb{Z}[\pi\times\rho])\\ \tau(X\times F) = \tau(X)\otimes\chi(F) + \chi(X)\otimes\tau(F) \in Wh(\pi\times\rho)\ .\end{cases}$$

[]

Given *-invariant subgroups $\begin{cases}S \subseteq \tilde{K}_0(\mathbb{Z}[\pi])\\ S \subseteq Wh(\pi)\end{cases}$, $\begin{cases}T \subseteq \tilde{K}_0(\mathbb{Z}[\rho])\\ T \subseteq Wh(\rho)\end{cases}$,
$\begin{cases}U \subseteq \tilde{K}_0(\mathbb{Z}[\pi\times\rho])\\ U \subseteq Wh(\pi\times\rho)\end{cases}$ such that $\begin{cases}[P\otimes Q] \in U\\ \tau(f)\otimes 1, 1\otimes\tau(g) \in U\end{cases}$ for $\begin{cases}[P] \in S,\ [Q] \in T\\ \tau(f) \in S,\ \tau(g) \in T\end{cases}$
there is defined a product in L-theory

$$\otimes\ :\ L^S_m(\pi)\otimes L^n_T(\rho) \longrightarrow L^U_{m+n}(\pi\times\rho)\ \ ;\ \ (C,\psi)\otimes(D,\theta) \longmapsto (C\otimes D, \psi\otimes\theta)$$

with the following geometric interpretation.

<u>Proposition 5.2</u> (Ranicki [23]) If $(f,b):M \longrightarrow X$ is a normal map of $\begin{cases}\text{finitely dominated}\\ \text{finite}\end{cases}$ m-dimensional geometric Poincaré complexes with $\pi_1(X) = \pi$ and $\begin{cases}[M]-[X] \in S \subseteq \tilde{K}_0(\mathbb{Z}[\pi])\\ \tau(M)-\tau(X) \in S \subseteq Wh(\pi)\end{cases}$, and if F is a $\begin{cases}\text{finitely dominated}\\ \text{finite}\end{cases}$ n-dimensional geometric Poincaré complex with $\pi_1(F) = \rho$ and $\begin{cases}[F] \in T \subseteq \tilde{K}_0(\mathbb{Z}[\rho])\\ \tau(F) \in T \subseteq Wh(\rho)\end{cases}$, then the quadratic signature of the product normal map of $\begin{cases}\text{finitely dominated}\\ \text{finite}\end{cases}$ (m+n)-dimensional geometric Poincaré complexes

$$(g,c) = (f,b)\times 1\ :\ M\times F \longrightarrow X\times F$$

is given by

$$\sigma^U_*(g,c) = \sigma^S_*(f,b)\otimes\sigma^*_T(F) \in L^U_{m+n}(\pi\times\rho)\ ,$$

the product of $\sigma^S_*(f,b) \in L^S_m(\pi)$ and $\sigma^*_T(F) \in L^n_T(\rho)$.

[]

The methods of Ranicki [26] apply to the products of algebraic Poincaré complexes, giving the following analogues of Propositions 2.2, 5.2:

Proposition 5.3 i) The product of a finitely dominated m-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi]$ (C,ψ) and a round finite n-dimensional symmetric Poincaré complex over $\mathbb{Z}[\rho]$ (D,θ) is an $(m+n)$-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi\times\rho]$ $(C\otimes D,\psi\otimes\theta)$ with canonical round finite structure, and torsion

$$\tau(C\otimes D,\psi\otimes\theta) = [C]\otimes\tau(D,\theta) \in K_1(\mathbb{Z}[\pi\times\rho])$$

the product of $[C]\in K_0(\mathbb{Z}[\pi])$ and $\tau(D,\theta)\in K_1(\mathbb{Z}[\rho])$.

ii) Given *-invariant subgroups $S\subseteq \widetilde{K}_0(\mathbb{Z}[\pi])$, $T\subseteq K_1(\mathbb{Z}[\rho])$, $U\subseteq Wh(\pi\times\rho)$ such that $S\otimes T\subseteq U$ there is defined a product in L-theory

$$\otimes : L^S_m(\pi)\otimes L^n_{rT}(\rho) \longrightarrow L^U_{m+n}(\pi\times\rho) \ ; \ (C,\psi)\otimes(D,\theta)\longmapsto (C\otimes D,\psi\otimes\theta) \ .$$

If $(f,b):M\longrightarrow X$ is a normal map of finitely dominated n-dimensional geometric Poincaré complexes with $\pi_1(X)=\pi$ and $[M]-[X]\in S\subseteq\widetilde{K}_0(\mathbb{Z}[\pi])$, and if F is a round finite n-dimensional geometric Poincaré complex with $\pi_1(F)=\rho$ and $\tau(F)\in T\subseteq K_1(\mathbb{Z}[\rho])$ then the product map of $(m+n)$-dimensional geometric Poincaré complexes with canonical (round) finite structure

$$(g,c) = (f,b)\times 1 : M\times F \longrightarrow X\times F$$

has quadratic signature

$$\sigma^U_*(g,c) = \sigma^S_*(f,b)\otimes\sigma^*_{rT}(F) \in L^U_{m+n}(\pi\times\rho)$$

the product of $\sigma^S_*(f,b)\in L^S_m(\pi)$ and $\sigma^*_{rT}(F)\in L^n_{rT}(\rho)$.

[]

An n-dimensional geometric Poincaré complex F is round simple if it is round finite and

$$\tau(F) \in \{\pm\rho\}\subseteq K_1(\mathbb{Z}[\rho]) \quad (\rho = \pi_1(F)) \ ,$$

so that $\tau(F) = 0 \in Wh(\rho)$ and the round simple symmetric signature $\sigma^*_{rs}(F)\in L^n_{rs}(\rho)$ is defined.

Proposition 5.3 shows in particular that for a round $\begin{cases}\text{finite}\\ \text{simple}\end{cases}$ n-dimensional geometric Poincaré complex F product with the round $\begin{cases}\text{finite}\\ \text{simple}\end{cases}$ symmetric signature $\begin{cases}\sigma^*_{rh}(F)\in L^n_{rh}(\rho)\\ \sigma^*_{rs}(F)\in L^n_{rs}(\rho)\end{cases}$ defines a morphism of

$$\begin{cases} -\otimes\sigma^*_{rh}(F) \ : \ L^p_m(\pi) \longrightarrow L^h_{m+n}(\pi\times\rho) \\ -\otimes\sigma^*_{rs}(F) \ : \ L^h_m(\pi) \longrightarrow L^s_{m+n}(\pi\times\rho) \ . \end{cases}$$

In the simple case these products define a map of generalized Rothenberg exact sequences

$$\begin{array}{ccccccccc} \ldots \longrightarrow & L^h_m(\pi) & \longrightarrow & L^p_m(\pi) & \longrightarrow & \hat{H}^m(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) & \longrightarrow & L^h_{m-1}(\pi) & \longrightarrow \ldots \\ & \downarrow{\scriptstyle -\otimes\sigma^*_{rs}(F)} & & \downarrow{\scriptstyle -\otimes\sigma^*_{rh}(F)} & & \downarrow{\scriptstyle -\otimes\tau(F)} & & \downarrow{\scriptstyle -\otimes\sigma^*_{rs}(F)} & \\ \ldots \longrightarrow & L^s_{m+n}(\pi\times\rho) & \longrightarrow & L^h_{m+n}(\pi\times\rho) & \longrightarrow & \hat{H}^{m+n}(\mathbb{Z}_2;Wh(\pi\times\rho)) & \longrightarrow & L^s_{m+n-1}(\pi\times\rho) & \longrightarrow \ldots \end{array}$$

with $\tau(F) \in \{\pm\rho\} \subseteq K_1(\mathbb{Z}[\rho])$. The map of exact sequences in the appendix of Munkholm and Ranicki [16] is the special case $F = S^1$. Moreover, the split injection

$$\bar{B}' = -\otimes\tau(S^1) \ : \ \hat{H}^m(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \hat{H}^{m+1}(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z}))$$

was identified there with the connecting map δ arising from a short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \longrightarrow Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(p^!) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow 0 \quad ,$$

with $Wh(p^!)$ the relative Whitehead group in the exact sequence of transfer maps

$$Wh(\pi) \xrightarrow{\tilde{p}_1^! = 0} Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(p^!) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \xrightarrow{\tilde{p}_0^! = 0} \tilde{K}_0(\mathbb{Z}[\pi\times\mathbb{Z}])$$

associated to the trivial S^1-bundle

$$S^1 \longrightarrow E = K(\pi,1) \times S^1 \xrightarrow{p \ = \ \text{projection}} B = K(\pi,1)$$

and $\mathbb{Z}_2$ acting by duality involutions. The relationship between transfer maps and duality in algebraic K-theory will be studied in Lück and Ranicki [13] for any fibration $F \longrightarrow E \xrightarrow{p} B$ with the fibre F a finitely dominated n-dimensional geometric Poincaré complex. In particular, there will be defined a duality involution $*:K_1(p^!) \longrightarrow K_1(p^!)$ on the relative K-group $K_1(p^!)$ in the transfer exact sequence

$$K_1(\mathbb{Z}[\pi_1(B)]) \xrightarrow{p_1^!} K_1(\mathbb{Z}[\pi_1(E)]) \longrightarrow K_1(p^!)$$
$$\longrightarrow K_0(\mathbb{Z}[\pi_1(B)]) \xrightarrow{p_0^!} K_0(\mathbb{Z}[\pi_1(E)]) \ ,$$

as well as assorted transfer maps $p^!:L_m(\pi_1(B)) \longrightarrow L_{m+n}(\pi_1(E))$ in algebraic L-theory. If F is round simple and $\pi_1(B)$ acts on F by self equivalences $F \longrightarrow F$ with $\tau = 0 \in Wh(\pi_1(E))$ (e.g. if p is a PL bundle with a round manifold fibre) then there is also defined a transfer exact sequence

$$Wh(\pi_1(B)) \xrightarrow{\tilde{p}_1^!} Wh(\pi_1(E)) \longrightarrow Wh(p^!)$$
$$\longrightarrow \tilde{K}_0(\mathbb{Z}[\pi_1(B)]) \xrightarrow{\tilde{p}_0^!} \tilde{K}_0(\mathbb{Z}[\pi_1(E)])$$

with a duality involution $*: Wh(p^!) \longrightarrow Wh(p^!)$ on the relative Whitehead group. The connecting maps δ in Tate $\mathbb{Z}_2$-cohomology arising from the short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$0 \longrightarrow \mathrm{coker}(\tilde{p}_1^!) \longrightarrow Wh(p^!) \longrightarrow \ker(\tilde{p}_0^!) \longrightarrow 0$$

and the transfer maps in L-theory together define a morphism of exact sequences

$$\begin{array}{ccccccccc}
\dots \longrightarrow & L_m^h(\pi) & \longrightarrow & L_m^{\ker(\tilde{p}_0^!)}(\pi) & \longrightarrow & \hat{H}^m(\mathbb{Z}_2;\ker(\tilde{p}_0^!)) & \longrightarrow & L_{m-1}^h(\pi) & \longrightarrow \dots \\
& \downarrow p^! & & \downarrow p^! & & \downarrow \delta & & \downarrow p^! & \\
\dots \longrightarrow & L_{m+n}^{\mathrm{im}(\tilde{p}_1^!)}(\Pi) & \longrightarrow & L_{m+n}^h(\Pi) & \longrightarrow & \hat{H}^{m+n}(\mathbb{Z}_2;\mathrm{coker}(\tilde{p}_1^!)) & \longrightarrow & L_{m+n-1}^{\mathrm{im}(\tilde{p}_1^!)}(\Pi) & \longrightarrow \dots
\end{array}$$

$$(\pi = \pi_1(B),\ \Pi = \pi_1(E)) .$$

In the case of the trivial fibration

$$F \longrightarrow E = B \times F \xrightarrow{p\ =\ \mathrm{projection}} B$$

(with the fibre F a round simple Poincaré complex, as before) the algebraic K-theory transfer maps are zero

$$p_i^! = -\otimes[F] = 0 : K_i(\mathbb{Z}[\pi]) \longrightarrow K_i(\mathbb{Z}[\pi\times\rho])$$
$$(i = 0,1\ \ \rho = \pi_1(F))$$

so that $\tilde{p}_i^! = 0$. Also, the algebraic L-theory transfer maps are given by the products with the round symmetric signatures

$$p^! = -\otimes\sigma^*_{rh}(F) : L_m^p(\pi) \longrightarrow L_{m+n}^h(\pi\times\rho)$$
$$p^! = -\otimes\sigma^*_{rs}(F) : L_m^h(\pi) \longrightarrow L_{m+n}^s(\pi\times\rho) ,$$

and δ is given by product with the torsion $\tau(F) \in \{\pm\rho\} \subseteq K_1(\mathbb{Z}[\rho])$

$$\delta = -\otimes\tau(F) : \hat{H}^m(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \hat{H}^{m+n}(\mathbb{Z}_2;Wh(\pi\times\rho))$$

as in the case $F = S^1$ considered in [16].

§6. The L-groups of a polynomial extension

There are 4 ways of extending an involution $a \mapsto \bar{a}$ on a ring A to an involution on the Laurent polynomial extension ring $A[z,z^{-1}]$, sending z to one of $z, z^{-1}, -z, -z^{-1}$. In each case it is possible to express $L_*(A[z,z^{-1}])$ (and indeed $L^*(A[z,z^{-1}])$) in terms of $L_*(A)$, and to relate such an expression to splitting theorems for manifolds – see Chapter 7 of Ranicki [24] for a general account of algebraic and geometric splitting theorems in L-theory. Only the case

$$A = \mathbb{Z}[\pi] \quad , \quad \bar{z} = z^{-1}$$

is considered here, for which $A[z,z^{-1}] = \mathbb{Z}[\pi][z,z^{-1}]$.

The geometric splittings of the L-groups $L_*(\pi\times\mathbb{Z})$ depend on the realization theorem of $\begin{cases} \text{Wall [35]} \\ \text{Shaneson [29]} \\ \text{Pedersen and Ranicki [18]} \end{cases}$, by which every element of $\begin{cases} L_n^s(\pi) \\ L_n^h(\pi) \\ L_n^p(\pi) \end{cases}$ ($n \geqslant 5$, π finitely presented) is the $\begin{cases} \text{simple} \\ \text{finite} \\ \text{projective} \end{cases}$ rel∂ surgery obstruction $\begin{cases} \sigma_*^s(f,b) \\ \sigma_*^h(f,b) \\ \sigma_*^p(f,b) \end{cases}$ of a normal map

$$(f,b) : (M,\partial M) \longrightarrow (X,\partial X)$$

from a compact n-dimensional manifold with boundary $(M,\partial M)$ to a $\begin{cases} \text{simple} \\ \text{finite} \\ \text{finitely dominated} \end{cases}$ n-dimensional geometric Poincaré pair $(X,\partial X)$ equipped with a reference map $X \longrightarrow K(\pi,1)$, and such that the restriction $\partial f = f| : \partial M \longrightarrow \partial X$ is a $\begin{cases} \text{simple} \\ - \\ - \end{cases}$ homotopy equivalence.

A morphism of groups

$$\phi : \pi \longrightarrow \Pi$$

induces functorially morphisms in the L-groups, given geometrically by

$$\phi_! : L_n^q(\pi) \longrightarrow L_n^q(\Pi) ;$$

$$\sigma_*^q((M,\partial M) \xrightarrow{(f,b)} (X,\partial X) \longrightarrow K(\pi,1))$$

$$\longmapsto \sigma_*^q((M,\partial M) \xrightarrow{(f,b)} (X,\partial X) \longrightarrow K(\pi,1) \xrightarrow{\phi} K(\Pi,1))$$

$$(q = s,h,p) ,$$

and algebraically by

$$\phi_! : L_n^q(\pi) \longrightarrow L_n^q(\Pi) ; \sigma_*^q(f,b) \longmapsto \mathbb{Z}[\Pi]\otimes_{\mathbb{Z}[\pi]}\sigma_*^q(f,b) .$$

In general $\phi_!$ will be written ϕ.

The geometric splitting of Shaneson [29]

$$L_n^s(\pi \times \mathbb{Z}) = L_n^s(\pi)\oplus L_{n-1}^h(\pi)$$

was obtained in the form of a split exact sequence

$$0 \longrightarrow L_n^s(\pi) \xrightarrow{\bar{\epsilon}} L_n^s(\pi \times \mathbb{Z}) \xrightarrow{B} L_{n-1}^h(\pi) \longrightarrow 0$$

with $\bar{\epsilon}$ the split injection of L-groups induced functorially from the split injection of groups $\bar{\epsilon}:\pi \rightarrowtail \pi \times \mathbb{Z}$. The split surjection B was defined geometrically by

$$B : L_n^s(\pi \times \mathbb{Z}) \twoheadrightarrow L_{n-1}^h(\pi) ;$$

$$\sigma_*^s((M,\partial M) \xrightarrow{(f,b)} (X,\partial X) \times S^1 \longrightarrow K(\pi,1) \times S^1 = K(\pi\times\mathbb{Z},1))$$

$$\longmapsto \sigma_*^h((N,\partial N) \xrightarrow{(g,c)} (X,\partial X) \longrightarrow K(\pi,1))$$

using the splitting theorem of Farrell and Hsiang [5],[7] to represent every element of $L_n^s(\pi\times\mathbb{Z})$ as the rel∂ simple surgery obstruction $\sigma_*^s(f,b)$ of an n-dimensional normal map $(f,b):(M,\partial M) \longrightarrow (X,\partial X) \times S^1$ with $(X,\partial X)$ a finite (n-1)-dimensional geometric Poincaré pair, such that f is transverse regular at $(X,\partial X) \times \{pt.\} \subset (X,\partial X) \times S^1$ with the restriction defining an (n-1)-dimensional normal map

$$(g,c) = (f,b)| : (N,\partial N) = f^{-1}((X,\partial X) \times \{pt.\}) \longrightarrow (X,\partial X)$$

with $\partial f:\partial M \longrightarrow \partial X \times S^1$ a simple homotopy equivalence and $\partial g:\partial N \longrightarrow \partial X$ a homotopy equivalence. There was also defined in [29] a splitting map for B

$$\overline{B}' : L^h_{n-1}(\pi) \rightarrowtail L^s_n(\pi \times \mathbb{Z}) \ ;$$

$$\sigma^h_*((M,\partial M) \xrightarrow{(f,b)} (X,\partial X) \longrightarrow K(\pi,1))$$

$$\longmapsto \sigma^s_*((M,\partial M) \times S^1 \xrightarrow{(f,b)\times 1} (X,\partial X) \times S^1$$

$$\longrightarrow K(\pi,1) \times S^1 = K(\pi\times\mathbb{Z},1))$$

$$(= \sigma^h_*(f,b)\otimes\sigma^*_r(S^1) \text{ by Proposition 5.3 ii}))$$

Let $\varepsilon' : L^s_n(\pi\times\mathbb{Z}) \twoheadrightarrow L^s_n(\pi)$ be the geometric split surjection determined by $\overline{\varepsilon}, B, \overline{B}'$, so that there is defined a direct sum system

$$L^s_n(\pi) \underset{\varepsilon'}{\overset{\overline{\varepsilon}}{\rightleftarrows}} L^s_n(\pi \times \mathbb{Z}) \underset{\overline{B}'}{\overset{B}{\rightleftarrows}} L^h_{n-1}(\pi) \quad .$$

Although it was claimed in Ranicki [20] that ε' coincides with the split surjection induced functorially from the split surjection of groups $\varepsilon : \pi\times\mathbb{Z} \longrightarrow \pi$ (or equivalently $\mathbb{Z}[\pi][z,z^{-1}] \longrightarrow \mathbb{Z}[\pi] \ ; \ z \longmapsto 1$) it does not do so in general. This may be seen by considering the composite

$$\varepsilon\overline{B}' \ : \ L^h_{n-1}(\pi) \xrightarrow{\overline{B}'} L^s_n(\pi\times\mathbb{Z}) \xrightarrow{\varepsilon} L^s_n(\pi) \ ,$$

which need not be zero. A generic element

$$\sigma^h_*((f,b) : (M,\partial M) \longrightarrow (X,\partial X)) \in L^h_{n-1}(\pi)$$

is sent by $\overline{B}'$ to

$$\overline{B}'(\sigma^h_*(f,b)) = \sigma^s_*((g,c) = (f,b)\times 1_{S^1} : (M,\partial M)\times S^1 \longrightarrow (X,\partial X)\times S^1)$$
$$\in L^h_n(\pi \times \mathbb{Z}) \ .$$

Now (g,c) is the boundary of the $(n+1)$-dimensional normal map

$$(f,b) \times 1_{(D^2,S^1)} \ : \ (M,\partial M)\times(D^2,S^1) \longrightarrow (X,\partial X)\times(D^2,S^1)$$

such that the target

$$(X,\partial X)\times(D^2,S^1) = (X\times D^2, X\times S^1 \cup_{\partial X\times S^1} \partial X\times D^2)$$

is a finite $(n+1)$-dimensional geometric Poincaré pair with simple boundary and

$$\tau((X,\partial X)\times(D^2,S^1)) = \tau(X,\partial X)\otimes\chi(D^2) + \chi(X)\otimes\tau(D^2,S^1)$$
$$= \tau(X,\partial X) \in Wh(\pi)$$

(by the relative version of Proposition 5.1). It follows that

$\varepsilon\bar{B}'\sigma_*^h(f,b) \in L_n^s(\pi)$ is the image of

$$\tau((X,\partial X)\times(D^2,S^1)) = \tau(X,\partial X)$$
$$\in \hat{H}^{n-1}(\mathbb{Z}_2;Wh(\pi)) = \hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi))$$

under the map $\hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi)) \longrightarrow L_n^s(\pi)$ in the Rothenberg exact sequence

$$\ldots \longrightarrow L_{n+1}^h(\pi) \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi)) \longrightarrow L_n^s(\pi) \longrightarrow L_n^h(\pi) \longrightarrow \ldots \; .$$

The discrepancy between ε and ε' will be expressed algebraically in Proposition 6.2 below; it is at most 2-torsion, and is 0 if $Wh(\pi) = 0$.

Novikov [17] initiated the development of analogues for algebraic L-theory of the techniques of Bass, Heller and Swan [2] and Bass [1] for the algebraic K-theory of polynomial extensions. In Ranicki [19],[20] the methods of [17] (which neglected 2-torsion) were refined to obtain for any group π algebraic isomorphisms

$$\begin{cases} \beta_L = \begin{pmatrix} \varepsilon \\ B \end{pmatrix} : L_n^s(\pi\times\mathbb{Z}) \longrightarrow L_n^s(\pi)\oplus L_{n-1}^h(\pi) \\ \beta_L = \begin{pmatrix} \varepsilon \\ B \end{pmatrix} : L_n^h(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi)\oplus L_{n-1}^p(\pi) \end{cases}$$

with inverses

$$\begin{cases} \beta_L^{-1} = (\bar{\varepsilon}\ \bar{B}) : L_n^s(\pi)\oplus L_{n-1}^h(\pi) \longrightarrow L_n^s(\pi\times\mathbb{Z}) \\ \beta_L^{-1} = (\bar{\varepsilon}\ \bar{B}) : L_n^h(\pi)\oplus L_{n-1}^p(\pi) \longrightarrow L_n^h(\pi\times\mathbb{Z}) \end{cases} ,$$

by analogy with the isomorphism of [2]

$$\beta_K : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi)\oplus\tilde{K}_0(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])\oplus\widetilde{Nil}(\mathbb{Z}[\pi])$$

recalled in §3 above. The isomorphisms β_L define the <u>algebraically significant</u> splitting

$$\begin{cases} L_n^s(\pi\times\mathbb{Z}) = L_n^s(\pi)\oplus L_{n-1}^h(\pi) \\ L_n^h(\pi\times\mathbb{Z}) = L_n^h(\pi)\oplus L_{n-1}^p(\pi) \; . \end{cases}$$

As already indicated above this does not in general coincide with the geometric splitting of $L_n^s(\pi\times\mathbb{Z})$ due to Shaneson [29], although the split surjection $B:L_n^s(\pi\times\mathbb{Z}) \twoheadrightarrow L_{n-1}^h(\pi)$ of [29] agrees with the algebraic B of [20].

Pedersen and Ranicki [18,§4] claimed to be giving a geometric interpretation of the algebraically significant splitting $L_*^h(\pi\times\mathbb{Z}) = L_*^h(\pi)\oplus L_{*-1}^p(\pi)$. However, the composite

$$\varepsilon\bar{B}' : L_{n-1}^p(\pi) \xrightarrow{\bar{B}'} L_n^h(\pi\times\mathbb{Z}) \xrightarrow{\varepsilon} L_n^h(\pi)$$

of the geometric split injection

$$\bar{B}' : L_{n-1}^p(\pi) \longrightarrow L_n^h(\pi\times\mathbb{Z}) \ ;$$

$$\sigma_*^p((f,b):(M,\partial M)\longrightarrow(X,\partial X))$$

$$\longmapsto \sigma_*^h((f,b)\times 1_{S^1} : (M,\partial M)\times S^1 \longrightarrow (X,\partial X)\times S^1)$$

$$(= \sigma_*^p(f,b)\otimes\sigma_r^*(S^1) \text{ by Proposition 5.3 ii)})$$

and the algebraic split surjection $\varepsilon: L_n^h(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi)$ need not be zero: there is defined a finitely dominated null-bordism with $\pi_1(X\times D^2) = \pi_1(X) = \pi$

$$(f,b)\times 1_{(D^2,S^1)} : (M,\partial M)\times(D^2,S^1) \longrightarrow (X,\partial X)\times(D^2,S^1)$$

of the relative (homotopy) finite surgery problem

$$(f,b)\times 1_{S^1} : (M,\partial M)\times S^1 \longrightarrow (X,\partial X)\times S^1 \ ,$$

with finiteness obstruction

$$[X\times D^2] = [X] \in \tilde{K}_0(\mathbb{Z}[\pi]) \ .$$

It follows that $\varepsilon\bar{B}'\sigma_*^p(f,b)\in L_n^h(\pi)$ is the image of $[X]\in\hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) = \hat{H}^{n+1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi]))$ under the map $\hat{H}^{n+1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi]))\longrightarrow L_n^h(\pi)$ in the generalized Rothenberg exact sequence

$$\ldots\longrightarrow L_{n+1}^p(\pi)\longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi]))\longrightarrow L_n^h(\pi)\longrightarrow L_n^p(\pi)\longrightarrow\ldots \ .$$

Thus $\bar{B}'$ and ε do not in general belong to the same direct sum system. In fact ε belongs to the algebraically significant direct sum decomposition of $L_n^h(\pi\times\mathbb{Z})$ described above, while $\bar{B}'$ belongs to the geometrically defined direct sum decomposition

$$L_n^h(\pi) \underset{\varepsilon'}{\overset{\bar{\varepsilon}}{\rightleftarrows}} L_n^h(\pi\times\mathbb{Z}) \underset{\bar{B}'}{\overset{B}{\rightleftarrows}} L_{n-1}^p(\pi)$$

with B as defined in [18,§4] and ε' the split surjection determined by $\bar{\varepsilon},B,\bar{B}'$. It is the latter direct sum system which is meant when referring to "the geometric splitting $L_*^h(\pi\times\mathbb{Z}) = L_*^h(\pi)\oplus L_{*-1}^p(\pi)$ of [18]".

Define the <u>geometrically significant</u> splitting

$$\begin{cases} L_n^s(\pi\times\mathbb{Z}) = L_n^s(\pi)\oplus L_{n-1}^h(\pi) \\ L_n^h(\pi\times\mathbb{Z}) = L_n^h(\pi)\oplus L_{n-1}^p(\pi) \end{cases}$$

to be the one given by the algebraic isomorphism

$$\begin{cases} \beta_L' = \begin{pmatrix} \varepsilon' \\ B \end{pmatrix} : L_n^s(\pi\times\mathbb{Z}) \longrightarrow L_n^s(\pi)\oplus L_{n-1}^h(\pi) \\ \beta_L' = \begin{pmatrix} \varepsilon' \\ B \end{pmatrix} : L_n^h(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi)\oplus L_{n-1}^p(\pi) \end{cases}$$

with inverse

$$\begin{cases} \beta_L'^{-1} = (\bar{\varepsilon}\ \bar{B}') : L_n^s(\pi)\oplus L_{n-1}^h(\pi) \longrightarrow L_n^s(\pi\times\mathbb{Z}) \\ \beta_L'^{-1} = (\bar{\varepsilon}\ \bar{B}') : L_n^h(\pi)\oplus L_{n-1}^p(\pi) \longrightarrow L_n^h(\pi\times\mathbb{Z}) \end{cases} ,$$

where

$$\begin{cases} \bar{B}' = -\otimes\sigma_r^*(S^1) : L_{n-1}^h(\pi) \rightarrowtail L_n^s(\pi\times\mathbb{Z}) \\ \bar{B}' = -\otimes\sigma_r^*(S^1) : L_{n-1}^p(\pi) \rightarrowtail L_n^h(\pi\times\mathbb{Z}) \end{cases}$$

and

$$\begin{cases} \varepsilon' = \varepsilon(1-\bar{B}'B) : L_n^s(\pi\times\mathbb{Z}) \twoheadrightarrow L_n^s(\pi) \\ \varepsilon' = \varepsilon(1-\bar{B}'B) : L_n^h(\pi\times\mathbb{Z}) \twoheadrightarrow L_n^h(\pi) \end{cases} .$$

<u>Proposition 6.1</u> The geometric splitting $\begin{cases} L_n^s(\pi\times\mathbb{Z}) = L_n^s(\pi)\oplus L_{n-1}^h(\pi) \\ L_n^h(\pi\times\mathbb{Z}) = L_n^h(\pi)\oplus L_{n-1}^p(\pi) \end{cases}$ of $\begin{cases} \text{Shaneson [29]} \\ \text{Pedersen and Ranicki [18]} \end{cases}$ is the geometrically significant splitting in algebra.

[]

The algebraically significant split injections $\begin{cases} \bar{B}: L_*^h(\pi) \rightarrowtail L_{*+1}^s(\pi\times\mathbb{Z}) \\ \bar{B}: L_*^p(\pi) \rightarrowtail L_{*+1}^h(\pi\times\mathbb{Z}) \end{cases}$ were defined in Ranicki [20] using the forms and formations of Ranicki [19]; for example

$$\bar{B} : L_{2i}^p(\pi) \rightarrowtail L_{2i+1}^h(\pi\times\mathbb{Z}) ;$$

$$(Q,\psi) \longmapsto (M\oplus M, \psi\oplus-\psi; \Delta, (1\oplus z)\Delta)\oplus(H_{(-)^i}(N); N, N)$$

sends a projective non-singular $(-)^i$-quadratic form over $\mathbb{Z}[\pi]$ (Q,ψ)

to a free non-singular $(-)^i$-quadratic formation over $\mathbb{Z}[\pi\times\mathbb{Z}] = \mathbb{Z}[\pi][z,z^{-1}]$ with $M = Q[z,z^{-1}]$ the induced f.g. projective $\mathbb{Z}[\pi\times\mathbb{Z}]$-module, $\Delta = \{(x,x)\in M\oplus M \mid x\in M\}\subset M\oplus M$ the diagonal lagrangian of $(M\oplus M,\psi\oplus-\psi)$, and $H_{(-)^i}(N) = (N\oplus N^*,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix})$ the $(-)^i$-hyperbolic (alias hamiltonian) form on a f.g. projective $\mathbb{Z}[\pi\times\mathbb{Z}]$-module N such that $M\oplus N$ is a f.g. free $\mathbb{Z}[\pi\times\mathbb{Z}]$-module. The geometrically significant split injections $\begin{cases}\bar{B}' : L^h_*(\pi) \rightarrowtail L^s_{*+1}(\pi\times\mathbb{Z})\\ \bar{B}' : L^p_*(\pi) \rightarrowtail L^h_{*+1}(\pi\times\mathbb{Z})\end{cases}$ were defined in §10 of Ranicki [22] using algebraic Poincaré complexes. It is easy to translate from complexes to forms and formations (or the other way round); for example, in terms of forms and formations

$$\bar{B}' : L^p_{2i}(\pi) \rightarrowtail L^h_{2i+1}(\pi\times\mathbb{Z}) ;$$
$$(Q,\psi) \longmapsto (M\oplus M,\psi\oplus-\psi;\Delta,(1\oplus z)\Delta)\oplus(H_{(-)^i}(N);N,N^*) ,$$

making apparent the difference between $\bar{B}$ and $\bar{B}'$ in this case.

For any group π the exact sequence

$$0 \longrightarrow \hat{H}^0(\mathbb{Z}_2;K_0(\mathbb{Z})) \longrightarrow L^1_{rh}(\pi) \longrightarrow L^1_h(\pi) \longrightarrow 0$$

splits, with the injection

$$\hat{H}^0(\mathbb{Z}_2;K_0(\mathbb{Z})) = \mathbb{Z}_2 \rightarrowtail L^1_{rh}(\pi) \ ;\ 1 \longmapsto \mathbb{Z}[\pi]\otimes_{\mathbb{Z}}\varepsilon\sigma^*_r(S^1)$$

split by the rational semicharacteristic

$$L^1_r(\pi) \twoheadrightarrow \mathbb{Z}_2 \ ;\ (C,\phi) \longmapsto \chi_{1/2}(\mathbb{Z}\otimes_{\mathbb{Z}[\pi]}C;\mathbb{Q}) .$$

By the discussion at the end of Ranicki [22,§10]

$$L^1(\mathbb{Z}) = L^1(\{1\})\oplus L^0(\{1\}) = \mathbb{Z}_2\oplus\mathbb{Z} ,$$

with $(0,1) = \sigma^*(S^1)\in L^1(\mathbb{Z})$ the symmetric signature of S^1. Let $\sigma^*_q(S^1)\in L^1_r(\mathbb{Z})$ be the image of $\sigma^*(S^1)\in L^1(\mathbb{Z})$ under the splitting map $L^1(\mathbb{Z}) \rightarrowtail L^1_r(\mathbb{Z})$, so that $\sigma^*_q(S^1) = (1-\bar{\varepsilon}\varepsilon)\sigma^*_r(S^1)$ and $\varepsilon\sigma^*_q(S^1) = 0\in L^1_r(\{1\})$. The algebraically significant injections are defined by

$$\begin{cases}\bar{B} = -\otimes\sigma^*_q(S^1) \ :\ L^h_n(\pi) \rightarrowtail L^s_{n+1}(\pi\times\mathbb{Z})\\ \bar{B} = -\otimes\sigma^*_q(S^1) \ :\ L^p_n(\pi) \rightarrowtail L^h_{n+1}(\pi\times\mathbb{Z}) \end{cases} .$$

Now

$$\sigma^*_r(S^1) - \sigma^*_q(S^1) = \bar{\varepsilon}\varepsilon\sigma^*_r(S^1) \in L^1_r(\mathbb{Z}) ,$$

so that

$$\begin{cases} \overline{B}' - \overline{B} = -\otimes(\sigma_r^*(S^1) - \sigma_q^*(S^1)) = -\otimes\overline{\varepsilon}\varepsilon\sigma_r^*(S^1) : L_n^h(\pi) \longrightarrow L_{n+1}^s(\pi\times\mathbb{Z}) \\ \overline{B}' - \overline{B} = -\otimes(\sigma_r^*(S^1) - \sigma_q^*(S^1)) = -\otimes\overline{\varepsilon}\varepsilon\sigma_r^*(S^1) : L_n^p(\pi) \longrightarrow L_{n+1}^h(\pi\times\mathbb{Z}) \end{cases} .$$

By analogy with the map of algebraic K-groups defined in §3

$$\omega = -\otimes\tau(-1) : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi)$$

define maps of algebraic L-groups

$$\begin{cases} \omega = -\otimes\varepsilon\sigma_r^*(S^1) : L_n^h(\pi) \longrightarrow L_{n+1}^s(\pi) \\ \omega = -\otimes\varepsilon\sigma_r^*(S^1) : L_n^p(\pi) \longrightarrow L_{n+1}^h(\pi) \end{cases} ,$$

where $\varepsilon\sigma_r^*(S^1) = (1,1) \in L_r^1(\{1\}) = \mathbb{Z}_2\oplus\mathbb{Z}_2$. As $\varepsilon\tau(S^1) = \tau(-1) \in K_1(\mathbb{Z}) = \mathbb{Z}_2$ the various maps ω together define a morphism of generalized Rothenberg exact sequences

$$\begin{array}{ccccccccc} \dots \longrightarrow & L_n^h(\pi) & \longrightarrow & L_n^p(\pi) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) & \longrightarrow & L_{n-1}^h(\pi) & \longrightarrow \dots \\ & \downarrow\omega & & \downarrow\omega & & \downarrow\omega & & \downarrow\omega & \\ \dots \longrightarrow & L_{n+1}^s(\pi) & \longrightarrow & L_{n+1}^h(\pi) & \longrightarrow & \hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi)) & \longrightarrow & L_n^s(\pi) & \longrightarrow \dots . \end{array}$$

<u>Proposition 6.2</u> The algebraically and geometrically significant split injections of L-groups differ by

$$\begin{cases} \overline{B}' - \overline{B} = \overline{\varepsilon}\omega : L_n^h(\pi) \xrightarrow{\omega} L_{n+1}^s(\pi) \xrightarrow{\overline{\varepsilon}} L_{n+1}^s(\pi\times\mathbb{Z}) \\ \overline{B}' - \overline{B} = \overline{\varepsilon}\omega : L_n^p(\pi) \xrightarrow{\omega} L_{n+1}^h(\pi) \xrightarrow{\overline{\varepsilon}} L_{n+1}^h(\pi\times\mathbb{Z}) \end{cases} .$$

The split surjections differ by

$$\begin{cases} \varepsilon' - \varepsilon = \omega B : L_n^s(\pi\times\mathbb{Z}) \xrightarrow{B} L_{n-1}^h(\pi) \xrightarrow{\omega} L_n^s(\pi) \\ \varepsilon' - \varepsilon = \omega B : L_n^h(\pi\times\mathbb{Z}) \xrightarrow{B} L_{n-1}^p(\pi) \xrightarrow{\omega} L_n^h(\pi) \end{cases} .$$

The L-theory maps ω factor as

$$\begin{cases} \omega : L_n^h(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Wh(\pi)) = \hat{H}^{n+2}(\mathbb{Z}_2;Wh(\pi)) \longrightarrow L_{n+1}^s(\pi) \\ \omega : L_n^p(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) = \hat{H}^{n+2}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_{n+1}^h(\pi) \end{cases} .$$

The K-theory map ω is the sum of the composites

$$\hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_{n-1}^h(\pi) \longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2;Wh(\pi)) = \hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi))$$

$$\hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) = H^{n+2}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_{n+1}^h(\pi) \longrightarrow \hat{H}^{n+1}(\mathbb{Z}_2;Wh(\pi)) .$$

<u>Proof</u>: Let $\begin{cases} L_n^{h,s}(\pi) \\ L_n^{p,h}(\pi) \end{cases}$ $(n \geqslant 0)$ be the relative cobordism group of $\begin{cases} \text{(finite,simple)} \\ \text{(finitely dominated,finite)} \end{cases}$ n-dimensional quadratic Poincaré pairs over $\mathbb{Z}[\pi]$ $(f:C \longrightarrow D, (\delta\psi,\psi) \in Q_n(f))$, so that there is defined an exact sequence

$$\begin{cases} \ldots \longrightarrow L_n^s(\pi) \longrightarrow L_n^h(\pi) \longrightarrow L_n^{h,s}(\pi) \longrightarrow L_{n-1}^s(\pi) \longrightarrow \ldots \\ \ldots \longrightarrow L_n^h(\pi) \longrightarrow L_n^p(\pi) \longrightarrow L_n^{p,h}(\pi) \longrightarrow L_{n-1}^h(\pi) \longrightarrow \ldots \end{cases}$$

and there are defined isomorphisms

$$\begin{cases} L_n^{h,s}(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Wh(\pi)) ; \\ \qquad (f:C \longrightarrow D, (\delta\psi,\psi)) \longmapsto \tau((1+T)(\delta\psi,\psi)_0 : C(f)^{n-*} \longrightarrow D) \\ L_n^{p,h}(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) ; \ (f:C \longrightarrow D, (\delta\psi,\psi)) \longmapsto [D] . \end{cases}$$

Product with the 2-dimensional symmetric Poincaré pair $\sigma^*(D^2,S^1)$ over $\mathbb{Z}$ defines isomorphisms of relative L-groups

$$\begin{cases} -\otimes\sigma^*(D^2,S^1) : L_n^{h,s}(\pi) \longrightarrow L_{n+2}^{h,s}(\pi) \\ -\otimes\sigma^*(D^2,S^1) : L_n^{p,h}(\pi) \longrightarrow L_{n+2}^{p,h}(\pi) \end{cases} ,$$

corresponding to the canonical 2-periodicity isomorphisms of the Tate $\mathbb{Z}_2$-cohomology groups

$$\begin{cases} \hat{H}^n(\mathbb{Z}_2;Wh(\pi)) \longrightarrow \hat{H}^{n+2}(\mathbb{Z}_2;Wh(\pi)) \\ \hat{H}^n(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow \hat{H}^{n+2}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) . \end{cases}$$

The boundary of $\sigma^*(D^2,S^1)$ is $\varepsilon\sigma_r^*(S^1)$.

[]

In particular, the algebraic and geometric splitting maps in L-theory differ in 2-torsion only, since $2\omega = 0$ (cf. Proposition 3.3).

The splitting maps in the algebraic and geometric splittings of $Wh(\pi\times\mathbb{Z})$ given in §3 and the duality involutions * are such that

$$\begin{aligned} &\bar{\varepsilon}* = *\bar{\varepsilon} : Wh(\pi) \longrightarrow Wh(\pi\times\mathbb{Z}) \\ &\varepsilon* = *\varepsilon , \ \varepsilon'* = *\varepsilon' : Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi) \\ &B* = -*B : Wh(\pi\times\mathbb{Z}) \longrightarrow \tilde{K}_0(\mathbb{Z}[\pi]) \\ &\bar{B}* = -*\bar{B} , \ \bar{B}'* = -*\bar{B}' : \tilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi\times\mathbb{Z}) \\ &\bar{\Delta}_\pm * = *\bar{\Delta}_\mp : \widetilde{Nil}(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi\times\mathbb{Z}) \\ &\Delta_\pm * = *\Delta_\mp : Wh(\pi\times\mathbb{Z}) \longrightarrow \widetilde{Nil}(\mathbb{Z}[\pi]) . \end{aligned}$$

The involution $*: Wh(\pi\times\mathbb{Z}) \longrightarrow Wh(\pi\times\mathbb{Z})$ interchanges the two $\widetilde{Nil}$ summands, so that they do not appear in the Tate $\mathbb{Z}_2$-cohomology groups and there are defined two splittings

$$\hat{H}^n(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z})) = \hat{H}^n(\mathbb{Z}_2;Wh(\pi))\oplus\hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \quad ,$$

the algebraically significant direct sum decomposition

$$\hat{H}^n(\mathbb{Z}_2;Wh(\pi)) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} \hat{H}^n(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z})) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} \hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi]))$$

and the geometrically significant direct sum decomposition

$$\hat{H}^n(\mathbb{Z}_2;Wh(\pi)) \underset{\varepsilon'}{\overset{\bar{\varepsilon}}{\rightleftarrows}} \hat{H}^n(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z})) \underset{\bar{B}'}{\overset{B}{\rightleftarrows}} \hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) .$$

Proposition 6.3 The Rothenberg exact sequence of a polynomial extension

$$\ldots \longrightarrow L_n^s(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi\times\mathbb{Z}) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z})) \longrightarrow L_{n-1}^s(\pi\times\mathbb{Z}) \longrightarrow \ldots$$

has two splittings as a direct sum of the exact sequences

$$\ldots \longrightarrow L_n^s(\pi) \longrightarrow L_n^h(\pi) \longrightarrow \hat{H}^n(\mathbb{Z}_2;Wh(\pi)) \longrightarrow L_{n-1}^s(\pi) \longrightarrow \ldots \quad ,$$

$$\ldots \longrightarrow L_{n-1}^h(\pi) \longrightarrow L_{n-1}^p(\pi) \longrightarrow \hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_{n-2}^h(\pi) \longrightarrow \ldots ,$$

an algebraically and a geometrically significant one.

[]

The split injection of exact sequences in the appendix of Munkholm and Ranicki [16] is the geometrically significant injection

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & L_{n-1}^h(\pi) & \longrightarrow & L_{n-1}^p(\pi) & \longrightarrow & \hat{H}^{n-1}(\mathbb{Z}_2;\tilde{K}_0(\mathbb{Z}[\pi])) & \longrightarrow & L_{n-2}^h(\pi) & \longrightarrow \ldots \\
& \downarrow \bar{B}' & & \downarrow \bar{B}' & & \downarrow \bar{B}' & & \downarrow \bar{B}' & \\
\ldots \longrightarrow & L_n^s(\pi\times\mathbb{Z}) & \longrightarrow & L_n^h(\pi\times\mathbb{Z}) & \longrightarrow & \hat{H}^n(\mathbb{Z}_2;Wh(\pi\times\mathbb{Z})) & \longrightarrow & L_{n-1}^s(\pi\times\mathbb{Z}) & \longrightarrow \ldots
\end{array}$$

As for algebraic K-theory (cf. the discussion just after Proposition 3.3) it is tempting to identify the geometrically significant split surjection $\begin{cases} \varepsilon' : L_n^s(\pi\times\mathbb{Z}) \longrightarrow L_n^s(\pi) \\ \varepsilon' : L_n^h(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi) \end{cases}$ with the split surjection of L-groups induced functorially by the split surjection of rings with involution

$$\eta : \mathbb{Z}[\pi][z,z^{-1}] = \mathbb{Z}[\pi\times\mathbb{Z}] \longrightarrow \mathbb{Z}[\pi] \ ; \ \sum_{j=-\infty}^{\infty} a_j z^j \longmapsto \sum_{j=-\infty}^{\infty} a_j(-1)^j$$

and indeed

$$\begin{cases} \varepsilon'|(=1) = \eta| : \mathrm{im}(\bar{\varepsilon}:L_n^s(\pi)\rightarrowtail L_n^s(\pi\times\mathbb{Z})) \longrightarrow L_n^s(\pi) \\ \varepsilon'|(=1) = \eta| : \mathrm{im}(\bar{\varepsilon}:L_n^h(\pi)\rightarrowtail L_n^h(\pi\times\mathbb{Z})) \longrightarrow L_n^h(\pi) \end{cases} .$$

However, $\eta\sigma_r^*(S^1) = (1,0) \neq 0 \in L_r^1(\{1\}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (since the underlying $\mathbb{Z}$-module chain complex is $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$) and in general

$$\begin{cases} \varepsilon'|(=0) \neq \eta| : \mathrm{im}(\bar{B}' = -\otimes\sigma_r^*(S^1):L_{n-1}^h(\pi)\rightarrowtail L_n^s(\pi\times\mathbb{Z})) \longrightarrow L_n^s(\pi) \\ \varepsilon'|(=0) \neq \eta| : \mathrm{im}(\bar{B}' = -\otimes\sigma_r^*(S^1):L_{n-1}^p(\pi)\rightarrowtail L_n^h(\pi\times\mathbb{Z})) \longrightarrow L_n^h(\pi) \end{cases}$$

so that

$$\begin{cases} \varepsilon' \neq \eta : L_n^s(\pi\times\mathbb{Z}) \longrightarrow L_n^s(\pi) \\ \varepsilon' \neq \eta : L_n^h(\pi\times\mathbb{Z}) \longrightarrow L_n^h(\pi) \end{cases} .$$

For $q = s,h,p$ the type q total surgery obstruction groups $\mathcal{S}_*^q(X)$ were defined in Ranicki [21] for any topological space X to fit into an exact sequence

$$\ldots \longrightarrow H_n(X;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*^q} L_n^q(\pi_1(X)) \longrightarrow \mathcal{S}_n^q(X) \longrightarrow H_{n-1}(X;\underline{\mathbb{L}}_0) \longrightarrow \ldots ,$$

with $\underline{\mathbb{L}}_0$ an algebraic 1-connective Ω-spectrum such that

$$\pi_*(\underline{\mathbb{L}}_0) = L_*(\{1\})$$

and σ_*^q an algebraic version of the Quinn assembly map. If X is a $\begin{cases}\text{simple}\\ \text{finite}\\ \text{finitely dominated}\end{cases}$ n-dimensional geometric Poincaré complex the total surgery obstruction $\begin{cases} s(X) \in \mathcal{S}_n^s(X) \\ s(X) \in \mathcal{S}_n^h(X) \\ s(X) \in \mathcal{S}_n^p(X) \end{cases}$ is defined, and is such that $s(X) = 0$ if (and for $n \geqslant 5$ only if) $\begin{cases} X \\ X \\ X\times S^1 \end{cases}$ is $\begin{cases}\text{simple}\\ -\\ -\end{cases}$ homotopy equivalent to a compact $\begin{cases} n\text{-} \\ n\text{-} \\ (n+1)\text{-} \end{cases}$ dimensional topological manifold. For a compact n-dimensional topological manifold M with $n \geqslant 5$ the exact sequence

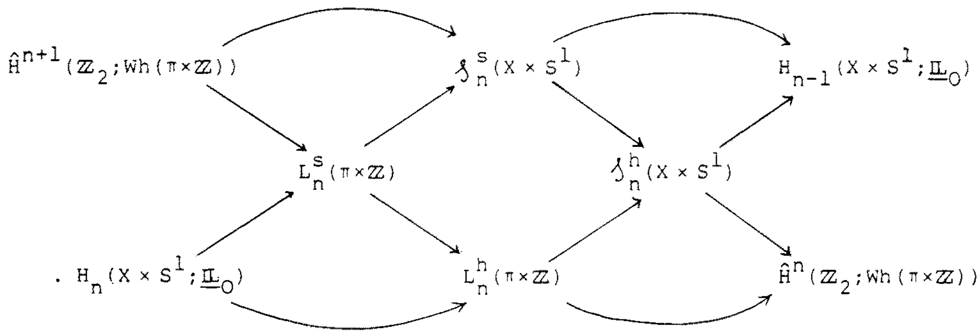
$$\ldots \longrightarrow H_{n+1}(M;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*^q} L_{n+1}^q(\pi_1(M)) \longrightarrow \mathcal{S}_{n+1}^q(M) \longrightarrow H_n(M;\underline{\mathbb{L}}_0) \xrightarrow{\sigma_*^q} L_n^q(\pi_1(M))$$

is isomorphic to the type q Sullivan-Wall surgery exact sequence

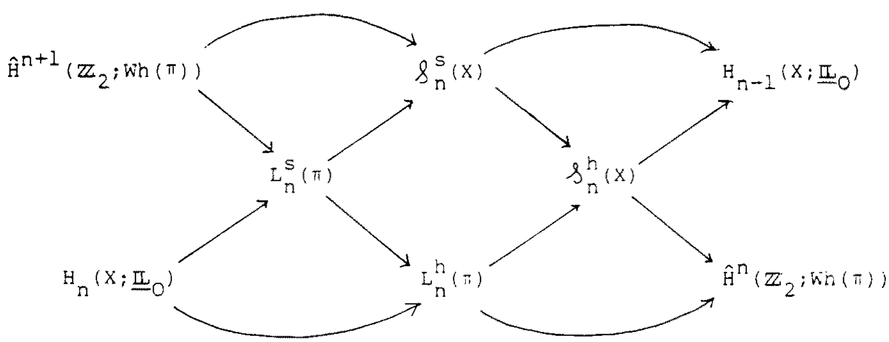
$$\cdots \longrightarrow [M\times D^1, M\times S^0; G/TOP, *] \xrightarrow{\theta^q} L^q_{n+1}(\pi_1(M)) \longrightarrow \mathcal{S}^{qTOP}(M)$$
$$\longrightarrow [M, G/TOP] \xrightarrow{\theta^q} L^q_n(\pi_1(M))$$

with θ^q the type q surgery obstruction map and $\mathcal{S}^{qTOP}(M)$ the type q topological manifold structure set of M.

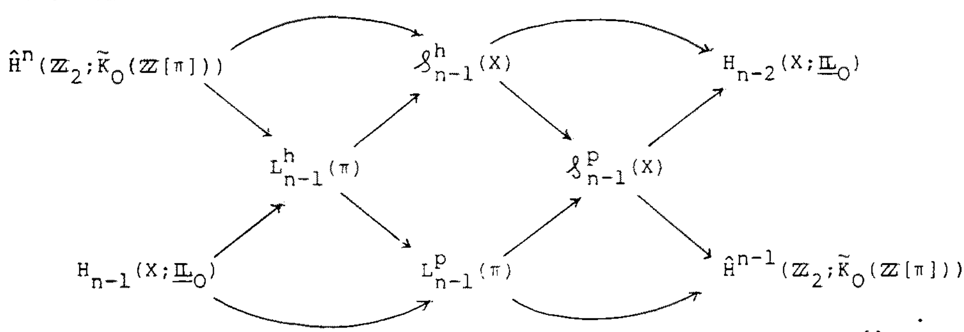
Proposition 6.4 For any connected space X with $\pi_1(X) = \pi$ the commutative braid of algebraic surgery exact sequences of a polynomial extension



has a geometrically significant splitting as a direct sum of the braid



and the braid



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It is appropriate to record here (in the terminology of this paper) a footnote from the preprint version of Cappell and Shaneson [3]: "it is not completely obvious that the maps given in Ranicki [20] give a splitting

$$L_n^s(\pi\times\mathbb{Z}) = L_n^s(\pi)\oplus L_{n-1}^h(\pi)$$

respected by the surgery map

$$\theta^s : [M\times S^1, G/TOP] = [M\times D^1, M\times S^0; G/TOP,*]\oplus[M,G/TOP] \longrightarrow L_{n+1}^s(\pi\times\mathbb{Z})$$

with M a compact n-dimensional topological manifold and $\pi = \pi_1(M)$."

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