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MATRIX REPRESENTATIONS OF ARTIN GROUPS

CRAIG C. SQUIER

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ABSTRACT. We define matrix representations of Artin groups over a 2-variable Laurent-polynomial ring and show that in the rank 2 case, the representations are faithful. In the special case of Artin's braid group, our representation is a version of the Burau representation and our faithfulness theorem is a generalization of the well-known fact that the Burau representation of B_3 is faithful.

In [**4**], Brieskorn and Saito coined the phrase "Artin groups" to denote a certain class of groups, defined by generators and relations, which stand in relationship to arbitrary Coxeter groups much as Artin's braid group B_n [**1**] stands in relationship to the symmetric group S_n. One of the nice features of Coxeter groups is that they have "standard" representations [**6**] as groups of matrices over the real numbers preserving a suitably defined bilinear form and that, moreover, these representations are faithful (see [**3**]). Our purpose here is to show the existence of analogous matrix representations of Artin groups over Laurent-polynomial rings preserving similarly defined sequilinear forms. Unfortunately, except in the simplest cases, the question of faithfulness of these Artin group representations remains open.

In §1, we define Artin groups G_M (by representation), a Hermitian form J, and unitary reflections for each given generator of G_M; these are defined using a given Coxeter matrix M. In §2, we show that the reflections associated to generators of G_M define a matrix representation of G_M (Theorem 1) and that when the presentation of G_M involves 2 generators, this representation is faithful (Theorem 2). We note that in the special case of the braid groups our representation is a version of the Burau representation ([**5**] or see [**2**]). The results below are first, a generalization to arbitrary Artin groups of the author's observation [**10**] that the Burau representation of B_n is unitary and second, a generalization to arbitrary rank 2 Artin groups of the well-known fact (see [**9** or **2**]) that the Burau representation of B_3 is faithful.

1. Definitions. Let n be a positive integer. A (rank n) *Coxeter matrix* M will be an $n \times n$ symmetric matrix $M = [m(i,j)]$ each of whose entries $m(i,j)$ is a positive integer or ∞ such that $m(i,j) = 1$ if and only if $i = j$. Out of a Coxeter matrix M, we shall build some presentations and some forms.

To define the presentations, let $X = \{x_1, \ldots, x_n\}$ be a finite set. For m a positive integer, define the symbol $\langle xy \rangle^m$ by the formula

$$\langle xy \rangle^m = \begin{cases} (xy)^k & \text{if } m = 2k, \\ (xy)^k x & \text{if } m = 2k+1. \end{cases}$$

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Let M be an $n \times n$ Coxeter matrix. G_M will denote the abstract group defined by generators $X = \{x_1, \ldots, x_n\}$ and relations all $\langle x_i x_j \rangle^{m(i,j)} = \langle x_j x_i \rangle^{m(i,j)}$ for $1 \leq i < j \leq n$. Throughout, the case $m(i,j) = \infty$ will stand for "no relation". G_M is the *Artin group* determined by M. W_M will denote G_M modulo the addition relations all $x_i^2 = 1$. Note that in the presence of the relations $x_i^2 = 1$, the defining relations of G_M take the form $(x_i x_j)^{m(i,j)} = 1$. W_M is called the *Coxeter group* determined by M. For the basic properties of Coxeter groups, see [**3** or **6**]. For a study of Artin groups and their relationship to Coxeter groups, see [**4**].

We define a symmetric bilinear form J_1 associated to W_M and a Hermitian form J associated to G_M. To motivate the definitions of J, we begin by recalling the (well-known— see [**3**]) definition of $J_1 = J_1(M)$: J_1 is the $n \times n$ matrix $[c_{ij}]$ where $c_{ij} = -2\cos(\pi/m(i,j))$. Here, we adopt the convention that $\pi/\infty = 0$ so that if $m(i,j) = \infty$ then $c_{ij} = -2$. Note that each $c_{ii} = 2$. Let V denote an n-dimensional vector space over $\mathbf{R}$ with basis $\{e_1, \ldots, e_n\}$. Identify each $v \in V$ with the column vector consisting of the coordinates of v with respect to the basis $\{e_1, \ldots, e_n\}$ of V. With this convention, if $v \in V$, let v' denote the transpose of v and, for $u, v \in V$, define $\langle u, v \rangle_1 = u' J_1 v$. Thus, J_1 defines a symmetric bilinear form on V. We use J_1 to define a matrix representation ρ_1 of W_M on V: if $v \in V$ and $x_i \in X$ define

$$(\rho_1(x_i))(v) = v - \langle e_i, v \rangle_1 e_i.$$

It is well known (again see [**3**]) that ρ_1 is a faithful linear representation of W_M.

To define J, let Λ denote the Laurent-polynomial ring $\mathbf{R}[s, s^{-1}, t, t^{-1}]$, where s and t are indeterminates over $\mathbf{R}$. Define $J = J(M)$ to be the $n \times n$ matrix $[a_{ij}]$ over Λ, where

$$a_{ij} = \begin{cases} -2s\cos(\pi/m(i,j)), & i < j, \\ 1 + st, & i = j, \\ -2t\cos(\pi/m(i,j)), & i > j. \end{cases}$$

Note that J_1 may be obtained from J by substituting $s = t = 1$.

To define analogues of the representation ρ_1 of W_M defined above, we introduce an analogue of complex conjugation in the Laurent-polynomial ring Λ: if $x \in \mathbf{R}$ then, as usual, $\overline{x} = x$; also, $\overline{s} = s^{-1}$ and $\overline{t} = t^{-1}$, extended to Λ additively and multiplicatively. Note that if complex numbers of norm 1 are substituted for s and t then we recover ordinary complex conjugation.

We extend the definition of conjugation to matrices entrywise and, if A is a matrix over Λ, we define $A^* = \overline{A}'$. For example, note that $J^* = s^{-1}t^{-1}J$.

Let V denote a free Λ-module with basis $\{e_1, \ldots, e_n\}$ and, as above, identify each $v \in V$ with its column vector of coordinates. If $u, v \in V$ define $\langle u, v \rangle = u^* J v$. Finally, we define ρ: if $v \in V$ and $x_i \in X$ define

$$(\rho(x_i))(v) = v - \langle e_i, v \rangle e_i.$$

We shall see below that ρ provides a matrix representation of the Artin group G_M.

Note that $(\rho(x_i))(v - s^{-1}t^{-1}\langle e_i, v \rangle e_i) = v$. It follows that each $\rho(x_i)$ acts invertibly on V. In fact, each $\rho(x_i)$ is a pseudo-reflection in the sense of [**3**]. Also, for each $x_i \in X$ and each $u, v \in V$, we have

$$\langle \rho(x_i)(u), \rho(x_i)(v) \rangle = \langle u, v \rangle.$$

Combining this observation with Theorem 1 below, we conclude that ρ is a representation of G_M in a group of unitary matrices.

2. Theorems. In this section, we show that the function ρ defined (on generators) above extends to a representation of the Artin groups G_M and that when $n=2$, this representation is faithful. (The second result includes the fact that the Burau representation of B_3 is faithful—see [**9** or **2**].)

To prove that ρ defines a representation of G_M, we need to show that ρ respects the defining relations of G_M. An important observation is the following

LEMMA. $\det J \neq 0$.

PROOF. In $\det J$, the coefficients of $(st)^n$ is 1, so $\det J \neq 0$. $\square$

In particular, $\langle -,- \rangle$ is nondegenerate: if $u \in V$ satisfies $\langle u, v\rangle = 0$ for all $v \in V$, then $u = 0$.

At this point, it is convenient to introduce the field-of-quotients F of Λ. F is a rational function field over $\mathbf{R}$. Extend the definition of conjugation to F. Letting V_F denote the F-vector space $V \otimes_\Lambda F$, extend $\langle -,- \rangle$ to V_F and also view ρ as a linear transformation on V_F. Note that since $\langle -,- \rangle$ is nondegenerate, if $u \in V_F$ satisfies $u \neq 0$, then $u^\perp = \{v \in V_F | \langle u, v\rangle = 0\}$ is an $(n-1)$-dimensional subspace of V_F. Also note that $\rho(e_i)$ is the identity on $e_i^\perp$. Given i, j satisfying $1 \leq i < j \leq n$, let V_{ij} denote the subspace of V_F spanned by e_i and e_j, and let $V_{ij}^\perp = e_i^\perp \cap e_j^\perp$. We need the following

LEMMA. $V_{ij} \cap V_{ij}^\perp = \{0\}$.

PROOF. Let $v = v_i e_i + v_j e_j \in V_{ij}$ where $v_i, v_j \in \Lambda$. If $v \in V_{ij}^\perp$, then $\langle e_i, v\rangle = \langle e_j, v\rangle = 0$ which leads to the following system of linear equations:

$$v_i(1+st) - 2v_j s\cos(\pi/m) = 0,$$
$$-2v_i t\cos(\pi/m) + v_j(1+st) = 0,$$

where m denotes $m(i,j)$. Since the determinant of the coefficient matrix is $\neq 0$ in Λ, the only solution is $v_i = v_j = 0$, so $v = 0$, as required. $\square$

Noting that the defining relations of G_M each involve exactly two generators, in order to show that ρ respects the defining relations of G_M, it suffices to show that each $\langle x_i x_j\rangle^{m(i,j)} = \langle x_j x_i\rangle^{m(i,j)}$ holds under ρ on the subspace V_{ij} of V_F.

Let a denote the matrix of x_i and b the matrix of x_j with respect to the basis e_i, e_j of V_{ij}. Writing m for $m(i,j)$, it follows that

$$a = \begin{pmatrix} -st & 2s\cos(\pi/m) \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 2t\cos(\pi/m) & -st \end{pmatrix}.$$

Thus it suffices to prove

LEMMA. *The matrices a and b above satisfy $\langle ab\rangle^m = \langle ba\rangle^m$.*

PROOF. Adjoin a square root q of st^{-1} to F and let

$$R = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}.$$

It is easy to check that $R^2 = I$ and $b = RaR$. It follows that $\langle ab\rangle^m = \langle ba\rangle^m$ if and only if $(aR)^m = (Ra)^m$. Clearly, $s^{-1}q(aR)$ and $s^{-1}q(Ra)$ have determinant 1 and trace 2 $\cos(\pi/m)$. It follows that $(s^{-1}q(aR))^m = (s^{-1}q(Ra))^m = -I$, as required. $\square$

Thus we have the following theorem.

THEOREM 1. *The function ρ extends to a representation of G_M in $GL_n(\Lambda)$.*

PROOF. Each relation $\langle x_i x_j \rangle^{m(i,j)} = \langle x_j x_i \rangle^{m(i,j)}$ holds under ρ on V_{ij} by the lemma and therefore on all of V_F since x_i and x_j are each the identity on $V_{ij}^{\perp}$. □

Except in the two-generator case, we do not know if the representation ρ is faithful. Here is the proof in the two-generator case. Let A and B, respectively, denote the matrices obtained by substituting $s = 1$ and $t = -1$ in a and b above.

LEMMA. *The matrix group generated by A and B has presentation $\langle AB \rangle^m = \langle BA \rangle^m$ and*

$$\begin{aligned} (AB)^m &= 1 \qquad (m \text{ even}), \\ (AB)^{2m} &= 1 \qquad (m \text{ odd}). \end{aligned}$$

PROOF. View A and B as linear fractional transformations acting on the upper half-plane. Using the fact that the matrix AB has determinant 1 and trace $2\cos(\pi(1 - (2/m)))$, it follows that AB satisfies $(AB)^m = (-1)^m I$. Thus, it suffices to prove that the group of linear fractional transformations generated by A and B has defining relations $\langle AB \rangle^m = \langle BA \rangle^m$ and $(AB)^m = 1$.

We prove this last fact by exhibiting the group generated by A and B as a subgroup of finite index in a suitable triangle group. Let R_1, R_2 and R_3 be transformations of the upper half-plane defined by

$$\begin{aligned} R_1 &= \text{reflection in the imaginary axis } x = 0, \\ R_2 &= \text{reflection in the axis } x = \cos(\pi/m), \\ R_3 &= \text{reflection in the unit circle.} \end{aligned}$$

Then R_1, R_2 and R_3 generate a $(2, m, \infty)$ triangle group with presentation (see [**7**]):

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_3)^2 = (R_2 R_3)^m = 1.$$

Noting that $R_1(z) = -\bar{z}$, $R_2(z) = -\bar{z} + 2\cos(\pi/m)$ and $R_3(z) = 1/\bar{z}$, it follows that, as linear fractional transformations, $A = R_2 R_1$ and $B = R_3 R_1 R_2 R_3$. It can be checked that the subgroup of the triangle group generated by A and B is normal and has index 2 when m is odd and index 4 when m is even. A routine application of the Reidemeister-Schreier algorithm produces the required presentation of the group generated by A and B. □

THEOREM 2. *The group of matrices generated by a and b has presentation $\langle ab \rangle^m = \langle ba \rangle^m$.*

PROOF. By the Lemma, the substitution produces a group with a presentation consisting of the desired relation together with a further relation $c = 1$ where $c = (ab)^m$ when m is even and $c = (ab)^{2m}$ when m is odd. In either case, c is a central element in the group defined by $\langle ab \rangle^m = \langle ba \rangle^m$. It follows that any additional relation between a and b must be a nonzero power of c. But any nonzero power of c has determinant a nonzero power of $-st$ and is therefore not the identity. Thus the matrix group generated by a and b has presentation $\langle ab \rangle^m = \langle ba \rangle^m$, as desired. □

3. Remarks. The (reduced) Burau representation of B_n (see [**2**]) may be obtained by substituting $s = 1$ in the representation ρ of B_n that arises above. In fact, the representation ρ itself is equivalent to the Burau representation: it is possible to conjugate the image of ρ by a diagonal matrix that, in each $\rho(x_i)$, "moves the t's above the diagonal" and "leaves the s's alone". The matrices that result have the property that their entries depend only on the product st. A similar conjugation is possible whenever the Coxeter graph Γ_M of M is a forest (Γ_M has vertices X and an edge connecting x_i and x_j provided $m(i,j) \geq 3$). In these cases, the representations ρ of G_M is conjugate to a representation over the Laurent-polynomial ring $\mathbf{R}[st, (st)^{-1}] \subseteq \Lambda$. In the case of B_n, the representation that results is the Burau representation.

In general, the question of the faithfulness of ρ remains open. The only known cases seem to be those that follow easily from Theorem 2: G_M is a direct product of rank 1 or 2 Artin groups (equivalently, Γ_M is a disjoint union of vertices and pairs of vertices connected by an edge). Much effort has been devoted (unsuccessfully) to trying to determine whether or not the Burau representation of B_4 is faithful. One other case that might be worth investigating is M defined by each $m(i,j) = \infty$, so that G_M is a free group.

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DEPARTMENT OF MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901