$\varphi(C_1) = \lim_{k\to\infty} h_k \dots h_1(C_1) = M$. If one is careful in the selection of the γ's φ will be a homeomorphism on $\dot{C}_1$ and $\varphi^{-1}\varphi(\dot{C}_1)$ will be $\dot{C}_1$. Since $\varphi(\dot{C}_1)$ is dense in M, $\varphi(\dot{C}_1)$ has dimension $\leq n - 1$.

COROLLARY. *Let* M^n *be a closed manifold and* K^n *a submanifold with bi-collared boundary. Then there is a map* $\varphi: K^n \xrightarrow{\text{onto}} M$ *such that* $\varphi \mid \dot{K}^n$ *is a homeomorphism,* $\varphi^{-1}\varphi(\dot{K}^n) = \dot{K}^n$, $\varphi(\dot{K}^n)$ *is of dimension* $\leq n - 1$.

COROLLARY. *Let* M *be a compact manifold with boundary* B. *Then there is a map* $\varphi: B \times [0, 1]$ *onto* M *such that* $\varphi \mid B \times 1 = 1$, $\varphi \mid B \times (0, 1]$ *is a homeomorphism,* $\varphi^{-1}\varphi(B \times 0) = B \times 0$ *and* $\varphi(B \times 0)$ *is of dimension* $\leq n - 1$.

These are corollaries in the sense that their proof precisely parallels that of the theorem.

COROLLARY. *Suppose* $R = \varphi(B \times 0)$ *in the preceding corollary. Let* $\pi: B \times [0, 1] \to B \times 0$ *by* $\pi(x, t) = (x, 0)$. *Let* $g: B \xrightarrow{\text{onto}} R$ *by* $g(x) = \varphi\pi\varphi^{-1}(x)$. *Then the mapping cylinder* C_g *is homeomorphic to* M. *In fact* M *is a regular neighborhood (in the sense of Hu's generalization of Whitehead's regular neighborhood) of* R.

Question. During the course of the proof of Lemma 1, polygonal arcs were constructed. This is done because subarcs of a polygonal arc are cellular. Is it true that every subarc of a cellular arc is cellular?

Andrew Ranicki

On Fibering Certain 3-Manifolds

John Stallings

1. Let E be the total space of a fiber space, whose base space is the 1-sphere C and whose fiber F is pathwise connected. From the exact homotopy sequence,

$$\pi_2(C) \to \pi_2(F) \to \pi_1(E) \to \pi_1(C) \to \pi_0(F)$$

noting the facts $\pi_2(C) = 0$, $\pi_0(F) = 0$, and $\pi_1(C) = Z$, we obtain the result that $\pi_1(E)$ has the normal subgroup $\pi_1(F)$ with quotient group Z.

If E is a compact 3-manifold, it is reasonable to expect F to be a compact 2-manifold; in particular, $\pi_1(F)$ would be finitely generated.

Now a converse is, to some extent, true. That is, we can show

THEOREM 1. *If* E *is a compact 3-manifold, and if* $\pi_1(E)$ *has a finitely generated normal subgroup* G, *whose quotient group is* Z, *then* G *is in fact the fundamental group of a 2-manifold* T *embedded in* E.

THEOREM 2. *If the hypotheses of Theorem* 1 *hold, and if* G *is not* $Z/2Z$, *and if* E *is irreducible (that is, every tame 2-sphere in* E *bounds a 3-cell), then* E *is the total space of a fiber space with base space a circle and with fiber the manifold* T *of Theorem* 1.

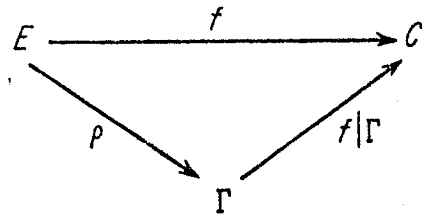
Note. Our assumption that E is irreducible is our way of avoiding the Poincaré conjecture.

2. Let $\varphi: \pi_1(E) \to Z$ be a homomorphism. Noting that $\pi_1(C) = Z$ and that C is aspherical, where C is a circle, we can obtain various maps (all mutually homotopic) $f: E \to C$, which induce φ. Let E and C be triangulated; let f be simplicial; let $p \in C$ be a point which is not a vertex in C's triangulation. Then $f^{-1}(p) = T$ is clearly a 2-manifold, which may not be connected; let the components of T be $T_1, T_2, \dots, T_n$.

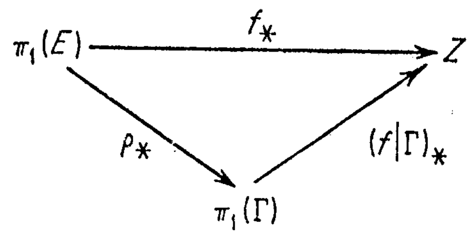
Pick points $a_i \in T_i$ and points b_j, one in each component of $M - T$. If T_i is in the boundary of the component of $M - T$ containing b_j, select an arc A_{ij} in that component joining a_i to b_j; do this so that $A_{ij} \cap A_{ik} = a_i$ and $A_{ij} \cap A_{kj} = b_j$.

If M_j approaches T_i from both sides, we have to choose two arcs A_{ij} and A'_{ij} whose union forms a circle cutting through T_i. The union of all these selected arcs A_{ij} is a 1-dimensional complex in E, which will be called Γ.

One easily finds a retraction $\rho : E \to \Gamma$ such that $\rho^{-1}(a_i) = T_i$ for all i. Thus, if we follow this retraction by the map f onto C, we obtain a map homotopic to f; that is, the following diagram is homotopy-commutative.



Passing to fundamental groups we get a commutative diagram; f^* is just



the same as φ; ρ_* is a retraction just as ρ is; and $\pi_1(\Gamma)$ is a free group. Hence ρ_* maps the kernel of f_* *onto* the kernel of $(f \mid \Gamma)_*$.

Now suppose the hypothesis of Theorem 1 holds, that the kernel of f_* is finitely generated and that f_* maps *onto* Z. Then we have proved that the kernel of $(f \mid \Gamma)_*$ is finitely generated.

But there is (by a well-known group-theoretic result) only one finitely generated normal subgroup of infinite index in any free group. Thus $(f \mid \Gamma)_*$ is an isomorphism. This enables us to see that $f \mid \Gamma$ is homotopic to a map $g : \Gamma \to C$ so that $g^{-1}(p)$ consists of just one of the points a_i.

Then $g\rho$ is a map of E into C, homotopic to f, and such that $(g\rho)^{-1}(p) = T_i$, some one of the components of $f^{-1}(p)$.

3. Now suppose $f : E \to C$ induces φ and that $f^{-1}(p) = T$ is a connected 2-manifold. If the injection $\pi_1(T) \to \pi_1(E)$ has a trivial kernel, then it follows from the loop theorem that there is a 2-disk $\Delta \subset E$ such that $\Delta \cap T = \partial \Delta$, where $\partial \Delta$ is a simple closed curve not contractible on T. We can find a 3-cell D containing Δ, such that $D \cap T$ is an annular neighborhood, both on D and T, of $\partial \Delta$. A simple construction will now give a map $g : E \to C$, homotopic to f, with $g^{-1}(p) = (T \cup D) - D \cap T$; this is a 2-manifold which may not be connected.

4. The simplification of Section 3 above raises the Euler characteristic of $f^{-1}(p)$ by two. If $f^{-1}(p)$ has been disconnected, we perform the simplification of Section 2; this discards a 2-manifold which is not a sphere, and thus lowers the Euler characteristic by at most one. The end result is to raise the characteristic of $f^{-1}(p)$; since this is bounded above, we eventually cannot simplify any further.

Thus, *there exists a map* $f : E \to C$ *which induces* φ, *such that* $f^{-1}(p) = T$ *is a connected 2-manifold such that the injection* $\pi_1(T) \to \pi_1(M)$ *is one-to-one.*

It should have been remarked that during each simplification, we could continue to have f simplicial with respect to triangulations of E and C (the triangulations of E will vary), and that p remains a non-vertex.

5. Clearly the image of $\pi_1(T)$ in $\pi_1(M)$ is contained in the kernel of f_*. It is, in fact, all of that kernel. The argument is exactly that of Neuwirth in his thesis: If there is anything else in the kernel of f_*, then a geometric construction shows that the kernel of f_* is the union of a strictly increasing sequence of groups, and so could not be finitely generated.

Thus is Theorem 1 established.

6. Now split E along the manifold T of Theorem 1. In this manner, we obtain a manifold M, in whose boundary are two copies T_0 and T_1 of T. It follows from earlier remarks that the maps $\pi_1(T_0) \to \pi_1(M)$ and $\pi_1(T_1) \to \pi_1(M)$ are isomorphisms.

The proof of Theorem 2 now involves two cases, depending on whether T has non-vacuous boundary or not. The case for $\partial T = \phi$ is reduced to the case when ∂T is not empty.

In the case when ∂T is not empty, ∂M is connected. Otherwise there would be a component of ∂E which did not intersect T; this component cannot be a sphere, since we assumed every sphere of E bounds a 3-cell; it cannot be anything else, since anything else would contribute to $H_1(M)$ so that $H_2(T_0) \to H_2(M)$ would not be onto.

Let $\chi(X)$ denote the Euler characteristic of X. We compute $\chi(\partial M)$. $\chi(\partial M) = 2\chi(M)$ for any compact 3-manifold M. In the case at hand, it is easy to see $\chi(M) \geq \chi(T)$. And since $M = T_0 + T_1 +$ (other pieces with Euler characteristic ≤ 0). We obtain the results that $\chi(\partial M) = 2\chi(T)$ and $\chi(M) = \chi(T)$.

Further analysis along this line shows that each component of $\partial M - (T_0 \cup T_1)$ is an annulus joining a component of ∂T_0 to one of ∂T_1.

7. Now T_0 is a retract of M. Since T_0 is aspherical and $\pi_1(T_0) \to \pi_1(M)$ is an isomorphism, there is no obstruction to constructing such a retraction; and according to Section 6, we can change this to a retraction $r : M \to T_0$ which is piecewise-linear and such that $r^{-1}(\partial T_0) = \partial M - Int\,(T_0 \cup T_1)$ and for each $x \in \partial T_0$, $r^{-1}(x)$ is a line segment.

8. Let $Q_1, \ldots, Q_n$ be a number of disjoint arcs on T_0, in general position with respect to r, such that $Q_i \subset T_0$ and such that $T_0 - \bigcup Q_i$ is a 2-cell. Such a set of arcs can be found on any connected 2-manifold with boundary.

We now consider $r^{-1}(Q_1), \ldots, r^{-1}(Q_n)$. The union of these is a non-connected 2-manifold in M. We now perform the sort of simplifications on this manifold which we did in Section 3. That is, we take disks D in M which intersect our manifold $\bigcup r^{-1}(Q_i)$ only along ∂D, which is not contractible on $\bigcup r^{-1}(Q_i)$. Then, noting that T_0 is aspherical, we deform r so as to cause $\bigcup r^{-1}(Q_i)$ to split along D.

This process cannot be carried on indefinitely, and we eventually reach a point where each component R_j of $r^{-1}(Q_i)$ has the property $\pi_1(R_j) \to \pi_1(M)$ is a monomorphism; since R_j is mapped by r onto the contractible Q_i and since $r_*: W_1(M) \to \pi_1(T_0)$ is an isomorphism, it follows that R_j is simply connected.

9. Call D_i the component of $r^{-1}(Q_i)$ which contains Q_i. Then D_i must be a disk. If we perform the modifications of Section 8 nicely, the intersection of D_i with T_1 will be an arc Q'_i.

Then $T_1 - \bigcup Q_i$ will be a disk, the only alternative being some disconnected set; for homological reasons, the alternative is impossible.

10. Let us now construct a homeomorphism $h: M \to T_0 \times [0, 1]$. On T_0 we define h as the inclusion onto $T_0 \times 0$.

$\partial M - (T_0 \cup T_1)$ is just homeomorphic to $(\partial T_0) \times [0, 1]$. Hence we can define h on $\partial M - (T_0 \cup T_1)$. We can assume that in so doing $D_i \cap (\partial M - (T_0 \cup T_1))$ goes into two vertical lines.

Then we can define h on each D_i so as to map D_i onto $Q_i \times [0, 1]$.

h has already been defined on ∂T_1 and Q'_i. Thus we can define h on the remaining 2-cell of T_1 to map T_1 onto $T_0 \times 1$.

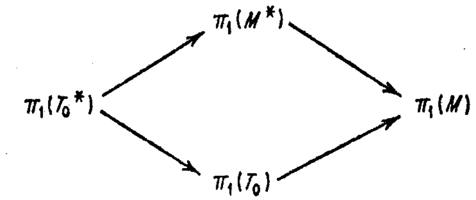
Extend the definition of f to a thin neighborhood of each D_i. By computing the Euler characteristic we find that what is left in M is bounded by a 2-sphere. From the hypothesis of Theorem 2, we conclude that this is a 3-cell, and we map it by h onto the remaining 3-cell in $T_0 \times [0, 1]$.

M is thus homeomorphic to $T \times [0, 1]$. We get E by some identification of $T \times 0$ with $T \times 1$. This immediately proves Theorem 2.

11. Now in case T has no boundary, we have eliminated, by hypothesis, the chance that T is a 2-sphere or a projective plane. Hence there is on T_0 a 2-sided simple closed curve C which does not separate T_0. We play the game of Section 8, using the curve C instead of the arcs Q_i. Eventually, we find an annulus K in M which connects C to a curve C on T_1.

Remove a tubular neighborhood fK from M. The resulting manifold is called M^* and $T_i \cap M^* = T_i^*$. We need to show now that $\pi_1(T_0^*) \to \pi_1(M^*)$ is an isomorphism in order to perform the construction of Sections 7–10.

Consider the diagram



Both the bottom homomorphisms are monomorphisms, and hence the kernel of $\pi_1(T_0^*) \to \pi_1(M^*)$ is trivial.

To show the map is onto, we use a geometric argument. Take the base point of all spaces concerned to be $P \in T_0^*$. An element of $\pi_1(M^*)$ is represented by a loop in $M - K$. Since $\pi_1(T_0) \to \pi_1(M)$ is an isomorphism, this loop is homotopic to a loop on T_0; make this homotopy in general position with respect to K; we then have a map $f: I \times I \to M$ such that $f(I \times 0) = f(I \times 1) = p$, and $f(0 \times I) \subset M - K$, and $f(1 \times I) \subset T_0$. $f^{-1}(K)$ consists of various curves and arcs not meeting three sides of $I \times I$.

We can modify f to remove an innermost closed curve of $f^{-1}(K)$ just using the fact that $\pi_1(K) \to \pi_1(M)$ is a monomorphism.

To remove an arc A of $f^{-1}(K)$ which bounds a part of $I \times I$ disjoint from the three sides we have mapped into $M - K$, we slide A along K until $f(A) \subset C$. The edge of the disk which A bounds is now mapped into T_0. Now we deform f so that it pushes this little disk over to $f(A)$ and then slightly further. A no longer will appear in $f^{-1}(K)$.

Doing this over and over we make $f^{-1}(K) = \emptyset$.

Thus we prove $\pi_1(T_0^*) \to \pi_1(M^*)$ is onto.

12. Hence as before we get M^* homeomorphic to $T_0^* \times I$. By pasting together two annuli on ∂M^* and two annuli on $(T_0^* \times I)$, we get M homeomorphic to $T_0 \times I$.

This completes the proof of Theorem 2 in all cases.

13. The outstanding problem presented by Theorem 2 is characterizing those 3-manifolds with fundamental group $\pi_1(E) = Z + Z_2$. In such an E there is a projective plane carrying the group Z_2, by Theorem 1. The difficulty is that Section 11 cannot be carried out here. If C is a 1-sided curve, the argument involving the Loop theorem is no longer valid; the disks we get, along which we try to split over one-sided 2-manifolds, have singularities along their boundaries.

This seems to be a hard problem.

14. Another question is whether two manifolds, E_1, E_2 as described in Theorem 2, which have isomorphic fundamental groups, are homeomorphic. This is probably true if the manifolds have no boundary; in the case of manifolds with boundary, a more complicated condition is required.

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Summary of Results on Contractible Open Manifolds

1. DEFINITIONS

An *open* n-*manifold* is a non-compact topological space that can be triangulated by a countable locally finite complex which is a combinatorial manifold without boundary (that is, the link of each vertex is a combinatorial $(n - 1)$-sphere). We shall be concerned only with orientable manifolds which are either open or *closed* (compact, without boundary).

2. STATEMENT OF RESULTS

If M^3 is a homotopy 3-sphere, then $M - \{\text{point}\}$ is a contractible open 3-manifold. Hence, to avoid the Poincaré conjecture, let us consider W-*spaces* (contractible open 3-manifolds each of whose compact subsets can be embedded in S^3). Whitehead [7] was the first to give an example of a W-space different from E^3. Whitehead's construction was very general in the following sense:

THEOREM 1. *Let* U *be a* W-*space. Then,* $U = \sum_{i=1}^{\infty} T_i$ *where* T_i *is a cube with handles,* $T_i \subseteq T_{i+1}$, *and* $j : T_i \to T_{i+1}$ *is* ~ 0.

From one of the PWZ theorems [5], we have immediately, using a theorem of Brown:

THEOREM 2. *If* U *is a* W-*space, then* $U \times E^1 = E^4$*—see* [1, 2].

THEOREM 3. *If* U_1, U_2, *are* W-*spaces, then* $U_1 \times U_2 = E^6$*—see* [1, 2].

In each case, we can express the product as a monotone union of open n-cells. It can be shown that the Poincaré conjecture is equivalent to the assertion that each contractible open 3-manifold is a W-space.

Question. Can we prove the statement: A contractible open n-manifold times E^1 is E^{n+1}.

We can almost answer the above question [4]:

THEOREM 4. *If* M^n *is a contractible open manifold, then* $M^n \times E^2 = E^{n+2}$.

To be sure that Theorem 2 applies to more than one space, note the following:

THEOREM 5. *There exist uncountably many topologically different contractible open subsets of* S^3*—see* [3].

The proof of Theorem 5 is an exercise in applying a theorem of Schubert on how one solid torus can wrap around inside another solid torus—see [6].

R. H. Bing has raised the question as to whether each W-space can be embedded in E^3. J. M. Kister and the author have recently shown that an example constructed by Bing cannot be embedded in E^3. It is easy to modify the examples given in [3] to make sure that no one of them can be embedded in E^3.

THEOREM 6. *There exist uncountably many topologically different* W-*spaces which cannot be embedded in* E^3.

The proof of the fact that the examples above cannot be embedded in E^3 is geometric. It would be of interest to obtain an algebraic proof. This might lead to an answer for the following:

Question. Is there a W-space which can be embedded in no closed 3-manifold?

We list some other questions which may prove to be interesting.

Question. Let M^3 be a closed 3-manifold with infinite fundamental group and such that $\pi_2(M) = 0$. If $\tilde{M}$ is the universal covering space of M, then $\tilde{M}$ is a contractible open 3-manifold. What are the properties of such covering spaces?

Question. The same question as above for the universal cover of the complement in S^3 of an unsplittable link.