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A topological proof of Grushko's theorem on free products

By
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In my thesis [6], there is a proof of Grushko's theorem [1] about generators of a free product; this is what I want to describe here. The proof uses simple topology and a combinatorial argument; it has the advantage that there is no complicated cancellation procedure. The proof makes it clear that the core of the argument is of an algorithmic nature, so that it has a certain effectiveness.

The topology in this proof is only a guise for Brandt groupoids, which are represented as sets of homotopy classes of paths in cell-complexes. HIGGINS [2] has developed the groupoid technique and used it on the Kurosh subgroup theorem. In the future I plan to publish a proof of Grushko's theorem using an algebraic construction slightly different from the groupoid.

An algebraicization of the proof given here is, though, straight-forward. Constructions in groupoids, analogous to adding cells to complexes, are certainly possible. The only topological theorem we need is VAN KAMPEN's [8] and in a groupoid-theoretic proof this theorem is totally irrelevant. We also use the fact that a direct limit of inclusions of CW-complexes is a CW-complex; this corresponds to the easy lemma that a direct limit of groupoids is a groupoid. Perhaps the study of groupoids alone is logically simpler, but for me the use of topological models helps with the over-all comprehension.

Besides Grushko's theorem, we shall here obtain WAGNER's extension [9] to infinitely generated groups. We also obtain a very interesting analogous theorem for free products with amalgamation.

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1. Terminology

The spaces we consider are CW-complexes. By a *path* in X is meant a map $P: [0, 1] \to X$, such that $P(0)$ and $P(1)$ are 0-cells of X; $P(0)$ and $P(1)$ are the *left* and *right end points* of P. Paths P and Q are said to be *homotopic*, if there is a homotopy between them which leaves fixed the end points. Homotopy classes of paths in X are called *path-classes*; each path-class has unique left and right end points.

If α and β are path-classes in X, and if the right end point of α is the left end point of β, then a product $\alpha\beta$ is defined. This gives the set of path-classes in X the structure of a category; an object is a 0-cell of X; a map is a path-class. Furthermore, given any path-class α there is an inverse path-class α^{-1}. Thus the path-classes in X form a groupoid.

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Each oriented 1-cell in X determines a path-class represented by a path along that 1-cell. Every path-class in X is a finite product of such path-classes and their inverses.

The choice of 0-cell in X gives X a *base-point*. The set of those path-classes in X, both of whose end points coincide with the base-point, is a group, denoted $\pi_1(X)$, called the *fundamental group* of X.

Let J be an index set. A *J-ad* $(X; \{A_\alpha\})$ consists of a complex X and a set of subcomplexes $\{A_\alpha\}$ indexed over $\alpha \in J$, such that $X = \bigcup_\alpha A_\alpha$, and such that for any $\alpha \neq \beta$, $A_\alpha \cap A_\beta = \bigcap_\gamma A_\gamma$. X is supposed to be provided with a basepoint in $\bigcap_\gamma A_\gamma$.

A *map* of J-ads $(X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ is a map $f: X \to Y$ which sends base-point to base-point, which for all n sends the n-skeleton of X into the n-skeleton of Y, and which for all $\alpha \in J$ sends A_α into B_α.

In a J-ad $(X; \{A_\alpha\})$, a *loop* is a path in X, whose end points coincide and lie in $\bigcap_\alpha A_\alpha$. A *tie* is a path in X, whose end points lie in different components of $\bigcap_\alpha A_\alpha$.

Let us call the elements of J *colors*. A *monochromatic* path is one whose image is contained completely in some one A_α.

Let a map $f: (X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ be given. A tie P in X is termed *binding*, if there exists α such that $P[0, 1] \subset A_\alpha$ and $f P$ is homotopic in B_α to a path in $\bigcap_\beta B_\beta$.

If P represents a path-class η in X and $f: X \to Y$ is given, then $f P$ represents a path-class denoted by $f_*(\eta)$ in Y. The definition that P is a binding tie could be phrased thus: For some α, there is a path-class ϑ in A_α; if f' denotes the restriction of f to a map $A_\alpha \to B_\alpha$, then $f'_*(\vartheta) = j_*(\zeta)$, where j is the inclusion $\bigcap_\beta B_\beta \subset B_\alpha$ and ζ is a path-class in $\bigcap_\beta B_\beta$.

2. Construction in case of binding ties

Let $f: (X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ be a map of J-ads. Let P be a binding tie with color α; thus $P[0, 1] \subset A_\alpha$ and $f P$ is homotopic in B_α to a path in $\bigcap_\beta B_\beta$.

By Δ we denote an abstract 2-cell, whose boundary is the union of two 1-cells Γ_1 and Γ_2 which intersect only in their end points. Identify Γ_1 with $[0, 1]$, so that P is a map of Γ_1 into $A_\alpha \subset X$.

By X', we denote the union of X and Δ, identifying $t \in \Gamma_1$ with $P(t) \in X$. Then X' is a CW-complex containing X and two additional cells Γ_2 and Δ; and X is a deformation retract of X'; in fact the inclusion $X \subset X'$ induces an isomorphism on the groupoids of path-classes.

By A'_α, we denote the union of A_α and Δ, attaching Δ to A_α along Γ_1 by means of P. For $\beta \neq \alpha$, by A_β, we denote the union of A_β and Γ_2, identifying the end points of the arc Γ_2 with the end points of P. Then $(X'; \{A'_\gamma\})$ is a J-ad which contains $(X; \{A_\gamma\})$. What we are interested in is $\bigcap_\gamma A'_\gamma$. This intersection clearly consists of $\bigcap_\gamma A_\gamma$ plus the arc Γ_2 which joins distinct components of $\bigcap_\gamma A_\gamma$.

Since $f P$ is homotopic in B_α to a path in $\bigcap_\gamma B_\gamma$, there is a map $g\colon \Delta \to B_\alpha$ such that $g = f P$ on Γ_1 and $g(\Gamma_2) \subset \bigcap_\gamma B_\gamma$. The map $f \cup g\colon X \cup \Delta \to Y$ respects the identification which produced X', and therefore defines a map $f'\colon X' \to Y$. Clearly $f'(A'_\beta) \subset B_\beta$. And so f' is a map of J-ads $(X'; \{A'_\gamma\}) \to (Y; \{B_\gamma\})$.

2.1. **Lemma.** *Let $f\colon (X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ be a map of J-ads. Then there is a J-ad $(X'; \{A'_\alpha\})$ containing $(X; \{A_\alpha\})$, such that X is a deformation retract of X', and such that each component of $\bigcap_\alpha A'_\alpha$ consists of components of $\bigcap_\alpha A_\alpha$ joined by arcs. And there is a map of J-ads $f'\colon (X'; \{A'_\alpha\}) \to (Y; \{B_\alpha\})$ which extends f, such that with respect to f' there are no binding ties.*

The proof is by induction on the number of components of $\bigcap_\alpha A_\alpha$, if this number is finite. If the number of such components is one, there is no binding tie, and so we take $f' = f$. The inductive step is the construction described above which reduces the number of components by one, whenever there is a binding tie; if there is no binding tie, of course, we take $f' = f$.

If there are infinitely many components in $\bigcap_\alpha A_\alpha$, we can still proceed, eliminating binding ties one by one, in a transfinite induction. When we come to a limit ordinal, we take the direct limit of the earlier constructions; as we remarked earlier, CW-complexes have the handy property that they behave well under direct limit. This process cannot go on indefinitely; for example it is easy to see that it must terminate at an ordinal number of cardinality at most that of the set of partitions of the set of components of $\bigcap_\alpha A_\alpha$.

2.2. *Remark.* We are interested mostly in the case that the components of $\bigcap_\alpha A_\alpha$ are points. Then in the end result, the components of $\bigcap_\alpha A'_\alpha$ will be *trees*; they will have trivial fundamental group. And so by VAN KAMPEN's theorem [*8*] the fundamental group of each component of A'_α will be a free factor of the fundamental group of X, conjugated by any path-class whose endpoints are the base-points of X and of that component of A'_α. (We are assuming that the component of A'_α belongs to the same component of X as the base point of X.)

3. The algorithm which finds a binding tie

Let $(X; \{A_\alpha\})$ be a J-ad. Each loop or tie is homotopic to a product of paths which run once across single 1-cells. This will be a product of monochromatic paths. By grouping these paths into maximal monochromatic blocks, our original path P is homotopic to a product $P_1 \cdot P_2 \cdots P_n$ of monochromatic paths, such that P_i and P_{i+1} have different colors for all i. Hence the end points of each P_i must belong to $\bigcap_\alpha A_\alpha$; for the left end point of P_1 and the right end point of P_n this happens because P is a tie or loop.

In each component of $\bigcap_\alpha A_\alpha$ choose a base point, such that among these base points occur the left end point of P_1 and the right end point of P_n. For $1 \leqq i < n$, let Q_i be a path in $\bigcap_\alpha A_\alpha$ which joinx the right end point of P_i to the base point in its component. Then the path P is homotopic to

$$(P_1 Q_1) \cdot (Q_1^{-1} P_2 Q_2) \ldots (Q_{n-1}^{-1} P_n).$$

Each term here is monochromatic; and if end points of a term are in the same component of $\bigcap_\alpha A_\alpha$, they coincide; hence each term is either a loop or a tie.

So we have proved:

3.1. **Lemma.** *Each tie or loop in a J-ad* $(X; \{A_\alpha\})$ *is homotopic to a product of monochromatic loops and ties* $P_1 P_2 \cdots P_n$, *whose end points are among a set of base points, one per component of* $\bigcap_\alpha A_\alpha$.

Now we come to the combinatorial fact which will imply Grushko's theorem.

3.2. **Theorem.** *Let* $f\colon (X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ *be a map of J-ads, where X is connected. Suppose* $\bigcap_\alpha B_\alpha$ *is a single point, so that* $\pi_1(Y)$ *is naturally the free product of the set of groups* $\{\pi_1(B_\alpha)\}$. *Suppose that the induced map* $f_*: \pi_1(X) \to \pi_1(Y)$ *is onto. Then if* $\bigcap_\alpha A_\alpha$ *is not connected, there is a binding tie.*

Proof. There is a tie Q in X, whose path-class η *is such that* $f_*(\eta) = 1$, *the trivial element of* $\pi_1(Y)$. For there is a tie P whose left end point is the base point of X. The path-class represented by P will be called ϑ. Since $f_*: \pi_1(X) \to \pi_1(Y)$ is onto, there is a loop in X, based at the base point, call it L, representing $\lambda \in \pi_1(X)$, such that $f_*(\lambda) = f_*(\vartheta)$. Then $P^{-1} L$ is the desired tie Q, since $f_*(\vartheta^{-1}\lambda) = 1$.

By 3.1., we can suppose Q is a product of monochromatic loops and ties

$$Q = Q_1 Q_2 \ldots Q_n.$$

There is such a tie $Q = Q_1 Q_2 \ldots Q_n$, *representing* $\eta = \eta_1 \eta_2 \ldots \eta_n$, *such that* $f_*(\eta) = 1$, *such that for all i it is true that* Q_i *and* Q_{i+1} *have different colors, and such that for all i if* Q_i *is a loop then* $f_*(\eta_i) \neq 1$. For, if Q_i and Q_{i+1} have the same color, then $Q_i Q_{i+1}$ is still a monochromatic loop or tie; we then obtain a representation of Q as a product of fewer terms $Q_1 Q_2 \ldots Q_{i-1}(Q_i Q_{i+1}) \cdots Q_n$. And if Q_i is a loop and $f_*(\eta_i) = 1$, then $Q' = Q_1 Q_2 \ldots Q_{i-1} Q_{i+1} \cdots Q_n$ is a tie, a product of fewer terms; and if Q' represents η', then

$$f_*(\eta') = f_*(\eta_1) f_*(\eta_2) \ldots f_*(\eta_{i-1}) \cdot 1 \cdot f_*(\eta_{i+1}) \ldots f_*(\eta_n) = f_*(\eta) = 1.$$

After a finite number of such reductions we obtain the desired Q. The number n has to remain $\geqq 1$ since the end points of Q are distinct and remain unchanged by these reductions.

Now we have the equation in $\pi_1(Y)$, $1 = f_*(\eta) = f_*(\eta_1) f_*(\eta_2) \ldots f_*(\eta_n)$. The terms $f_*(\eta_i)$ and $f_*(\eta_{i+1})$ lie in different factors $\pi_1(B_\alpha)$ for all i; and $n \geqq 1$. Therefore some term $f_*(\eta_i) = 1$; otherwise we would have a reduced word of length $\geqq 1$ in this free product, representing 1. Now Q_i cannot be a loop, for we have avoided such trivial loops. Hence Q_i is a tie; it is monochromatic. And it is binding since $f_*(\eta_i) = 1$; that is, $f Q_i$ is nullhomotopic in Y; but because $\pi_1(Y)$ is a free product, $f Q_i$ must be null-homotopic in the factor B_α into which it is mapped. Q.E.D.

A similar, but weaker, result is true for free products with amalgamation.

3.3. **Theorem.** *Let* f: $(X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$ *be a map of J-ads, where* X *is connected. Suppose that* $C = \bigcap_\alpha B_\alpha$ *has only one* 0*-cell, and that for each* α, $\pi_1(C) \to \pi_1(B_\alpha)$ *is an embedding; so that* $\pi_1(Y)$ *is naturally isomorphic to the free product of the set of groups* $\{\pi_1(B_\alpha)\}$ *with amalgamated subgroup* $\pi_1(C)$. *Suppose that the induced map* f_*: $\pi_1(X) \to \pi_1(Y)$ *is onto. In addition, suppose every component of every* A_α *has trivial fundamental group. Then if* $\bigcap_\alpha A_\alpha$ *is not connected, there is a binding tie.*

Proof. The important extra hypothesis is that each component of each A_α have trivial fundamental group. This says that each monochromatic loop in X is null-homotopic; so that by 3.1 if $\bigcap_\alpha A_\alpha$ is connected, then $\pi_1(X)$ is trivial. Thus the hypothesis that $\bigcap_\alpha A_\alpha$ is not connected is, in any interesting case, superfluous.

As in the proof of 3.2, we find a tie in X, called Q, such that $f\,Q$ is homotopic to a path in $\bigcap_\alpha B_\alpha$. First find a tie P starting at the base point, representing ϑ in the groupoid of path-classes in X; since $\bigcap_\alpha B_\alpha$ has only one 0-cell, $f\,P$ is a loop in Y. Because f_* is onto there is an element $\lambda \in \pi_1(X)$ such that $f_*(\lambda) = f_*(\vartheta)$. Then $\vartheta^{-1}\lambda$ is represented by a tie Q, and $f_*(\vartheta^{-1}\lambda) = 1$.

We can, by 3.1, suppose Q is a product $Q_1 Q_2 \dots Q_n$ of monochromatic loops and ties. In this expression, remove all the loops since they must, by our extra hypothesis, be null-homotopic. Amalgamate Q_i and Q_{i+1} if they have the same color; remove any resulting loops; and so on. Finally we find a tie Q which is a product of monochromatic ties only, $Q_1 Q_2 \dots Q_n$; let η and η_i be the path-classes which Q and Q_i represent; this expression of Q as a product will have the properties that $f_*(\eta) = 1$ and that $f_*(\eta_i)$ and $f_*(\eta_{i+1})$ will, for all i, lie in different factors $\pi_1(B_\alpha)$. Then we have $1 = f^*(\eta) = f_*(\eta_1) f_*(\eta_2) \dots f_*(\eta_n)$ and $n \geqq 1$.

If none of the terms $f_*(\eta_i)$ lay in the amalgamating subgroup π_1(C), then from the elementary theory of free products with amalgamation [*4*], a product of such terms, where adjacent terms lie in different factors, could not be trivial. Hence there is i such that $f_*(\eta_i)$ belongs to $\pi_1(C)$. That is to say, $f\,Q_i$ is homotopic in Y to a loop in $C = \bigcap_\alpha B_\alpha$. But in an free product with amalgamation such as this, the map $\pi_1(B_\alpha) \to \pi_1(Y)$ is an embedding; hence $f\,Q_i$ is homotopic, in the B_α into which it is mapped, to a loop in C. Hence Q_i is a binding tie. Q.E.D.

Note that we cannot get rid of loops Q_i for which $f_*(\eta_i)$ belongs to $\pi_1(C)$ unless we have the hypothesis that the components of A_α have trivial fundamental group. For example, suppose $Q = Q_1 Q_2 Q_3$, where $f_*(Q_2) \in \pi_1(C)$, and $1 = f_*(\eta) = f_*(\eta_1) f_*(\eta_2) f_*(\eta_3)$. After deleting Q_2 we have $Q' = Q_1 Q_3$ and $f_*(\eta') = f_*(\eta_1\,\eta_2\,\eta_3 \cdot \eta_3^{-1}\,\eta_2^{-1}\,\eta_3) = f_*(\eta_3)^{-1} f_*(\eta_2^{-1}) f_*(\eta_3)$. This will not generally belong to $\pi_1(C)$ unless $\pi_1(C)$ is a *normal* subgroup of $\pi_1(Y)$. The case of free products with normal amalgamating subgroup is little different from the case of plain free products.

4. The major theorems

4.1. **Theorem** of GRUSHKO and WAGNER. *Let φ: $\Gamma \to *_\alpha \Pi_\alpha$ be a homomorphism of the free group Γ onto the free product of groups $\{\Pi_\alpha\}$. Then Γ is itself a free product, $\Gamma = *_\alpha \Gamma_\alpha$, such that $\varphi(\Gamma_\alpha) \subset \Pi_\alpha$.*

Proof. Let B_α be a 2-dimensional complex determined by a presentation of Π_α. We identify Π_α with $\pi_1(B_\alpha)$. Let Y be the union of the complexes $\{B_\alpha\}$, identifying all the base points to a single point. Then $\pi_1(Y)$ may be considered the same as $*_\alpha \Pi_\alpha$.

Let Γ have a free basis $\{\gamma_\tau\}$, where τ ranges over some index set. Consider for a moment a single element γ_τ. Then $\varphi(\gamma_\tau) = a_1 a_2 \ldots a_n$, where each a_i belongs to one of the groups Π_α.

Let S_τ denote a 1-sphere, divided into n 1-cells, called, starting at a base point and proceeding around S_τ, $W_1, W_2, \ldots, W_n$. Define f on W_i to be a path in B_α such that $f|W_i$ represents a_i. Then f defines a map from S_τ into Y.

Let X denote the union of all the S_τ, identifying all base points together. The union of all the maps f above defines a map f: $X \to Y$. We identify Γ with $\pi_1(X)$; then φ is the same as the map f_*: $\pi_1(X) \to \pi_1(Y)$.

To each of the segments W_i of S_τ we associate an index α such that $f|W_i$ is a loop in B_α. Define A_α to be the union of all the 0-cells of X, and all such segments W_i to which α is associated. Then $(X; \{A_\alpha\})$ and $(Y; \{B_\alpha\})$ are J-ads, for the index set J of the free product $*_\alpha \Pi_\alpha$; and f is a map of J-ads. Finally, $\bigcap_\alpha A_\alpha$ is just the 0-skeleton of X.

By 2.1, there is a J-ad $(X'; \{A'_\alpha\})$, containing $(X; \{A_\alpha\})$, with X a deformation retract of X', and an extension f' of f, such that in X' there are no binding ties. We can still identify φ: $\Gamma \to *_\alpha \Pi_\alpha$ with f'_*: $\pi_1(X') \to \pi_1(Y)$.

By 3.2, $\bigcap_\alpha A'_\alpha$ must be connected. By 2.2 then $\bigcap_\alpha A'_\alpha$ is a tree. By VAN KAMPEN's theorem [8], $\pi_1(X')$ is the free product of the groups $\pi_1(A'_\alpha)$. To complete the proof we define $\Gamma_\alpha = \pi_1(A'_\alpha)$.

4.2. *Remarks.* In case Γ is of finite rank, and if we can effectively determine whether any element of Π_α is trivial, and if for each element of $*_\alpha \Pi_\alpha$ we can effectively find a pre-image in Γ, then this proof and the free product structure of Γ is effective. For, by 3.2, we can discover binding ties by an algorithm, whenever they exist.

My original version of this proof involved 3-dimensional manifolds and has been expanded into one of the proofs of Kneser's Conjecture [5], [7]. In fact, with hindsight we can see that a proof of Grushko's theorem could have been derived from the discussion in KNESER's paper [3].

4.3. **Theorem** (for amalgamated free products). *Let φ: $\Gamma \to (*_\alpha \Pi_\alpha)_\Sigma$ be a homomorphism of a free group onto the free product of groups $\{\Pi_\alpha\}$ with amalgamated subgroup Σ. Then there is an element x of some free basis of Γ, and an index α, such that $\varphi(x) \in \Pi_\alpha$.*

Proof. Let C_α be a complex with $\pi_1(C_\alpha) = \Pi_\alpha$. Let D be such that $\pi_1(D) = \Sigma$. Choosing D to be two-dimensional, we can find maps g_α: $D \to C_\alpha$ inducing the inclusions $\Sigma \subset \Pi_\alpha$. Let B_α be the mapping cylinder of g_α. These mapping cylinders intersect exactly in D; their union will be called Y. We can identify $\pi_1(Y)$ with $(*_\alpha \Pi_\alpha)_\Sigma$.

We realize Γ as $\pi_1(X)$, where X is one-dimensional, and find a J-ad $(X; \{A_\alpha\})$ and a map f: $(X; \{A_\alpha\}) \to (Y; \{B_\alpha\})$, inducing φ, just as in the proof of 4.1. Again $\bigcap_\alpha A_\alpha$ will be a discrete set of points.

By 2.1, we can assume there are no binding ties in X, and by 2.2 that all components of $\bigcap_\alpha A_\alpha$ have trivial fundamental group. By 3.3 some component of some A_α has non-trivial fundamental group, which by 2.2 is a free factor of $\pi_1(X)$ conjugated by a path-class.

Let z be an element of a free basis of one of the non-trivial fundamental groups of components of an A_α. Let ρ be a path-class in X connecting the base point of that component to the base point of X. Then $\rho^{-1} z \rho$ belongs to a basis of $\pi_1(X)$.

Assuming D has only one 0-cell, which we can always arrange, $f_*(\rho)$ belongs to $\pi_1(Y)$. Since $f_*: \pi_1(X) \to \pi_1(Y)$ is onto, there is $\sigma \in \pi_1(X)$ such that $f_*(\sigma) = f_*(\rho)$.

Now $x = \sigma(\rho^{-1} z \rho) \sigma^{-1}$ belongs to a basis of $\pi_1(X)$, and $f_*(x) = f_*(z) \in \pi_1(B_\alpha)$, since z belongs to the fundamental group of some component of A_α.

4.4. *Remark.* Of course, Γ is always the free product of $\{\varphi^{-1}(\Pi_\alpha)\}$ *with amalgamated subgroup* $\varphi^{-1}(\Sigma)$. However, one cannot quite conclude from 4.3 that Γ is itself an ordinary free product $*_\alpha \Gamma_\alpha$ in such a way that $\varphi(\Gamma_\alpha) \subset \Pi_\alpha$. Here is a simple counterexample.

The free group Γ on two symbols $\{x, y\}$ is the free product of its subgroups $\{x, y^2 x^2 y x^{-2} y^{-2}\}$ and $\{x^2, y^2\}$ with amalgamated subgroup $\{x^2, y^2 x^2 y^2 \times x^{-2} y^{-2}\}$. To see this, let Π_1 be the free group with basis $\{a, b\}$, Π_2 that with basis $\{c, d\}$, Σ that with basis $\{e, f\}$. Embed Σ in Π_1 by $e \to a^2$ and $f \to b^2$; embed Σ in Π_2 by $e \to c$ and $f \to dc\,dc^{-1} d^{-1}$. Then $(\Pi_1 * \Pi_2)_\Sigma$ has a presentation $\{a, b, c, d$: $a^2 = c, b^2 = dc\,dc^{-1} d^{-1}\}$.

The map φ: $\Gamma \to (\Pi_1 * \Pi_2)_\Sigma$ is given by $\varphi(x) = a$ and $\varphi(y) = c^{-1} d^{-1} bdc$. The inverse map ψ: $(\Pi_1 * \Pi_2) \to \Gamma$ is given by $\psi(a) = x$, $\psi(b) = y^2 x^2 y x^{-2} y^{-2}$; $\psi(c) = x^2$, $\psi(d) = y^2$. Then φ and ψ are inverses of each other.

If now Γ splits into a free product $\Gamma_1 * \Gamma_2$ with Γ_1 contained in $\{x, y^2 x^2 y \times x^{-2} y^{-2}\}$ and Γ_2 contained in $\{x^2, y^2\}$, then Γ_2 must be trivial since $\{x^2, y^2\}$ contains no basis elements. This would imply Γ is generated by $\{x, y^2 x^2 y \times x^{-2} y^{-2}\}$, which is ridiculous since y is not generated by those elements.

4.5. *Remark.* One can derive certain results from 4.3. For example, let $\langle x_1, \ldots, x_n \rangle$ denote the smallest normal subgroup of Γ containing $\{x_1, \ldots, x_n\}$. Then there is a sequence $\{x_1, \ldots\}$ of elements of Γ, of length n if Γ has rank n, countable if Γ has infinite rank; such that for each k, $\Gamma / \langle x_1, \ldots, x_k \rangle$ is free and has a basis one of whose elements is represented by x_{k+1}; and such that for each k, there is α such that $\varphi(x_k) \in \Pi_\alpha$. (I do not know whether this can be continued transfinitely.)

To prove this, we find x_1 by 4.3, mapping into Π_α. Then $\Gamma/\langle x_1\rangle$ is a free group mapping onto $(*_\alpha \Pi'_\alpha)_{\Sigma'}$, where Π'_α and Σ' are the images of Π_α and Σ under the map which collapses the normal subgroup generated by $\varphi(x_1)$. So by 4.3, we find a basis element y of $\Gamma/\langle x_1\rangle$ which maps into some factor Π'_β; we can easily find a representative of y in Γ, called x_2, which maps into Π_β. And we continue in this way until the first infinite ordinal ω; it then is not clear (to me) whether $\Gamma/\langle x_1, x_2, \ldots\rangle$ is free.

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