Whitehead torsion of free products

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0. There is a covariant functor from groups to abelian groups known as Whitehead torsion. It has been of interest mainly in combinatorial and differential topology. Our main result is that, if $A \longrightarrow A*B \longleftarrow B$ represents $A*B$ as the free product of the groups A and B, then the induced diagram of Whitehead torsions, $\mathcal{WT}(A) \longrightarrow \mathcal{WT}(A*B) \longleftarrow \mathcal{WT}(B)$, represents $\mathcal{WT}(A*B)$ as the direct sum of $\mathcal{WT}(A)$ and $\mathcal{WT}(B)$. Or, as we shall say to save space, $\mathcal{WT}(A*B) = \mathcal{WT}(A) + \mathcal{WT}(B)$.

The proof is purely algebraic and will be developed in a broader algebraic context.

My original proof was a geometric monstrosity and may be forgotten because of the cleaner algebraic proof in this paper. The basic tools used are S. Gersten's theorem [4] on K_1 of a free ring, P. Cohn's description of the free product of rings [2], and a few trivial algebraic tricks.

1. The functor K_1[1]

Let Λ be a ring with 1. Let $\mathcal{G}$ denote the set of square, invertible matrices over Λ; in $\mathcal{G}$ there is a composition law $\oplus$, described thus:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

"I" will denote the identity matrix of any size. Then $\approx$ will denote the equivalence relation on $\mathcal{G}$ generated by:

(1) $A \approx A \oplus I$; and

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[1] *On the notation.* In Whitehead's paper [9], the symbol $\mathfrak{A}$ is used for what we call $\mathcal{G}$; the symbols $\mathfrak{A}/\mathfrak{S}$ are used for what we call $\mathcal{W}(\Lambda)$; also, $\tau(\alpha)$ has the same meaning as ours in § 7; and T is used for what we call $\mathcal{WT}(\Pi)$. Our guess is that Whitehead's T stood for "torsion," since the values of τ lie in T, and $\tau(\alpha)$ is called the "torsion of α." More recent notation has been derived from the analogy to the groups of vector bundles over a space; thus $K^1(\Lambda)$, $K_1(\Lambda)$, $K^{-1}(\Lambda)$ are used for what we call $\mathcal{W}(\Lambda)$; such things as $\bar{K}_1(\Lambda)$ and $\widetilde{K}^1(\Lambda)$ are used for our $K_1(\Lambda)$. The symbols $\mathrm{Wh}(\Pi)$ have also been used to denote Whitehead's T and our $\mathcal{WT}(\Pi)$.

In our opinion, the basic functor, which we call K_1, should have a simple, reasonable symbol; the other functors, which have only isolated uses, deserve more bizarre symbolism. The use of the subscript "1" is defensible only because there is another functor K_0 (or $\bar{K}_0$ or $\widetilde{K}^0$ or "projective class group") which sometimes occurs as the target of a boundary operator from some source K_1 (but this phenomenon does not occur in this paper).

(2) $A \approx EA$, where E is any matrix of the form

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \text{ or } \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}.$$

This equivalence relation is compatible with the operation $\oplus$; the set of equivalence classes forms a commutative group $\mathfrak{W}(\Lambda)$; the equivalence relation is the same as that generated by (1). $A \approx A \oplus I$; and

(3) $AB \approx BA$ if A and B have the same size;

it also follows that $AB \approx A \oplus B$. Finally, if $\varphi : \Lambda \to \Gamma$ is a map of rings (with "$\varphi(1) = 1$" included in the definition of map of rings), then the correspondence $A \to A^{\varphi}$, where A is a Λ-matrix and A^{φ} the Γ-matrix whose entries are the images of those of A by φ, induces a homomorphism $\mathfrak{W}(\varphi)$: $\mathfrak{W}(\Lambda) \to \mathfrak{W}(\Gamma)$; so that $\mathfrak{W}$ is a covariant functor from the category of rings to the category of commutative groups. These facts are well-known and due to Whitehead [9].

Let R denote a fixed ring. We shall call Λ an *R-ring*, if Λ is a ring containing R, and the inclusion $R \subset \Lambda$ is a map of rings, and there exists a map of rings $\varepsilon : \Lambda \to R$ such that for all $r \in R$, $\varepsilon(r) = r$; the map ε is not part of the structure of R, only its existence is necessary for Λ to be an R-ring. If $\varphi : \Lambda \to \Gamma$ is a map of rings, and if Λ and Γ are R-rings, and if for all $r \in R$, $\varphi(r) = r$, then we call φ a *map of R-rings*. Clearly, R-rings and their maps form a category.

Define $K_1(\Lambda; R)$ to be the cokernel of the map induced from the inclusion, $\mathfrak{W}(R) \to \mathfrak{W}(\Lambda)$. Since we shall not in this paper have occasion to consider more than one ground ring R at a time, we abbreviate the notation to $K_1(\Lambda)$. From $\mathfrak{W}$, K_1 inherits the fact that it is a covariant functor, from R-rings to commutative groups.

2. Free products

The word "free product" is used to denote the direct sum in a category whose objects have multiplicative structures. For example, we say that the R-ring C is described as the free product of R-rings A and B, if there are given maps $\alpha : A \to C$ and $\beta : B \to C$, such that for any R-ring X and maps $\alpha' : A \to X$ and $\beta' : B \to X$, there is one and only one consistent map $\gamma : C \to X$. We abbreviate this sentence to $C = A*B$.

Similarly, if G and H are groups, we may speak of their free product $G*H$.

The *group ring* of a group G, with coefficient ring R, denoted $R(G)$, is an R-ring Λ, together with a homomorphism of G into the group of units of Λ, such that given any homomorphism of G into the group of units of an arbitrary R-ring X, there is one and only one consistent map $\Lambda \to X$.

It is clear by abstract argument, then, that $R(G*H) = R(G)*R(H)$.

There is a well-known construction for free products of groups [7, p. 11].

If we consider only the more restrictive category of *R-algebras*, that is, R-rings Λ such that every element of R commutes with every element of Λ, then the analogous concept of the group *algebra* has a simple construction. Specifically, R will be a commutative ring and $R(G)$ will be the free R-module with basis $\{G\}$, in which $r \in R$ is identified with $r \cdot 1$. It has multiplication defined by $(r_1 g_1)(r_2 g_2) = (r_1 r_2)(g_1 g_2)$.

Z will denote the integers. It is obvious that every Z-ring is a Z-algebra. Hence the integral group ring always exists, by the above construction. Therefore we conclude that for certain Z-rings, namely $Z(G)$ and $Z(H)$, their free product as Z-rings exists, namely, $Z(G)*Z(H) = Z(G*H)$.

We now wish to prove the existence of free products of R-rings in general, and to establish a particularly convenient description of it.

(The nature of group rings, for arbitrary R and G, seems interesting and is open.)

3. Graded rings

Let S be a semigroup with identity element, usually written multiplicatively. That Σ is an *S-graded ring*, means that to each $w \in S$ there is associated an abelian group Σ_w, and that to each $u, v \in S$ there is associated a homomorphism $\mu_{u,v} : \Sigma_u \otimes \Sigma_v \to \Sigma_{uv}$; such that the obvious associative law holds, and such that in Σ_1 there is a two-sided unit element for this multiplication. We shall confuse this concept of Σ with the direct sum, over all $w \in S$, of Σ_w, together with its multiplication defined *via* $\mu_{u,v}$ for all u and v in S. Then Σ is an ordinary ring.

In case of the additive semigroup N of non-negative integers, we use additive notation on the indices.

One important example of an N-graded ring is the *tensor ring of a bimodule*. Let M be a bimodule over the ring R. We define a graded ring $T(M)$ as follows:

$$T(M)_0 = R$$

$$T(M)_{n+1} = M \otimes_R T(M)_n$$

$$\mu_{p,q} : T(M)_p \otimes T(M)_q \to T(M)_p \otimes_R T(M)_q = T(M)_{p+q}$$

is the natural map, using the natural associativity of the tensor product.

That $T(M)$ is a graded ring is obvious. It is an R-ring since $T(M)$ can be retracted onto R by sending $T(M)_n$ to 0 for $n > 0$. And it has the following defining universal property: There is a map of bimodules $\alpha : M \to T(M)$, such that for any R-ring X and any map of bimodules $\alpha' : M \to X$ there is one and only one consistent map of R-rings $\beta : T(M) \to X$.

It is now seen that we can construct the free product of two tensor rings easily; namely, $T(M_1)*T(M_2) = T(M_1 + M_2)$, where "+" denotes the direct sum of bimodules, which can be explicitly realized as a set of ordered pairs, $M_1 \times M_2$.

3.1. Ring graded on F

F will denote to free semigroup on two symbols $\{a, b\}$. F is the set of all finite words in $\{a, b\}$, including the empty word 1, with juxtaposition as multiplication.

Suppose that A and B are two R-bimodules. Define a ring Λ graded on F (it will turn out that Λ is just $T(A + B)$ with a more complicated grading) thus:

$$\begin{aligned} \Lambda_1 &= R \\ \Lambda_{aw} &= A \otimes_R \Lambda_w \\ \Lambda_{bw} &= B \otimes_R \Lambda_w \,. \end{aligned}$$

[Roughly, Λ_w is just w with occurrences of a replaced by A and b replaced by B and $\otimes_R$ inserted between terms.]

$$\mu_{u,v} : \Lambda_u \otimes \Lambda_v \to \Lambda_u \otimes_R \Lambda_v = \Lambda_{uv}$$

is the natural map.

A little thought will show that this is the free product of $T(A)$ and $T(B)$. With this gradation we shall denote the ring by $T(A, B)$.

3.2. The semigroup G

G will denote the semigroup on two symbols $\{a, b\}$ with the relations $aa = a$ and $bb = b$. We shall now describe the free product of two R-rings Λ and Γ as a G-graded ring. Let us remark that given any word in the symbols $\{a, b\}$, by applying reductions which change any segment aa to a and any segment bb to b, we eventually obtain a reduced word, one with no segment of the form aa or bb. And this reduced word is independent of the chain of reductions used.

Now, let Λ and Γ be R-rings. Let $\varepsilon_\Lambda : \Lambda \to R$ and $\varepsilon_\Gamma : \Gamma \to R$ be retractions. Let A denote $\varepsilon_\Lambda^{-1}(0)$ and B denote $\varepsilon_\Gamma^{-1}(0)$. Then A and B are R-bimodules, and as bimodules, $\Lambda = R + A$ and $\Gamma = R + B$. Further the multiplications on Λ and Γ define associative maps $A \otimes_R A \to A$ and $B \otimes_R B \to B$.

Now, Λ can be recovered from A (and Γ from B similarly) and this multiplication by taking the tensor ring $T(A)$ and factoring modulo the relations which say that an element equals its image under the map $T(A)_2 = A \otimes_R A \to A = T(A)_1$.

Therefore, without further ado or proof, it is clear that $\Lambda * \Gamma$ may be described as $T(A) * T(B)$ modulo the relations which identify an element with its image under the multiplications $A \otimes_R A \rightarrow A$ and $B \otimes_R B \rightarrow B$. Using the gradation $T(A, B)$ these are maps $T(A, B)_{aa} \rightarrow T(A, B)_a$ and $T(A, B)_{bb} \rightarrow T(A, B)_b$.

Let us observe what these relations do to the gradation on $T(A, B)$. If u is obtained, for example, by reducing a segment aa of w to a, then the multiplication $A \otimes_R A \rightarrow A$ identifies $T(A, B)_w$ with its image in $T(A, B)_u$ under the map consisting of the appropriate tensor product of identity maps and multiplication. If u is a reduced word, obtained from w by two different sequences of reductions, then the relations introduced by the two sequences of multiplications will be exactly the same, *because of associativity.*

Hence, adding the relations to $T(A, B)$ makes every element of $T(A, B)_w$ equivalent to an element of $T(A, B)_u$, where u is the reduced version of w, and no two elements of $T(A, B)_u$ are identified.

This shows that $\Lambda * \Gamma$ can be described as a G-graded ring. The components $(\Lambda * \Gamma)_g$ are isomorphic as R-bimodules to $T(A, B)_u$ where u is the reduced word representing $g \in G$. The multiplicative structure is a mixture of the multiplicative structures of Λ and Γ and tensor product.

This is the structure theorem we wish. We may write it mnemonically thus:

If $\Lambda = R + A$ *and* $\Gamma = R + B$, *then*

$$\Lambda * \Gamma = R + A + B + A \otimes B + B \otimes A + A \otimes B \otimes A + B \otimes A \otimes B + \cdots$$

where $\otimes$ *means* $\otimes_R$. *The multiplicative structure is determined by multiplying components by the tensor product and then collapsing if possible using the multiplications* $A \otimes A \rightarrow A$ *and* $B \otimes B \rightarrow B$ *derived from* Λ *and* Γ.

3.2.1. Let us observe the very important fact that $\Lambda * \Gamma$ contains as a subring the tensor ring of $A \otimes_R B$. And that this may be explicitly described as:

$R +$ (left ideal generated by $A \otimes_R B$) $\cap$ (right ideal generated by $A \otimes_R B$) $= T(A \otimes_R B)$.

For, the left ideal generated by $A \otimes_R B$ consists of all $T(A, B)_{wab}$ where wab is reduced; and the right ideal generated by $A \otimes_R B$ consists of all $T(A, B)_{abw}$ where abw is reduced. Their intersection then is exactly the set of all $T(A, B)_{(ab)^n}$ for $n = 1, 2, \cdots$. And multiplication between these components is just that found in the tensor ring.

3.3. A remark about groups

This construction of the free product of rings gives a rather unusual construction for the free product of groups. Let Π_1 and Π_2 be multiplicative groups; construct $Z(\Pi_1)$ and $Z(\Pi_2)$; then by the above method construct $Z(\Pi_1) * Z(\Pi_2)$. Define the group Δ to be the group of those units in $Z(\Pi_1) * Z(\Pi_2)$ which are generated by Π_1 and Π_2. It is claimed that Δ is the free product of Π_1 and Π_2:

Let X be any test group and $\alpha : \Pi_1 \to X$, $\beta : \Pi_2 \to X$ homomorphisms. These extend, by the universal property of group rings, to maps $Z(\Pi_1) \to Z(X)$ and $Z(\Pi_2) \to Z(X)$, and by the universal property of free product of rings, to a map $Z(\Pi_1) * Z(\Pi_2) \to Z(X)$. This, restricted to Δ, maps Δ into X, since Δ is generated by Π_1 and Π_2 and these are mapped into X.

Therefore, Δ is indeed the free product of Π_1 and Π_2. The reader might find it amusing to work on a few simple examples. For example, if we express $aba^{-1}b^{-1}$ as a sum of homogeneous components, we get

$$1 - a^*(b^{-1})^* + b^*(a^{-1})^* + a^*b^*(a^{-1})^* + b^*(a^{-1})^*(b^{-1})^* + a^*b^*(a^{-1})^*(b^{-1})^*$$

where x^* denotes $x - 1$.

It would seem that the homogeneous components in the G-grading are related to the free derivatives in the sense of Fox [3].

4. Higman's trick

G. Higman [6], in work on the Whitehead torsion of an infinite cyclic group, performed basically the trick leading to simplification which follows.

Consider a matrix M whose entries belong to a free product $\Lambda * \Gamma$ of R-rings. Evidently, we may write each entry as a sum of terms each of which is the product of elements of Λ and Γ. We perform an inductive simplification of M by changing M to $M \oplus I$ and multiplying on the left or the right by elementary matrices. We reduce the number of terms which have been written as the longest products until we obtain a matrix M' in which every entry is the sum of an element of Λ and an element of Γ. This reduction process can be described thus:

$$\begin{pmatrix} \& & \& & \& \\ \& & \% + xy & \& \\ \& & \& & \& \end{pmatrix} \approx \begin{pmatrix} \& & \& & \& & 0 \\ \& & \% + xy & \& & 0 \\ \& & \& & \& & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} \& & \& & \& & 0 \\ \& & \% + xy & \& & 0 \\ \& & \& & \& & 0 \\ 0 & y & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} \& & \& & \& & 0 \\ \& & \% & \& & -x \\ \& & \& & \& & 0 \\ 0 & y & 0 & 1 \end{pmatrix}$$

Hence, if M is an invertible matrix over $\Lambda * \Gamma$, it is equivalent in the group $\mathfrak{W}(\Lambda * \Gamma)$ to a matrix of the form $M_1 + M_a + M_b$, where the entries of M_1 belong to R, the entries of M_a to A, the entries of M_b to B; in what follows the subscripts 1, a, b will always have this meaning.

Recall that $A = \varepsilon_\Lambda^{-1}(0)$ and $B = \varepsilon_\Gamma^{-1}(0)$, where ε_Λ and ε_Γ are retractions of Λ and Γ onto R. Applying the map $\varepsilon_\Lambda * \varepsilon_\Gamma$ to $\Lambda * \Gamma$ maps our invertible matrix $M_1 + M_a + M_b$ into M_1 and maps $\Lambda * \Gamma$ onto R. Therefore M_1 is invertible over R. Multiplying $M_1 + M_a + M_b$ by M_1^{-1} changes the class of the matrix only by an element of $\mathfrak{W}(R)$, and therefore represents the same element of $K_1(\Lambda * \Gamma)$. Hence:

Every element of $K_1(\Lambda * \Gamma)$ *is represented by an invertible matrix of the form*

$$I + N_a + N_b .$$

5. Another trick

On applying the maps (identity)$*\varepsilon_\Gamma$ and $\varepsilon_\Lambda*$(identity) to the matrix $I + N_a + N_b$ we obtain $I + N_a$ and $I + N_b$, and we discern that these are matrices invertible over Λ and Γ, respectively. Their inverses are evidently of the form $I + P_a$ and $I + P_b$, respectively.

The following identity is the trick referred to:

$$I + N_a + N_b = (I + N_a)(I - P_a P_b)(I + N_b)$$

which it is suggested that the reader derive from the equations

$$(I + N_a)(I + P_a) = I = (I + P_b)(I + N_b) .$$

Now the matrix in the middle, $I - P_a P_b$, must therefore be invertible over $\Lambda * \Gamma$, with inverse $I + Q$. Where do the entries of Q lie? We note:

$$(I + Q)(I - P_a P_b) = I \quad \text{and} \quad (I - P_a P_b)(I + Q) = I .$$

This is the same as to say:

$$Q = (I + Q)P_a P_b \quad \text{and} \quad Q = P_a P_b (I + Q) .$$

The entries of $P_a P_b$ lie in $A \otimes_R B$. Hence, from the first equation, we conclude that the entries of Q lie in the left ideal generated by $A \otimes_R B$; and, from the second equation, that they lie in the right ideal.

Therefore, recalling observation 3.2.1, we know that $I - P_a P_b$ is an invertible matrix over the subring $T(A \otimes_R B)$ of $\Lambda * \Gamma$.

Hence:

$K_1(\Lambda * \Gamma)$ *is generated by the images, under the obvious maps, of* $K_1(\Lambda)$, $K_1(\Gamma)$, *and* $K_1(T(A \otimes_R B))$.

6. Fundamental theorems

LEMMA. *Let H be a covariant functor from R-rings to abelian groups, such that $H(R) = 0$. Then the maps $\Lambda \longrightarrow \Lambda * \Gamma \longleftarrow \Gamma$ induce an embedding of $H(\Lambda) + H(\Gamma)$ in $H(\Lambda * \Gamma)$.*

The proof is easy.

6.1. THEOREM. *Let A and B be the kernels of retractions of the R-rings Λ and Γ onto R. If $K_1(T(A \otimes_R B)) = 0$, then $K_1(\Lambda * \Gamma) = K_1(\Lambda) + K_1(\Gamma)$.*

PROOF. By 5 and the hypothesis that $K_1(T(A \otimes_R B)) = 0$, we know that $K_1(\Lambda * \Gamma)$ is generated by the image of $K_1(\Lambda) + K_1(\Gamma)$. By the Lemma, the map $K_1(\Lambda) + K_1(\Gamma) \to K_1(\Lambda * \Gamma)$ is an embedding. Hence the natural equality.

We now recall a result of Gersten [4] that, if $K_1(R(x)) = 0$, then K_1 of a "free" ring is zero; and this is in particular true if R is a principal ideal domain.

The terminology means this. $R(x)$ is the tensor ring of an R-bimodule isomorphic to R itself. A "free" ring is the tensor ring of an R-bimodule isomorphic to the direct sum of several copies of R; such an R-bimodule we shall call "free".

A very brief sketch of the proof of Gersten's result follows.

The "free" ring Λ can be thought of as a polynomial ring over R in non-commuting variables $\{x_1, \cdots, x_n\}$, which, however, commute freely with R.

An invertible matrix over Λ can, by Higman's trick, be made equivalent in $K_1(\Lambda)$ to a matrix

$$I + N_1 x_1 + \cdots + N_n x_n .$$

The inverse of this matrix can be written explicitly in the ring of formal power series; since the inverse exists in the polynomial ring, all but a finite number of its coefficients are zero. Hence we find that there is a number k such that any product of more than k of the matrices $\{N_i\}$ is zero.

It then follows that the matrix is the product of matrices of the form $I + P_i w_i$, where P_i are nilpotent R-matrices and w_i are words in the x's. To see this, multiply $I + N_1 x_1 + \cdots + N_n x_n$ by $(I + N_1 x_1)^{-1} \cdots (I + N_n x_n)^{-1}$ and verify that the homogeneous components of this matrix are twice as nilpotent. Then perform the same sort of trick on this matrix, and so on.

Such matrices $I + Pw$ are the images of invertible matrices $I + Px$ under a map of $R(x)$ into the "free" ring. Hence, K_1 *("free" ring) is generated by the images, under all R-ring maps $R(x) \to$ "free" ring, of $K_1(R(x))$.*

Finally, if R is a principal ideal domain, and we consider a matrix $I + Px$, where P is nilpotent, this is similar (and hence equivalent in $\mathfrak{W}(R(x))$) to a matrix $I + Qx$ where Q is in upper triangular form; such matrices are obviously the product of elementary matrices.

From 6.1 and Gersten's result, we clearly have:

6.2. THEOREM. *If $A \otimes_R B$ is "free" as an R-bimodule, and if $K_1(R(x)) = 0$, then $K_1(\Lambda * \Gamma) = K_1(\Lambda) + K_1(\Gamma)$.*

Now, one easily sees that the functors K_1 and T commute with direct limit. We might call a direct limit of "free" R-bimodules semi-"free". So that we can replace the condition on $A \otimes_R B$ by requiring that it merely be semi-"free".

In case R is a commutative ring, we can use the theorem of Lazard [8] to deduce that semi-"free" R-bimodules are the same as *flat* R-modules. Furthermore, Bass, Heller, and Swan [1] have shown that rings R which are *regular* have the property that $K_1(R(x)) = 0$; this extends our rather easy remark that this is so when R is a principal ideal domain. We can therefore conclude, for example:

6.2.1. COROLLARY: *If R is a regular, commutative ring, and if A and B are kernels of retractions of the R-rings Λ and Γ onto R, and if $A \otimes_R B$ is a flat R-module; then, $K_1(\Lambda * \Gamma) = K_1(\Lambda) + K_1(\Gamma)$.*

A special case of this, in proof of which we do not need to use these more general results of Lazard, Bass, Heller, and Swan, is:

6.3. COROLLARY. *If Λ and Γ are Z-rings which are Z-torsion-free, then $K_1(\Lambda * \Gamma) = K_1(\Lambda) + K_1(\Gamma)$.*

Now, let Π denote a group, and consider the category of Z-rings. There is a homomorphism $\Pi \to K_1(Z(\Pi))$ determined by associating to $\pi \in \Pi$ the 1×1 matrix (π). The quotient of $K_1(Z(\Pi))$ by the image of Π under this homomorphism has been called T by Whitehead. To emphasize its functorial character, we call it $\mathfrak{WT}(\Pi)$, the *Whitehead torsion* of Π. $\mathfrak{WT}$ is clearly a covariant functor from groups to commutative groups, such that $\mathfrak{WT}(\text{trivial group}) = 0$.

It therefore follows a remark like the Lemma at the beginning of this section, that $\mathfrak{WT}(\Pi_1) + \mathfrak{WT}(\Pi_2)$ is embedded in $\mathfrak{WT}(\Pi_1 * \Pi_2)$. To show the two are equal, it is therefore enough to show $\mathfrak{WT}(\Pi_1 * \Pi_2)$ is generated by $\mathfrak{WT}(\Pi_1)$ and $\mathfrak{WT}(\Pi_2)$. Since $\mathfrak{WT}$ is a quotient group of K_1, it is enough to show that $K_1(Z(\Pi_1 * \Pi_2))$ is generated by $K_1(Z(\Pi_1))$ and $K_1(Z(\Pi_2))$.

But, by 2, $Z(\Pi_1 * \Pi_2) = Z(\Pi_1) * Z(\Pi_2)$. And by 6.3,

$$K_1(Z(\Pi_1) * Z(\Pi_2)) = K_1(Z(\Pi_1)) + K_1(Z(\Pi_2)) .$$

Hence:

6.4. THEOREM. *The Whitehead torsion of a free product of groups is equal to the direct sum of the Whitehead torsions of the factors.*

7. Conclusion

From 6.4 and Higman's result that the Whitehead torsion of an infinite cyclic group is zero, it evidently follows that *the Whitehead torsion of a free group F is zero* and that $\mathfrak{WT}(\Pi * F)$ *is naturally isomorphic to* $\mathfrak{WT}(\Pi)$.

This suggests that the Whitehead torsion might be conceived of as a sort of "obstruction" thus:

Let $\varphi : F \longrightarrow \Pi$ be a homomorphism of a free group onto Π; let M_F and M_Π be free modules over $Z(F)$ and $Z(\Pi)$, of the same rank; pick some epimorphism $M_F \longrightarrow M_\Pi$. Given an automorphism $\alpha : M_\Pi \longrightarrow M_\Pi$, there is always an endomorphism $\tilde{\alpha} : M_F \longrightarrow M_F$ covering it. When can α be covered by an *auto*-morphism? The answer is, stably, if and only if the torsion of α, $\tau(\alpha) \in \mathcal{WT}(\Pi)$ is zero. Thus $\tau(\alpha)$ appears to be very much like a K-theoretic characteristic class in the realm of vector bundles.

It seems that there is much more of significance in this analogy, cf. Gersten's thesis [5].

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