CHAPTER 20

Topological Rigidity Theorems

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Contents

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1. Introduction

Two topological manifolds which are properly homotopy equivalent may easily fail to be homeomorphic. Since the 1930s the lens spaces $L(7,1)$ and $L(7,2)$ have been known to be homotopy equivalent but not simple homotopy equivalent [29], for example, and the topological invariance of Whitehead torsion [24] shows that manifolds which are not simple homotopy equivalent are not homeomorphic.

If we can prove for a manifold M^n that any manifold which is properly homotopy equivalent to M is homeomorphic to M then we say that M^n is topologically rigid, by analogy to results in geometry which assert that any deformation of a geometric structure is equivalent to the original. Obstructions to modifying a homotopy equivalence to a homeomorphism are found in the Whitehead group and in surgery theory, and rigidity arguments tend to mingle geometry and algebra to show that these obstructions vanish. Results in the subject usually specialize to a geometrically defined class of manifolds.

Most topological rigidity results concern manifolds which are aspherical (connected, with contractible universal covering spaces). The homotopy theory of these spaces is particularly simple: a continuous map between aspherical CW complexes is a homotopy equivalence if and only if it induces an isomorphism of fundamental groups. The primal example of an aspherical manifold is Euclidean space; the most familiar closed aspherical manifolds are the tori $T^n = (S^1)^n$.

CONJECTURE 1.1 (*Borel Conjecture*). Every homotopy equivalence $f : M \to N$ of closed aspherical manifolds is homotopic to a homeomorphism.

Conjecture 1.1 is reported by F.T. Farrell and W.-C. Hsiang [108] to have entered the oral culture by the 1960s as A. Borel's response to [134]. In that study of solvmanifolds (coset spaces $\Gamma\backslash S$, where S is a solvable Lie group) Mostow showed that if two closed solvmanifolds have isomorphic fundamental groups then they are diffeomorphic. By the early to middle 1960s it was known that the connected sum of an exotic sphere and a torus may carry a nonstandard differentiable structure [17, Corollary 2.8], so smooth rigidity could not be expected. The classification of PL homotopy tori in the late 1960s (see Section 2.3) rules out piecewise linear rigidity claims.

The truth of Conjecture 1.1 for surfaces is classical (see Section 2.1) and by the late 1960s it was known that if two irreducible, sufficiently large 3-manifolds are homotopy equivalent then they are homeomorphic [169]. Surgery theory is the technology relevant to the rigidity problems in high dimensions, with satellite algebraic problems on K- and L-groups, and the following generalization of Conjecture 1.1 arises naturally when the rigidity conjecture is translated into a claim about the long exact sequence of topological surgery.

CONJECTURE 1.2 (*Generalized Borel Conjecture*). If $f : (M, \partial M) \to (N, \partial N)$ is a homotopy equivalence of compact aspherical manifolds which restricts to a homeomorphism of boundaries then f is homotopic rel boundary to a homeomorphism.

This generalization is more and more frequently referred to as "the Borel Conjecture", because of parallels with the Novikov Conjecture and because surgical methods usually yield Conjecture 1.2 if they prove Conjecture 1.1. Some work has also been done on versions of these problems for open manifolds: if noncompact aspherical manifolds are properly homotopy equivalent, are they homeomorphic?

Another problem concerning the topology of aspherical manifolds was discussed in 1969 by Wall [176] as a mate to the spherical spaceform problem [166,127]:

PROBLEM 1.1 (*The topological Euclidean space form problem*). Classify all free proper actions of discrete groups on Euclidean space.

This is a descendant of Hilbert's Eighteenth Problem on crystallographic groups [133], which was solved by Bieberbach (see Section 4.1). Problem 1.1 is usually understood to concern both existence and uniqueness. A rigidity or uniqueness claim in this context would conclude that two actions of a group Γ on $\mathbb{R}^n$ are topologically conjugate and would be a case of the Borel Conjecture or its proper counterpart.

Existential results should assume algebraic properties of the group Γ and deduce that there is a free proper action of Γ on some $\mathbb{R}^n$; they might also give information about appropriate values of n and the action (we especially want to know if $\Gamma\backslash\mathbb{R}^n$ may be a closed manifold). See Sections 3.1 and 9 for some results and a statement of the existence problem.

After M. Davis produced examples of closed manifolds whose universal covering spaces are contractible but non-Euclidean [38] the scope of investigations into aspherical manifolds enlarged, and developments in geometric group theory and on the Novikov conjecture have further altered our view of the rigidity problem.

2. Examples and classical facts

Positive results in a few cases of low dimension or tractable fundamental group shaped later approaches to rigidity problems. This section sketches the cases of 2-manifolds and n-dimensional tori, as well as the most important simply connected manifolds. Work on 3-manifolds has also influenced high-dimensional arguments, especially splitting theorems, and those results are reported in later sections.

2.1. *Example 1: Surfaces*

The classification of closed surfaces is often sketched in elementary topology courses and yields the strongest rigidity conclusions known in any dimension, since asphericity is not assumed: If closed 2-manifolds M and N are homotopy equivalent then they are homeomorphic. Moreover, any homotopy equivalence of closed surfaces deforms to a homeomorphism.

We may argue for the statements above with an outline going back to Riemann: dissect a target surface along nicely imbedded circles, use transversality to dissect the source manifold along preimages of these circles, and then modify the map to simplify it towards a homeomorphism. This line of attack, sometimes called splitting, is a mainstay of

3-manifold topology [169] and has also been used and studied in higher dimensions, where it may be obstructed (see Section 6).

Riemann surfaces with elliptic points are orbifolds rather than manifolds, but have rigidity properties similar to those of 2-manifolds when we consider maps which preserve singular data (stratified or equivariant maps). Once again, complex analysis and geometry provide better information on stratified rigidity questions for 2-orbifolds than we expect to find in other dimensions (Section 10.2).

2.2. *Classical results*

Several classical facts guide and constrain rigidity theorems in higher dimensions. Rigid manifolds with nontrivial fundamental group might be expected to have rigid universal covering spaces (although we are learning that less control over the rigidifying homeomorphisms might be possible than one would like [5]), so we consider here the simplest compact and noncompact manifolds.

Milnor's discovery of exotic smooth structures on spheres [132] led to the classification of manifold structures of different kinds. Topological versions of tools familiar from Smale's generalized Poincaré conjecture show that there are no exotic topological structures on spheres, except perhaps in dimension three (see [154, Section 4.13] for $n \geqslant 5$ and [92, Chapter 7] for $n = 4$):

THEOREM 2.1 (Topological rigidity of spheres). *Let $n \neq 3$. If a closed manifold M^n is homotopy equivalent to S^n then M^n is homeomorphic to S^n.*

Note that degree theory shows that every homotopy equivalence on an n-sphere is homotopic to a homeomorphism, so spheres are topologically rigid except perhaps in dimension 3.

However, there are homotopy types which contain many non-homeomorphic closed and simply connected manifolds, and these are classified by surgery theory (see Section 5.2 and the chapter of this volume on surgery). In addition, closed and simply connected manifolds may admit self-homotopy equivalences which do not deform to homeomorphisms (a sample computation with modern tools, following an idea which goes back to Novikov, appears in [149, Example 20.4]).

Stallings's characterization of Euclidean space [163] leads to a counterpart to the topological rigidity of spheres.

THEOREM 2.2 (Characterization of Euclidean space). *Let $n \geqslant 5$. If M^n is a noncompact contractible manifold which is simply connected at infinity then M^n is homeomorphic to $\mathbb{R}^n$.*

The characterization theorem, the invariance of simple connectedness at infinity under proper homotopy equivalence, and proper degree theory yield a rigidity statement.

COROLLARY 2.1 (Rigidity of Euclidean space). *If $f : M^n \to \mathbb{R}^n$ is a proper homotopy equivalence of open manifolds and $n \geqslant 5$ then f is properly homotopic to a homeomorphism.*

2.3. *Example 2: The torus*

One of the simplest noncontractible aspherical manifolds, the n-dimensional torus, illustrates the range of issues which appear under the heading of rigidity. We begin with the properties of the torus T^n as a flat Riemannian manifold.

Let g_0 denote the Riemannian metric on a product of n circles of radius 1. If g is any other flat Riemannian metric on T^n then there is an affine diffeomorphism of T^n which transforms g_0 to g, so up to affine equivalence, there is only one flat n-torus (see Section 4). By considering the lengths of shortest closed geodesics one sees easily that there are uncountably many isometrically distinct flat Riemannian metrics on T^n. The isometry classification of two-dimensional flat tori presented in elementary complex analysis is a model for the first steps in the study of deformations and rigidity for discrete subgroups of Lie groups (also Section 4).

The classification of PL structures on T^n for $n \geqslant 5$ was an early success of surgery on manifolds with nontrivial fundamental group [109,174,175]. This result of Wall and Hsiang–Shaneson played an essential role in the last stages of the Kirby–Siebenmann theorem on triangulating topological manifolds [121]. Although the set of PL structures on T^n contains more than one element, passing to a 2^n-sheeted covering space renders all these PL structures equivalent. Smooth structures on the torus have a similar lifting property, and it seems to be an open question whether distinct PL (respectively smooth) structures on most aspherical manifolds should agree after lifting to some finite-sheeted covering space.

Any topological manifold homotopy equivalent to T^n must be homeomorphic to the standard torus. The extension of surgery theory from the smooth and PL categories to the category of topological manifolds (see Section 5) leads to an easy proof of this assertion using splitting theorems of Bass–Heller–Swan and Shaneson (Section 6). In fact, more is known [122, pp. 264–283]:

THEOREM 2.2 (Rigidity of tori). *Let $f : M^{n+k} \to T^n \times D^k$ be a homotopy equivalence which restricts to a homeomorphism of the boundaries. If $n + k \geqslant 5$ then f is homotopic to a homeomorphism.*

A quotient space of T^n by a finite group acting smoothly and freely admits a flat Riemannian metric, and all flat Riemannian manifolds admit such a description. The analysis of flat manifolds by Farrell and Hsiang (Section 7.3) mixes Frobenius reciprocity (Section 5.4) and a circle of ideas known as controlled topology. The covering projection $T^n \to T^n$ which has degree $r > 0$ in each circle factor has counterparts for every closed Riemannian flat manifold. These expanding self maps suggest vanishing proofs which represent algebraic data by geometric constructions with associated size information, assuming that the algebraic element under consideration is trivial if a size estimate is sufficiently small: this is the idea at the core of controlled topology (Section 7).

If a finite group Q acts smoothly and properly discontinuously but not freely on T^n then the quotient space is a flat orbifold. Structure and rigidity results in this situation may consider T^n/Q as a stratified space or may emphasize the equivariant topology of the action (Q, T^n). We have quite a bit of information about the flat case (see especially [35,36]) but still have a lot to do in general (Section 10).

3. Constructions and basic properties of aspherical manifolds

3.1. *Basic properties*

The fundamental groups of aspherical manifolds have a number of algebraic properties. If M^n is any aspherical manifold then its fundamental group has cohomological dimension n or less. This is the only constraint in the most general case:

THEOREM 3.1 (Johnson [176]). *Let Γ be a countable discrete group. Γ is of finite cohomological dimension if and only if there is a proper smooth free action of Γ on $\mathbb{R}^n$ for all sufficiently large n.*

If M^n is compact then $\Gamma = \pi_1(M)$ satisfies additional constraints. Because a compact manifold has the homotopy type of a finite CW complex [122], the fundamental group Γ of such a manifold is of type FL [18,156].

DEFINITION 3.1. Let Γ be a countable group.
(a) Γ is of *type FL* if and only if the trivial $\mathbb{Z}\Gamma$ module $\mathbb{Z}$ has a finite-length resolution by finitely generated free $\mathbb{Z}\Gamma$-modules.
(b) Γ is of *type FP* if and only if the trivial $\mathbb{Z}\Gamma$ module $\mathbb{Z}$ has a finite-length resolution by finitely generated projective $\mathbb{Z}\Gamma$-modules.

If Γ is the fundamental group of a closed aspherical manifold then, in addition to being of type FL, Γ is a Poincaré duality group: we will return to this condition in Section 9. The fundamental groups of some noncompact aspherical manifolds lie in a larger class, known as duality groups; nonuniform arithmetic subgroups of semisimple linear Lie groups are the core examples. K. Brown's textbook [18] is a good guide to the finiteness properties cited above, along with duality groups and Poincaré duality groups.

3.2. *Coset spaces and curvature conditions*

Differential geometry is the most classical source of aspherical manifolds, mainly through the following result:

THEOREM 3.2 (The Cartan–Hadamard theorem). *If V^n is a simply connected smooth manifold with a complete Riemannian metric of nonpositive sectional curvature then for every $v \in V$ the exponential map* $\exp\colon T_v V \to V$ *is a diffeomorphism.*

COROLLARY 3.1. *Every complete Riemannian manifold of nonpositive sectional curvature is aspherical.*

The almost flat manifolds studied by Gromov [97] are also aspherical, as are many almost nonpositively curved manifolds [94]. Gromov's characterization of almost flat manifolds leads into Lie groups and coset spaces:

DEFINITION 3.2. A smooth manifold M^n is *almost flat* if it admits a sequence of complete Riemannian metrics g_i and a number $D > 0$ such that for all i we have an upper bound $\operatorname{diam}(M, g_i) \leqslant D$, while the sectional curvatures of the g_i converge to 0 uniformly.

DEFINITION 3.3. An *infranilmanifold* is a double coset space $M = \Gamma\backslash G/K$, where G is a Lie group which is virtually connected and virtually nilpotent, K is a maximal compact subgroup of G, and Γ is a discrete subgroup of G.

THEOREM 3.3 (Gromov [97]). *A smooth manifold is almost flat if and only if it is an infranilmanifold.*

More generally, if G is any virtually connected Lie group and $K < G$ is a maximal compact subgroup then G/K is diffeomorphic to a Euclidean space, while any discrete subgroup $\Gamma < G$ acts properly discontinuously by left translation on G/K. If Γ is torsion-free then this is a free action and $\Gamma\backslash G/K$ is an aspherical manifold with fundamental group Γ, while if Γ contains elements of finite order then these elements fix points of G/K and $\Gamma\backslash G/K$ is an orbifold.

The subgroup Γ of the Lie group G is said to be *uniform* or *cocompact* if $\Gamma\backslash G$ is compact in the quotient topology, while Γ is said to be a *lattice* in G if $\Gamma\backslash G$ has finite volume with respect to any left-invariant volume form on G. (Note that both these properties depend upon the imbedding of Γ in G.) Two subgroups A and B of a group G are said to be *commensurable* if $A \cap B$ is of finite index in both A and B.

DEFINITION 3.4. Let G be an algebraic subgroup of GL_n, defined over the field $\mathbb{Q}$ of rational numbers. Thus, G is defined by a set $\mathcal{S}$ of polynomial equations in the entries of $n \times n$ matrices and in the inverse of the determinant, and these equations have rational coefficients.

(a) If $\mathbb{F}$ is any extension field of $\mathbb{Q}$ then $G_{\mathbb{F}}$ denotes the set of solutions of $\mathcal{S}$ with entries in $\mathbb{F}$. ($G_{\mathbb{F}}$ is known as the set of $\mathbb{F}$-points of G; it is a subgroup of $\mathrm{GL}_n(\mathbb{F})$.)
(b) $G_{\mathbb{Z}}$ denotes $G_{\mathbb{Q}} \cap \mathrm{GL}_n(\mathbb{Z})$.
(c) A subgroup Γ of $G_{\mathbb{Q}}$ is *arithmetic* if Γ and $G_{\mathbb{Z}}$ are commensurable.
(d) A group Δ is arithmetic if it can be imbedded as an arithmetic subgroup of $G_{\mathbb{Q}}$ for some $\mathbb{Q}$-arithmetic subgroup G of GL_n.

For example, the group $\mathrm{SL}_n(\mathbb{Z})$ of matrices with integral entries and determinant 1 is an arithmetic subgroup of $\mathrm{SL}_n(\mathbb{Q})$. Serre's survey [157] is highly recommended to topologists looking into arithmetic groups, along with the relevant sections of Brown's book [18]. Note that $\mathrm{SL}_n(\mathbb{Z})$ contains elements of finite order, such as permutation matrices; this is common in arithmetic groups, but they are nonetheless a source of aspherical manifolds thanks to the following result (see [155], [15, pp. 113–115], or [147, pp. 93–95]):

THEOREM 3.4 (Selberg's lemma). *If Δ is a finitely generated subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *then Δ contains a torsion-free subgroup of finite index.*

The discrete subgroups constructed in the proof of the next theorem [15] are arithmetic:

THEOREM 3.5 (Borel). *Every connected semisimple Lie group G contains a discrete uniform subgroup.*

The work of Margulis shows that in many semisimple Lie groups every lattice is arithmetic with respect to some $\mathbb{Q}$-structure [129,130]. Much earlier, Mal'cev had done as much for lattices in nilpotent Lie groups [128].

3.3. *Torus actions and singular fiberings*

If K is a compact, connected Lie group acting effectively on an aspherical manifold then K must be a torus (see Theorem 10.1). Constructions of torus actions are also an effective means of producing aspherical manifolds, often as twisted products

$$M^{n+k} = V^k \times_Q T^n,$$

where V is an aspherical manifold and Q is a discrete group.

These actions are closely related to the Seifert fibered spaces of 3-manifold topology and have been thoroughly studied by Conner, Raymond, Lee, and others [31,32,123,124]. Singular fibering constructions have also been studied with fibers other than tori [117,151].

The features of singular fiberings which have been relevant for rigidity investigations are associated to combinatorial descriptions of these spaces as a stratified system of bundles over a base orbifold of lower dimension, and to the topology of the base orbifold. Constructability questions often pass through an obstruction in group cohomology, which Conner, Lee, and Raymond have analyzed to good effect. See the chapter in this volume by Lee and Raymond for more detailed information.

3.4. *Groups generated by reflections*

Section 1 cited the work of M. Davis on aspherical manifolds constructed from groups generated by reflections. We sketch this important construction here and recommend the chapter by Davis in this volume as well as the papers [38,39].

The basic version of these constructions begins with a manifold with triangulated boundary, $(X, \partial X)$ and associates to the top-dimensional simplices of ∂X generators for a Coxeter system; copies of X are glued together along these simplices to produce a space U on which Γ acts with fundamental domain X. Coxeter groups had long been considered in spaces of constant curvature [16,114]; see especially the work of Vinberg [168].

A *Coxeter system* is a pair (Γ, V), where Γ is a group and V is a set of elements $v_i \in \Gamma$ such that Γ has a presentation in terms of the elements v_i and $m(i, j) \in \mathbb{Z}\ (>0) \cup \{\infty\}$

$$\Gamma = \langle v_i \in \Gamma:\ \text{for each } i,\ v_i^2 = e \text{ and for } i \neq j\ (v_i v_j)^{m(i,j)} = e \text{ if } m(i, j) < \infty \rangle.$$

If a group Γ admits such a description then we say that it is a Coxeter group. The space U is constructed by this method: if X is a space with a family of subspaces X_i indexed by a set V and if (Γ, V) is a Coxeter system with generators V then we define a space $U = (\Gamma \times X)/\sim$, where the equivalence relation is generated by $(g, x) \sim (h, y)$ if and only if $x = y$, $x \in X_i$ and $g^{-1}h = \gamma_i$. Γ acts on U by left translation in the Γ factor, and $\Gamma \backslash U \cong X$.

Davis applies this construction to a contractible manifold X with boundary; the resulting space $U = U(X, \Gamma)$ is then a contractible manifold if the triangulation of ∂X is sufficiently fine, Γ is an infinite Coxeter group, and each $m(i, j)$ is finite (many of the well-studied examples take $m(i, j) = 2$). If ∂X is a nonsimply connected homology sphere then U is often not simply connected at infinity, so that U is not homeomorphic to Euclidean space, by Theorem 2.2. Because every Coxeter group Γ has a torsion-free subgroup Γ_0 of finite index, Davis produces a closed manifold $M = \Gamma_0 \backslash U$ with contractible, non-Euclidean universal covering space U. The reader should be aware that this sketch has omitted some important details in the proof of the following result [38]:

THEOREM 3.6 (Davis's aspherical manifolds). *In every dimension greater than or equal to four there exist closed aspherical manifolds not covered by Euclidean space.*

The construction of U sketched above reappears in the argument for the following useful observation [39, pp. 213–215].

THEOREM 3.7 (Davis's doubling trick). *If X is an aspherical manifold with triangulated boundary then X is a retract of a closed aspherical manifold.*

Note that by taking regular neighborhoods in Euclidean space we see that there exists a compact aspherical manifold with fundamental group Γ if and only if Γ is the fundamental group of a finite, aspherical simplicial complex. This consequence of Theorem 3.7 appears in [39, pp. 215–217]:

THEOREM 3.8. *If the Novikov Conjecture (respectively the Integral Novikov Conjecture) holds for the fundamental group of every closed aspherical manifold then it holds for every group Γ admitting a finite $K(\Gamma, 1)$.*

Bizhong Hu has used similar arguments for fundamental groups of polyhedra of nonpositive curvature in the sense of Alexandrov [112]. Moussong's work [137] on nonpositively curved polyhedral metrics for Coxeter complexes leads to contractible topological manifolds V^n with complete polyhedral metrics of nonpositive curvature such that V^n is not homeomorphic to $\mathbb{R}^n$. Note also that these examples show that the polyhedral counterpart of the Cartan–Hadamard Theorem (3.2 above) is false.

See Section 10.2 for rigidity results in this context.

3.5. *Polyhedra of nonpositive curvature*

Throughout this section we consider curvature in the sense of Alexandrov and Toponogov, which applies to polyhedra rather than manifolds, as in Gromov's influential paper on hyperbolic groups [98]. See the chapter in this volume by Davis for details.

Gromov introduced a construction called *hyperbolization*, which modifies a polyhedron to produce a polyhedron which is nonpositively curved. This idea has been worked out with care and some improvements by Davis and Januszkiewicz [41] and Charney and Davis [28]: we warn the reader that Gromov's original treatment of relative hyperbolization is not adequate for applications and Charney–Davis should be consulted for details.

Hyperbolization constructions proceed by a cell replacement construction: every k-cell in a polyhedron is replaced by a canonical space of the same dimension which has nontrivial fundamental group and the combinatorics of the polyhedron are reproduced in the attaching data for these canonical pieces. Some observations about this kind of construction are elementary: Lemma 3.1 on aspherical pasting shows that if every canonical piece is aspherical and the attaching maps are built with a little care and induce injections on fundamental groups then we should produce an aspherical polyhedron modeled on the combinatorics of the original complex. It is also not surprising that this construction can be arranged so that the n-dimensional canonical piece has a preferred map onto an n-cell, and so that the hyperbolized polyhedron has an essential map onto the original space. Considerable attention is required to manage curvature during the rebuilding process, especially in relative hyperbolization, which leaves a subcomplex intact during the enlargement process.

The first contribution of hyperbolization is statistical: a large number of groups are fundamental groups of finite polyhedra with nonpositive or negative curvature, including many triangulated manifolds or Poincaré complexes. Another application, which depends on relative hyperbolization, was announced by Gromov and improved by Charney and Davis [28]:

THEOREM 3.9. *Every triangulable manifold is cobordant to a triangulable manifold of strictly negative curvature.*

3.6. *Aspherical complements*

A number of open manifolds obtained by deletion operations are known to be aspherical. We include pointers to some of these spaces, since any proper version of the Borel Conjecture must encompass them.

Let K be a polyhedral circle in S^3. The asphericity of the knot complement $S^3 \setminus K$ was the testbed and motivation for some of the fundamental results of 3-manifold topology [106,152].

Following work of Arnol'd and Deligne, many hyperplane arrangements in Euclidean space have been shown to have aspherical complements [140, Chapter 5].

J.H.C. Whitehead exhibited a contractible open manifold which is not Euclidean space by taking the complement of a certain compactum in the 3-sphere [152,184].

3.7. *Moduli spaces of surfaces*

A class of examples arises in Riemann surface theory which is closely related to manifolds of nonpositive curvature, but which seems not to overlap with them.

Let Σ_g^2 be a closed, orientable surface of genus g and let $\mathcal{T}_g$ denote the Teichmüller space of marked hyperbolic structures on Σ_g^2; an important result in surface theory asserts that $\mathcal{T}_g \cong \mathbb{R}^{6g-g}$. The group of outer automorphisms of the fundamental group of our surface is the quotient of the group of all automorphisms by the normal subgroup of inner automorphisms, $\mathrm{Out}(\pi_1(\Sigma_g^2)) = \mathrm{Aut}(\pi_1(\Sigma_g^2))/\mathrm{Inn}(\pi_1(\Sigma_g^2))$. There is an action of $\mathrm{Out}(\pi_1(\Sigma_g^2))$ on $\mathcal{T}_g$ which changes markings; this action is properly discontinuous, but not free, since the group of outer automorphisms contains finite subgroups.

$\mathrm{Out}(\pi_1(\Sigma_g^2))$ has many of the properties of arithmetic groups. In particular, this group has a torsion-free subgroup Γ of finite index, and $\mathcal{T}_g/\Gamma$ is an open aspherical manifold which is known to have good compactifications [101–103].

The mapping class group $\mathrm{Out}(\pi_1(\Sigma_g^2))$ is studied in algebraic geometry and geometric group theory, as well as in the topology of aspherical manifolds.

More generally, if M^n is any aspherical manifold then $\mathrm{Out}(\pi_1(M^n))$ acts on the structure set $\mathcal{S}^{\mathrm{TOP}}(M^n)$ (see Definition 5.1) and plays an important role in the study of the group of homeomorphisms of M^n [32].

3.8. *Branched coverings and pasting constructions*

Constructions which alter or combine aspherical manifolds to produce new ones include the formation of fiber bundles with aspherical base and fiber, certain group actions (see Section 3.3), and the two constructions discussed below.

Branched coverings are familiar in the geometry of surfaces and 3-manifolds. They were used by Gromov and Thurston [100] in higher dimensions to build manifolds with metrics of nonpositive curvature which are close to hyperbolic manifolds although they admit no Riemannian metric of constant negative sectional curvature.

Pasting constructions can be done within the class of aspherical complexes, using the following lemma which goes back to [184].

LEMMA 3.1 (Aspherical pasting lemma). *Suppose that X is a finite complex such that each component of X is aspherical. Suppose that A_0 and A_1 are aspherical subcomplexes of X such that each inclusion $h_i : A_i \hookrightarrow X$ induces an injection on the fundamental group. If there is a homotopy equivalence $\phi : A_0 \to A_1$ then every component of the adjunction space $Y = X \cup (A_0 \times [0,1])$ formed by attaching $A_0 \times \{0\}$ to X by h_0 and attaching $A_0 \times \{1\}$ to X by $h_1 \circ \phi$ is aspherical.*

This result suggests the splitting strategy explored in Section 6 and gives the fundamental groupoid of Y the structure of an HNN extension or free product with amalgamations (i.e., the fundamental groupoid of a graph of groups). For this reason, work with pasting and splitting of aspherical manifolds tends to have close connections with combinatorial group theory.

Gromov and Piatetski-Shapiro used a pasting construction to hybridize two arithmetic groups and produce new non-arithmetic hyperbolic manifolds [99]. (See also the discussion in [130, Appendix C].)

4. Rigidity theorems in geometry

Two celebrated theorems provide models for a number of rigidity results in geometry and equivariant topology. Bieberbach's solution to Hilbert's Eighteenth Problem on crystallographic groups begins with metric data, namely a flat Riemannian metric, and asserts that a homotopy equivalence determines metric structure up to a weaker equivalence relation, namely affine isomorphism. (A flat metric can be rescaled by a constant, so affine equivalence is the tightest geometric equivalence relation which might coincide with homotopy equivalence.) Mostow's rigidity theorem for hyperbolic manifolds deforms a homotopy equivalence to an isometry, but the most natural setting for the theorem works with discrete subgroups of Lie groups rather than metric structures on manifolds.

4.1. *Bieberbach's theorem*

THEOREM 4.1 (Bieberbach). *If $f: M \to N$ is a homotopy equivalence between closed, connected, flat Riemannian manifolds then f is homotopic to an affine diffeomorphism.*

A diffeomorphism of flat Riemannian manifolds is affine if and only if it carries geodesics to geodesics; we amplify to give a global view of Bieberbach's theorem. The universal covering space $\widetilde{M}^n$ of a complete flat Riemannian manifold M^n is isometric to Euclidean space in the standard metric, and the action of the fundamental group of M^n on $\widetilde{M}^n$ by deck transformations is identified with the action of a discrete subgroup of the group $\mathrm{E}(n)$ of rigid Euclidean motions on $\mathbb{R}^n$. Theorem 4.1 is part of the structure theory of crystallographic groups [185, Chapter 3] developed by Bieberbach, which builds upon the fact that $\mathrm{E}(n)$ is a split extension

$$1 \longrightarrow \mathbb{R}^n \longrightarrow \mathrm{E}(n) \longrightarrow \mathrm{O}(n) \longrightarrow 1, \tag{4.1}$$

in which $\mathrm{O}(n)$ acts on $\mathbb{R}^n$ by the tautological representation. The corresponding group $\mathrm{Aff}(n)$ of affine motions enlarges the quotient in the split extension:

$$1 \longrightarrow \mathbb{R}^n \longrightarrow \mathrm{Aff}(n) \longrightarrow \mathrm{GL}(n) \longrightarrow 1,$$

and the inclusion $\mathrm{E}(n) \hookrightarrow \mathrm{Aff}(n)$ is a morphism of split extensions. If M_1^n and M_2^n are homotopy equivalent compact flat Riemannian manifolds, with fundamental groups Γ_1, $\Gamma_2 < \mathrm{E}(n)$ then Bieberbach shows that Γ_1 and Γ_2 are conjugate subgroups of $\mathrm{Aff}(n)$.

4.2. *Mostow rigidity*

The first version of Mostow's rigidity theorem appeared in [135]. Recall that a *hyperbolic manifold* is a complete Riemannian manifold whose sectional curvature is everywhere -1.

THEOREM 4.2 (Mostow rigidity theorem for hyperbolic manifolds). *If $f: M \to N$ is a homotopy equivalence between closed hyperbolic manifolds of dimension three or more then f is homotopic to an isometry.*

Mostow rigidity does not require constant curvature [136]. A *locally symmetric space* is a double coset space $\Gamma\backslash G/K$, where G is a semisimple Lie group, K is a maximal compact subgroup, and Γ is a discrete subgroup of G which acts on G/K as a group of covering transformations. A left-invariant Riemannian metric on G which is right K-invariant descends to a Riemannian metric on the double coset space.

THEOREM 4.3 (Mostow rigidity theorem for locally symmetric spaces). *Let M and N be compact locally symmetric spaces of non-positive sectional curvature. If the universal covering space $\widetilde{M}$ cannot be written as a metric product $M_1 \times M_2$, where one of the factors is of dimension one or two, then any homotopy equivalence $f: M \to N$ is homotopic to a diffeomorphism g, which may be assumed to be an isometry after the Riemannian metric on M is renormalized.*

Precursors to Mostow's rigidity theorem established rigidity for particular discrete subgroups in linear groups [155] or concluded that a discrete subgroup of a linear group is locally rigid: any continuous deformation carries Γ to a conjugate subgroup of G, as in the work of Weil [178–180]. The main construction of these papers of Weil leads to a description of the Zariski tangent space of the deformation space for a discrete subgroup of a linear group [126].

Mostow's rigidity theorem has been extremely influential, leading notably to the super-rigidity and arithmeticity results of Margulis [129,130,187] and the analysis of Riemannian manifolds of nonpositive curvature and high rank [7,8].

5. Algebraic and topological foundations

Topological rigidity theorems are customarily proved in two or three main stages, passing through K-theory to a study of the structure set in topological surgery. A vanishing theorem for the Whitehead group of $\pi_1(M^n)$ is usually the first sign of progress. Although conventional wisdom claims that an idea which computes K-theory or proves a K-theoretic version of the Novikov Conjecture should eventually succeed in surgery computations, the gap between K- and L-theory arguments can be hard to bridge.

5.1. *K-theory obstructions*

The first obstruction to deforming a homotopy equivalence $f: M^n \to N^n$ to a homeomorphism is the obstruction to simple homotopy equivalence, i.e., the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(M))$.

The classes of manifolds for which rigidity theorems have been proved to date have much stronger vanishing properties than $\mathrm{Wh}(\pi_1(M)) \cong 0$. These are associated to a sequence of "Whitehead groups" $\mathrm{Wh}_i(\mathbb{Z}\Gamma)$ such that $\mathrm{Wh}_0(\mathbb{Z}\Gamma) \cong \widetilde{K}_0(\mathbb{Z}\Gamma)$, $\mathrm{Wh}_1(\mathbb{Z}\Gamma) \cong \mathrm{Wh}(\Gamma)$, and $\mathrm{Wh}_2(\mathbb{Z}\Gamma)$ agrees with the group defined by Hatcher–Wagoner in their study of pseudoisotopy.

The Loday assembly map for K-theory [125,170],

$$A: H_*\big(BG; \mathbb{K}(\mathbb{Z})\big) \to K_*(\mathbb{Z}G),$$

is part of the exact sequence in homotopy groups for a fibration of spectra

$$\mathbb{H}\big(BG;\ \mathbb{K}(\mathbb{Z})\big) \to \mathbb{K}(\mathbb{Z}G) \to \mathbb{W}\mathrm{h}^{\mathbb{Z}}(G)$$

in which $\pi_i(\mathbb{W}\mathrm{h}^{\mathbb{Z}}(G))$ agrees with the algebraically defined Whitehead groups $\mathrm{Wh}_i(\mathbb{Z}G)$ for $i = 0, 1, 2$ and may be taken as the definition of the Whitehead groups more generally. The Whitehead space thus measures the difference between the algebraic K-theory of G and a homology theory; this difference comes down to the possible failure of the excision axiom (Mayer–Vietoris sequences) for K-theory and is involved in splitting problems (Section 6).

K-theoretic versions of the Novikov Conjecture concern the assembly map A described above [90, pp. 32–33]. One of the most broadly applicable results in the realm of the Novikov and Borel Conjectures was proved in [14] using Waldhausen's A-theory [171], cyclic homology, and the authors' cyclotomic trace:

THEOREM 5.1 (Bökstedt–Hsiang–Madsen). *If Γ is a group such that $H_i(B\Gamma)$ is finitely generated for all i then*

$$A: H_*\big(B\Gamma; \mathbb{K}(\mathbb{Z})\big) \otimes \mathbb{Q} \to K_*(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$$

is a split injection.

Conjectures discussed in Section 10 are posed for pseudoisotopy and A-theory as well as K-theory.

5.2. *Surgery theory*

Surgery on topological manifolds was a relatively late addition to a project which began in the smooth category and was shaped by smoothing and triangulation questions as well as classification problems. Although the topological theory is more complicated to establish than the smooth and PL versions, it has some formal properties which are particularly useful in rigidity arguments.

DEFINITION 5.1. The *structure set* of a compact manifold with boundary, M, is denoted $\mathcal{S}^{\mathrm{TOP}}(M, \partial M)$ or $\mathcal{S}(M, \partial M)$ and is defined to be the set of equivalence classes of pairs (N, f) where N is a compact manifold with boundary and $f:(N, \partial N) \to (M, \partial M)$ is a homotopy equivalence which restricts to a homeomorphism of boundaries; two pairs (N_1, f_1), (N_2, f_2) are equivalent if and only if there is a homeomorphism $h: N_1 \to N_2$ such that $f_2 \circ h$ is homotopic to f_1.

The Generalized Borel Conjecture (Conjecture 1.2) is conveniently phrased this way: If M is a compact aspherical manifold then $|\mathcal{S}^{\mathrm{TOP}}(M, \partial M)| = 1$. Structure sets for the categories of differentiable and piecewise linear manifolds are also defined and indicated by different decorations: $\mathcal{S}^{\mathrm{DIFF}}$, $\mathcal{S}^{\mathrm{PL}}$. In these terms, Milnor's celebrated theorem [132] reads as $|\mathcal{S}^{\mathrm{DIFF}}(S^7)| \neq 1$.

The Loday assembly map discussed above has a counterpart in surgery, introduced by Quinn. To a space X there is associated a spectrum $\mathbb{L}_\bullet(X)$ such that the homotopy groups of this spectrum are Wall's L-groups, $\pi_i(\mathbb{L}_\bullet(X)) \cong L_i(\mathbb{Z}[\pi_1(X)])$, and with a homotopy equivalence

$$\mathbb{L}_\bullet(\text{pt.})_0 \simeq L_0(\mathbb{Z}) \times G/TOP,$$

where G/TOP is the classifying space for topological reductions of stable spherical bundles. The $L_0(\mathbb{Z})$ factor in the 0-component of the surgery spectrum has become a matter of great interest [19], but to produce the long exact sequence of surgery we work with the 1-connective cover of $\mathbb{L}_\bullet(\text{pt.})$, denoted by $\mathbb{L}_\bullet$, which has $(\mathbb{L}_\bullet)_0 \simeq G/TOP$. The assembly map in surgery, $A: \mathbb{H}(X;\ \mathbb{L}_\bullet) \to \mathbb{L}_\bullet(X)$, is a map of spectra which appears in a cofibration sequence

$$\mathbb{H}(X;\ \mathbb{L}_\bullet) \to \mathbb{L}_\bullet(X) \to \mathbb{S}_\bullet(X). \tag{5.1}$$

There are versions of this construction for L-groups with different decorations, but since we work with manifolds whose Whitehead groups vanish we may ignore the distinction between surgery up to homotopy equivalence (L_*^h) and surgery up to simple homotopy equivalence (L_*^s).

If M^n is a closed n-dimensional manifold then the long exact sequence of homotopy groups in Equation (5.1) is identified by Poincaré duality with the surgery exact sequence of Browder–Sullivan–Wall [175, Chapter 10]. In particular, $\mathbb{S}_i(M) = \pi_i(\mathbb{S}_\bullet(M))$ may sometimes be identified with the topological structure set:

$$\mathcal{S}^{\mathrm{TOP}}\big(M^n \times D^k,\ \partial\big) = \mathbb{S}_{n+k+1}(M) \quad (k \geqslant 0).$$

The fact that the long exact sequence for topological surgery on a manifold is an exact sequence of Abelian groups is a considerable aid in computations. The classification argument for classifying tori [109,174,175] is an example of the difficulties imposed in categories other than TOP by their less robust surgery sequences.

5.3. *Siebenmann periodicity*

One of the important features of surgery in the topological category is a periodicity result due to Siebenmann [122, pp. 277–283] with corrections by Nicas [138].

THEOREM 5.2 (Siebenmann periodicity). *If $(M^n, \partial M)$ is a manifold with boundary and if $n \geqslant 6$ then there is an injection $\mathcal{S}^{\mathrm{TOP}}(M^n) \to \mathcal{S}^{\mathrm{TOP}}(M^n \times D^4, \partial)$, which is an isomorphism if $\partial M \neq \emptyset$.*

5.4. *Induction theorems*

The induction theorems needed here are modeled on the Frobenius reciprocity theorem for group representations. Dress [42] proved such theorems for the K- and L-groups and from homotopy theory we obtain similar results for the homological term in the long exact sequence of surgery theory [1, Chapter 4]. Induction methods were crucial in the spherical space form problem [166,127], and may be most familiar to topologists in that setting. It is important to note that versions of these theorems are valid for an infinite group Γ, provided we have a good finite quotient group of Γ.

The sketch below largely follows Nicas' memoir [138], which should be consulted for details.

Let M be a connected manifold with universal covering space $p: \widetilde{M} \to M$. Choose a basepoint $\widetilde{x}_0$ of $\widetilde{M}$, thus determining an action of $\Gamma = \pi_1(M, p(\widetilde{x}_0))$ on $\widetilde{M}$. To each subgroup $\Delta < \Gamma$ we associate the covering space $p_\Delta : \Delta\backslash\widetilde{M} \to M$. The inclusion $\Delta \hookrightarrow \Gamma$ and covering projection p_Δ determine transfer homomorphisms in K-groups, L-groups, and $H_*(-; \mathbb{L})$, which induce a transfer in the topological structure set.

Some transfer homomorphisms admit nice geometric descriptions associated to geometric interpretations of the groups involved. In particular, the transfer on $\mathcal{S}(M^n, \partial M)$ is simply pullback: If $[f] = [(V, \partial V) \to (M, \partial M)]$ is an the equivalence class of a homotopy equivalence of manifolds, restricting to a homeomorphism on the boundary, then $p_\Delta^*([f])$ is the equivalence class of the pullback or lift of f to $(V_\Delta, \partial(V_\Delta)) \to (M_\Delta, \partial(M_\Delta))$.

The K- and L-groups are covariant functors, where morphisms associated to a group homomorphism $\phi : G \to H$ are usually defined by tensoring an RG-module with RH, which is viewed as an (RG, RH)-bimodule. If ϕ is the inclusion of a finite-index subgroup then ϕ_* is modeled on the construction of induced representations, while the transfer morphism ϕ^* is modeled on the restriction operation in representation theory. Nicas [138] must work with the kernel and cokernel of the assembly map in the surgery exact sequence in order to draw conclusions about the topological structure set, whose group structure is essential for the argument.

Our goal is a test for triviality of elements x of the structure set $\mathcal{S}(M)$ of a manifold M. The assay for triviality is a collection of transferred images $p_\Delta^*(x) \in \mathcal{S}(M_\Delta)$, running over a family of finite-index subgroups Δ of $\pi_1(M)$: if each transferred element is trivial then we hope to conclude that x is trivial, i.e., represented by a homeomorphism. The family of subgroups used in the test is indexed by a collection of subgroups of a finite quotient group of $\pi_1(M)$.

Let C_k denote the cyclic group of order k and let C denote the infinite cyclic group.

DEFINITION 5.2. Let p be a prime number.

(a) A finite group G is *p-elementary* if it is the direct product of a cyclic group C_k and a p-group P, $G = C_k \times P$. G is *elementary* if it is p-elementary for some prime p.

(b) A finite group G is *p-hyperelementary* if it is a semidirect product of a cyclic group and a p-group, $G = C_k \times_\alpha P$. G is *hyperelementary* if it is p-hyperelementary for some prime p.

The next result is a weak version of [138, Proposition 6.2.9].

THEOREM 5.3 (Nicas). *Let M^n be a compact, connected manifold, where $n \geqslant 5$, let $\Gamma = \pi_1(M)$, and let $\psi : \Gamma \to Q$ be an epimorphism onto a finite group.*

Let $\mathcal{A}$ be the set of 2-hyperelementary subgroups of Q, let $\mathcal{B}$ be the set of p-elementary subgroups of Q where p runs through all the odd prime numbers, and let $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$.

Suppose that $x \in \mathcal{S}(M, \partial M)$ is an element of the topological structure set of $(M, \partial M)$. If $p^*_{\psi^{-1}(H)}(x) = 0$ for every $H \in \mathcal{C}$ then $x = 0$.

Nicas' result depends upon the work of Dress [42], which describes K- and L-groups as modules over functors with readily established induction properties, following the approach of J.A. Green to induced representations [96]. The next theorem is a statement of a weakened form of Dress' results adapted for our applications.

THEOREM 5.4 (Dress). *Let M^n be a compact, connected manifold, where $n \geqslant 5$, let $\Gamma = \pi_1(M)$, and let $\psi : \Gamma \to Q$ be an epimorphism onto a finite group. Let $\mathcal{D}$ be the set of elementary subgroups of Q.*

If $x \in \widetilde{K}_0(\mathbb{Z}\Gamma)$ then $x = 0$ if and only if the transfer $p^*_{\psi^{-1}(H)}(x) = 0$ for every $H \in \mathcal{D}$.

If $y \in \widetilde{\mathrm{Wh}}(\Gamma)$ then $y = 0$ if and only if the transfer $p^*_{\psi^{-1}(H)}(y) = 0$ for every $H \in \mathcal{D}$.

5.5. *Pseudoisotopy*

Pseudoisotopy is a tool for studying manifolds through their homeomorphism groups.

DEFINITION 5.3. A *topological pseudoisotopy* on a manifold M is a homeomorphism $h : M \times I \to M \times I$ such that $H|M \times \{0\}$ is inclusion. The space of all topological pseudoisotopies on M is denoted by $P(M)$.

K-theory and pseudoisotopy were shown to be linked by Hatcher and Wagoner [105]. Deeper connections emerged with the study of the stabilization operation $P(M) \to P(M \times I)$ obtained by forming the product $h \times \mathrm{Id}_I$. The space of *stable pseudoisotopies*, $\mathcal{P}(M)$, is the direct limit of the $P(M \times I^k)$ under stabilization. This space and its counterpart for diffeomorphisms of smooth manifolds have been shown by Hatcher to be Ω-spectra [104].

Waldhausen's A-theory is the context in which K-theory and pseudoisotopy seem to make their nearest approach [171, 172]. We are unable to say much about this subject here, but rigidity theorems and conjectures cited below will involve pseudoisotopy.

6. Splitting and fibering problems

Dissection arguments are a mainstay of classification results in topology, with classical roots in the study of a Riemann surface by cutting it open along curves. Waldhausen proved topological rigidity by cut and paste methods for a large family of aspherical 3-manifolds [169]:

THEOREM 6.1 (Waldhausen). *Let M^3 and N^3 be compact, connected, sufficiently large, irreducible* 3*-manifolds. If $f:(M,\partial M)\to(N,\partial N)$ is a homotopy equivalence which restricts to a homeomorphism of boundaries then f is homotopic to a homeomorphism.*

Classification or rigidity arguments which induce over a sequence of cut-and-paste constructions usually begin with a description of the target manifold N via manifolds with boundary obtained by cutting along codimension-one, two-sided submanifolds; we seek to transport this structure in N to the source manifold M, and then to solve a relative version of the homeomorphism problem in the induction step, as in the conclusion and proof of Waldhausen's theorem above. The step in which a dissection of N is transported to M involves more than transversality, since at minimum we seek to deform a homotopy equivalence of single spaces $M^n\to N^n$ to a homotopy equivalence of pairs $(M^n,V^{n-1})\to(N^n,W^{n-1})$.

6.1. *Splitting problems in K-theory*

Let N^n be a manifold with a a codimension-one submanifold W. This submanifold is said to be two-sided if it has an open neighborhood U such that $(U,W)\cong(W\times(-1,1),W\times\{0\})$.

Let N^n be a manifold with a connected, two-sided submanifold W^{n-1}. A *splitting problem* is a homotopy equivalence $f:M^n\to N^n$; a solution to the splitting problem is a map g which is homotopic to f, transverse to W^{n-1} (so that $V=g^{-1}(W)$ is a two-sided, codimension-one submanifold of M), and so that $g:(M^n,V^{n-1})\to(N^n,W^{n-1})$ is a homotopy equivalence of pairs. One often says that a homotopy equivalence f *splits* along W if it may be deformed to a map g as above.

Obstructions for splitting problems arise in both K-theory and surgery, and may be interpreted as exotic terms in generalized Mayer–Vietoris sequences for those theories. These exotic terms are K-theoretic groups in their own right, called Nil groups; see [9, 170] for more information.

Waldhausen followed his results on sufficiently large 3-manifolds, which imply the vanishing of Whitehead groups for fundamental groups of such manifolds, with an algebraic study of K-theoretic splitting obstructions [170]. This study analyzes Whitehead groups for HNN extensions and free products with amalgamations, and we describe those results below: a finite induction with these methods allows Waldhausen to deduce a vanishing theorem which may be posed in terms of graphs of groups [158].

A ring R is regular coherent if every finitely presented right R-module admits a finite resolution by finitely generated projective R-modules, and R is regular Noetherian if it is regular coherent and every finitely generated R-module is finitely presented. A group G is regular coherent if for every regular Noetherian ring R the group ring RG is regular coherent. (For example, finitely generated free Abelian groups have regular Noetherian group rings, while surface groups are regular coherent.)

Let $\mathcal{C}$ be the smallest class of groups containing the trivial group, closed under the construction of finite graphs of groups with with regular coherent edge groups, and closed under filtering direct limits.

THEOREM 6.2. *If R is regular Noetherian and G is a group in $\mathcal{C}$ then $\mathbb{W}\mathrm{h}^R(G)$ is contractible.*

COROLLARY 6.1. *If G is the fundamental group of a compact irreducible 3-manifold which is sufficiently large then $\mathrm{Wh}_i(\mathbb{Z}G) = 0$ for all $i \geqslant 0$.*

6.2. *Fibering obstructions*

Bass, Heller, and Swan [9] anticipated Waldhausen's splitting theorem in K-theory in the important special case of a group of the form $G = H \times C_\infty$, where C_∞ denotes the infinite cyclic group. This result and its extension from direct to semidirect products [51] are part of the analysis of a geometric problem related to splitting:

PROBLEM 6.1 (*The Fibering Problem*). When is a manifold M^n homeomorphic to the total space of a bundle over a circle with closed manifold fiber?

Irreducible 3-manifolds which fiber over the circle are well understood [162,106], and those results motivated the study of fibering problems in higher dimensions.

The high-dimensional fibering problem was solved by Farrell in his 1967 thesis [47], which exhibited complete obstructions in K-theory (see also [160]). The argument uses h-cobordism methods to produce a product structure on an infinite cyclic covering space. The fibering theorem was extended to the boundary dimensions [181] after Freedman's seminal work on 4-manifolds [91,92]. (Recall that restrictions on the fundamental group are currently required for 4-dimensional surgery: see [92].)

6.3. *Splitting problems in surgery*

The surgical aspect of splitting problems was examined by Wall [175, Chapters 11 and 12] and Cappell [21–23], following earlier work of Shaneson [159] and Wall on L-groups of products with an infinite cyclic group.

In [23] Cappell proved a splitting theorem under the assumptions that K-theory obstructions vanish and that $\pi_1(V^{n-1})$ is a square-root closed subgroup of $\pi_1(M^n)$. (A subgroup $H < G$ is *square-root closed* if $g \in G$ and $g^2 \in H$ imply $g \in H$.) This result and extensions of it in the algebraic theory of surgery (see Ranicki's work) have been used to prove cases of both the Novikov Conjecture and Borel Conjecture ([139] is one of several examples).

Cappell's announcements [21,22] define counterparts to the Nil groups of K-theory, called UNil groups.

6.4. *Nil and UNil groups*

The UNil groups, like the Nil's, appear as exotic terms in Mayer–Vietoris sequences, and if they are nontrivial for splitting data (M^n, V^{n-1}) then one expects the structure set of M^n to contain more than one element, and indeed to be large [22].

Cappell shows that the UNil groups are sometimes nontrivial by analyzing the free product of two cyclic groups of order two, $D_\infty = C_2 * C_2$. Since this infinite dihedral group is a crystallographic group, one should not be surprised to encounter difficulties as rigidity investigations move from manifolds to orbifolds (Section 10). The Borel Conjecture predicts that all UNil groups vanish for a torsion-free fundamental group and no evidence is presently known against this claim.

Some order is emerging in the picture of Nil, UNil, and related groups in K-theory [33, 48,49], but more work is needed.

7. Controlled topology

The next few paragraphs are a guide to parts of the rapidly growing literature on controlled topology, emphasizing topics associated with rigidity theorems. See the chapter on homology manifolds in this volume for more detail and a broader perspective.

Connell and Hollingsworth [30] published in the late 1960s a description of Whitehead torsion in terms of *geometric modules*: such an object is a based free module M over a ring, together with a function from the basis to a metric space (X, d). This function gives a notion of support in X for an element of M, and we say that a module element is ε-small if its support has diameter less than ε. We get a similar notion of the diameter of an endomorphism of M. Connell and Hollingsworth reduced the topological invariance of Whitehead torsion to certain problems about geometric modules and morphisms, but were unable to resolve these.

In the late 1970s Ferry [84,85] and Chapman and Ferry [27] proved approximation theorems, including results that show that a sufficiently controlled homotopy equivalence must be a simple equivalence. The resulting Thin h-Cobordism Theorem (see Theorem 7.1) was at the heart of the rigidity program for flat manifolds. Chapman later formulated counterparts of the finiteness obstruction and Whitehead torsion for controlled maps and spaces in response to Quinn's work [25,26].

Also in the late 1970s, Farrell and Hsiang began work on flat manifolds other than the torus with sharp results on K-theory and partial results on surgery [52], proving topological rigidity of flat Riemannian manifolds with holonomy groups of odd order.

Quinn's 1979 paper [143] studied ends of spaces and maps with controlled methods; this paper also revived the geometric modules of Connell and Hollingsworth and established some of their conjectures (see also [146,93] and [50, Lectures 9–11]). Quinn continued his influential work on ends in [144,145] and Quinn's student Yamasaki has worked on K-theory of crystallographic groups as well as foundations [150,186].

The completion of the proof of topological rigidity for flat Riemannian manifolds [56] by Farrell and Hsiang covered almost flat manifolds as well. The argument depended upon fiberwise control results established in [56] and implicit in [143]. These fiberwise results are precursors of the foliated control theorems employed by Farrell and Jones in their work on manifolds of nonpositive curvature [58,60–62,64,65,67,69].

Another vein of controlled argument emphasizes bounded structures and is represented by [2,3,141], and a number of other papers. The bounded theory has better categorical properties than ε-controlled algebra.

7.1. *Control theorems*

Recall that an h-cobordism is a manifold W^{n+1} whose boundary is partitioned as a disjoint union $\partial W = M_0^n \coprod M_1^n$, where both inclusions $i_j : M_j \hookrightarrow W$ are homotopy equivalences. (We often call M_0 the *base* of the h-cobordism.) It follows that there exist deformation retractions $r_j : W \times I \to W$ such that $r_j(w, 0) = w$ for every $w \in W$, $r_j(w, 1) \in M_j$ for every $w \in W$, and $r_j(x, t) = x$ for every $x \in M_j$ and every $t \in I$.

DEFINITION 7.1. Let (W^{n+1}, M_0^n, M_1^n) be an h-cobordism and let d be a metric on M_0. W is *ε-controlled* provided there exist deformation retractions r_0 and r_1 of W to M_0 and M_1, as above, such that for every $w \in W$ the paths ρ_w and σ_w defined by

$$\rho_w(t) = r_0\big(r_0(w, t),\ 1\big) \quad (t \in I)$$

and

$$\sigma_w(t) = r_0\big(r_1(w, t),\ 1\big) \quad (t \in I)$$

have diameter less than ε in M_0.

The controlled vanishing theorem for Whitehead torsion proved by Ferry in 1977 has been extremely influential [84]:

THEOREM 7.1 (Thin h-cobordism theorem). *For each closed Riemannian manifold M^n of dimension $n > 4$ there exists a real number $\varepsilon > 0$ such that any ε-controlled h-cobordism W with base M has trivial Whitehead torsion.*

A similar result was proved by Chapman and Ferry for homeomorphisms [27,85].

DEFINITION 7.2. Let M_0 be a Riemannian manifold with metric d, and let ε be a positive real number. Suppose that $f : M_1 \to M_0$ is a homotopy equivalence. We say that f is an *ε-equivalence* if there exist a homotopy inverse $g : M_0 \to M_1$, a homotopy $F : M_0 \times I \to M_0$ from $f \circ g$ to Id_{M_0}, and a homotopy $G : M_1 \times I \to M_1$ from $g \circ f$ to Id_{M_1} so that for every $x \in M_0$ the path ρ_x defined by

$$\rho_x(t) = F(x, t) \quad (t \in I)$$

has diameter less than ε in M_0 and for every $y \in M_1$ the path σ_y defined by

$$\sigma_y(t) = f\big(G(y,t)\big) \quad (t \in I)$$

has diameter less than ε in M_1.

THEOREM 7.2 (Chapman–Ferry α-approximation theorem). *Let N^n be a compact Riemannian manifold of dimension $n \geqslant 5$. There is an $\varepsilon > 0$ such that any ε-equivalence $f : (M, \partial M) \to (N, \partial N)$ which restricts to a homeomorphism of boundaries is homotopic to a homeomorphism.*

The "α" of the title denotes an open cover of N; the full strength of the theorem works with open covers rather than metric conditions, and applies to noncompact spaces. This style of controlled vanishing or rigidity result has also been established for the projective class group $\widetilde{K}_0$ and the structure set of topological surgery [56].

7.2. *Properties of flat manifolds*

The rigidity argument presented in Section 7.3 was largely motivated by the theorem of Epstein and Shub [46] which asserts that every closed flat Riemannian manifold supports an expanding diffeomorphism. This suggests an attack on invariants which can be represented by geometric modules or bounded morphisms of geometric modules: transfer an element x through a diffeomorphism h which is expanding enough that the diameter of $h^*(x)$ is so small that a control theorem implies $h^*(x) = 0$.

Unfortunately, there is no reason for transfer by an expanding map to induce an isomorphism in K-theory or surgery. A more elaborate argument is required, which uses Frobenius induction and multiple covering spaces of flat manifolds, as well as a vanishing result for sufficiently expansive covering projections.

Flat manifolds and orbifolds of dimension n are quotients of $\mathbb{R}^n$ by discrete subgroups $\Gamma < \mathrm{E}(n)$, where $\mathrm{E}(n)$ is the group of Euclidean isometries (Section 4.1). In general, such a Γ is called a *crystallographic group*; if Γ is torsion-free and cocompact in $\mathrm{E}(n)$ then Γ is called a *Bieberbach group*.

Bieberbach showed [185, Chapter 3] that the group extension decomposition of the group $\mathrm{E}(n)$ of isometries of $\mathbb{R}^n$ exhibited in Equation (4.1) is preserved in cocompact discrete subgroups $\Gamma < \mathrm{E}(n)$: Γ has a maximal normal free Abelian subgroup A, which consists of translations, and Γ is an extension

$$1 \to A \to \Gamma \to G \to 1, \tag{7.1}$$

where $A \cong C^n$ is a free Abelian group and G is a finite group.

The finite group G is known as the *holonomy group* of Γ, and although Γ is imbedded in the split extension of Equation (4.1), Equation (7.1) is not usually a split extension. Auslander and Kuranishi showed that every finite group appears as the holonomy group of some Bieberbach group [185, Theorem 3.4.8].

Every abstract group Γ possessing a normal free Abelian subgroup of finite index is realized as a crystallographic group, so rigidity theorems proved for flat manifolds by geometric methods may be recast as theorems concerning a certain class of abstract groups.

7.3. *Topological rigidity of flat manifolds*

The vanishing theorem of Farrell and Hsiang for K-theory obstructions [52, Theorem 3.1] was the first general result on rigidity questions for flat manifolds, and serves as a starting point for later arguments.

THEOREM 7.3 (Farrell–Hsiang). *If Γ is a Bieberbach group then* $\mathrm{Wh}(\Gamma) = 0$ *and* $\widetilde{K}_0(\mathbb{Z}\Gamma) = 0$.

The outline of the proof begins with the observation that every crystallographic group has a non-elementary finite quotient group, so we may apply Theorem 5.4. One might expect an induction over the order of the holonomy group of Γ to yield the conclusion, but the existence of expanding maps on flat manifolds [46] is one aspect of a fact which is awkward for the induction argument: A flat manifold may be a covering space of itself. We say that a self-map is infinitesimally s-expansive if its differential satisfies $||Df(v)|| = s||v||$ for every tangent vector v.

Farrell and Hsiang are obliged to prove a structural result for Bieberbach groups Γ [52, Theorem 1.1] and a quantitative vanishing result for expansive covering projections [52, Theorem 2.3], which shows that if an expansive self-map of a closed flat manifold is infinitesimally s-expansive for sufficiently large s then the associated transfer annihilates the Whitehead and projective class groups.

The main argument is a double induction, running over both the rank of the translation subgroup of Γ and the order of the holonomy group. (Note that the seed of the induction is the free Abelian case, $\Gamma \cong C^n$, which is settled by the Bass–Heller–Swan theorem [9].) The induction step uses a non-elementary finite quotient G of Γ, which is chosen so that if $H < G$ is covered by a subgroup $\Gamma_H < \Gamma$ which is isomorphic to Γ then the associated covering projection is infinitesimally s-expansive for s large enough to annihilate the K-groups. We have arranged matters so that for each elementary subgroup H of G, either (a) we reduce to a case already settled by the induction hypothesis or (b) the subgroup of Γ lying over H is isomorphic to Γ and we are in the infinitesimally s-expansive case for large s. Both alternatives are won cases, so Theorem 5.4 finishes the argument.

The analogous argument for surgery theory was less satisfactory in [52] because of difficulties with splitting obstructions. Improvements in [57] give a rigidity theorem.

THEOREM 7.4 (Farrell–Hsiang). *Let M^n be a closed aspherical manifold whose fundamental group is virtually nilpotent and let E^{n+k} be the total space of a D^k-bundle over M^n. If $n + k > 4$ then $\mathcal{S}(E^{n+k}) = 0$; in particular, if $n > 4$ then M is topologically rigid.*

The scheme adopted for this argument, which covers almost flat manifolds (Definition 3.2) as well as flat manifolds, was called a "fibering apparatus", and has been adopted in later papers by Lee and Raymond. The main feature of the scheme for the present account is the control theorem proved in [56], in which diameters are measured in the base of a fiber bundle whose fibers are already known to have good rigidity properties. The notion that control could be maintained in some directions and relaxed in others was the key to rigidity results for Riemannian manifolds of nonpositive curvature.

7.4. *Properties of negatively curved manifolds*

Two features seen in a compact Riemannian manifold M^n of nonpositive sectional curvature have been exploited effectively by topologists. The first and most readily generalized of these is the compactification of the universal covering space $\widetilde{M}^n$ by adjoining an $(n-1)$-sphere at infinity defined by geodesic rays [43]. This boundary sphere will be denoted by $\partial_\infty M$ below.

The visibility sphere V_x at a point $x \in \widetilde{M}^n$ is the unit sphere in $T_x\widetilde{M}$, which we identify with the set of unit-speed geodesic rays based at x. With the Cartan–Hadamard theorem and a radial reparametrization of the exponential map based at x, we identify $\widetilde{M}$ with the interior of the disk D^n and we identify the visibility sphere V_x with the boundary of D^n, compactifying $\widetilde{M}$. This compactification, denoted here by $\mathrm{VC}(\widetilde{M})$ appears heavily dependent upon x and to make it more natural we introduce an identification of visibility spheres $V_x \cong V_y$. Two geodesic rays ρ, σ are asymptotes, or asymptotically equivalent, if there is a constant C such that for all $t \geqslant 0$ $d(\rho(t), \sigma(t)) \leqslant C$. If M is a complete Riemannian manifold of nonpositive sectional curvature then this equivalence relation on elements of visibility spheres at different basepoints makes the compactification of $\widetilde{M}$ by the sphere at infinity naturally enough that the isometry group $\mathrm{Isom}(\widetilde{M})$ acts by homeomorphisms upon $\mathrm{VC}(\widetilde{M})$.

The compactification of $\widetilde{M}^n$ to a disk by $\partial_\infty M$ and emulations of this compactification, as for word hyperbolic groups [98], play an important role in most approaches to the Novikov Conjecture for nonpositively curved manifolds and related groups. (See [88,89] for more information.) We want to use the fact that if $\sigma(t)$ $(-\infty < t < \infty)$ is a geodesic in $\widetilde{M}^n$ then this curve has well-defined endpoints in the sphere at infinity, $\sigma(-\infty), \sigma(\infty) \in \partial_\infty \widetilde{M}^n$

The second geometric feature to discuss is the geodesic flow Let SM denote the subbundle of the tangent bundle TM consisting of tangent vectors of length 1, so that SM has fiber S^{n-1}. Let $p: TM \to M$ be the projection in the tangent bundle and let $q = p|_{SM}$ be the projection in the unit tangent sphere bundle. Define a vector field Σ on TM by $\Sigma_v = (v, 0) \in T_v TM \cong T_{p(v)}M \oplus T_{p(v)}M$, where the first summand is horizontal and the second vertical in $T_v TM$. Σ is known as the *geodesic spray* and restricts to a vector field of length 1 everywhere on SM. The *geodesic flow* is the flow ϕ_t on SM of the vector field $\Sigma|_{SM}$.

LEMMA 7.1. *Let ϕ_t be the geodesic flow on the unit tangent sphere bundle SM of a complete Riemannian manifold.*

(b) *ϕ_t is a complete flow, i.e., the flow line through any initial point exists for all t.*

(b) *Integral curves of ϕ_t cover geodesics in the following sense: For any $v \in SM$ $t \mapsto q \circ \phi_t(v)$ is the geodesic in M through $q(v)$ with initial vector v.*

Anosov's study of the geodesic flow [4] showed that the following features of the flow imply many of its other properties, and tend to rigidify it among flows; the fact that the geodesic flow does have these properties essentially go back to Hadamard. Any smooth flow which satisfies (a)–(c) below is now called an *Anosov flow*.

LEMMA 7.2. *Let SM be the unit tangent sphere bundle of a complete Riemannian manifold of sectional curvature* $-b^2 \leqslant K \leqslant -a^2 < 0$.

If ϕ_t *is the geodesic flow on SM then*:

(a) *The tangent bundle TSM splits continuously into a Whitney sum of three* ϕ_t*-invariant subbundles,* $TSM = E^0 \oplus E^u \oplus E^s$, *where*
(b) E^0 *is tangent to the geodesic flow, and*
(c) *There exist positive constants C and* λ *such that*

$$||D_x\phi_t|_{E^s_x}|| \leqslant C\mathrm{e}^{-\lambda t} \quad \textit{and} \quad ||D_x\phi_{-t}|_{E^u_x}|| \leqslant C\mathrm{e}^{-\lambda t}.$$

The subbundles E^s and E^u are the *stable subbundle* and the *unstable subbundle*, respectively. In the most familiar example of this construction, the hyperbolic plane $M^2 = H^2_{\mathbb{R}}$ is realized as the unit disk, $SM = M^2 \times S^1$, and the 3-dimensional tangent space T_vSM splits as a sum of three line bundles. If $x = q(v) \in M^2$ and $\sigma(t)$ is the unit-speed geodesic through x with initial tangent vector $\mathrm{d}\sigma(t)/\mathrm{d}t|_{t=0} = v$ then the stable and unstable line bundles may be described in terms of horocycles through x, orthogonal to v, and passing through $\sigma(\infty)$ (in the stable case) or $\sigma(-\infty)$ (in the unstable case). An illustrated account of this example may be found in [11, Chapter 3].

The geodesic flow is a well-studied object in dynamics and geometry, largely because of its structural stability and ergodic properties (on manifolds of finite volume) and the fact that its periodic orbits are smooth closed geodesics. We need the much more basic properties of the geodesic flow stated in Lemma 7.2.

7.5. *Topological rigidity of nonpositively curved manifolds*

While the Farrell–Hsiang program on flat and almost flat manifolds was underway W.C. Hsiang suggested that the distinctive metric features of the geodesic flow might be the foundation for rigidity arguments addressing negatively curved manifolds. A topological counterpart to Mostow rigidity for hyperbolic manifolds (Theorem 4.2) was particularly sought.

This ambition was realized by a sequence of papers of Farrell and Jones which began in 1986 [58] and continued for at least ten years. The new ingredient from controlled topology is a vanishing theorem in the mode of Ferry–Chapman–Quinn, but with control hypotheses adapted to the squeezing/expanding behavior of an Anosov flow. Since the geodesic flow lives on the unit tangent sphere bundle SM rather than on M, a device is needed to move representative elements of obstruction groups from M to SM (we have in mind, for example, the Whitehead group of $\pi_1(M)$ and the realization of each element of this group as the Whitehead torsion of an h-cobordism based on M).

The *asymptotic transfer* lifts a path $\alpha : I \to M$ to a path $v\alpha : I \to SM$ such that $q \circ (v\alpha) = \alpha$. The notion of asymptotic equivalence for geodesic rays discussed in Section 7.4 is visualized directly within the visibility compactification: any geodesic ray has a well-defined endpoint in $\partial_\infty \widetilde{M}$, and two geodesic rays are asymptotes if and only if their endpoints in the sphere at infinity agree. The asymptotic transfer will first be defined for paths in $\widetilde{M}$; because the construction is equivariant for the action of the isometry group, it descends to M.

For each $v \in SM$ let γ_v be the geodesic in $\widetilde{M}$ such that $\dot{\gamma}_v(0) = v$, where the overdot indicates derivative with respect to the time parameter along the geodesic. (Recall that the geodesic equation is a second order differential equation, so we are solving the initial value problem $\gamma_v(0) = q(v)$, $\dot{\gamma}_v(0) = v$.) For each $v \in S\widetilde{M}$ and each $x \in \widetilde{M}$ let $v(x) \in S_x M$ be the unique unit vector at x such that γ_v and $\gamma_{v(x)}$ are asymptotes. We define the asymptotic transfer $v\alpha$ of the path α by

$$v\alpha(t) = v\big(\alpha(t)\big) \quad (t \in I).$$

The most important property of the asymptotic construction is a squeezing statement: The geodesic flow on SM shrinks $v\alpha$ in all directions normal to the flow lines, so that the flow deforms $v\alpha$ arbitrarily close to some flow line as $t \to +\infty$, while the length of the deformed curve remains bounded. (This metric statement is based on the fact that $v\alpha$ is contained in a leaf of E^s.)

DEFINITION 7.3. A path ρ in the unit tangent sphere bundle SM is (β, ε)-controlled if there is another path σ in SM such that:

(a) The image of σ is contained in an arc of length β inside a flow line of the geodesic flow.

(b) $d(\rho(t), \sigma(t)) < \varepsilon$ for every $t \in I$.

LEMMA 7.3. *Given $\beta > 0$ and $\varepsilon > 0$ there exists $s_0 > 0$ such that for every smooth path α in M of length less than β and for every vector $v \in S_{\alpha(0)}M$ the composite with the geodesic flow $\phi_s(v\alpha)$ is $(\sqrt{2}\beta, \varepsilon)$-controlled for every $s \geqslant s_0$.*

This is one of the foliated control theorems established by Farrell and Jones, specialized from a theorem in [61].

THEOREM 7.5 (Foliated control for h-cobordisms). *Given M^n, a closed manifold of strictly negative sectional curvature, where $n > 2$, and a positive real number β, there exists $\varepsilon > 0$ such that every (β, ε)-controlled h-cobordism with base SM has trivial Whitehead torsion.*

The argument for this theorem follows the model of Ferry [84], after SM is equipped with a "long-thin cell structure" adapted to the dynamics of the geodesic flow. Using this foliated control theorem, an argument for the vanishing of $\mathrm{Wh}(\pi_1(M))$ may be built on the following outline:

(1) Represent $x \in \mathrm{Wh}(\pi_1(M))$ by a smooth h-cobordism W with base M.

(2) Take smooth deformation retractions of W onto its top and base and let β bound the arc lengths of the deformation paths or tracks for these homotopies (as measured in M).
(3) Let $\mathcal{W}$ be the total space of the pullback of $q : SM \to M$ to W (pull back over the retraction onto the base of the h-cobordism). $\mathcal{W}$ is now an h-cobordism, which we equip with top and base retractions whose tracks are asymptotic transfers of the tracks in M.
(4) Take ε from Theorem 7.5 for the data $\mathcal{W}$ and $\sqrt{2}$. Flow forward with the geodesic control long enough to make $(\mathcal{W}, SM)$ $(\sqrt{2}\beta, \varepsilon)$-controlled.
(5) Apply Theorem 7.5 to conclude that the asymptotically transferred h-cobordism has trivial Whitehead torsion.

A difficulty appears at the end of the argument: we do not know if asymptotic transfer induces an isomorphism in Whitehead groups, and in fact the known transfer theorems for algebraic K-theory don't allow us to draw such a conclusion because we are transferring through a fiber bundle projection where the spherical fiber has Euler characteristic 0 or 2, depending on n. Farrell and Jones modified the construction so that transfer is performed in a related, but noncompact, bundle whose fiber has Euler characteristic 1.

Arguments of this kind have led to a number of conclusions, including these from [77, 80].

THEOREM 7.6 (Rigidity of nonpositively curved manifolds). *Let M^n be a closed Riemannian manifold of nonpositive sectional curvature.*
(a) $\mathrm{Wh}(\pi_1(M)) = 0$.
(b) *If* $m + n \geqslant 5$ *then* $\mathcal{S}(M^n \times D^k, \partial) = 0$.

Subsequent developments have required modifications of the asymptotic transfer to a "focal transfer" which focuses on a point at finite distance from a lifted path rather than on a point at infinity. Noncompact manifolds have also been studied by Farrell and Jones with these methods, and some of this work remains in progress as this chapter is written. Their paper on isomorphism conjectures [78] considers orbifolds as well as manifolds (see Section 10) and refines our understanding of the role of geodesic flow. For many computations it seems likely that we can view elements of obstruction groups as concentrated along closed loops (or along periodic orbits for the geodesic flow in a negatively curved manifold).

Geometers and dynamicists are currently investigating rigidity properties of geodesic flows (to what degree does the dynamics of the flow determine the Riemannian metric?). We quote only one of these results. C.B. Croke, P. Eberlein, and B. Kleiner have established the following rigidity result [37].

THEOREM 7.7 (Croke, Eberlein, and Kleiner). *Let M and N be compact Riemannian manifolds of sectional curvature $K \leqslant 0$, such that M has dimension three or more and rank at least two. If there is a C^0 conjugacy F between the geodesic flows on the unit tangent sphere bundles of M and N then there exists an isometry $G : M \to N$ that induces the same isomorphism as F on fundamental groups.*

8. Exotic structures

8.1. *Manifold structures*

We now know that the full range of variations of geometric structures can be realized in aspherical manifolds, often with interesting side conditions.

Davis and Hausmann showed that there are closed aspherical manifolds with no smooth or PL structure [40]:

THEOREM 8.1 (Davis–Hausmann). (a) *For each $n \geqslant 13$ there exists an aspherical closed PL manifold of dimension n which does not have the homotopy type of a smooth manifold.*

(b) *For each $n \geqslant 8$ there exists an aspherical closed topological manifold of dimension n which is not homeomorphic to a closed PL manifold.*

Farrell and Jones have found exotic smooth structures on compact or noncompact manifolds of negative curvature. In the compact case [71] they take a connected sum of a hyperbolic manifold and an exotic sphere, but in the noncompact case a connected sum is known to be inadequate for the creation of nonstandard smooth structures. In [79] Farrell and Jones modify smooth structures with a noncompact counterpart to Dehn surgery, modifying a properly imbedded tube.

In both the compact and noncompact cases, Farrell and Jones show that some of the exotic smooth structures they produce can be equipped with a Riemannian metric of negative sectional curvature. This construction limits the possible scope of rigidity theorems or pinching theorems in differential geometry and also leads to some new results on harmonic maps.

8.2. *Exotic harmonic maps*

A *harmonic map* between smooth manifolds is a critical point for an energy functional which is roughly the mean square of the covariant derivative. These maps are solutions of nonlinear elliptic partial differential equations and have good existence and regularity properties in several different contexts. (An example which is essentially familiar to topologists is a harmonic map from a closed manifold to the circle S^1: such a map pulls back $d\theta$ to a harmonic 1-form in the sense of Hodge theory.)

Differential geometers had conjectured that a harmonic homotopy equivalence between closed manifolds of nonpositive curvature must always be a homeomorphism. Some of the evidence for such a claim is found in the role of harmonic map techniques in recent work on rigidity and arithmeticity done by K. Corlette, C. Simpson, M. Gromov and R. Schoen, and others. Farrell and Jones used the classic existence results of Eells and Sampson on harmonic maps to a nonpositively curved target and their constructions of exotic smooth structures to produce singular harmonic homotopy equivalences [83].

THEOREM 8.2. *Let $n \geqslant 11$ and suppose that M^n is a closed smooth manifold with a Riemannian metric g such that either (M, g) has negative sectional curvature or (M, g) is a flat torus. Then there exists a second Riemannian metric h on M and a harmonic homotopy equivalence $f : (M, h) \to (M, g)$ such that f is not injective.*

8.3. *Homology manifolds*

The ENR homology manifolds introduced by Bryant, Ferry, Mio, and Weinberger [19] suggest non-manifold variations on some of the questions raised above.

If M is a closed aspherical manifold with trivial Whitehead group and if the surgery assembly map $A : H_i(M; \mathbb{L}) \to L_i(\mathbb{Z}\pi_1(M))$ is an isomorphism for all i then M is rigid in the class of ENR homology manifolds: there is no exotic ENR homology manifold with the homotopy type of M, and every closed manifold homotopy equivalent to M is homeomorphic to M [86].

An ENR homology manifold version of the realization problem for Poincaré duality groups has also been suggested. There seems to be no progress to report as this chapter is written.

9. Poincaré duality groups

This notion, mentioned in Section 3.1, was introduced by Bieri [13] and Johnson and Wall [116]. The abbreviation "PD^n group" is often used in discussing Poincaré duality groups.

DEFINITION 9.1 (*Poincaré duality group*). A *Poincaré duality group* of formal dimension n is a countable group Γ of type FP such that

(a) $D = H^n(\Gamma; \mathbb{Z}\Gamma)$ is an infinite cyclic Abelian group and
(b) there is a class $z \in H_n(\Gamma; D)$ so that the cap product map

$$- \cap z : H^i(\Gamma; M) \to H_{n-i}(\Gamma; D \otimes M)$$

is an isomorphism for all Γ-modules M and all integers i.

See Definition 3.1 for the FP property, and see Brown [18, Section VIII.10] for more information on Poincaré duality groups. (The handbook survey [10] on group cohomology may also be useful.) Brown shows that if (a) and (b) hold Γ is necessarily of type FP [18, p. 222], but we include the extra hypothesis for clarity.

The following theorem combines work of Eilenberg–Ganea and Wall [18, Section VIII.7]. Here $\mathrm{cd}(\Gamma)$ denotes the cohomological dimension of the discrete group Γ.

THEOREM 9.1. *Let Γ be an arbitrary group and let $n = \max(3, \mathrm{cd}(\Gamma))$. Then there exists an n-dimensional $K(\Gamma, 1)$-complex Y. If Γ is finitely presented and of type FL (respectively FP) then Y can be taken to finite (respectively finitely dominated).*

Note that the fundamental group of a finitely dominated complex is necessarily finitely presented [173, Lemma 1.3], but that the homological finiteness properties FL and FP do not directly lead to a finite presentation for a group. The long-standing question raised by this situation was resolved by Bestvina and Brady [12]:

THEOREM 9.2 (Bestvina–Brady). *There are groups of type FP which are not finitely presented.*

Building upon the construction of Bestvina and Brady, M. Davis has announced examples of groups which satisfy Poincaré duality (conditions (a) and (b) in Definition 9.1) but which do not have finitely dominated classifying spaces, because these groups are not finitely presented.

The following question has been posed in different flavors at different times [176,177]. The recent developments summarized above oblige us to assume finite presentability.

PROBLEM 9.1 (*Realization problem for Poincaré duality groups*). Is every finitely presented Poincaré duality group the fundamental group of a closed aspherical manifold?

The attack on this question has produced good results in dimension 2, due to Eckmann and his collaborators [45,44], and some information on special classes of groups [6,115] for which induction arguments work well and group extensions are closely related to geometry. Some information has been obtained in dimension 3 [167,107], largely through splitting and fibering arguments.

Surgery theory has an existential aspect which gives a scheme for deciding whether a finite complex that satisfies Poincaré duality is homotopy equivalent to a manifold. The language of the structure set (Definition 5.1) extends to this situation and we seek to show that $|\mathcal{S}^{\mathrm{TOP}}(K(\Gamma, 1))| \neq 0$ if Γ is a Poincaré duality group with finite $K(\Gamma, 1)$. The most general forms of the Borel Conjecture have implications for Problem 9.1, but do not seem to be established except for groups we already know are realized by aspherical manifolds.

A characterization and approximate realization of Poincaré duality groups appears in [165]:

THEOREM 9.3. *Let Γ be a finitely presented group of finite cohomological dimension. Γ is a Poincaré duality group if and only if there exists a closed PL manifold M with fundamental group Γ and universal cover $\widetilde{M}$ such that $\widetilde{M}$ is homotopy equivalent to a finite complex.*

If Γ has a finite $K(\Gamma, 1)$-complex then the space $\widetilde{M}$ of the theorem may be taken to be homotopy equivalent to a sphere of arbitrarily high dimension. The question of whether or not a finitely presented Poincaré duality group is of type FL and so has a finite Eilenberg–MacLane space remains a considerable obstacle in the realization problem.

The number of known Poincaré duality groups grew dramatically in the 1980's with the introduction of Davis's Coxeter group construction (see Section 3.4 and [38]) and Gromov's definition and constructions of hyperbolic groups (Section 3.5).

For example, Theorem 3.7 implies that if π is a discrete group with a finite $K(\pi, 1)$ complex then π appears as a retractive subgroup in many Poincaré duality groups [131]. The Davis–Hausmann examples in Theorem 8.1 provided the first examples of Poincaré duality groups which cannot be realized by smooth closed aspherical manifolds.

Although these developments seem to raise the difficulty of the realization problem, as this is written few if any potential counterexamples are known for Problem 9.1.

The essential task in confirming that a construction yields a PD^n group is eased by recognition criteria for Poincaré duality groups, including this one [18, Theorem VIII.10.1]:

LEMMA 9.1. *Let Γ be a group of type FP. Γ is a* PD^n *group if and only if $H^i(\Gamma; \mathbb{Z}\Gamma) = 0$ for all $i \neq n$ and $H^n(\Gamma; \mathbb{Z}\Gamma) = \mathbb{Z}$.*

The topological meaning of Lemma 9.1 is relatively clear if Γ is a group with a finite $K(\Gamma, 1)$ complex $B\Gamma$, with universal covering space $E\Gamma$. In this case we have isomorphisms $H^i(\Gamma; \mathbb{Z}\Gamma) \cong H^i(B\Gamma; \mathbb{Z}\Gamma) \cong H^i_c(E\Gamma; \mathbb{Z})$, and the cohomology with compact supports of the contractible space $E\Gamma$ may be further identified with Čech cohomology of the end of $E\Gamma$. A topologist may thus wish to think of a PD^n group (with a finite classifying complex) as a group whose end is a Čech cohomology $(n-1)$-sphere: managing this computation during a uniformly described geometric construction is the key to Davis' examples (Section 3.4).

10. Group actions and stratified spaces

10.1. *Lie group actions*

Conner and Raymond showed that effective actions of compact Lie groups on closed aspherical manifolds are rather constrained [31, Theorem 5.6]:

THEOREM 10.1. *If (G, M) is a compact connected Lie group acting effectively on a closed aspherical manifold then*
(a) *G is a toral group with* $\dim(G)$ *at most the rank of the center of $\pi_1(M)$;*
(b) *all isotropy groups are finite; and*
(c) *the Euler characteristic $\chi(M) = 0$.*

Several equivariant or fiberwise rigidity theorems for torus actions on aspherical manifolds are known, usually passing through structure theorems on these actions to conclude that if T^k acts smoothly and effectively on closed aspherical manifolds M^n and N^n and if M^n and N^n are homotopy equivalent, then these manifolds are equivariantly homeomorphic [31,32,123,124].

Another genre of uniqueness or rigidity theorem for group actions grows out of dynamics and Margulis super-rigidity. Zimmer and his students have worked with noncompact group actions and cocycle versions of superrigidity [187] while Katok and Spatzier have made progress on extending notions from the dynamics of Anosov flows to group actions [118].

10.2. *Finite groups, finite subgroups, and stratified rigidity*

Finite groups enter into the study of aspherical manifolds in several ways. The first is exemplified by an arithmetic group Γ acting on the contractible manifold G/K: although Γ has a finite-index subgroup which is torsion-free, arithmetic groups often contain elements of finite order, which act with fixed points. The quotient $\Gamma\backslash G/K$ is then an orbifold, rather than a manifold, and rigidity assertions about the action of Γ should engage with the stratified nature of this orbifold and the special role of finite subgroups.

The best studied class of groups from this point of view is the crystallographic groups. Yamasaki [186] did computations in the L-groups of crystallographic groups, following proposals of Quinn. Connolly and Kozniewski have worked on rigidity problems, obtaining both rigidity results [35] if the holonomy group is of odd order and some more technical hypotheses are satisfied. They have also found examples of the failure of equivariant rigidity [36] for crystallographic groups which have holonomy groups of even order.

Prassidis and Spieler [142] proved equivariant rigidity for every properly discontinuous, cocompact action (Γ, V) of a Coxeter group Γ on a contractible, boundaryless manifold V such that fixed point sets of finite subgroups are contractible and such that 3-dimensional fixed point sets are manifolds. This result was anticipated by the equivariant rigidity result of Rosas [153] in the right-angled case (all the exponents $m(i, j) = 2$ in the Coxeter presentation of Γ). (It is easier in the right-angled case than in general to describe the action of Γ on a universal Γ-space in terms of reflections across submanifolds. These groups also tend to split along square-root closed subgroups.) Prassidis and Spieler also obtain results on the equivariant Whitehead groups of such a manifold.

The formulation of rigidity conjectures is still in flux for a cocompact action of a discrete group Γ, containing torsion, on a contractible manifold V^n. Optimists have expressed a hope that if Γ contains no elements of even order and the fixed sets $\mathrm{Fix}(\Delta, V)$ are contractible (and perhaps the images of those fixed sets have well behaved fundamental groups, e.g., not $C_2 * C_2$), then (Γ, V) might be rigid. Pessimists point to the gap conditions of equivariant surgery as a sign of the distance remaining between known results and those hopes.

The Farrell–Jones isomorphism conjectures [78] are posed in terms of the classifying spaces for proper actions considered by Serre [156] and Connolly–Koźniewski [34]. Farrell and Jones enlarge the scope of these spaces by building a classifying space for actions with a prescribed class of isotropy subgroups (earlier work assumed finite isotropy, while Farrell and Jones emphasize virtually cyclic isotropy). The isomorphism conjectures are too complicated to state here and appear in [78, Section 1.7]: of particular interest is the attempt to formulate generalizations of the Borel or Novikov conjectures for stable pseudoisotopy theory, K-theory, and L-theory simultaneously.

A second context in which finite groups appear in the study of aspherical manifolds is in Nielsen-type problems: If M^n is a closed aspherical manifold and $G < \mathrm{Out}(\pi_1(M^n))$ is a finite group of outer automorphisms, does G lift to act on M^n? Conner and Raymond have studied these questions for some aspherical manifolds [32].

Finally, aspherical orbifolds may be regarded as stratified spaces, without special attention to the origins of the stratification in a group action. Much of the literature on stratified topology is actually concerned with equivariant problems, but foundational work on the topology of stratified spaces and related problems has been done by F. Quinn and B. Hughes, while stratified characteristic classes (especially in intersection homology) have been considered by R. MacPherson, S. Cappell, and J. Shaneson. The book of Goresky and MacPherson [95] serves as a guide to work on stratified spaces which derives from Thom and the algebraic geometers, in addition to making important contributions of its own. Equivariant surgery and related problems are a highly developed subject to which Petrie, Dovermann, Lück, Rothenberg, Schultz, and many others have contributed. Weinberger's notes on stratified spaces [182] are a snapshot of a rapidly developing subject.

11. Remarks on the literature

We begin with three general sources which have been especially useful in preparing this chapter. Two sets of lecture notes present the methods and viewpoint of Farrell, Hsiang, and Jones on the Borel Conjecture [73,50]. The mid-1990s understanding of the Novikov Conjecture is covered very well in the volumes [88,89]; the summary and historical survey [90] in the first volume may be recommended with particular enthusiasm.

A number of tools and problems in manifold topology are discussed in Ferry's book [87] and in the updated version of Kirby's problem list [120]. Ranicki's writings are the main sources on the algebraic theory of surgery and his algebraic reformulations of splitting obstructions should be consulted along with the original sources. Wall's book [175] remains the skeleton on which experts organize almost every discussion of non-simply connected surgery theory, but we still lack a satisfactory treatment of surgery for beginners. This is especially true for surgery in the topological category; see [122, pp. 264–289] for some of the details and see Ranicki's work, especially [149], for the current formulation of topological surgery. Freedman and Quinn should be consulted for four-dimensional surgery [92]. The two volumes of *Surveys on Surgery Theory* (Princeton University Press, 2000 and 2001) are a roadmap to the state of the art in surgery and its applications.

Apart from the work of M. Bökstedt, W.C. Hsiang, and I. Madsen (Theorem 5.1, [14]) on K-theory Novikov conjectures for groups with finitely generated homology in every dimension, the settled cases of the Borel Conjecture or Novikov Conjecture mix topology with geometrically or combinatorially defined classes of groups. See the chapters in this volume on geometric topics in group theory for more information on groups satisfying curvature conditions. Brown's book [18] is recommended for the homological and finiteness properties of groups mentioned above, and a forthcoming book of Geoghegan is probably the best reference on ends of groups.

ENR homology manifolds are discussed in [183] (Chapter 21 of this volume) and [19, 20]. These spaces, like much touched on in this chapter, remain the subject of vigorous investigation.

References

[1] J.F. Adams, *Infinite Loop Spaces*, Annals of Math. Studies, Vol. 90, Princeton University Press, Princeton (1978).

[2] D.R. Anderson, F.X. Connolly, S.C. Ferry and E.K. Pedersen, *Algebraic K-theory with continuous control at infinity*, J. Pure Appl. Alg. **94** (1994), 25–47.

[3] D.R. Anderson and H.J. Munkholm, *Boundedly Controlled Topology*, Lecture Notes in Math., Vol. 1323, Springer, Berlin (1988).

[4] D.V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1960).

[5] O. Attie, J. Block and S. Weinberger, *Characteristic classes and distortion of diffeomorphisms*, J. Amer. Math. Soc. **5** (1992), 919–921.

[6] L. Auslander and F.E.A. Johnson, *On a conjecture of C.T.C. Wall*, J. London Math. Soc. **14** (1976), 331–332.

[7] W. Ballmann, M. Brin and P. Eberlein, *Structure of manifolds of nonpositive curvature. I*, Ann. of Math. **122** (1985), 171–203.

[8] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser, Boston (1985).

[9] H. Bass, A. Heller and R. Swan, *The Whitehead group of a polynomial extension*, Inst. Hautes Études Sci. Publ. Math. **22** (1964), 64–79.

[10] D.J. Benson and P.H. Kropholler, *Cohomology of groups*, Handbook of Algebraic Topology, North-Holland, Amsterdam (1995), 917–950.

[11] T. Bedford, M. Keane and C. Series, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford University Press, Oxford (1991).

[12] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.

[13] R. Bieri, *Gruppen mit Poincaré Dualität*, Comment. Math. Helv. **47** (1972), 373–396.

[14] M. Bökstedt, W.C. Hsiang and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465–540.

[15] A. Borel, *Compact Clifford–Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.

[16] N. Bourbaki, *Groupes et Algèbres de Lie, Ch. 4–6*, Hermann, Paris (1968).

[17] W. Browder, *On the action of* $\theta^n(\partial\pi)$, Differential and Combinatorial Topology, Princeton University Press, Princeton (1965), 23–36.

[18] K.S. Brown, *Cohomology of Groups*, Graduate Texts in Math., Vol. 87, Springer, Berlin (1982).

[19] J.L. Bryant, S.C. Ferry, W. Mio and S. Weinberger, *Topology of homology manifolds*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), 324–328.

[20] J.L. Bryant, S.C. Ferry, W. Mio and S. Weinberger, *Topology of homology manifolds*, Ann. of Math. **143** (1996), 435–467.

[21] S.E. Cappell, *Unitary nilpotent groups and Hermitian K-theory. I*, Bull. Amer. Math. Soc. **80** (1974), 1117–1122.

[22] S.E. Cappell, *Manifolds with fundamental group a generalized free product. I*, Bull. Amer. Math. Soc. **80** (1974), 1193–1198.

[23] S.E. Cappell, *A splitting theorem for manifolds*, Invent. Math. **33** (1976), 69–170.

[24] T.A. Chapman, *Topological invariance of Whitehead torsion*, Amer. J. Math. **93** (1974), 488–497.

[25] T.A. Chapman, *Controlled boundary and h-cobordism theorems*, Trans. Amer. Math. Soc. **280** (1983), 73–95.

[26] T.A. Chapman, *Controlled Simple Homotopy Theory and Applications*, Lecture Notes in Math., Vol. 1009, Springer, Berlin (1983).

[27] T.A. Chapman and S. Ferry, *Approximating homotopy equivalences by homeomorphisms*, Amer. J. Math. **101** (1979), 567–582.

[28] R.M. Charney and M.W. Davis, *Strict hyperbolization*, Topology **34** (1995), 329–350.

[29] M.M. Cohen, *A Course in Simple-Homotopy Theory*, Springer, New York (1973).

[30] E.H. Connell and J. Hollingsworth, *Geometric groups and Whitehead torsion*, Trans. Amer. Math. Soc. **140** (1969), 161–181.

[31] P.E. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds (Athens, GA, 1969), J.C. Cantrell and C.H. Edwards, Jr., eds, Markham, Chicago (1970), 227–264.

[32] P.E. Conner and F. Raymond, *Deforming homotopy equivalences to homeomorphisms in aspherical manifolds*, Bull. Amer. Math. Soc. **83** (1977), 36–85.

[33] F.X. Connolly and M.O.M. Da Silva, *The groups* $N^r K_0(\mathbb{Z}\pi)$ *are finitely generated* $\mathbb{Z}[\mathbb{N}^r]$ *modules if* π *is a finite group*, K-Theory **9** (1995), 1–11.

[34] F. Connolly and T. Koźniewski, *Finiteness properties of classifying spaces of proper* Γ*-actions*, J. Pure Appl. Alg. **41** (1986), 17–36.

[35] F. Connolly and T. Koźniewski, *Rigidity and crystallographic groups, I*, Invent. Math. **99** (1990), 25–48.

[36] F. Connolly and T. Koźniewski, *Examples of lack of rigidity in crystallographic groups*, Algebraic Topology, Poznań 1989, S. Jackowski, B. Oliver and K. Pawalowski, eds, Lecture Notes in Math., Vol. 1474, Springer, Berlin (1991), 139–145.

[37] C.B. Croke, P. Eberlein and B. Kleiner, *Conjugacy and rigidity for nonpositively curved manifolds of higher rank*, Topology **35** (1996), 273–286.

[38] M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Annals of Math. **117** (1983), 293–324.

[39] M.W. Davis, *Coxeter groups and aspherical manifolds*, Algebraic Topology, Aarhus 1982, I. Madsen, B. Oliver, eds, Lecture Notes in Math., Vol. 1051, Springer, Berlin (1984), 197–221.

[40] M. Davis and J.-C. Hausmann, *Aspherical manifolds without smooth or PL structures*, Algebraic Topology (Arcata, 1986), Lecture Notes in Math., Vol. 1370, Springer, New York (1989), 135–142.

[41] M.W. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. **34** (1991), 347–388.

[42] A.W.M. Dress, *Induction and structure theorems for orthogonal representations of finite groups*, Annals of Math. **102** (1975), 291–325.

[43] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–109.

[44] B. Eckmann and P. Linnell, *Poincaré duality groups of dimension two II*, Comment Math. Helv. **58** (1983), 111–114.

[45] B. Eckmann and H. Müller, *Poincaré duality groups of dimension two*, Comment Math. Helv. **55** (1980), 510–520.

[46] D. Epstein and M. Shub, *Expanding endomorphisms of flat manifolds*, Topology **7** (1968), 139–141.

[47] F.T. Farrell, *The obstruction to fibering a manifold over a circle*, Indiana Univ. Math. J. **21** (1971/1972), 315–346.

[48] F.T. Farrell, *The nonfiniteness of Nil*, Proc. Amer. Math. Soc. **65** (1977), 215–216.

[49] F.T. Farrell, *The exponent of UNil*, Topology **18** (1979), 305–312.

[50] F.T. Farrell, *Surgical Methods in Rigidity*, Tata Institute, Bombay (1996).

[51] F.T. Farrell and W.C. Hsiang, *Manifolds with $\pi_1 = G \times_\alpha T$*, Amer. J. Math. **95** (1973), 813–848.

[52] F.T. Farrell and W.C. Hsiang, *The topological-Euclidean space form problem*, Invent. Math. **45** (1978), 181–192.

[53] F.T. Farrell and W.C. Hsiang, *The Whitehead group of poly-(finite or cyclic) groups*, J. London Math. Soc. (2) **24** (1981), 308–324.

[54] F.T. Farrell and W.C. Hsiang, *On Novikov's conjecture for nonpositively curved manifolds. I*, Ann. of Math. **113** (1981), 199–209.

[55] F.T. Farrell and W.C. Hsiang, *The stable topological-hyperbolic space form problem for complete manifolds of finite volume*, Invent. Math. **69** (1982), 155–170.

[56] F.T. Farrell and W.C. Hsiang, *A structure set analogue of Chapman–Ferry–Quinn theory*, Continua, Decompositions, Manifolds (Austin, TX, 1980), Univ. Texas Press, Austin (1983), 172–190.

[57] F.T. Farrell and W.C. Hsiang, *Topological characterization of flat and almost flat Riemannian manifolds M^n ($n \neq 3, 4$)*, Amer. J. Math. **105** (1983), 641–672.

[58] F.T. Farrell and L.E. Jones, *Algebraic K-theory of hyperbolic manifolds*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 115–119.

[59] F.T. Farrell and L.E. Jones, *Algebraic K-theory of spaces stratified fibered over hyperbolic orbifolds*, Proc. Nat. Acad. Sci. USA **83** (1986), 5364–5366.

[60] F.T. Farrell and L.E. Jones, *h-cobordisms with foliated control*, Bull. Amer. Math. Soc. (N.S.) **15** (1986), 69–72. *Erratum*, Bull. Amer. Math. Soc. (N.S.) **16** (1987), 177.

[61] F.T. Farrell and L.E. Jones, *K-theory and dynamics. I*, Ann. of Math. **124** (1986), 531–569.

[62] F.T. Farrell and L.E. Jones, *K-theory and dynamics. II*, Ann. of Math. **126** (1987), 451–493.

[63] F.T. Farrell and L.E. Jones, *Algebraic K-theory of discrete subgroups of Lie groups*, Proc. Nat. Acad. Sci. USA **84** (1987), 3095–3096.

[64] F.T. Farrell and L.E. Jones, *Foliated control with hyperbolic leaves*, K-Theory **1** (1987), 337–359.

[65] F.T. Farrell and L.E. Jones, *Foliated control theory. I, II*, K-Theory **2** (1988), 357–430.

[66] F.T. Farrell and L.E. Jones, *The surgery L-groups of poly-(finite or cyclic) groups*, Invent. Math. **91** (1988), 559–586.

[67] F.T. Farrell and L.E. Jones, *Topological rigidity for hyperbolic manifolds*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 277–282.

[68] F.T. Farrell and L.E. Jones, *Implication of the geometrization conjecture for the algebraic K-theory of 3-manifolds*, Geometry and Topology (Athens, GA, 1985), C. McCrory and T. Shifrin, eds, Dekker, New York (1987), 109–113.

[69] F.T. Farrell and L.E. Jones, *A topological analogue of Mostow's rigidity theorem*, J. Amer. Math. Soc. **2** (1989), 257–370.

[70] F.T. Farrell and L.E. Jones, *Compact negatively curved manifolds (of dim $\neq 3, 4$) are topologically rigid*, Proc. Nat. Acad. Sci. USA **86** (1989), 3461–3463.
[71] F.T. Farrell and L.E. Jones, *Negatively curved manifolds with exotic smooth structures*, J. Amer. Math. Soc. **2** (1989), 899–908.
[72] F.T. Farrell and L.E. Jones, *Rigidity and other topological aspects of compact nonpositively curved manifolds*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 59–64.
[73] F.T. Farrell and L.E. Jones, *Classical Aspherical Manifolds*, CBMS Regional Conference Series in Mathematics, Vol. 75, Amer. Math. Soc., Providence, RI (1990).
[74] F.T. Farrell and L.E. Jones, *Foliated control without radius of injectivity restrictions*, Topology **30** (1991), 117–142.
[75] F.T. Farrell and L.E. Jones, *Computations of stable pseudoisotopy spaces for aspherical manifolds*, Algebraic Topology Poznań 1989, Lecture Notes in Math., Vol. 1474, Springer, Berlin (1991), 59–74.
[76] F.T. Farrell and L.E. Jones, *Rigidity in geometry and topology*, Proceedings of the International Congress of Mathematicians (Kyoto, 1990), Math. Soc. Japan, Tokyo (1991), 653–663.
[77] F.T. Farrell and L.E. Jones, *Stable pseudoisotopy spaces of compact nonpositively curved manifolds*, J. Differential Geom. **34** (1991), 769–834.
[78] F.T. Farrell and L.E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249–297.
[79] F.T. Farrell and L.E. Jones, *Nonuniform hyperbolic lattices and exotic smooth structures*, J. Differential Geom. **38** (1993), 235–261.
[80] F.T. Farrell and L.E. Jones, *Topological rigidity for compact nonpositively curved manifolds*, Differential Geometry, Proc. Symp. Pure. Math., Vol. 54, Part 3, Amer. Math. Soc., Providence, RI (1993), 229–274.
[81] F.T. Farrell and L.E. Jones, *Complex hyperbolic manifolds and exotic smooth structures*, Invent. Math. **117** (1994), 57–74.
[82] F.T. Farrell and L.E. Jones, *The lower algebraic K-theory of virtually infinite cyclic groups*, K-Theory **9** (1995), 13–30.
[83] F.T. Farrell and L.E. Jones, *Some non-homeomorphic harmonic homotopy equivalences*, Bull. London Math. Soc. **28** (1996), 177–182.
[84] S. Ferry, *The homeomorphism group of a compact Hilbert cube manifold is an ANR*, Ann. of Math. **106** (1977), 101–119.
[85] S. Ferry, *Homotoping ϵ-maps to homeomorphisms*, Amer. J. Math. **101** (1979), 567–582.
[86] S. Ferry, *Controlled Topology and the Characterization of Topological Manifolds*, NSF-CBMS Conference Lecture Notes, Knoxville (1994).
[87] S.C. Ferry, *Topological Manifolds and Polyhedra*, Oxford University Press, to appear.
[88] S.C. Ferry, A. Ranicki and J. Rosenberg, eds, *Novikov Conjectures, Index Theorems and Rigidity, Vol. 1*, London Math. Soc. Lecture Notes, Vol. 226, Cambridge University Press, Cambridge (1995).
[89] S.C. Ferry, A. Ranicki and J. Rosenberg, eds, *Novikov Conjectures, Index Theorems and Rigidity, Vol. 2*, London Math. Soc. Lecture Notes, Vol. 227, Cambridge University Press, Cambridge (1995).
[90] S.C. Ferry, A. Ranicki and J. Rosenberg, *A history and survey of the Novikov conjecture*, Novikov Conjectures, Index Theorems and Rigidity, Vol. 1, S.C. Ferry, A. Ranicki and J. Rosenberg, eds, London Math. Soc. Lecture Notes, Vol. 226, Cambridge University Press, Cambridge (1995), 7–66.
[91] M.H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.
[92] M.H. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton Mathematical Series, Vol. 39, Princeton University Press, Princeton (1990).
[93] M.H. Freedman and Z. Wang, *Controlled linear algebra*, Prospects in Topology, F. Quinn, ed., Annals of Math. Studies, Vol. 138, Princeton University Press, Princeton (1995), 138–156.
[94] K. Fukaya and T. Yamaguchi, *Almost nonpositively curved manifolds*, J. Differential Geom. **33** (1991), 67–90.
[95] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Springer, Berlin (1988).
[96] J.A. Green, *Axiomatic representation theory for finite groups*, J. Pure Appl. Algebra **1** (1971), 41–77.
[97] M. Gromov, *Almost flat manifolds*, J. Differential Geom. **13** (1978), 231–241.
[98] M. Gromov, *Hyperbolic groups*, Essays in Group Theory, S.M. Gersten, ed., Math. Sci. Res. Inst. Publ., Vol. 8, Springer, New York (1987), 75–263.

[99] M. Gromov and I. Piatetski-Shapiro, *Non-arithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. **66** (1988), 93–103.

[100] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987), 1–12.

[101] J. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), 157–176.

[102] W.J. Harvey, *Boundary structure for the modular group*, Riemann Surfaces and Related Topics (Stony Brook, 1978), Ann. of Math. Studies, Vol. 97, Princeton University Press, Princeton (1978), 245–251.

[103] W.J. Harvey, *Geometric structure of surface mapping class groups*, Homological Group Theory, C.T.C. Wall, ed., London Math. Soc. Lecture Note Series, Vol. 36, Cambridge University Press, Cambridge (1979), 255–269.

[104] A. Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, Algebraic and Geometric Topology (Stanford, 1976), Proc. Symp. Pure Math., Vol. 32, Amer. Math. Soc., Providence, RI (1978), 2–21.

[105] A. Hatcher and J. Wagoner, *Pseudoisotopies of compact manifolds*, Astérisque **6** (1973).

[106] J. Hempel, 3-*Manifolds*, Annals of Mathematics Studies, Vol. 86, Princeton University Press, Princeton (1976).

[107] J.A. Hillman, *The Algebraic Characterization of* 4-*Manifolds*, London Math. Soc. Lecture Note Series, Vol. 198, Cambridge University Press, Cambridge (1994).

[108] W.C. Hsiang, *Borel's conjecture, Novikov's conjecture and the K-theoretic analogues*, Algebra, Analysis and Geometry (Taipei, 1988), World Scientific, Teaneck, NJ (1989), 39–58.

[109] W.C. Hsiang and J. Shaneson, *Fake tori*, Topology of Manifolds (Athens, GA, 1969), J.C. Cantrell and C.H. Edwards, Jr., eds, Markham, Chicago (1970), 18–51.

[110] W.-C. Hsiang and C.T.C. Wall, *On homotopy tori II*, Bull. London Math. Soc. **1** (1969), 341–342.

[111] B. Hu, *Whitehead groups of finite polyhedra with nonpositive curvature*, J. Differential Geom. **38** (1993), 501–517.

[112] B. Hu, *Relations between simplicial and manifold nonpositive curvature*, Geometric Group Theory (Proceedings of a Special Research Quarter at The Ohio State University, Spring 1992), R. Charney, M. Davis and M. Shapiro, eds, Ohio State University Math. Research Institute Publications, No. 3, de Gruyter, Berlin (1995).

[113] B. Hughes and A. Ranicki, *Ends of Complexes*, Cambridge Tracts in Mathematics, No. 123, Cambridge University Press, Cambridge (1996).

[114] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990).

[115] F.E.A. Johnson, *On the realisability of polysurface groups*, J. Pure Appl. Algebra **15** (1979), 235–241.

[116] F.E.A. Johnson and C.T.C. Wall, *On groups satisfying Poincaré duality*, Ann. of Math. **96** (1972), 592–598.

[117] Y. Kamishima, K.B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, Quart. J. Math. Oxford (2) **34** (1983), 433–452.

[118] A. Katok and R.J. Spatzier, *First cohomology of Anosov actions of higher rank Abelian groups and applications to rigidity*, Institut des Hautes Études Scientifiques. Publications Mathématiques/ **79** (1994), 131–156.

[119] M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. of Math. **77** (1963), 504–537.

[120] R. Kirby, *Problems in low-dimensional topology*, Geometric Topology (Athens, GA, 1993), AMS/IP Studies in Advances Mathematics, No. 2, Part 2, Amer. Math. Soc./International Press, Providence, RI (1997), 35–473.

[121] R.C. Kirby and L.C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742–749.

[122] R.C. Kirby and L.C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Mathematics Studies, Vol. 88, Princeton University Press, Princeton (1977).

[123] K.B. Lee and F. Raymond, *Geometric realization of group extensions by the Seifert construction*, Combinatorics and Algebra, C. Greene, ed., Contemporary Math., Vol. 34, Amer. Math. Soc., Providence, RI (1984), 353–411.

[124] K.B. Lee and F. Raymond, *The role of Seifert fiber spaces in transformation groups*, Group Actions on Manifolds, R. Schultz, ed., Contemporary Math., Vol. 36, Amer. Math. Soc., Providence, RI (1985), 367–425.

[125] J.-L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 309–377.

[126] A. Lubotzky and A.R. Magid, *Varieties of representations of finitely presented groups*, Mem. Amer. Math. Soc. **336** (1985).

[127] I. Madsen, C.B. Thomas and C.T.C. Wall, *The topological spherical form problem – II. Existence of free actions*, Topology **15** (1976), 375–382.

[128] A. Malcev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32. Amer. Math. Soc. Transl. Series 1, Vol. 9 (1962), 277–307.

[129] G.A. Margulis, *Discrete groups of motions of manifolds of non-positive curvature*, Proc. Int. Congress. Math. (Vancouver, 1974), Amer. Math. Soc. Transl. **109** (1977), 33–45.

[130] G.A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer, Berlin (1991).

[131] G. Mess, *Examples of Poincaré duality groups*, Proc. Amer. Math. Soc. **110** (1990), 1145–1146.

[132] J.W. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. **64** (1956), 399–405.

[133] J.W. Milnor, *Hilbert's Problem* 18: *On crystallographic groups, fundamental domains, and on sphere packing*, Mathematical Developments Arising From Hilbert Problems, Proc. Symp. Pure Math. 28, Amer. Math. Soc., Providence, RI (1976), 491–506.

[134] G.D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. **60** (1954), 1–27.

[135] G.D. Mostow, *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.

[136] G.D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Annals of Math. Studies, Vol. 78, Princeton University Press, Princeton (1973).

[137] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. Thesis, Ohio State University (1988).

[138] A.J. Nicas, *Induction theorems for groups of homotopy manifold structures*, Mem. Amer. Math. Soc. **267** (1982).

[139] A.J. Nicas and C.W. Stark, *K-theory and surgery of codimension-two torus actions on aspherical manifolds*, J. London Math. Soc. (2) **31** (1985), 173–183.

[140] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer, Berlin (1992).

[141] E.K. Pedersen and C. Weibel, *A nonconnective delooping of algebraic K-theory*, Algebraic and Geometric Topology (Rutgers, 1983), A. Ranicki, N. Levitt and F. Quinn, eds, Lecture Notes in Math., Vol. 1126, Springer, Berlin (1985), 166–181.

[142] S. Prassidis and B. Spieler, *Rigidity of Coxeter Groups*, Preprint, Vanderbilt University (1996).

[143] F. Quinn, *Ends of maps*, I, Ann. of Math. **110** (1979), 275–331.

[144] F. Quinn, *Ends of maps*, II, Invent. Math. **68** (1982), 353–424.

[145] F. Quinn, *Ends of maps*. III: *Dimensions* 4 *and* 5, J. Differential Geom. **17** (1982), 503–521.

[146] F. Quinn, *Geometric algebra*, Algebraic and Geometric Topology (Rutgers, 1983), A. Ranicki, N. Levitt and F. Quinn, eds, Lecture Notes in Math., Vol. 1126, Springer, Berlin (1985), 182–198.

[147] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergebnisse der Math., Vol. 68, Springer, Berlin (1972).

[148] A.A. Ranicki, *Exact Sequences in the Algebraic Theory of Surgery*, Princeton University Press, Princeton (1981).

[149] A.A. Ranicki, *Algebraic L-Theory and Topological Manifolds*, Cambridge Tracts in Mathematics, Vol. 102, Cambridge University Press, Cambridge (1992).

[150] A. Ranicki and M. Yamasaki, *Controlled K-Theory*, Topology Appl. **61** (1995), 1–59.

[151] F. Raymond and D. Wigner, *Construction of aspherical manifolds*, Geometric Applications of Homotopy Theory, Lecture Notes in Math., Vol. 657, Springer, Berlin (1978), 408–422.

[152] D. Rolfsen, *Knots and Links*, Publish or Perish, Wilmington, DE (1976).

[153] E. Rosas, *Rigidity theorems for right angled reflection groups*, Trans. Amer. Math. Soc. **308** (1988), 837–848.

[154] T.B. Rushing, *Topological Embeddings*, Pure and Applied Math., Vol. 52, Academic Press, London (1973).

[155] A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to Function Theory (Int. Colloq. on Function Theory, Bombay, 1960), Tata Institute, Bombay (1960).

[156] J.-P. Serre, *Cohomologie des groupes discrets*, Prospects in Mathematics, Ann. of Math. Studies, Vol. 70, Princeton University Press, Princeton (1971), 77–169.

[157] J.-P. Serre, *Arithmetic groups*, Homological Group Theory, C.T.C. Wall, ed., London Math. Soc. Lecture Note Series, Vol. 36, Cambridge University Press, Cambridge (1979), 105–136.
[158] J.-P. Serre, *Trees*, Springer, Berlin (1980).
[159] J.L. Shaneson, *Wall's surgery obstruction groups for* $\mathbb{Z} \times G$, Ann. of Math. **90** (1969), 296–334.
[160] L. Siebenmann, *A total Whitehead obstruction to fibering over the circle*, Comment. Math. Helv. **45** (1970), 1–48.
[161] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. **74** (1961), 391–406.
[162] J. Stallings, *On fibering certain* 3*-Manifolds*, Topology of 3-Manifolds and Related Topics, M.K. Fort, Jr., ed., Prentice-Hall, Englewood Cliffs, NJ (1962), 95–100.
[163] J. Stallings, *The piecewise linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.
[164] C.W. Stark, *L-theory and graphs of free Abelian groups*, J. Pure Appl. Algebra **47** (1987), 299–309.
[165] C.W. Stark, *A characterization of virtual Poincaré duality groups*, Michigan Math. J. **42** (1995), 99–102.
[166] C.B. Thomas and C.T.C. Wall, *The topological spherical space form problem I*, Compositio Math. **23** (1971), 101–114.
[167] V.G. Turaev, *Three-dimensional Poincaré complexes*, Math. USSR Sbornik **67** (1990), 261–282.
[168] E.B. Vinberg, *Hyperbolic reflection groups*, Russian Math. Surveys **40** (1985), 31–75.
[169] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. **87** (1968), 56–88.
[170] F. Waldhausen, *Algebraic K-theory of generalized free products*, Ann. of Math. **108** (1978), 135–256.
[171] F. Waldhausen, *Algebraic K-theory of topological spaces*, I, Algebraic and Geometric Topology (Stanford, 1976), Proc. Symp. Pure Math., Vol. 32, Amer. Math. Soc., Providence, RI (1978), 35–60.
[172] F. Waldhausen, *Algebraic K-theory of spaces, a manifold approach*, Current Trends in Algebraic Topology, Part 1 (London, Ontario, 1981), Canad. Math. Soc. Conf. Proc., Vol. 2, Amer. Math. Soc., Providence, RI (1982), 141–184.
[173] C.T.C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math. **81** (1965), 56–69.
[174] C.T.C. Wall, *On homotopy tori and the annulus theorem*, Bull. London Math. Soc. **1** (1969), 95–97.
[175] C.T.C. Wall, *Surgery on Compact Manifolds*, Academic Press, London (1970).
[176] C.T.C. Wall, *The topological space-form problems*, Topology of Manifolds (Athens, GA, 1969), J.C. Cantrell and C.H. Edwards, Jr., eds, Markham, Chicago (1970), 319–331.
[177] C.T.C. Wall, *List of problems*, Homological Group Theory, C.T.C. Wall, ed., LMS Lecture Notes, Vol. 36, Cambridge University Press, Cambridge (1979), 369–394.
[178] A. Weil, *Discrete subgroups of Lie groups* I, Ann. of Math. **72** (1960), 369–384.
[179] A. Weil, *Discrete subgroups of Lie groups* II, Ann. of Math. **75** (1962), 578–602.
[180] A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.
[181] S. Weinberger, *On fibering four- and five-manifolds*, Israel J. Math. **59** (1987), 1–7.
[182] S. Weinberger, *The Topological Classification of Stratified Spaces*, University of Chicago Press, Chicago (1994).
[183] S. Weinberger, Homology manifolds, Handbook of Geometric Topology, R.J. Daverman and R.B. Sher, eds, Elsevier, Amsterdam (2002), 1085–1102.
[184] J.H.C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. **6** (1935), 268–279.
[185] J.A. Wolf, *Spaces of Constant Curvature*, (4th edn), Publish or Perish, Berkeley (1977).
[186] M. Yamasaki, *L-groups of crystallographic groups*, Invent. Math. **88** (1987), 571–602.
[187] R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Math., Vol. 81, Birkhäuser, Boston (1984).