# PARALLEL TRANSPORT AND CLASSIFICATION OF FIBRATIONS

by

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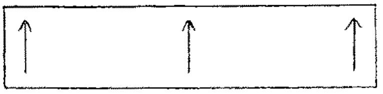
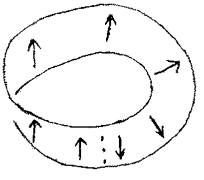
The simplest example of parallel transport is the field of (parallel) vertical vectors on $I \times I$:



and the simplest non-trivial example occurs when we form this strip into a Moebius band:



clearly distinguishing the Moebius band from the cylinder.

The idea of parallel transport originates in differential geometry where geometric structure such as curvature is revealed by parallel

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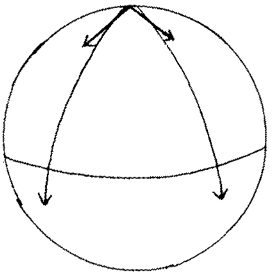
transporting tangent vectors along curves:



Essentially the same idea occurs in covering space theory where a loop in the space covered determines a deck transformation or permutation of the sheets of the covering. [Veblen and Whitehead] suggested the greater generality of fibre bundles as a setting. We shall look at fiber spaces as well.

We begin formally.

Provisional Definition: For a fibre space $F \longrightarrow E \longrightarrow X$, a (parallel) transport is a map

$$\tau : F \times \Omega X \longrightarrow F$$

such that

1) the trivial loop acts as the identity

2) each loop acts as a homotopy equivalence

3) $\tau$ is transitive (i.e., $\tau(f, \lambda + \mu) = \tau(\tau(f,\lambda),\mu)$) or reasonably close to it.

Classically and intuitively we would expect strict transitivity: transporting the fibre around one loop and then another should be the same as transporting it around the sum of the two loops. For fibre spaces we lack such precision as we can see by constructing $\tau$ from the Covering Homotopy Property.

Consider

$$\begin{array}{ccc} F \times \Omega X & \xrightarrow{f_o} & E \\ \downarrow & & \downarrow \\ F \times \Omega X & \xrightarrow[g_t]{} & X \end{array}$$

where $f_o(y,\lambda) = y$ and $g_t(y,\lambda) = \lambda(t)$. The CHP gives us

$f_t: F \times \Omega X \longrightarrow E$ with $f_1: F \times \Omega X \longrightarrow F$; in fact, we can assume

$f_t(y,e) = y$ where $e$ is the trivial loop. We set $\tau = f_1$ and achieve 1 and 2.

The lifting $f_t$ is not unique, but any two are homotopic. (They are homotopic within $E$ to $f_o$ by a homotopy whose image in $X$ is homotopy trivial and thus the homotopy can be deformed to be fibre preserving, i.e., $f_1$ and $f_1'$ are homotopic in $F$.) The same reason applies to show $\tau(\tau \times 1) \simeq \tau \quad (1 \times m): F \times \Omega X \times \Omega X \longrightarrow F$ where $m$ is loop addition [Hilton].

One can in fact say more, but we need a language with which to say it. One approach is to consider the adjoint map $\operatorname{ad} \tau : \Omega X \longrightarrow F^F$. (We will not worry about the function space topology but rather always use continuity in reference to $\tau$ rather than $\operatorname{ad} \tau$). The transitivity of $\tau$ is equivalent to the multiplicativity of $\operatorname{ad} \tau$. The homotopy condition above is equivalent to $\operatorname{ad} \tau$ being an H - map. In general for maps of one associative H - space to another we have the notion of strong homotopy multiplicativity.

<u>Definition</u>. If $Y$ and $Z$ are topological monoids, a map $f: Y \longrightarrow Z$ is s.h.m. (strongly homotopy multiplicative) if any of the following conditions are satisfied:

a) There exist maps $f_n : Y^n \times I^{n-1} \longrightarrow Z$ such that $f_1 = f$ and

$$f_n(y_1, \cdots, y_n, t_1, \cdots, t_{n-1}) =$$

$$f_{n-1}(\cdots, y_i y_{i+1}, \cdots, \hat{t}_i, \cdots) \quad \text{if} \quad t_i = 0$$

$$f_i(y_1, \cdots, y_i, t_1, \cdots, t_{i-1}) \cdot f_{n-i-1}(y_{i+1}, \cdots, y_n, t_{i+1}, \cdots, t_{n-1})$$

$$\text{if} \quad t_i = 1 .$$

b) $Sf : SY \longrightarrow SZ$ extends to $BY \longrightarrow BZ$.

c) There exists a commutative diagram

$$\begin{tikzcd} WY \arrow[d] \arrow[dr, "h"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

where $WY \longrightarrow Y$ is the standard retraction [Floyd] and $h$ is a homomorphism.

d) $f$ can be factored up to homotopy as $Y \longrightarrow Y_1 \longleftarrow Y_2 \longrightarrow \cdots \longrightarrow Z$ where the $Y_i$ are also monoids and the maps are homomorphisms and the maps $Y_{2i} \longrightarrow Y_{2i-1}$ are homotopy equivalences.

In particular we can ask if $\operatorname{ad} \tau : \Omega X \longrightarrow F^F$ is shm. Repeated use of the CHP provides the adjoint maps

4) $$\tau_n : F \times (\Omega X)^n \times I^{n-1} \longrightarrow F$$

as desired. Details are given in [10]. The significance of these maps is that they completely determine the fibration as we now indicate.

Let us back up a little. If a group $G$ acts on a space $Y$, we can look at the orbit space $Y/G$. If $G \longrightarrow Y \longrightarrow Y/G$ is not a principal $G$-bundle, we can replace it, up to homotopy, by one, namely $G \longrightarrow EG \times Y \longrightarrow EG \times_G Y = Y_G$ where $EG$ is the universal (contractible) $G$-bundle.

For any fibre space $F \longrightarrow E \longrightarrow X$, we have the fibration (up to homotopy) $\Omega X \longrightarrow F \longrightarrow E$ which suggests trying to identify $E$ as $F_{\Omega X}$

in some sense. The lack of transitivity is a problem, so let us look at $Y_G$ in more detail. One way of describing $Y_G$ is a realization of the simplicial space

$$\begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} Y \times G \times G \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} Y \times G \ \underset{\text{proj}}{\overset{\text{action}}{\rightrightarrows}} \ Y \ .$$

In May's notation, the realization is $B(Y,G,*)$, though we have not mentioned degeneracies and prefer to avoid their use, cf. [7].

Now suppose that we have a sh - action of a monoid $G$ on $Y$ (i.e., maps $m_n : Y \times G^n \times I^{n-1} \longrightarrow Y$ adjoint to an shm - map). Form $\coprod_{n \geq 0} Y \times G^n \times I^n$ and factor by the following equivalence relation:

$$\begin{aligned} (y,g_1,\cdots,g_n,t_1,\cdots,t_n) &\sim (\cdots,g_i g_{i+1},\cdots,\hat{t}_i,\cdots) && \text{if} \quad t_i = 0 \\ &\sim (m_i(y,\cdots,g_i,t_1,\cdots,t_{i-1})g_{i+1},\cdots,g_n,t_{i+1},\cdots) && \text{if} \quad t_i = 1 \end{aligned}$$

Again call the result $Y_G$ or $B(Y,G,*)$.

In particular all this applies to a transport $\tau$ for $F \longrightarrow E \longrightarrow B$.

<u>Theorem</u>: Let $\{\tau_i\}$ be a family of maps satisfying 1), 2), 3) and 4). The map $B(F, \Omega X,*) \longrightarrow B(*, \Omega X,*) = B\Omega X$ is a quasifibration with fibre $F$. (With extra connective tissue, Fuchs has been able to

build an equivalent Dold fibration [3].)

If $\tau_i$ is obtained from $F \to E \to B$ using the CHP as indicated above, then $E \to B$ is weakly fibre homotopy equivalent to $B(F, \Omega X, *) \to B(*, \Omega X, *)$.

If $\{\tau_i\}$ is arbitrary as above and $\{\tau_i'\}$ is constructed from $B(F, \Omega X, *) \to B(*, \Omega X, *)$ using the CHP, then $\{\tau_i\}$ is homotopic to $\{\tau_i'\}$.

Thus $\{\tau_i\}$ is a complete invariant of $E \to B$; homotopy classes of transports classify fibrations.

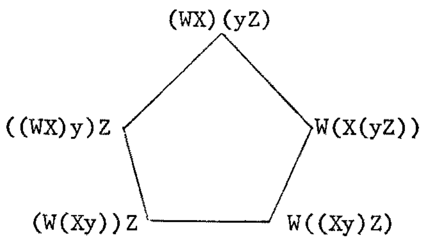
The usual way of classifying fibrations is by homotopy classes of maps $X \to BH(F)$ where $H(F)$ is the monoid of self-homotopy equivalences of $F$. Now $\{\mathrm{ad}\,\tau_i\}$ is an shm-map of $\Omega X$ into $H(F)$ and hence induces a map at the $B$ level. We have thus

$$X \simeq B\Omega X \xrightarrow{B\mathrm{ad}\tau} BH(F)$$

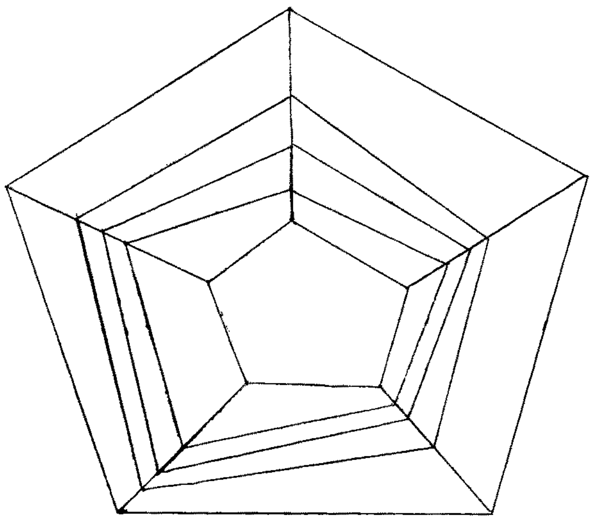
<u>Theorem</u>. For a suitable choice of the equivalence $X \simeq B\Omega X$, the classifying map above is the usual one [11].

Here we should note that we assume $X$ has the homotopy type of a CW-complex in order to assert $X \simeq B\Omega X$. I am unaware of any study of more general topological conditions (e.g., perhaps weakly locally contractible and paracompact) which would guarantee the same equivalence.

Remarks on operads: Within the context of this conference, it is appropriate to mention the relation between the structures we have been studying and the concept of operads. Our transport $\{\tau_i\}$ is a collection of higher homotopies i.e., maps $F \times (\Omega X)^n \times I^{n-1} \to F$, whereas an operad action is of the form $Y^n \times M(n) \longrightarrow Y$, where $M(n)$ is a parameter space frequently more complicated than a cube, though often contractible in cases of current interest. An "ancient" example are my complexes $K_n$ e.g., $K_3 = I$ but $K_4$ is a pentagon




and $K_5$ a polyhedron with 6 pentagonal and 3 quadrilateral faces. Malraison has a function space equivalent of $K_n$, readily described in terms of maps $[0,1] \xrightarrow{f} [0,n]$. Thinking of $f^{-1}(i)$ as dividing $[0,1]$ into $n$ pieces, we can see the relevance to loop spaces by using loops parameterized from 0 to 1 and the classical addition of loops. The corresponding $K_n$ structures can be pictured



One reason for studying $\{K_n\}$-spaces rather than strict monoids is that the definition is homotopy invariant. If $X \simeq Y$ and $X$ is a monoid, $Y$ need not admit an equivalent monoid structure (cf. Exotic multiplications on $S^3$ [Slifker]) but $Y$ will admit an equivalent $\{K_n\}$-structure (usually called strongly homotopy associative - s.h.a.).

Now recall that an operad is, among other things, a category; where defined, composition is associative. It makes sense to talk of $M \longrightarrow \mathrm{End}\, X$ being shm rather than a strict morphism, a sh-functor rather than a strict functor. Again if $X \simeq Y$ and $X$ is an $M$-space, then $Y$ is at least an sh-$M$-space (Lada). Alternatively Boardman asserts $Y$ is a $WM$-space where $WM$ is his construction, presented here by Floyd.

Floyd has also pointed out that a WM - space $X$ can be replaced up to homotopy by an M- space. Lada has given an alternative description of this process, namely $B(M,M,X)$ where $B$ is constructed using cubes as above. Actually Lada, following May, usedthe associated triple $MX$ which is just the free gadget

$$MX = \coprod M(n) \times_{\Sigma_n} X^n/\sim$$

where the equivalence is given entirely in terms of degeneracies $d_i : M(n) \longrightarrow M(n-1)$ corresponding to $X^{n-1} \longrightarrow X^n$ by inserting the base point in the i-th coordinate.

In comparing operads by morphisms $M \longrightarrow M^1$ which are homotopy equivalences on each component, we find the inverse maps $M^1 \longrightarrow M$ are at least shm. Finally since operads have associative compositions, we can generalize to sh - operads having operads act on operads.

Since the conference, I have seen work of Segal in which he has related $E_\infty$ - $\Sigma$ - operads to his $\Gamma$ - structures and given an alternate approach to the last two paragraphs using essentially form d above for handling sh - morphisms.

To come back to more concrete objects, I will consider briefly the "local" approach to classification. Here local refers to structure defined on a space in terms of an open cover $\{U_\alpha\}$. For example, a fibre bundle $p:E \longrightarrow B$ is defined in terms of local product structures:

$$\begin{tikzcd}[column sep=tiny] p^{-1}(U_\alpha) \arrow[dr] & \equiv & U_\alpha \times F \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

A fibre space over a nice base [1] can be defined in terms of local equivalences:

$$\begin{tikzcd}[column sep=tiny] p^{-1}(U_\alpha) \arrow[dr] & \simeq & U_\alpha \times F \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

A foliation is defined in terms of special local coordinates:

$$U_\alpha \equiv R^k \times R^{n-k}$$

Now an open cover $\{U_\alpha\}$ gives rise to a simplicial space $U$:

$$\begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} \{U_\alpha \cap U_\beta \cap U_\gamma\}_{\alpha,\beta,\gamma} \;\begin{smallmatrix}\to\\\to\\\to\end{smallmatrix}\; \{U_\alpha \cap U_\beta\}_{\alpha,\beta} \;\begin{smallmatrix}\to\\\to\end{smallmatrix}\; \{U_\alpha\}$$

where all intersections are non-empty. (If desired, think of $\{U_\alpha\}$ as a category $U$ with $\mathrm{Ob}U = \coprod U_\alpha$, the disjoint union and $\mathrm{Mor}\, U$ given by $\mathrm{Mor}\,(x \in U_\alpha, y \in U_\beta) = \emptyset$ unless $x = y$ in which case $\mathrm{Mor}\,(x,x) = x$). There is an obvious map $BU = |U| \longrightarrow X$ which if $X$ is paracompact is a homotopy equivalence. (The pictures in [9] are quite indicative.)

Now local structures of the sorts considered above imply compatability on the overlaps.

For example, a fibre bundle involves transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$ (where $G$ is the group of the bundle) such that on $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

This is a morphism $U \longrightarrow G$ and hence induces $X \simeq BU \longrightarrow BG$. Classification can be verified directly if we choose the appropriate realization, namely Milnor's which has built in a nice "universal" open cover.

To be precise, for a category $C$, consider the subset of

$$BC \subset \Delta^\infty \times C^\infty$$

consisting of pairs $(\vec{t}, \{g_{ij}\})$ s/t $\vec{t} \in \Delta^\infty$ and $i,j$ runs over all pairs such that $t_i t_j \neq 0$, $g_{ij} \in \operatorname{Mor} C$ except $g_{ii} \in \operatorname{Ob} C$ and if $t_i t_j t_k \neq 0$ then $g_{ij}g_{jk} = g_{ik}$. Topologize this space by the limit of the quotient topologies of the maps $\Delta^n \times C^{[n]} \longrightarrow BC$ where $C^{[n]}$ denotes composable $n$-tuples and the map is given by

$$(s_0, \cdots, s_n, g_1, \cdots, g_n) \longrightarrow (\vec{t}, \{g_{ij}\})$$

where $t_{k_j} = s_j$ for some $k_0 < k_1 < \cdots < k_n$ and $g_{k_i k_j} = g_{i+1} \cdots g_j$.

The universal cover of $BU$ is given by $\{t_i^{-1}(0,1]\}$ and the $g_{ij}$ coordinates regarded as functions $U_i \cap U_j \longrightarrow \mathrm{Mor}\, C$ are universal transition functions [9]. (Strictly speaking, the $U_i$ are only point-finite, but following [1] or [6] we can deform the original $t_i$ to functions $\bar{t}_i$ which are locally finite, so the associated $\bar{t}_i^{-1}(0,1]$ are also.)

We now describe the classification procedure. If $X$ is paracompact, we can now restrict attention to countable locally finite coverings $\{U_i\}$. The "1 - cocycle" condition $g_{ij}g_{jk} = g_{ik}$ says that $x \longmapsto \{g_{ij}(x)\}$ induces a map $BU \longrightarrow BC$. Conversely given a map $X \xrightarrow{f} BC$, define a local structure on $X$ in terms of the covering $\{f^{-1}(U_i)\}$ by $\gamma_{ij}(x) = g_{ij} \circ f\ (x)$ for $x \in U_i \cap U_j$.

Starting from any $f: X \longrightarrow BC$, we obtain $X \longrightarrow B\{f^{-1}U\} \longrightarrow BC$. If we use $t_i \circ f$ as the partition of unity subordinate to $\{f^{-1}U_i\}$, the composite is given by

$$x \longmapsto (\cdots, t_i \circ f(x), \cdots, g_{ij} \circ f(x))$$

but this is precisely how one would represent $f$ in terms of coordinates

$t_i$ and $g_{ij}$. In the other direction, if we start with a cocycle $\gamma_{ij}$ on a numerable covering $U$ with associated partition of unity $\lambda_i$, then

$$X \longrightarrow BU \longrightarrow BC$$

is given by

$$x \longrightarrow (\lambda_i(x), \gamma_{ij}(x))$$

and this pulls back the universal example to the open cover $\lambda_i^{-1}(0,1] \subset U_i$ with transition functions $\gamma_{ij}(x)$.

Since the same bundle gives rise to different 1-cocycles as we vary the cover or choice of local coordinates, we must also consider equivalence classes of bundles. Following Steenrod [12], two fibre bundles $E_i \longrightarrow X$ for $i = 1,2$ are equivalent if the union of the corresponding families of transition functions can be extended to a 1-cocycle on the union of the corresponding coverings. i.e., if $BU \longrightarrow BC$ and $BV \longrightarrow BC$ extend to $B(U,V)$. The corresponding partitions of unity give rise to maps $\lambda: X \longrightarrow BU \longrightarrow B(U,V)$ and $\mu: X \longrightarrow BV \longrightarrow B(U,V)$ These are homotopic via $t\lambda + (1-t)\mu$ where this really means the obvious linear homotopy in terms of the $\Delta^\infty$ coordinates. Thus the classifying maps $X \longrightarrow BC$ for equivalent transition functions are homotopic.

Now for bundles, a bundle $E \longrightarrow X \times I$ is equivalent to $E_0 \times I \longrightarrow X \times I$ where $E_0 = E|X \times 0$; hence homotopic maps induce equivalent bundles. Thus we have the result:

> Equivalence classes of $G$-bundles over $X$ are in 1-1 correspondence with homotopy classes of maps $X \longrightarrow BG$.

In general, the notion of equivalence must be weakened so as to insure that a structure on $X \times I$ implies the equivalence of the restrictions to $X \times t$. This is the approach which works for foliations [4].

For fibre spaces, we have one additional subtlety; we have $g_{\alpha\beta}g_{\beta\gamma}$ only homotopic to $g_{\alpha\gamma}$ in $H(F)$. As discovered by Wirth [14], a specific choice of homotopy

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \times I \longrightarrow H(F)$$

is crucial to the classification as are higher homotopies

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \times I^{n-1} \longrightarrow H(F).$$

In other words, we have an shm-map $U \longrightarrow H(F)$ and hence a classifying map

$$X \simeq BU \longrightarrow BH(F)$$

for paracompact $X$.

Thus whether through the local or the global (e.g., CHP and transport) approach, we see that classification of fibre spaces involves shm-maps. Once again, we can return to strict morphisms by enlarging the operative objects, e.g., $W\Omega X$ or $WU$, but it is the shm-maps which are the immediate consequences of the defining properties of fibre spaces.

## BIBLIOGRAPHY


1. A. Dold, Partitions of unity in the theory of fibrations, Ann. of Math. (2) 78 (1963), 223-255. MR 27 #5264.

2. E. Floyd, this conference.

3. M. Fuchs, A modified Dold-Lashof construction that does classify H -principle fibrations (to appear).

4. A Haefliger, Homotopy and Integrability, Lecture Notes in Mathematics 197.

5. P. Hilton, Homotopy theory and duality, Gordon and Breach, New York, 1965 MR 33 #6624.

6. J. W. Milnor, Construction of universal bundles. II, Ann. of Math. (2) 63 (1956), 430-436.

7. G. Segal, Categories and Cohomology theories.

8. J. D. Stasheff, "Parallel" transport in fibre spaces, Bol. Soc. Mat. Mexicana (2) 11 (1966), 68-84 MR 38 #5219.

9. J. D. Stasheff, Appendices to Bott's lectures on Foliations, Lecture Notes in Math 279.

10. N. E. Steenrod, The classification of sphere bundles, Ann. of Math (2) 45 (1944), 294-311.

11. O. Veblen and J. H. C. Whitehead, The foundations of differential Geometry, Cambridge University Press, 1932.

12. J. F. Wirth, Fibre spaces and the higher homotopy cocycle relations, Thesis, Notre Dame, Ind., 1965.