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A CLASSIFICATION THEOREM FOR FIBRE SPACES

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LET $[X, Y]$ denote the set of homotopy classes of maps of X into Y. Let $LF(X)$ denote the set of fibre homotopy equivalence classes of Hurewicz fibrings $p : E \to X$ with fibres of the homotopy type of F.

CLASSIFICATION THEOREM. *If F is a finite CW-complex, there is a space B_H such that $[\ , B_H]$ and $LF(\ )$ are naturally equivalent as functors from the category of CW-complexes and homotopy classes of maps to the category of sets and functions.*

We regard $LF(\ )$ as a functor as follows: Given a map $f: X \to Y$ and a Hurewicz fibring $p : E \to Y$, the *induced Hurewicz fibring* $f^*p : E_f \to X$ is defined by taking $E_f = \{(x, e) \mid f(x) = p(e)\}$ and setting $(f^*p)\ (x, e) = x$. The induced function $LF(f) : LF(Y) \to LF(X)$ is given by $LF(f)\ [p] = [f^*p]$. If f is homotopic to g, then $LF(f) = LF(g)$ by Corollary 6.6 of [1].

The following fact about Hurewicz fibrings will often be of use to us. The technique used in the proof provided our original insight into the classification theorem.

PROPOSITION (0). *If $p : E \to B$ is a Hurewicz fibring and B and all the fibres have the homotopy type of CW-complexes, then so does E.*

This follows from the following special case:

PROPOSITION (1). *Let $p : E \to B$ be a Hurewicz fibring with fibres of the homotopy type of a CW-complex F. If $B = B' \cup e^n$ and $E' = p^{-1}(B')$ has the homotopy type of a CW-complex, then so does E.*

Proof. Let $\chi : e^n \to B$ be the characteristic map. Since e^n is contractible, the induced space E_χ is fibre homotopy equivalent to a product. Let $e^n \times F \underset{\psi}{\overset{\phi}{\rightleftarrows}} E_\chi$ be fibre homotopy inverses and let $\bar{\chi} : E_\chi \to E$ be the obvious map. Now let $v = \chi \circ \phi | S^{n-1} \times F$ and form $\mathscr{E} = e^n \times F \cup_v E'$. Clearly $\bar{\chi}\phi$ induces a map $\alpha : \mathscr{E} \to E$. To obtain an inverse, let $h_t : E_\chi \to E_\chi$ be a homotopy covering the identity such that $h_1 = 1$, $h_0 = \phi\psi$. Represent e^n as CS^{n-1}

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and for any cone CX, define $s : CX \to CX$ to be the deformation

$$s(t, x) = \begin{cases} (2t, x)\ 0 \leq 2t \leq 1, \text{ the vertex of the cone being } (0, x) \\ (1, x)\ 1 \leq 2t \leq 2. \end{cases}$$

Now construct a map $\beta : E \to \mathscr{E}$ extending the identity on E'. On $E - E'$, let β be given by $E - E' \xrightarrow{\bar{\chi}^{-1}} E_\chi \xrightarrow{\phi} e^n \times F \xrightarrow{s \times 1} e^n \times F \to \mathscr{E}$ when $\psi\bar{\chi}^{-1}$ lies over the top half of e^n, i.e. if $\psi\bar{\chi}^{-1}(e) = (t, x, y)$ with $x \in S^{n-1}$, $y \in F$ and $0 \leq 2t \leq 1$, then $\beta(e) = (2t, x, y)$. Otherwise when $\psi\bar{\chi}^{-1}(e) = (t, x, y)$ with $1 \leq 2t \leq 2$, let $\beta(e) = \bar{\chi}h_{2t-1}\bar{\chi}^{-1}$. We see that β is well defined and it is not hard to check that α and β are mutual homotopy inverses rel E'.

If E' were itself a CW-complex, so would $\mathscr{E}$ be, assuming v deformed to be cellular. More generally, if $E' \xrightarrow{\gamma} \mathscr{E}' \xrightarrow{\delta} E'$ are homotopy inverses and $\mathscr{E}'$ a CW-complex, then $\mathscr{E}$ has the homotopy type of the CW-complex $e^n \times F \cup_{\gamma v} \mathscr{E}'$. Thus E is sure to have the homotopy type of a CW-complex.

We will prove Proposition (0) by repeated application of Proposition (1) using the following technique. For any map $f : E \to B$, a CW-complex, we can consider E_w, the space obtained using the weak topology of the decomposition $E_w = \bigcup E_\alpha$ where the E_α are the inverse images $f^{-1}(e_\alpha)$ of the closed cells e_α of B. (It is easy to show that if B is locally finite, E_w is homeomorphic to E.) Now assume $p : E \to B$ is a Hurewicz fibring and all the fibres have the homotopy type of a CW-complex F. The argument for Proposition (1) extends to prove that E_w has the homotopy type of a CW-complex when B is obtained from B' by simultaneously attaching any number of cells, for example when B is the n-skeleton and B', the $(n-1)$-skeleton of a CW-complex. Moreover, as shown in an appendix, under these conditions we can prove:

PROPOSITION (2). *E_w has the homotopy type of E.*

Proposition (0) now follows by induction on the skeleta of B.

Although we are interested in studying Hurewicz fibrings, we must for technical reasons introduce *quasi-fibrings*, that is, maps $p : E \to B$ which have the property that $p_* : \pi_i(E, p^{-1}(x)) \to \pi_i(B, x)$ is an isomorphism for all i, all $x \in B$ and all choices of base point in $p^{-1}(x)$. The essential facts about quasi-fibrings are given in [3].

DEFINITION (3). *If $p_i : E_i \to B$, $i = 0, 1$ are quasi-fibrings, a map $f : E_0 \to E_1$ is a* map over B *if $p_1 f = p_0$ and is a* weak fibre equivalence *if it is a map over B and a weak homotopy equivalence. The quasi-fibrings p_0 and p_1 are* quasi-equivalent *if there is a sequence of quasi-fibrings $p_1, p_2 \ldots, p_{n-1}, p_n = p_0$ such that for each i, p_i is weakly fibre homotopy equivalent to p_{i+1} or p_{i+1} is weakly fibre homotopy equivalent to p_i.*

Denote by $QF(X)$ the set of quasi-equivalence classes of quasi-fibrings $p : E \to X$ with fibres of the weak homotopy type of F and total spaces of the homotopy type of CW-complexes.

PROPOSITION (4). *Each class in $QF(X)$ is represented by a Hurewicz fibring.*

Proof. Given any map $p : E \to X$, the associated Hurewicz fibring $\mathrm{Hur}(p) : \mathrm{Hur}(E) \to X$ is obtained by taking $\mathrm{Hur}(E) \subset E \times X^I$ to be $\{(e, \lambda) | \lambda(0) = p(e)\}$ and setting $\mathrm{Hur}(p)(e, \lambda) = \lambda(1)$. The original space E is imbedded in $\mathrm{Hur}(E)$ by $j(e) = (p(e), \lambda_e)$ where $\lambda_e(t) = e$; E is in fact a deformation retract of $\mathrm{Hur}(E)$. If p is a quasi-fibring, j is a weak fibre equivalence.

Unfortunately it is difficult to regard $QF(X)$ as a functor since in general the map induced from a quasi-fibring need not be a quasi-fibring. However, we do have:

PROPOSITION (5). *If $p_i : E_i \to B$ are quasi-fibrings and $f : E_0 \to E_1$ is a weak fibre equivalence then the associated Hurewicz fibrings* $\mathrm{Hur}(p_0)$ *and* $\mathrm{Hur}(p_1)$ *are fibre homotopy equivalent, provided the spaces E_i have the homotopy type of CW-complexes.*

Proof. We have

$$\begin{array}{ccc} E_0 & \xrightarrow{f} & E_1 \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle j_1} \\ \mathscr{E}_0 & & \mathscr{E}_1 \end{array}$$

where $\mathscr{E}_i = \mathrm{Hur}(E_i)$. Let $k_0 : \mathscr{E}_0 \to E_0$ be an inverse for j_0, e.g. $k_0(e, \lambda) = e$. The composition $j_1 f k_0$ is not a map over B since k_0 is not, but since $p_0 k_0$ can be deformed to $\mathrm{Hur}(p_0)$, $j_1 f k_0$ can be deformed to a map $/$ over B. Since all the maps involved are at least weak equivalences, so is $/$; in fact, $/$ is a homotopy equivalence since $\mathscr{E}_i$ has the homotopy type of the CW-complex E_i. Hence by Theorem (6.1) of [1], $/$ is a fibre homotopy equivalence.

COROLLARY (6). *$QF(X)$ and $LF(X)$ are isomorphic if X and F are CW-complexes.* (We could regard QF as a functor by identifying $QF(X)$: with $LF(X)$: we prefer to work directly with LF.)

There are certain ways of getting new fibrings from old which are useful to us.

The associated principal map Prin(p) (cf. [2])

Let F be a locally compact space. Given a quasi-fibring $p : E \to B$ with fibres of the homotopy type of F, we construct the *associated principal map* $\mathrm{Prin}(p) : \mathrm{Prin}(E) \to B$. $\mathrm{Prin}(E)$ is the subspace of E^F (with the compact-open topology) consisting of maps $\varphi : F \to E$ such that φ is a homotopy equivalence between F and some fibre $p^{-1}(x)$. The map is given by $\mathrm{Prin}(p)(\varphi) = p(\varphi(F))$. The fibres of $\mathrm{Prin}(p)$ have the homotopy type of $H = H(F)$, the topological monoid of homotopy equivalences of F into itself. An operation $\mu : \mathrm{Prin}(E) \times H \to \mathrm{Prin}(E)$ is given by $\mu(\varphi, h) = \varphi \circ h$. (To ensure μ is continuous, we need F to be locally compact.)

I do not know if $\mathrm{Prin}(p)$ is in general a quasi-fibring. However,

LEMMA (7). *Prin(p) is a Hurewicz fibring if p is.*

Proof. Let $f : X \to \mathrm{Prin}(E)$ and $h : X \times I \to B$ such that $\mathrm{Prin}(p) \circ f = h | X \times 0 = h_0$. Consider the induced spaces E_h and E_{h_0}. Lemma (6.5) of [1] provides a map $R : E_{h_0} \times I \to E_h$ which from the information given can readily be seen to be a fibre homotopy equivalence. If we map $X \times F$ into E_{h_0} by sending (x, y) to $(x, f(x)(y))$, the composition $\theta : X \times F \times I \to$

$E_{h_0} \times I \to E_h \to E$ induces a weak homotopy equivalence between corresponding fibres and hence is adjoint to a homotopy $f' : X \times I \to \mathrm{Prin}(E)$ covering h. $[f'(x, t)(y) = \theta(x, y, t)]$.

A similar result will hold for a special type of quasi-fibring which is of crucial importance in our work.

The prolongation Prol(p)

Given a quasi-fibring $p : E \to B$ with fibres of the homotopy type of a space F, we imbed it in the *prolongation* $\mathrm{Prol}(p) : \mathrm{Prol}(E) \to \mathrm{Prol}(B)$. For any space X, CX will denote the cone on X. $\mathrm{Prol}(E) = C(\mathrm{Prin}(E)) \times F \cup_\nu E$ where $\nu : \mathrm{Prin}(E) \times F \to E$ is defined by $\nu(\varphi, f) = \varphi(f)$. $\mathrm{Prol}(B) = C(\mathrm{Prin}(E)) \cup_{p'} B$ where $p' = \mathrm{Prin}(p)$, and $\mathrm{Prol}(p)$ is defined as an extension of p by

$$\begin{array}{ccc} C(\mathrm{Prin}(E)) \times F & \longrightarrow & \mathrm{Prol}(E) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Prol}(p)} \\ C(\mathrm{Prin}(E)) & \longrightarrow & \mathrm{Prol}(B). \end{array}$$

LEMMA (8). *Prol(p) is a quasi-fibring if p is.*

Proof. The proof of Proposition (2.3) in [2] is readily adaptable. We need only observe that for each $\varphi \in \mathrm{Prin}(E)$, the map $\nu(\varphi, \;\;) : F \to E$ is a homotopy equivalence into a fibre.

LEMMA (9). *Prin(Prol(p)) is a quasi-fibring if Prin(p) is.*

Proof. Prin(Prol(p)) can be identified with Prol(Prin(p)) by identifying

$$\mathrm{Prin}(C(\mathrm{Prin}(E)) \times F) \text{ with } C(\mathrm{Prin}(E)) \times H.$$

The ultimate prolongation Ult(p) of a quasi-fibring

Given a quasi-fibring $p : E \to B$, relabel it $q_0 : D_0 \to B_0$. Inductively define $q_n : D_n \to B_n$ to be $\mathrm{Prol}(q_{n-1}) : \mathrm{Prol}(D_{n-1}) \to \mathrm{Prol}(B_{n-1})$ and let $\mathrm{Ult}(p) : \mathrm{Ult}(E) \to \mathrm{Ult}(B)$ be the limit of the quasi-fibrings $q_n : D_n \to B_n$.

LEMMA (10). *Ult(p) is a quasi-fibring if p is. Prin(Ult(p)) is a quasi-fibring if p is a Hurewicz fibring.*

Proof. The case for Ult(P) is covered explicitly in [2]. The argument applies equally well to Prin(Ult(p)) as the limit of the quasi-fibrings Prin(q_n) in light of Lemmas (7) and (9).

LEMMA (11). *Prin(Ult(E)) is aspherical.*

Proof. It is sufficient to prove $\mathrm{Prin}(D_n)$ is contractible in $\mathrm{Prin}(D_{n+1})$. The contraction is given by $k_t(\varphi)(f) = (t, \varphi, f) \in C(\mathrm{Prin}(D_n)) \times F$.

The universal example $u : UE \to B_H$

Consider the trivial fibring $\theta : F \to *$, a point. Prin(F) is just H so Prin(Ult(θ)) is a quasi-fibring with aspherical total space and fibre H. Following Dold and Lashof, we denote this quasi-fibring by $p_H : E_H \to B_H$ and call it the universal H-fibring. The analogy with the universal bundle of a topological group is underlined by our main theorem which asserts that B_H is a classifying space for a certain type of fibring.

The Hurewicz fibring Hur(Ult(θ)) we denote by $u : UE \to B_H$. We shall see that under suitable restrictions it is indeed the Universal Example of a Hurewicz fibring with fibres of the homotopy type of F. But first we verify:

PROPOSITION (12). *Let $f^*p : E_f \to X$ be induced by $f: X \to Y$ from a Hurewicz fibring $p : E \to Y$. If X, Y and E have the homotopy type of CW-complexes, then so does E_f.*

Proof. Let $M(f, p)$ denote the double mapping cylinder of f and p, i.e. $M(f, p) = X \times I \cup_f Y \cup_p E \times I$ where $(x, 1)$ is identified with $f(x)$ and $(e, 1)$ with $p(e)$. Consider the space $\mathscr{E} = \{\lambda : I \to M(f, p) | \lambda(0) \in X \times 0, \lambda(1) \in E \times 0\}$. I claim $\mathscr{E}$ has the same homotopy type as E_f. Explicit equivalences are given as follows: Define $\alpha : E_f \to \mathscr{E}$ by

$$\alpha(x, e)(t) = \begin{cases} (x, 2t) & 0 \leq 2t \leq 1. \\ (e, 2-2t) & 1 \leq 2t \leq 2. \end{cases}$$

Define $\gamma : \mathscr{E} \to E$ by $\gamma(\lambda) = \lambda(1)$ and $\not{p} : \mathscr{E} \to X$ by $\not{p}(\lambda) = \lambda(0)$. Since $f\not{p}$ is homotopic to $p\gamma$, the map γ may be deformed to γ' which will cover $f\not{p}$. Define an inverse for α by $\beta(\lambda) = (\not{p}(\lambda), \gamma'(\lambda))$. By Theorem 3 of [5] (cf. Corollary (3)), $\mathscr{E}$ has the homotopy type of a CW-complex, hence E_f does.

COROLLARY (13) (cf. [4; p. 7]). *The fibres of p have the homotopy type of CW-complexes if E and B do.*

THEOREM (14). *Let $p : E \to B$ be a Hurewicz fibring. If F is a finite CW-complex and B (and hence E) has the homotopy type of a CW-complex, then Hur(Ult(p)) has fibres of the homotopy type of F.*

Proof. Since j : Ult(E) $\to$ Hur(Ult(E)) is fibre preserving and a homotopy equivalence, it induces a weak equivalence between corresponding fibres. It is sufficient therefore, to verify that the fibres of Hur(Ult(p)) have the homotopy type of CW-complexes, for which, in the light of Corollary (13), it is enough that Ult(E) and Ult(B) have the homotopy type of CW-complexes. For this we need the fact that if B has the homotopy type of a CW-complex so does Prin(E), using Proposition (0) and the fact that H has the homotopy type of a CW-complex since F is compact [5, Corollary (2)]. It follows that C(Prin(E)) $\times F$ has the homotopy type of a CW-complex and hence so do Prol(E) and Prol(B). Since Prin(Prol E) can be identified with Prol(Prin E), the argument can be iterated. Passing to the limit, we verify the same thing for Ult(E) and Ult(B): they are of the homotopy type of CW-complexes. In particular this is true of B_H.

The transformation $S : [\ , B_H] \to LF(\ )$

We define $S(f) = LF(f)[u] = [f^*u : UE_f \to X]$ for any map $f: X \to B_H$.

The transformation $T : LF(\ ) \to [\ , B_H]$

Consider a commutative diagram
$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$
where p and p' are Hurewicz

fibrings and j induces equivalences between corresponding fibres. If we carry out our constructions on p and p', we obtain

$$
\begin{tikzcd}
E' \arrow[rrr] \arrow[dr] \arrow[ddd] & & & E \arrow[dl] \arrow[ddd] \\
& X' \arrow[r, "f"] \arrow[d] & X \arrow[d] & \\
& \mathrm{Ult}(X') \arrow[r, "\mathrm{Ult}(f)"] & \mathrm{Ult}(X) & \\
\mathrm{Ult}(E') \arrow[ur] \arrow[rrr] & & & \mathrm{Ult}(E). \arrow[ul]
\end{tikzcd}
$$

Since Prin(Ult(E')) and Prin(Ult(E)) are aspherical, Ult(f) is a weak homotopy equivalence. If X' and X have the homotopy type of CW-complexes, so do Ult(X') and Ult(X), assuming F is a finite CW-complex. Thus Ult(f) is a homotopy equivalence.

In particular this is true if X' is a point. In this case, Ult(X') is just B_H and we will denote the equivalence by $j: B_H \to \mathrm{Ult}(X)$. It induces an isomorphism $j_*: [Y, B_H] \to [Y, \mathrm{Ult}(X)]$ for any space Y. In fact for any point X' within a given path-component of X, we get the same isomorphism j_*. [This can be seen by considering the fibring E_μ induced over a path $\mu: I \to X$. E_μ is fibre homotopy equivalent to $F \times I$ since I is contractible, from whence it follows easily that Ult(X) is homotopy equivalent to $B_H \times I$. Thus Ult(μ) gives a homotopy between the corresponding maps j.] On each component, therefore, we define $T(p)$ by $j_* T(p) = [g]$ where g is the inclusion $X \subset \mathrm{Ult}(X)$. Returning to the figure for arbitrary p', we have similarly $j': B_H \to \mathrm{Ult}(X')$, $g': X' \subset \mathrm{Ult}(X')$ and $\mathrm{Ult}(f)j' = j$ so $j_* T(p') = \mathrm{Ult}(f)_* j_* T(p') = \mathrm{Ult}(f)_*[g'] = [\mathrm{Ult}(f)g'] = [gf]$. Thus if $X = X'$ and f is the identity, then $T(p') = T(p)$ so T passes to equivalence classes. Since $[gf] = f^*[g]$, T is natural.

The relation between T and S

THEOREM (15). *ST is the identity on LF( ).*

Proof. Let $p: E \to X$ be imbedded in Ult(p) as in defining T. The imbedding $E \to \mathrm{Ult}(E) \to \mathrm{Hur}(\mathrm{Ult}(E))$ will induce homotopy equivalences between corresponding fibres. Thus p is equivalent to g^* Hur(Ult(p)) by Theorem (6.3) of [1]. Similarly u is equivalent to j^* Hur(Ult(p)). Let $p' = \mathrm{Hur}(\mathrm{Ult}(p))$: we have $ST[p] = T(p)^*[u] = T(p)^*j^*[p'] = g^*[p'] = [p]$.

As for TS, suppose we attempt to evaluate $T(u)$. We would have

$$
\begin{tikzcd}
F \arrow[rrr] \arrow[dr] \arrow[ddd] & & & UE \arrow[dl] \arrow[ddd] \\
& * \arrow[r] \arrow[d] & B_H \arrow[d, "g"] & \\
& B_H \arrow[r, "j"'] & \mathrm{Ult}(B_H) & \\
\mathrm{Ult}(F) \arrow[ur] \arrow[rrr] & & & \mathrm{Ult}(UE) \arrow[ul]
\end{tikzcd}
$$

so that $T(u) = h \circ g$ where h is any inverse for j and hence $T(u)$ is a homotopy equivalence of B_H onto itself. [In fact j is homotopic to g so that $T(u) = 1$, but we do not need this.]

THEOREM (16). *TS is an automorphism on* $[\ ,B_H]$.

Proof. By naturality $TS[f] = T(f^*[u]) = f^*T[u] = T(u)_*[f]$.

Combining Theorems (15) and (16), we see that T is one-to-one and onto. This completes the proof of the classification theorem.

The use of the Dold and Lashof construction to define the functor T, I owe to I. M. James. The full strength of the present classification theorem and the efficiency of the proofs given is due to the patient insistence of several people, particularly J. Milnor, D. M. Kan and the referee.

APPLICATIONS

Let $F = S^n$ and let $B_{0(n+1)}$ be the classifying space for the orthogonal group on R^{n+1}. Since an orthogonal motion of R^{n+1} induces a homeomorphism of S^n onto itself, $0(n+1)$ can be regarded as a subgroup of H. Dold and Lashof have shown that there is a map $J: B_{0(n+1)} \to B_H$ which corresponds to identifying orthogonal bundles up to fibre homotopy equivalence. From our point of view, J can be defined as $T(\gamma^{n+1})$ where γ^{n+1} is the universal S^n-bundle $\gamma^{n+1}: E \to B_{0(n+1)}$.

THEOREM (17). *The induced map* $J^*: H^*(B_H; Z_p) \to H^*(B_{0(n+1)}; Z_p)$ *is onto for* $p = 2$ *or* 3.

For prime p and $n > 4$, J^* is not onto since $H^4(B_{0(n+1)}; Z_p) = Z_p$ while $H^4(B_H; Z_p) = 0$ since $\pi_i(B_H) \approx \pi_{i-1}(H)$ has no p-primary component for $i \leq 4$, being isomorphic to $\pi_{n+i-1}(S^n)$.

Proof for $p = 2$. We know that $H^*(B_{0(n+1)}; Z)$ is generated by the Stiefel–Whitney classes $W_i(\gamma^{n+1})$ which can be defined for a sphere fibring $p: E \to B$ by

$$W_i = \Phi^{-1} Sq^i \Phi(1)$$

where $\Phi: H^{j-1}(B) \to H^{j+n}(B, E) = H^{j+n}(M_p, E)$ is the Thom isomorphism [6, Theorem (1.3)] and M_p is the mapping cylinder of p. This isomorphism can be obtained from the Gysin sequence which exists for any fibring with fibre of the homotopy type of S^n. In particular, we can define $W_i(u)$ in this way and by naturality we have $J^*W_i(u) = W_i(\gamma^{n+1})$ so J^* is onto.

Proof for $p = 3$. This is almost the same. The Steenrod operations $\mathscr{P}_3^i$ replace the Sq^i, $H^*(B_{0(n+1)}; Z_p)$ is generated by the mod 3 reductions of the Pontrjagin classes p_i and one knows $p_i \equiv \Phi^{-1}\mathscr{P}_3^i\ \Phi(1) \bmod 3$ [7].

APPENDIX

Proof of Proposition (2). Let $i: E_w \to E$ denote the identity map from the weak to the original topology and let $p_w: E_w \to B$ be the obvious map. Recall that E_w is a deformation retract of $\mathrm{Hur}(E_w)$ and in fact the deformation retraction can be given by $h_t(e, \lambda) = (e, \lambda^t)$ where $\lambda^t(s) = \lambda(ts)$ for $0 \leq s \leq 1$ so that $p_w h_0 \simeq \mathrm{Hur}(p_w)$.

$$\begin{tikzcd}
\mathrm{Hur}(E_w) \arrow[r, shift right, "h_0"'] \arrow[dr, "\mathrm{Hur}(p_w)"'] & E_w \arrow[l, shift right, "j"'] \arrow[r, "i"] \arrow[d, "p_w"] & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

Thus $pih_0 \simeq \mathrm{Hur}(p_w)$ so ih_0 can be deformed to $g : \mathrm{Hur}(E_w) \to E$, a map over B. Now i is a weak homotopy equivalence (the topologies agree on compact subsets) and hence ih_0 and g are also. Thus g restricted to corresponding fibres is a weak homotopy equivalence.

If E_w has the homotopy type of a CW-complex, so does $\mathrm{Hur}(E_w)$ and hence so do the fibres of the latter by Corollary (13). Thus g restricted to corresponding fibres is a homotopy equivalence and hence g is a fibre homotopy equivalence by Theorem (6.3) of [1].

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