ANNALS OF MATHEMATICS
Vol. 50, No. 4, October, 1949

COHOMOLOGY INVARIANTS OF MAPPINGS[1]

BY N. E. STEENROD

(Received September 8, 1948)

1. Introduction

The Hopf invariant of a map of a $(2n - 1)$-sphere into an n-sphere [6] is shown herein to be a special case of a general bilinear operation involving an arbitrary map of one space into another, cohomology classes in the image space, and cup products of the Alexander-Čech-Whitney type. The new operations, called *functional cup products*, are invariants of the homotopy class of the function, and their vanishing is a necessary condition for the function to be inessential. A variety of examples are given in which these new products are non-zero in spite of the fact that the usual invariants vanish. These examples indicate that the functional cup products will play an important part in the study of extension problems and homotopy classification.

The Hopf invariant is most readily pictured for a map $f: S^3 \rightarrow S^2$ of a 3-sphere on a 2-sphere. Let f be simplicial, and let x_1, x_2 be interior points of simplexes of S^2. Then $f^{-1}(x_1), f^{-1}(x_2)$ are seen to be unions of simple closed polygons. Using orientations of S^2 and S^3, one assigns orientations to the edges of these polygons so that they become 1-cycles γ_1 and γ_2. The Hopf invariant of f is the linking number of γ_1 and γ_2.

The construction of the 1-cycles γ_1, γ_2 from the 0-cycles x_1, x_2 is a special case of the *inverse homomorphism* defined by Hopf for a map of one manifold on another. It is known that these inverse homomorphisms are dual to the induced homomorphisms of cohomology groups. The latter are defined for arbitrary spaces, while the former are not. It is apparent, therefore, that a translation of the Hopf invariant into the language of cohomology may lead to a more general operation.

This translation is readily found. The dual of the 0-cycle x_1 is a 2-cocycle u_1 which generates the cohomology group. To γ_1 corresponds f^*u_1 (the induced counter-image of u_1 in S^3). To calculate the linking number, one chooses a 2-chain Γ whose boundary is γ_1 and forms the intersection 0-cycle of Γ with γ_2. The dual of Γ is a 1-cochain a in S^3 whose coboundary δa is f^*u_1. It exists since every 2-cocycle on S^3 is a coboundary. The dual of the intersection is the cup product $a \smile f^*u_2$ which is a 3-cocycle of S^3. The third cohomology group of S^3, $H^3(S^3)$, is infinite cyclic. If z is a generating 3-cocycle of $H^3(S^3)$, then $a \smile f^*u_2 = hz$ for some integer h. Then h is the Kronecker index of the intersection of Γ with γ_2, i.e. h is the Hopf invariant.

If one attempts to carry over this procedure to a simplicial map $f: K' \rightarrow K$ of one complex into another several difficulties arise. First, if u is a p-cocycle on K, f^*u need not be a coboundary. We are therefore restricted to cocycles

[1] A summary of this paper appeared in a paper by the same title [9] (see bibliography).

representing elements of the kernel of $f^*: H^p(K) \to H^p(K')$. Assuming f^*u is a coboundary, choose a cochain a with $\delta a = f^*u$. Now let v be a q-cocycle on K. We can then form $a \smile f^*v$. The second difficulty is that $a \smile f^*v$ need not be a cocycle:

$$\delta(a \smile f^*v) = \delta a \smile f^*v = f^*u \smile f^*v = f^*(u \smile v).$$

(In the case of the 2-sphere, $u_1 \smile u_2$ was identically zero). The difficulty is resolved by assuming that $u \smile v = \delta b$ is a coboundary, then $a \smile f^*v - f^*b$ is in fact a cocycle. The third difficulty is that the cohomology class of this cocycle can be altered by altering the choice of a and b. It can be altered by any element of the image group $f^*H^{p+q-1}(K)$ by altering b, and by any element of $H^{p-1}(K') \smile f^*v$ by altering a. One obtains then a unique operation by factoring $H^{p+q-1}(K')$ by these two subgroups. (Both subgroups were zero for the map $S^3 \to S^2$).

We are thus led to a product, denoted by $u \smile_f v$, defined for cohomology classes u, v of dimensions p, q such that $f^*u = 0$ and $u \smile v = 0$. The value of the product is an element of the factor group of $H^{p+q-1}(K')$ by the union of the subgroups $f^*H^{p+q-1}(K)$ and $H^{p-1}(K') \smile f^*v$.

In studying the properties of these products the author first used the technique of simplicial complexes, maps cochains, etc., and the machinery of the singular and Čech theories for extending results to general spaces. This proved exceedingly cumbersome. A second definition was found which is invariant in form in that it employs only well-established properties of the cohomology groups and their homomorphisms. This is the definition to be used. In addition to providing simpler proofs, it divests the situation of the irrelevant machinery of complexes, and thereby fits into the present trend in algebraic topology.

The principal tool of this invariant method is a space attached to a function f called *the mapping cylinder of f*. It has been used extensively by J. H. C. Whitehead [10] and R. H. Fox [2]. Through its use, one can define new homology invariants of f. In particular, the homomorphisms of homology groups induced by f are imbedded in an exact sequence of groups. The homotopy type of the mapping cylinder is an invariant of the homotopy class of f. In this way the new invariants depend only on the homotopy type of f.

In a recent paper [8][2], the author extended the cup products by defining

[2] The author takes this opportunity to correct an error in this paper. On page 299, §6, it is assumed that the group G is paired with itself to G'. For the results that follow to be valid, this pairing must be commutative. The reason for this appears in the coboundary formula 5.1. There, G_1, G_2 were paired to G', and the pairings used in defining the terms $u \smile_{i-1} v$ and $v \smile_{i-1} u$ were tacitly assumed to be the same. Thus, when $G_1 = G_2$, the pairing must be commutative. This correction does not affect the subsequent applications since, in all such, the pairing is commutative. S. Eilenberg has pointed out to me that, if one assumes that the pairing is anti-commutative, then the coboundary formula still holds if the sign of the term $v \smile_{i-1} u$ is reversed. In this case, one obtains also an invariant set of squaring operations with the same properties except that the cases $p - i$ even and odd are interchanged.

cup-i products for all integers i. These led to invariant squaring operations. Just as the cup products lead to functional cup products, the squaring operations lead to *functional squares*.

One of the principal applications of the functional products is to the projection of a sphere bundle on its base space. It is shown that Gysin's homomorphism [4] is a special case of a functional product. His central isomorphism theorem is obtained from a more general theorem about manifolds with boundaries. It was a study of Gysin's paper which led to the present results.

Another application is to the mappings of spheres on spheres. The functional squares provide an additional method of showing that certain maps are essential. In particular, a map of an $(n + 1)$-sphere into an n-sphere is inessential if and only if a certain functional square is zero. Since these operations are computable, at least for simplicial maps, they provide an effective means of determining the homotopy class.

2. Review of homology and cohomology theory

We describe briefly the notations and principal properties of homology and cohomology theories. Since the two are similar, just, one of these, cohomology, is given in detail, and the changes to be made for homology are indicated at the end. For a full discussion see [1].

A *pair* (X, A) consists of a topological space X and a subspace A of X. If A is vacuous, the symbol $(X, 0)$ is abbreviated by X. A *map* $f: (X, A) \to (Y, B)$ is a continuous function from X to Y such that $f(A) \subset B$. We write $(X', A') \subset (X, A)$ if $X' \subset X$, $A' \subset A$ and X' has the subspace topology. The symbolism $f: (X', A') \subset (X, A)$ is read: f is the inclusion map of (X', A') into (X, A) (i.e. $f(x) = x$ for each $x \in X'$).

If $f: (X, A) \to (Y, B)$, and $(X', A') \subset (X, A)$ and $(Y', B') \subset (Y, B)$ are such that $f(X') \subset Y'$, $f(A') \subset B'$ then the map $f_1: (X', A') \to (Y', B')$ such that $f_1(x) = f(x)$ for $x \in X'$ is called the map *defined* by f.

Two maps $f_0, f_1: (X, A) \to (Y, B)$ are *homotopic*, $f_0 \simeq f_1$, if there exists a map $h: (X \times I, A \times I) \to (Y, B)$, where $I = [0, 1]$, such that $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$ for $x \in X$.

A cohomology theory H defined on a category of pairs of spaces and of maps assigns to each pair (X, A) and each integer $q \geqq 0$ an abelian group $H^q(X, A)$ called the q-dimensional cohomology group of X mod A. For convenience we define $H^q(X, A) = 0$ for $q < 0$. In addition it assigns a homomorphism

$$\delta: H^{q-1}(A) \to H^q(X, A)$$

called the coboundary operator. Furthermore it assigns to each map $f: (X, A) \to (Y, B)$ and each q a homomorphism

$$f^*: H^q(Y, B) \to H^q(X, A)$$

called the homomorphism induced by f.

For any pair (X, A) let

$$i: A \subset X, \qquad j: X \subset (X, A)$$

be inclusion maps. The infinite sequence

$$(2.1) \qquad \cdots \to H^{q-1}(A) \xrightarrow{\delta} H^q(X, A) \xrightarrow{j^*} H^q(X) \xrightarrow{i^*} H^q(A) \to \cdots$$

of groups and homomorphisms is called the *cohomology sequence* of (X, A), abbreviated: C.S. of (X, A).

The system H^q, f^*, δ of groups and homomorphisms have the following properties.

2.2. If $f: (X, A) \subset (X, A)$, then $f^*: H^q(X, A) \subset H^q(X, A)$.

2.3. If $f: (X, A) \to (Y, B)$, $g: (Y, B) \to (Z, C)$, then $(gf)^* = f^*g^*$.

2.4. If $f: (X, A) \to (Y, B)$, and $f_1: A \to B$ is the map defined by f, then, for each q, commutativity holds in the diagram

$$\begin{array}{ccc} H^{q-1}(A) & \xleftarrow{f_1^*} & H^{q-1}(B) \\ \downarrow \delta & & \downarrow \delta \\ H^q(X, A) & \xleftarrow{f^*} & H^q(Y, B). \end{array}$$

Explicitly, for each $u \in H^{q-1}(B)$, $f^*\delta u = \delta f_1^* u$.

2.5. If $f_0, f_1: (X, A) \to (Y, B)$ are homotopic, then $f_0^* = f_1^*$ for each dimension q.

2.6. The cohomology sequence of (X, A) is *exact*. Explicitly, in each group of the sequence (2.1), the image of the group on the left is the kernel of the homomorphism on the right.

2.7. If U is open in X and its closure $\overline{U}$ lies in the interior of A, then the inclusion map $f: (X - U, A - U) \subset (X, A)$ induces isomorphisms $f^*: H^q(X, A) \approx H^q(X - U, A - U)$. This property is referred to as invariance under *excision*.

2.8. If P is a space consisting of a single point then $H^q(P) = 0$ for $q \neq 0$.

The group $H^0(P)$ is called the *coefficient* group of the cohomology theory and is denoted by G.

Corresponding to a prescribed coefficient group there are various ways of constructing a cohomology theory. For example, the Čech theory is defined on the family of compact pairs (X, A) and all maps of such. We shall regard two cohomology theories as distinct if they have distinct coefficient groups even though they are constructed by the same process.

A homology theory likewise assigns to each q, (X, A) a group $H_q(X, A)$ called the q^{th} homology group of X mod A. However the boundary operator has reverse direction:

$$\partial: H_q(X, A) \to H_{q-1}(A).$$

In addition the homomorphism induced by a map $f: (X, A) \to (Y, B)$ has reverse direction:

$$f_*: H_q(X, A) \to H_q(Y, B).$$

The homology sequence of (X, A) is obtained by reversing the arrows in (2.1) lowering the indices and replacing δ, j^*, i^*, by ∂, j_*, i_*. The properties 2.2 to 2.8, if modified in a similar fashion, all hold for a homology theory.

3. Cup and cap products

We describe briefly the basic properties of the cup and cap products involving homology and cohomology. For full details see [1].

A *triad* $(X; A_1, A_2)$ consists of a space X and two subspaces A_1, A_2. A *map* $f: (X; A_1, A_2) \to (Y; B_1, B_1)$ is a continuous function from X to Y such that $f(A_i) \subset B_i$ $(i = 1, 2)$. For any such f, we denote by

$$f_i : (X, A_i) \to (Y, B_i) \ (i = 1, 2), \qquad f_3 : (X, A_1 \cup A_2) \to (Y, B_i \cup B_2)$$

the maps defined by f.

A cup product is a pairing of two cohomology theories ${}_1H$, ${}_2H$ to a third H in the following sense. If $(X; A_1, A_2)$ is a triad, and

$$u \in {}_1H^p(X, A_1), \qquad v \in {}_2H^q(X, A_2),$$

then their *cup product* $u \smile v$ is defined and is an element of $H^{p+q}(X, A_1 \cup A_2)$. These products have the following properties.

3.1. $u \smile v$ is bilinear.

3.2. If (X, A) is a pair, and $i: A \subset X$, and $u \in {}_1H^{p-1}(A)$, $v \in {}_2H^q(X)$, then, in $H^{p+q}(X, A)$, we have

$$\delta(u \smile i^*v) = \delta u \smile v.$$

3.3. Similarly, $u \in {}_1H^p(X)$ and $v \in {}_2H^{q-1}(A)$ implies

$$\delta(i^*u \smile v) = (-1)^p u \smile \delta v.$$

3.4. If $f: (X; A_1, A_2) \to (Y; B_1, B_2)$, and $u \in {}_1H^p(Y, B_1)$, $v \in {}_2H^q(Y, B_2)$, then

$$f_3^*(u \smile v) = f_1^*u \smile f_2^*v.$$

It is useful to extend the domain of definition of $u \smile v$ as follows. Let (X_1, A_1), (X_2, A_2) be two pairs such that X_1, X_2 are subspaces of a space X. Define the intersection of the two pairs to be a pair:

$$(X_1, A_1) \cap (X_2, A_2) = (X_1 \cap X_2, (X_1 \cap A_2) \smile (A_1 \cap X_2)).$$

Let h_1, h_2 be the inclusion maps

$$h_1 : (X_1, A_1) \cap X_2 \subset (X_1, A_1), \qquad h_2 : (X_2, A_2) \cap X_1 \subset (X_1, A_1).$$

If $u \in {}_1H^p(X_1, A_1)$ and $v \in {}_2H^q(X_2, A_2)$, define $u \smile v \in H^{p+q}((X_1, A_1) \cap (X_2, A_2))$ by

$$u \smile v = h_1^*u \smile h_2^*v. \tag{3.5}$$

This is clearly an extension of the original cup product and 3.1 still holds.

The analog of 3.4 for a map $f: (X; (X_1, A_1), (X_2, A_2)) \to (Y; (Y_1, B_1), (Y_2, B_2))$ obviously follows from the restricted form of 3.4. Under the hypotheses of 3.2 we have $u \smile v = u \smile i^*v$ by (3.5). Hence the conclusion of 3.2 takes on the simplified form

(3.2′) $$\delta(u \smile v) = \delta u \smile v.$$

Similarly, the conclusion of 3.3 can be written

(3.3′) $$\delta(u \smile v) = (-1)^p u \smile \delta v.$$

A cap product is a pairing of a cohomology theory ${}_1H$ with a homology theory ${}_2H$ to a homology theory H as follows. If $(X; A_1, A_2)$ is a triad, and

$$v \in {}_1H^q(X, A_1), \qquad z \in {}_2H_r(X, A_1 \cup A_2),$$

then their *cap product* $v \frown z \in H_{r-q}(X, A_2)$ is defined. These products have the following properties

3.6. $v \frown z$ is bilinear.

3.7. If (X, A) is a pair, and $i: A \subset X$, and $v \in {}_1H^q(X)$, $z \in {}_2H_r(X, A)$, then, in $H_{r-q-1}(A)$, we have

$$\partial(v \frown z) = (-1)^q i^*v \frown \partial z.$$

3.8. Similarly, if $v \in {}_1H^q(A)$ and $z \in {}_2H_r(X, A)$, then, in $H_{r-q-1}(X)$, we have

$$\delta v \frown z + (-1)^q i_*(v \frown \partial z) = 0.$$

3.9. If $f: (X; A_1, A_2) \to (Y; B_1, B_2)$, and $v \in {}_1H^q(Y, B_1)$, $z \in {}_2H_r(X, A_1 \cup A_2)$, then

$$f_{2*}(f_1^*v \frown z) = v \frown f_{3*}z.$$

Unlike the cup product, there is an ambiguity in the symbol $v \frown z$ in the following sense. If $v \in {}_1H^q(X, A_1)$ and $z \in {}_2H_r(X, A)$ and $A \supset A_1$, then $v \frown z \in H_{r-q}(X, A_2)$ is defined for each A_2 such that $A = A_1 \cup A_2$. Thus the full triad must be specified whenever $v \frown z$ is written. This ambiguity appears in the term $\delta v \frown z$ of 3.8.

If $(X; (X_1, A_1), (X_2, A_2))$ is a compound triad as above, and

$$v \in {}_1H^q(X_1, A_1), \qquad z \in {}_2H_r((X_1, A_1) \cap (X_2, A_2)),$$

we extend the definition of $v \frown z \in H_{r-q}(X_2, A_2)$ by defining

$$v \frown z = h_{2*}(h_1^*v \frown z).$$

Then 3.6 holds, 3.9 generalizes to maps of compound triads, and in 3.7 and 3.8 the i^*, i_* can be omitted.

4. Homomorphisms of exact sequences

The construction of the functional product, given in the next section, has a purely algebraic aspect which we consider now. The author wishes to acknowledge that he read the following algebraic construction in an unpublished manuscript of R. H. Fox who applied it in a situation entirely different from the ones contemplated in this paper.

Suppose we have two exact sequences of groups and a homomorphism of one into the other as shown in the diagram of Fig. 1. The commutativity relation

$$\begin{array}{ccccccccccc} \cdots \to & G_1 & \xrightarrow{g_1} & G_2 & \xrightarrow{g_2} & G_3 & \xrightarrow{g_3} & G_4 & \xrightarrow{g_4} & G_5 & \to \cdots \\ & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_4} & & \downarrow{\scriptstyle \phi_5} & \\ \cdots \to & H_1 & \xrightarrow{h_1} & H_2 & \xrightarrow{h_2} & H_3 & \xrightarrow{h_3} & H_4 & \xrightarrow{h_4} & H_5 & \to \cdots \end{array}$$

Fig. 1

$\phi_{i+1}g_i = h_i\phi_i$ is assumed to hold in each square. Define K_4 to be the intersection of the kernels of ϕ_4 and g_4 :

(4.1) $$u \in K_4 \quad \text{if} \quad g_4 u = 0 \quad \text{and} \quad \phi_4 u = 0.$$

Define L_2 to be the smallest subgroup of H_2 containing the images of h_1 and ϕ_2. Using $+$ in this sense,

(4.2) $$L_2 = \phi_2(G_2) + h_1(H_1).$$

LEMMA 4.3. *Let $u \in K_4$. There exist elements $u' \in G_3$ and $w \in H_2$ such that*

(4.4) $$h_2 w = \phi_3 u', \qquad g_3 u' = u.$$

In fact, for each u' such that $g_3 u' = u$, a w satisfying 4.4 exists. The set of elements w such that a u' exists satisfying 4.4 is a coset of L_2 in H_2 denoted by $\bar{\phi} u$. The mapping

$$\bar{\phi} : K_4 \to H_2/L_2$$

so defined is a homomorphism.

PROOF. Since $g_4 u = 0$, it follows from exactness of the upper line of Fig. 1 that a u' exists such that $g_3 u' = u$. For any such u', commutativity in the third square yields $h_3\phi_3 u' = \phi_4 g_3 u' = \phi_4 u = 0$ since $u \in K_4$. Then $h_3\phi_3 u' = 0$, and the exactness of the lower line imply the existence of a $w \in H_2$ satisfying 4.4.

Now let (w_1, u_1'), (w_2, u_2') be two pairs satisfying 4.4. Then $g_3(u_1' - u_2') = 0$. This and exactness provides an element $s \in G_2$ such that $g_2 s = u_1' - u_2'$. Then commutativity in the second square yields

$$\begin{aligned} h_2(w_1 - w_2 - \phi_2 s) &= h_2 w_1 - h_2 w_2 - h_2\phi_2 s = \\ \phi_3 u_1' - \phi_3 u_2' - \phi_3 g_2 s &= \phi_3(u_1' - u_2') - \phi_3(u_1' - u_2') = 0. \end{aligned}$$

This and exactness imply the existence of a $t \in H_1$ such that $h_1 t = w_1 - w_2 - \phi_2 s$. This shows that $w_1 - w_2 \in L_2$.

Suppose now that (w_1, u_1') satisfy 4.4, and $s \in G_2$, $t \in H_1$. Let $u_2' = u_1' - g_2 s$, and $w_2 = w_1 - h_1 t - \phi_2 s$. Then the exactness property $g_3 g_2 = 0$ yields $g_3 u_2' = g_3 u_1' = u$. Similarly, the exactness property $h_2 h_1 = 0$ yields

$$h_2 w_2 = h_2 w_1 - h_2\phi_2 s = \phi_3 u_1' - \phi_3 g_2 s = \phi_3(u_1' - g_2 s) = \phi_3 u_2'.$$

This shows that the w's satisfying 4.4 form an entire coset of L_2 in H_2.

Finally, suppose u_1, $u_2 \in K_4$. Choose (w_1, u_1') to satisfy 4.4 for u_1, and (w_2, u_2') to satisfy 4.4 for u_2. Since g_i, h_i, ϕ_i are all homomorphisms, it follows that $(w_1 + w_2, u_1' + u_2')$ satisfy 4.4 for $u_1 + u_2$. This completes the proof.

LEMMA 4.5. *The operation $\bar{\phi}$ defined in 4.3 is natural. Precisely, let T be a transformation of the ladder of Fig. 1 into a second such ladder (distinguished by affixing a prime throughout), thus $T_i : G_i \to G_i'$, $T_i' : H_i \to H_i$ so that commutativity holds in every square:*

$$T_{i+1}g_i = g_i'T_i, \qquad T_{i+1}'h_i = h_i'T_i', \qquad T_i'\phi_i = \phi_i'T_i.$$

Then T_4 maps K_4 into K_4', and T_2' maps L_2 into L_2' thereby inducing a map $T_2'' : H_2/L_2 \to H_2'/L_2'$. Finally, for each $u \in K_4$, $T_2''\bar{\phi}u = \bar{\phi}'T_4u$.

PROOF. If $u \in K_4$, then $g_4'T_4u = T_5g_4u = T_50 = 0$; and $\phi_4'T_4u = T_4'\phi_4u = T_4'0 = 0$. Therefore $T_4u \in K_4'$. If $w \in L_2$, choose $s \in G_2$ and $t \in H_1$ so that $w = h_1t + \phi_2s$. Then

$$T_2'w = T_2'h_1t + T_2'\phi_2s = h_1'T_1't + \phi_2'T_2s.$$

But this implies $T_2'w \in L_2$. Finally, if (w, u') satisfy 4.4 for $u \in K_4$, it follows quickly that $(T_2'w, T_3u')$ satisfy 4.4 for $T_4u \in K_4'$.

5. Construction of the functional cup product

Assume, as in §3, that ${}_1H$ and ${}_2H$ are two cohomology theories paired to a third H by a cup product, and suppose they are defined on the pair (X, A). Let $v \in {}_2H^q(X)$ be a fixed element. Using v and cup products, a homomorphism of the C.S. of (X, A) based on ${}_1H$ into that based on H is constructed as shown in Fig. 2 where $i: A \subset X, j: X \subset (X, A)$,

$$\begin{array}{ccccccccc} \to {}_1H^{p-1}(A) & \xrightarrow{\delta} & {}_1H^p(X, A) & \xrightarrow{j^*} & {}_1H^p(X) & \xrightarrow{i^*} & {}_1H^p(A) & \to \\ \downarrow\smile v & & \downarrow\smile v & & \downarrow\smile v & & \downarrow\smile v & \\ \to H^{r-1}(A) & \xrightarrow{\delta} & H^r(X, A) & \xrightarrow{j^*} & H^r(X) & \xrightarrow{i^*} & H^r(A) & \to \end{array}$$

Fig. 2

and $r = p + q$. Each vertical homomorphisn is obtained by forming the cup product of each element of the group with v.

To show that this is a homomorphism of the one sequence into the other, commutativity must be shown to hold in each square. This holds for the left hand square by 3.2. For the middle square, we have only to apply 3.4 to the inclusion map $(X; 0, 0) \subset (X; A, 0)$. For the right square, apply 3.4 to the map i: $(A; 0, 0) \subset (X; 0, 0)$ and observe that, by 3.5, $s \smile i^*v = s \smile v$ for $s \in {}_1H^p(A)$.

The situation of §4 now obtains and the operation described there can be applied. Because of the three stage effect in the cohomology sequence, it can be applied in three distinct ways. We shall concentrate on just one of these—the one obtained by allowing G_4 to correspond to ${}_1H^p(X)$. We translate the results of §4 into the new language.

Define ${}_1K^p(i, v)$ to be the subgroup of ${}_1H^p(X)$ which is the intersection of the kernels of i^* and v:

$$(5.1) \qquad u \in {}_1K^p(i, v) \quad \text{if} \quad i^*u = 0 \quad \text{and} \quad u \smile v = 0.$$

Define $L^{r-1}(i, v)$ to be the smallest subgroup of $H^{r-1}(A)$ containing the images of i^* and $\smile v$.

$$L^{r-1}(i, v) = i^*H^{r-1}(X) + {}_1H^{p-1}(A) \smile v, \qquad r = p + q. \tag{5.2}$$

THEOREM 5.3. *Let $u \in {}_1K^p(i, v)$. There exist elements $u' \in {}_1H^p(X, A)$ and $w \in H^{r-1}(A)$ such that*

$$\delta w = u' \smile v \quad \text{and} \quad j^*u' = u. \tag{5.4}$$

In fact, for each u' satisfying $j^*u' = u$, a w exists satisfying 5.4. *The set of those w's, for each of which a u' satisfying* 5.4 *exists, forms a coset of $L^{r-1}(i, v)$ in $H^{r-1}(A)$ denoted by $u \underset{i}{\curlyvee} v$. The operation $u \underset{i}{\curlyvee} v$ from ${}_1K^p(i, v)$ to $H^{r-1}(A)/L^{r-1}(i, v)$ is linear in u.*

The symbol $u \underset{i}{\curlyvee} v$ can be read: the i-cup-product of u and v. The $\underset{i}{\curlyvee}$ plays the role of the dot or cross of a bilinear operation.

THEOREM 5.5. *Let $T: (X', A') \to (X, A)$ and let $T_1 : X' \to X$ and $T_2 : A' \to A$ be the maps defined by T. Let $v \in {}_2H^q(X)$ be fixed. Then the induced homomorphism T_1^* maps ${}_1K^p(i, v)$ into ${}_2K^p(i, T_1^*v)$, and T_2^* maps $L^{r-1}(i, v)$ into $L^{r-1}(i, T_1^*v)$ thereby inducing a homomorphism:*

$$T_2^\# : H^{r-1}(A)/L^{r-1}(i, v) \to H^{r-1}(A)/L^{r-1}(i, T_1^*v).$$

Finally, for each $u \in {}_1K^p(i, v)$, we have

$$T_2^\#(u \underset{i}{\curlyvee} v) = (T_1^*u) \underset{i}{\curlyvee} (T_1^*v).$$

In order that 5.5 should follow from 4.5 it is necessary to check that the mapping T^*, T_1^*, T_2^* of the ladder of Fig. 2 into the corresponding ladder for (X', A') commutes with the operations of the two ladders. Commutativity with the horizontal homomorphisms was observed in 2.3, 2.4; and commutativity with vertical homomorphisms follows from 3.4.

As remarked above, the operation of §4 can be applied in three distinct ways to the homomorphism of Fig. 2. For example, if $u \in H_1^p(X, A)$ is such that $j^*u = 0$ and $u \smile v = 0$, then $u \underset{j}{\curlyvee} v$ is an element of the factor group of $H^{r-1}(X)$ by the subgroup spanning $j^*H^{r-1}(X, A)$ and $H_1^{p-1}(X) \smile v$. Theorems similar to 5.3 and 5.5 hold. This product is included in an extension of the $u \underset{i}{\curlyvee} v$ product given in §12. However this is not the case for the product $u \underset{\delta}{\curlyvee} v$ defined for $u \in {}_1H^p(A)$ such that $\delta u = 0$ and $u \smile v = 0$. We have no applications of this product similar to those to be made of $u \underset{i}{\curlyvee} v$.

The application of the cup product in defining $u \underset{i}{\curlyvee} v$ is unsymmetric. If $u \in {}_1H^p(X)$ is fixed, then the operation $u \smile$ maps the C.S. based on ${}_2H$ into that based on H. However commutativity breaks down due to the $(-1)^p$ in the coboundary formula 3.3 This is remedied by using instead the operation $(-1)^{pq}u \smile$ and we obtain a second i-cup-product of u and v. Suppose $j^*u' = u$ and $j^*v' = v$, and $\delta w = u' \smile v$. If 3.4 is applied to the inclusion map $(X; A, 0) \subset (X; A, A)$ we obtain $u' \smile v = u' \smile v' = u \smile v'$. Thus when the two products can be compared, they differ only in sign.

Assume, as in §3, that ${}_1H$ is a cohomology theory paired with a homology theory ${}_2H$ by a cap product to a homology theory H. Let $v \in {}_1H^q(X)$ be a fixed element. Using cap products we obtain the homomorphism shown in Fig. 3 of the H.S. of (X, A) based on ${}_2H$ into that based on H. To obtain

$$\begin{array}{ccccccccc}
\to & {}_2H_r(X) & \xrightarrow{j_*} & {}_2H_r(X, A) & \xrightarrow{\partial} & {}_2H_{r-1}(A) & \xrightarrow{i_*} & {}_2H_{r-1}(X) & \to \\
 & \downarrow{\scriptstyle \epsilon v \frown} & & \downarrow{\scriptstyle \epsilon v \frown} & & \downarrow{\scriptstyle \epsilon v \frown} & & \downarrow{\scriptstyle \epsilon v \frown} & \\
\to & H_{r-q}(X) & \xrightarrow{j_*} & H_{r-q}(X, A) & \xrightarrow{\partial} & H_{r-q-1}(A) & \xrightarrow{i_*} & H_{r-q-1}(X) & \to
\end{array}$$

Fig. 3

commutativity in the middle square, we choose $\epsilon = (-1)^{qr}$. Define ${}_2K_{r-1}(i, v)$ to consist of elements $z \in {}_2H_{r-1}(A)$ such that $i_*z = 0$ and $v \frown z = 0$. Define

$$L_{r-q}(i, v) = i_*H_{r-q}(A) + v \frown {}_2H_r(X).$$

Just as above, we obtain a product

$$v \mathbin{\underset{i}{\frown}} z \in H_{r-p}(X)/L_{r-p}(i, v), \qquad z \in {}_2K_{r-1}(i, v).$$

The analogs of 5.3 and 5.5 are left to the reader.

6. Further algebraic properties

The assumptions of §5 apply also in this section.

THEOREM 6.1. *Let $u \in {}_1H^p(X)$, and let $v_1, v_2 \in {}_2H^q(X)$ be such that $u \smile v_1 = u \smile v_2 = 0$ and $i^*u = i^*v_1 = i^*v_2 = 0$. Then*

$$u \mathbin{\underset{i}{\overset{\smile}{\vee}}} v_1 + u \mathbin{\underset{i}{\overset{\smile}{\vee}}} v_2 = u \mathbin{\underset{i}{\overset{\smile}{\vee}}} (v_1 + v_2),$$

all three terms being defined and elements of the group $H^{r-1}(A)/i^*H^{r-1}(X)$ where $r = p + q$.

PROOF. Since $s \smile v = s \smile i^*v$ for $s \in {}_1H^{p-1}(A)$, the vanishing of i^*v_1 and i^*v_2 implies that all three groups L^{r-1} reduce to $i^*H^{r-1}(X)$. Choose a u' such that $j^*u' = u$, (see 5.3). Then choose w_1 and w_2 so that $\delta w_1 = u' \smile v_1$ and $\delta w_2 = u' \smile v_2$. It follows that $\delta(w_1 + w_2) = u' \smile (v_1 + v_2)$. Thus 5.4 holds for $w_1 + w_2$, $v_1 + v_2$ in place of w and v. Hence w_1 belongs to the coset $u \mathbin{\underset{i}{\overset{\smile}{\vee}}} v_1$, w_2 belongs to the coset $u \mathbin{\underset{i}{\overset{\smile}{\vee}}} v_2$, and $w_1 + w_2$ to the coset $u \mathbin{\underset{i}{\overset{\smile}{\vee}}} (v_1 + v_2)$.

THEOREM 6.2. *Assume that the cohomology theories ${}_1H$ and ${}_2H$ coincide and that the cup product satisfies the commutation rule $a \smile b = (-1)^{pq} b \smile a$ where p and q are the dimensions of a and b. Let $u \in {}_1H^p(X)$, $v \in {}_2H^q(X)$ be such that $u \smile v = 0$ and $i^*u = i^*v = 0$. Then $v \smile u = 0$, and*

$$u \mathbin{\underset{i}{\overset{\smile}{\vee}}} v = (-1)^{pq}\, v \mathbin{\underset{i}{\overset{\smile}{\vee}}} u,$$

both products being defined and elements of $H^{r-1}(A)/i^*H^{r-1}(X)$ where $r = p + q$.

PROOF. Choose $u' \in {}_1H^p(X, A)$ and $v' \in {}_2H^q(X, A)$ so that $j^*u' = u$ and $j^*v' = v$. If 3.4 is applied to the inclusion map $(X; A, 0) \subset (X; A, A)$ we obtain

$$u' \smile v = u' \smile v' = u \smile v' = (-1)^{pq}\, v' \smile u.$$

Choose w so that $\delta w = u' \smile v$ and therefore $\delta(-1)^{pq}w = v' \smile u$. Thus w is an element of the coset $u \underset{i}{\smile} v$, and $(-1)^{pq}w$ is an element of the coset $v \underset{i}{\smile} u$.

The analog of 6.1 for cap products is

THEOREM 6.3. *Let $z \in {}_2H_{r-1}(A)$ be such that $i^*z = 0$. Let $v_1, v_2 \in {}_1H^q(X)$ be such that $v_1 \frown z = v_2 \frown z = 0$, and both v_1 and v_2 have cap product zero with ${}_2H_r(X)$. Then*

$$v_1 \underset{i}{\frown} z + v_2 \underset{i}{\frown} z = (v_1 + v_2) \underset{i}{\frown} z$$

all three terms being defined and elements of the group $H_{r-q}(X)/i_*H_{r-q}(A)$.

The proof is similar to that of 6.1. There is no analog of 6.2 for cap products.

7. Application to manifolds with boundary

Assume that X is a connected orientable manifold of dimension r with a non-vacuous regular boundary A (i.e. A is a connected orientable $(r - 1)$-manifold and A has a neighborhood in X which is a product space of A with a line segment).

Assume furthermore of the three cohomology theories ${}_1H$, ${}_2H$ and H and the cup product pairing of ${}_1H$, ${}_2H$ to H that the Lefschetz-Poincaré duality theorem in the relative form holds in (X, A). Precisely, if $p + q = r$, each of the three pairings

$$\begin{array}{llll} {}_1H^p(X, A) & \text{with } {}_2H^q(X) & \text{to } H^r(X, A), \\ {}_1H^p(X) & \text{with } {}_2H^q(X, A) & \text{to } H^r(X, A), \\ {}_1H^p(A) & \text{with } {}_2H^{q-1}(A) & \text{to } H^{r-1}(A) \end{array}$$

given by the cup product is completely orthogonal (i.e. either of the first two groups is the group of all continuous homomorphisms of the other into the third group).[3] This assumption is realized in many ways. For example, let G_1, G_2 be character groups of one another—one group discrete, the other compact; let ${}_1H$, ${}_2H$ be based on G_1, G_2 as coefficient group, and let H be based on the real numbers mod 1 as coefficients. Again, let all three cohomology theories be based on a field of coefficients (e.g. the rational numbers). In this case the adjective *continuous*, when applied to *homomorphism*, means *linear*. Finally, let all three cohomology theories be based on the integers as coefficients, and assume that there is no torsion in (X, A).

Since X is an r-manifold with a boundary it follows that $H^r(X) = 0$ and that both $H^{r-1}(A)$ and $H^r(X, A)$ are isomorphic to the coefficient group and

$$\delta: H^{r-1}(A) \approx H^r(X, A). \tag{7.1}$$

From $H^r(X) = 0$, we have $u \smile v = 0$ for any $u \in {}_1H^p(X)$, $v \in {}_2H^q(X)$ with $p + q = r$. Define ${}_1K^p$, ${}_2K^q$ to be the kernels of the homomorphisms $i^*: {}_1H^p(X) \to {}_1H^p(A)$ and $i^*: {}_2H^q(X) \to {}_2H^q(A)$. It follows that $u \underset{i}{\smile} v$ is defined for any $u \in {}_1K^p$, $v \in {}_2K^q$.

[3] This most useful form of the duality theorem has not appeared explicitly in the literature. However it is a direct consequence of the argument given by S. Lefschetz, *Algebraic Topology* (Colloq. Amer. Math. Soc., 1942) to prove his form of the theorem V, 32.2.

From 7.1 and the exactness of the C.S. of (X, A), it follows that $i^*H^{r-1}(X) = 0$. If $v \in {}_2K^q$, then $s \smile v = s \smile i^*v = 0$ for each $s \in {}_1H^{p-1}(A)$. This implies that $L^{r-1}(i, v) = 0$. Thus, if $u \in {}_1K^p$ and $v \in {}_2K^q$, then $u \underset{i}{\smile} v$ is defined and is an element of $H^{r-1}(A)$. By 5.3 and 6.1, the operation $u \underset{i}{\smile} v$ is a pairing of ${}_1K^p$ and ${}_2K^q$ to $H^{r-1}(A)$.

THEOREM 7.2. *Under the above assumptions the pairing* $u \underset{i}{\smile} v$ *of* ${}_1K^p$ *and* ${}_2K^q$ *to* $H^{r-1}(A)$ $(p + q = r)$ *is completely orthogonal.*

PROOF. Let $\phi\colon {}_1K^p \to H^{r-1}(A)$ be any continuous homomorphism. Then $\delta\phi\colon {}_1K^p \to H^r(X, A)$, and $\delta\phi j^*\colon {}_1H^p(X, A) \to H^r(X, A)$ are continuous homomorphisms. Since ${}_1H^p(X, A)$ and ${}_2H^q(X)$ are completely orthogonal, there exists a $v \in {}_2H^q(X)$ such that $\delta\phi j_* u = u \smile v$ for each $u \in {}_1H^p(X, A)$. If $s \in {}_1H^{p-1}(A)$, then $j_*\delta s = 0$ by exactness. Therefore $\delta s \smile v = \delta\phi j^*\delta s = 0$. By 3.2′, $\delta(s \smile i^*v) = \delta(s \smile v) = \delta s \smile v$. Therefore $\delta(s \smile i^*v) = 0$. By 7.1, it follows that $s \smile i^*v = 0$. Since this holds for every s, and ${}_1H^{p-1}(A)$ and ${}_2H^q(A)$ are completely orthogonal, it follows that $i^*v = 0$, i.e. $v \in {}_2K^q$. If $u \in {}_1K^p$, choose a u' such that $j^*u' = u$. Then $\delta\phi u = u' \smile v$, and this implies that $\phi u = u \underset{i}{\smile} v$. Thus, any ϕ can be realized as an i-cup product with a suitable v.

Conversely, let $\psi\colon {}_2K^q \to H^{r-1}(A)$ be any continuous homomorphism. By a similar argument, there exists a $u \in {}_1H^p(X)$ such that $u \smile v' = \delta\psi j^*v'$ for each $v' \in {}_2H^q(X, A)$. Using 3.3, we conclude that $i^*u = 0$, so that $u \in {}_1K^p$. Now let $v \in {}_2K^q$. Choose $u' \in {}_1H^q(X, A)$ and $v' \in {}_2H^q(X, A)$ so that $j^*u' = u$ and $j^*v' = v$. Applying 3.4 to the inclusion map $(X; A, 0) \subset (X; A, A)$ we obtain $u' \smile v = u' \smile v'$. Similarly $u \smile v' = u' \smile v'$. Hence $u' \smile v = u \smile v' = \delta\psi v$. It follows that $u \underset{i}{\smile} v = \psi v$. Thus any ψ can be realized as the i-cup product with a suitable u.

To complete the proof we must show that for any non-zero $v \in {}_2K^q$ there exists a $u \in {}_1K^p$ such that $u \underset{i}{\smile} v \neq 0$, and reciprocally. Suppose $v \neq 0$ is given. Since the pairing of ${}_1H^p(X, A)$ with ${}_2H^q(X)$ to $H^r(X, A)$ is orthogonal, there exists a $u' \in {}_1H^p(X, A)$ such that $u' \smile v \neq 0$. Let $u = j^*u'$. Then $u \underset{i}{\smile} v = \delta^{-1}(u' \smile v) \neq 0$. The other half of the argument is similar.

COROLLARY 7.3. *If the boundary A of the r-manifold X is a sphere, then* $u \underset{i}{\smile} v$ *is defined for all* $u \in {}_1H^p(X)$, $v \in {}_2H^q(X)$ $(p + q = r, p > 0, q > 0)$ *and provides a completely orthogonal pairing of* ${}_1H^p(X)$ *with* ${}_2H^q(X)$ *to* $H^{r-1}(A)$.

Since ${}_1H^p(A) = 0$ for $0 < p < r - 1$, it follows that the kernel ${}_1K^p$ of i^* is the entire group ${}_1H^p(X)$. For $p = r - 1$, this follows from 7.1 and exactness. Similarly ${}_2K^q = {}_2H^q(X)$ for $0 < q < r$. Thus 7.2 applies to give the corollary. The exceptions $p = 0, q = r$ and $p = r, q = 0$ can be eliminated by using the reduced 0-dimensional cohomology groups.

To obtain corresponding results for cap products we make explicit and more satisfactory assumptions. Let ${}_2H$ be the homology theory based on integer coefficients. Then ${}_2H_r(X, A)$ and ${}_2H_{r-1}(A)$ are infinite cyclic groups, and

$$(7.4) \qquad \partial\colon {}_2H_r(X, A) \approx {}_2H_{r-1}(A).$$

Choose a generator z of ${}_2H_r(X, A)$. Then ∂z generates ${}_2H_{r-1}(A)$. Let G be an arbitrary group and let ${}_1H$ and H be the cohomology and homology theory, respectively, based on G as coefficients. Let the cap product pairing of ${}_1H$ with ${}_2H$ to H be based on the natural pairing of the coefficient groups. It follows that the operation $\frown z$ maps ${}_1H^q(X)$ isomorphically onto $H_{r-q}(X, A)$, and maps ${}_1H^q(X, A)$ isomorphically onto $H_{r-q}(X)$. Finally, $\frown \partial z$ maps ${}_1H^{q-1}(A)$ isomorphically onto $H_{r-q}(A)$.

THEOREM 7.5. *Let ${}_1K^q$ be the kernel of i^*: ${}_1H^q(X) \to {}_1H^q(A)$. The operation which sends $v \in {}_1K^q$ into $v \widehat{i} \partial z$ is an isomorphism of ${}_1K^q$ onto $H_{r-q}(X)/i_*H_{r-q}(A)$.*

PROOF. Since A has dimension $r - 1$, we have ${}_2H_r(A) = 0$. This and 7.4 and exactness imply ${}_2H_r(X) = 0$. Since $i^*v = 0$, we have $i^*v \frown \partial z = v \frown \partial z = 0$. Thus the hypotheses of 6.3 (with ∂z in place of z) are satisfied. It follows that the operation $v \xrightarrow{\iota} v \widehat{i} \partial z$ is a homomorphism of ${}_1K^q$ into $H_{r-q}(X)/i_*H_{r-q}(A)$.

Suppose $v \widehat{i} \partial z = 0$. Recalling the definition (see Fig. 3), this means that a $w \in H_{r-q}(X)$ satisfying $j_*w = v \frown z$ is an image $w = i_*t$ for some $t \in H_{r-q}(A)$. Then $v \frown z = j_*w = 0$ by exactness of the H.S. of (X, A). Since the operation $\frown z$ is an isomorphism onto, it follows that $v = 0$. Thus $v \to v \widehat{i} \partial z$ has kernel zero.

Finally, let $w \in H_{r-q}(X)$. Since $\frown z$ maps, ${}_1H^q(X)$ onto $H_{r-q}(X, A)$, there exists a $v \in {}_1H^q(X)$ such that $\epsilon v \frown z = j_*w$. By 3.7, $i^*v \frown \partial z = \pm\partial(v \frown z) = \pm\partial j_*w = 0$. Since $\frown \partial z$ is an isomorphism, it follows that $i^*v = 0$; therefore $v \in {}_1K^q(X)$. Clearly, $v \widehat{i} \partial z$ is the coset containing w. This completes the proof.

COROLLARY 7.6. *If the boundary A of X is a sphere, then the operation $v \to v \widehat{i} \partial z$ maps ${}_1H^q(X)$ $(0 < q < r)$ isomorphically onto $H_{r-q}(X)$.*

This last isomorphism can be obtained directly as a composition of the isomorphism $\frown z$ of ${}_1H^q(X)$ onto $H_{r-q}(X, A)$ followed by the inverse of the isomorphism i_*: $H_{r-q}(X) \to H_{r-q}(X, A)$.

8. Mapping cylinders

Let $f: X' \to X$ be a map. Let $I = [0, 1]$, and let Y be the union of the separated spaces $X' \times I$ and X. We proceed to "match" the subset $X' \times 1$ with its image in X under f. This is done by means of an upper semi-continuous collection in Y. The elements of the collection are first, the single points (x', τ) for $x' \in X'$, $0 \leqq \tau < 1$, and, secondly, the sets $x \cup (f^{-1}(x) \times 1)$ for $x \in X$. The resulting decomposition space is denoted by X_f and is called the *space of the mapping cylinder*[4] of f. We agree to identify each $x' \in X'$ with $(x', 0) \in X_f$ and each $x \in X$ with $x \cup (f^{-1}(x) \times 1) \in X_f$. In this way X and X' are imbedded topologically in X_f. (These identifications lead to difficulty only in case X and X' have common points. In such cases the two homeomorphs of X and X' in X_f are denoted by X_1 and X'_0.) Let

$$i: X' \subset X_f, \qquad k: X \subset X_f$$

be these inclusion maps. Define a map

$$f': X_f \to X$$

[4] This notion is due to J. H. C. Whitehead [10].

by $f'(x', \tau) = f(x')$ for $x' \in X'$, $0 \leqq \tau < 1$, and $f'(x) = x$ for $x \in X$. We refer to f' as the map which *collapses* X_f into X. Clearly f' is an extension of f. We have, then, the diagram of spaces, pairs, and maps of Fig. 4

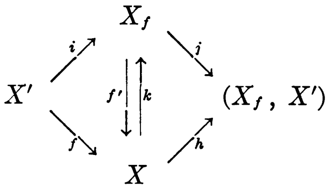

Fig. 4

where h, i, j, k are inclusion maps. The entire configuration is called the *mapping cylinder of f*. (abbreviated: MC(f)).

The basic properties of the MC(f) are:

8.1. $f'i = f$, and $jk = h$.

8.2. $f'k =$ the identity map $X \to X$.

8.3. $kf' \simeq$ the identity map $X_f \to X_f$.

8.4. The two maps f', k form a homotopy equivalence of X and X_f.

The first two statements are trivial. The fourth follows from the second and third. To prove the third, define

$$h(x', \tau, \theta) = (x', (1 - \theta)\tau + \theta) \qquad x' \in X', 0 \leqq \tau < 1, 0 \leqq \theta \leqq 1,$$

$$h(x, \theta) = x \qquad x \in X, 0 \leqq \theta \leqq 1.$$

Then h is a homotopy of the identity map of X_f into the map kf'.

Figure 5 shows the diagram of homology groups, induced homomorphisms and boundary operator associated with the MC(f). The H.S. of (X_f, X')

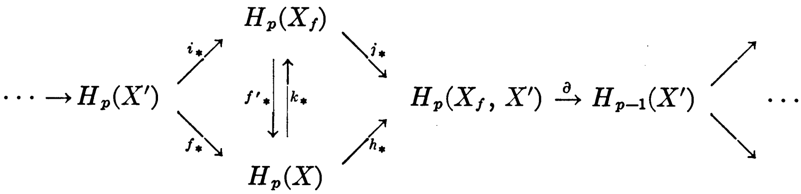

Fig. 5

is obtained by running along the upper line. It follows from 8.4 and 2.5 that *f'_* is an isomorphism onto and k_* is its inverse.* By 8.1, we have $f'_* i_* = f_*$ and $j_* k_* = h_*$. Thus the sequence of groups and homomorphisms obtained by running along the lower line of Fig. 5 is isomorphic to the H.S. of (X_f, X'), and under this isomorphism f_* corresponds to i_*. Thus, from the point of view of algebraic topology, the induced homomorphism i_* is completely equivalent to f_*. It has the advantage that it is bound naturally into an exact sequence.

Definition 8.5. For any homology theory H and map $f: X' \to X$ the *homology* *sequence of f* (abbreviated: H.S. of f) is the infinite sequence of groups and homomorphisms of Fig. 5. The *cohomology sequence of f* (abbreviated: C.S. of f) is defined analogously.

The lower line of Fig. 5 provides an imbedding of the infinite sequence of homomorphisms $f_*: H_p(X') \to H_p(X)$ ($p = 0, \pm 1, \cdots$) into an exact sequence. This can be done in many ways by interpolating groups H_p in place of $H_p(X_f, X')$ and suitable homomorphisms in place of h_*, ∂. The group H_p is only partially determined by the condition of exactness: it must be an extension of the factor group $H_p(X)/f_*H_p(X')$ by the kernel of $f_*: H_{p-1}(X') \to H_{p-1}(X)$. Any such set of extensions could be used. The important feature of the extensions chosen, $H_p(X_f, X')$, is that they are defined naturally and are additional algebraic invariants of f. The precise meaning of this naturality and invariance is clarified in the next section.

The following example shows that the H.S. of f is not determined by the homomorphisms f_* alone. Let X be a 2-sphere, and X' a projective plane. Let f map X' into a point x_0 of X. Let g map a projective line L of X' into x_0 and map the 2-cell $X' - L$ topologically onto $X - x_0$. Let the homology theory H be based on integer coefficients. Then $f_* = g_*$ in all dimensions, and are trivial except in the dimension 0. However it is readily seen that $H_2(X_f, X')$ is the direct sum of an infinite cyclic group and a cyclic group of order 2 while $H_2(X_g, X')$ is an infinite cyclic group.

Assume now that ${}_1H$, ${}_2H$ are cohomology theories paired by a cup product to a third H. Using the MC(f) we shall extend the functional product $u \underset{i}{\smile} v$ defined for inclusion maps in §5 to general maps. Let $f: X' \to X$. If $v \in {}_2H^q(X)$, define ${}_1K^p(f, v)$ to be the subgroup of elements u in ${}_1H^p(X)$ satisfying

$$u \smile v = 0, \qquad f^*u = 0. \tag{8.6}$$

Define

$$L^{r-1}(f, v) = f^*H^{r-1}(X) + {}_1H^{p-1}(X') \smile f^*v, \qquad r = p + q. \tag{8.7}$$

It is readily proved that f'^* maps, ${}_1K^p(f, v)$ isomorphically onto ${}_1K^p(i, f'^*v)$ (see 5.1), and $L^{r-1}(f, v) = L^{r-1}(i, f'^*v)$ (see 5.2). Thus we can make the

Definition 8.8. For each $u \in {}_1K^p(f, v)$ define $u \underset{f}{\smile} v$ by $u \underset{f}{\smile} v = (f'^*u) \underset{i}{\smile} (f'^*v)$.

Since f'^* is linear, the linearity of $u \underset{i}{\smile} v$ in u implies the linearity of $u \underset{f}{\smile} v$ in u. Similarly, the linearity in v, as expressed in 6.1, carries over to the extended product. The commutation rule of 6.2 with i replaced by f is also immediate.

The $\underset{i}{\frown}$ product also extends to an $\underset{f}{\frown}$ product. If $v \in {}_1H^q(X)$ is fixed, define ${}_2K_{r-1}(f, v)$ to be the subgroup of elements z of ${}_2H_{r-1}(X')$ such that

$$f^*v \frown z = 0, \qquad f_*z = 0. \tag{8.9}$$

Define

$$L_{r-q}(f, v) = f_*H_{r-q}(X') + v \frown {}_2H_r(X). \tag{8.10}$$

Then ${}_2K_{r-1}(f, v) = {}_2K_{r-1}(i, f'^*v)$ and k_* maps $L_{r-q}(f, v)$ isomorphically onto $L_{r-q}(i, f'^*v)$.

DEFINITION 8.11. For each $z \in {}_2K_{r-1}(f, v)$ define $v \frown_f z$ by $v \frown_f z = f'_*((f'^*v) \frown_i z)$. Then $v \frown_f z$ is linear in z. The analog of 6.3 is that linearity in v holds for v's satisfying $v \frown {}_2H_r(X) = 0$.

In 8.8 we have a second definition of $u \smile_f v$ in case f is an inclusion map $A \subset X$. It is necessary to check that 8.8 agrees with 5.3 in this case. If we compare 5.1 and 5.2 with 8.6 and 8.7 it is clear that the two products have the same domain and the same range. Denote by A_0 the portion of the MC(f) consisting of points (x, τ) with $x \in A$, $\tau = 0$. (We cannot identify A_0 with A since $A \subset X$). The collapsing map $f'\colon X_f \to X$ defines maps $f'_0\colon A_0 \to A$ and $f'_1\colon (X_f, A_0) \to (X, A)$. By 5.5, the product $u \smile_f v$, as defined in 5.3, has, as image under $f_0'^{\#}$, the product $(f'^*u) \smile_i (f'^*v)$. This last is the definition of $u \smile_f v$ given in 8.8. Since f'_0 is the natural identification of A_0 with A, the assertion is proved.

Observe that the definitions of the domains and ranges of the products $\smile_f$, $\frown_f$ are made without recourse to the MC(f). In particular, if f is a homeomorphism, all four reduce to zero. To obtain non-trivial products one must have an f whose f^* has non-trivial kernels, and has images which are proper subgroups.

9. Transformations of maps

If $f\colon X' \to X$ and $g\colon Y' \to Y$, a *transformation* $T\colon f \to g$ is a pair of maps (t', t):

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{9.1}$$

such that $tf(x') = gt'(x')$ for each $x' \in X'$.

We assign to T a map of the MC(f) into that of g as follows. Define

$$\begin{aligned} t''(x', \tau) &= (t'(x'), \tau), & x' \in X', 0 \leqq \tau < 1, \\ t''(x) &= t(x), & x \in X. \end{aligned} \tag{9.2}$$

Then $t''\colon X_f \to Y_g$, and $t'' \mid X = t$, and $t'' \mid X' = t'$. Denote by $\bar{t}\colon (X_f, X') \to (Y_g, Y')$ the map of the pairs defined by t''. Thus the four maps t, t', t'', $\bar{t}$ carry the spaces of Fig. 4 into the corresponding spaces of the MC(g). It is easily checked that these maps commute with the maps of the two mapping cylinders. (There are six trivial relations to verify.) If H is a homology theory, it follows that the induced homomorphisms t_*, t'_*, t''_*, $\bar{t}_*$ commute with the homomorphisms of the homology sequences of f and g. Thus T induces a homomorphism of the H.S. of f into that of g which we denote by T_*.

Transformations have properties similar to properties of maps. Thus $T\colon f \to f$ is called the *identity* if $t\colon X \subset X$, $t'\colon X' \subset X'$ are identities. By 9.2, t'' is likewise an identity. Hence

9.3. *If $T\colon f \to f$ is the identity, then T_* is the identity map of the* H.S. *of f.*

If $T_1 : f_1 \to f_2$ and $T_2 : f_2 \to f_3$, their *composition* $T_3 : f_1 \to f_3$ is the pair of maps $t_3 = t_2 t_1$, $t'_3 = t'_2 t'_1$. By 9.2, $t''_3 = t''_2 t''_1$. Hence

9.4. *If $T_1 : f_1 \to f_2$ and $T_2 : f_2 \to f_3$, then $(T_2 T_1)_* = T_{2*} T_{1*}$.*

If $T\colon f \to g$ is such that both t, t' are topological maps, define $T^{-1} = (t^{-1}, t'^{-1})$.

Then $T^{-1}: g \to f$ and TT^{-1}, $T^{-1}T$ are the identity maps of g, f respectively. In this case we say that f and g are *topologically equivalent* and T is an equivalence. By 9.3 and 9.4 we have

9.5. *If $T: f \to g$ is a topological equivalence, then T_* maps the* H.S. *of f isomorphically onto that of g.*

This last is a precise formulation of the statement that the H.S. of f is a topological invariant of f.

Similar propositions hold for cohomology.

Assume now the situation where cup products are defined. The invariance of the product $\underset{f}{\smile}$ is expressed in the following

THEOREM 9.6. *Let T be a transformation of a map f into a map g as in* 9.1; *and let $v \in {}_2H^q(Y)$. Then t^* maps ${}_1K^p(g, v)$ into ${}_1K^p(f, t^*v)$, and t'^* maps $L^{r-1}(g, v)$ into $L^{r-1}(f, t^*v)$ thereby inducing a homomorphism*

$$t'^{\#}: H^{r-1}(Y')/L^{r-1}(g, v) \to H^{r-1}(X')/L^{r-1}(f, t^*v).$$

Finally, for each $u \in {}_1K^p(g, v)$, we have

$$t'^{\#}(u \underset{g}{\smile} v) = (t^*u) \underset{f}{\smile} (t^*v).$$

PROOF. Now $u \smile v = 0$ implies, by 3.4, that $t^*u \smile t^*v = 0$; and $g^*u = 0$ and the commutativity relation $tf = gt'$ imply $f^*t^*u = 0$. This proves the first statement. An element of $L^{r-1}(g, v)$ has the form $g^*w + s \smile g^*v$ for some $w \in H^{r-1}(Y)$ and $s \in {}_1H^{p-1}(Y)$. Apply t^* to this element, use 3.4 and the relation $tf = gt'$. The result is $f^*t^*w + t'^*s \smile f^*t^*v$ which is an element of $L^{r-1}(f, t^*v)$. For the last statement, let $i: Y' \subset Y_g$ and $i': X' \subset X_f$ be inclusions. Applying 8.8 and the relation $tf' = g't''$, we have

$$(t^*u) \underset{f}{\smile} (t^*v) = (f'^*t^*u) \underset{i'}{\smile} (f'^*t^*v) = (t''^*g'^*u) \underset{i'}{\smile} (t''^*g'^*v).$$

Now apply 5.5 to the map $\bar{t}$ (in place of T) and the elements g'^*u, g'^*v (in place of u, v). Then the term on the right above becomes

$$t'^{\#}((g'^*u) \underset{i}{\smile} (g'^*v)) = t'^{\#}(u \underset{g}{\smile} v) \qquad \text{by 8.8.}$$

Two special cases of 9.6 are useful. In the first, let $X = Y$ and $t =$ identity. In the second, let $X' = Y'$ and $t' =$ identity.

COROLLARY 9.7. *Let $t': X' \to Y'$, and $g: Y' \to Y$. Let $u \in {}_1H^p(Y)$ and $v \in {}_2H^q(Y)$ be such that $u \smile v = 0$ and $g^*u = 0$. Then*

$$t'^{\#}(u \underset{g}{\smile} v) = u \underset{gt'}{\smile} v.$$

COROLLARY 9.8. *Let $f: X' \to X$, and $t: X \to Y$. Let $u \in {}_1H^p(Y)$ and $v \in {}_2H^q(Y)$ be such that $u \smile v = 0$ and $f^*t^*u = 0$. Then, if $t': X' \to X'$ is the identity,*

$$t'^{\#}(u \underset{tf}{\smile} v) = (t^*u) \underset{f}{\smile} (t^*v).$$

For cap products 9.6 has the analog.

THEOREM 9.9. *If* $T: f \to g$, *and* $v \in {}_1H^q(Y)$, *then* t'_* *maps* ${}_2K_{r-1}(f, t^*v)$ *into* ${}_2K_{r-1}(g, v)$, *and* t_* *induces a homomorphism*

$$t_{\#} : H_{r-q}(X)/L_{r-q}(f, t^*v) \to H_{r-q}(Y)/L_{r-q}(g, v).$$

For each $z \in {}_2K_{r-1}(f, t^*v)$, *we have*

$$t_{\#}((t^*v) \underset{f}{\frown} z) = v \underset{g}{\frown} (t'_*z).$$

10. Invariance under homotopy

Let f_0 and f_1 be two homotopic maps $X' \to X$ and let $f: X' \times I \to X$ be a homotopy of f_0 into f_1. For each $\theta \in I$, define $t'_\theta: X' \to X' \times I$ by $t'_\theta(x') = (x', \theta)$. Let $t: X \subset X$. Since $f(x, 0) = f_0(x)$, it follows that t_0, t form a transformation $T_0 : f_0 \to f$. Similarly, t'_1, t form $T_1 : f_1 \to f$. Associated with T_θ are the maps t''_θ and $\bar{t}_\theta$ (see 9.1).

THEOREM 10.1. *For* $\theta = 0$ *and* 1, $\bar{t}_\theta: (X_{f_\theta}, X') \to (X_f, X' \times I)$ *is a homotopy equivalence. Hence* $T_{\theta*}$ *induces an isomorphism of the* H.S. *of* f_θ *onto that of* f.

COROLLARY 10.2. *Homotopic maps have mapping cylinders which are homotopically equivalent. Hence their homology and cohomology sequences are isomorphic.*

COROLLARY 10.3. *If* $f_0 \simeq f_1$ *and either of* $u \underset{f_0}{\smile} v$, $u \underset{f_1}{\smile} v$ *are defined, then both are defined and they are equal. The same holds for* $v \underset{f_0}{\frown} z$ *and* $v \underset{f_1}{\frown} z$.

PROOF. Because of symmetry, it suffices to prove the case $\theta = 0$. It is readily verified that $\bar{t}_0$ is a homeomorphism of (X_{f_0}, X') with the subspace of $(X_f, X' \times I)$ consisting of X and points (x', θ, τ) for which $\theta = 0$. It suffices to exhibit a deformation retraction of $(X_f, X' \times I)$ into this subspace. Let I' be the τ-interval used in constructing the MC(f). Then X_f is obtained as an upper semi-continuous collection on the disjoint union $(X' \times I \times I') \cup X$. One readily constructs a deformation retraction of $I \times I'$ into the subset $(I \times 1) \cup (0 \times I')$ such that $I \times 0$ is deformed over itself into 0×0. For example, this is accomplished by radial projection from the point (2, 0) in the (θ, τ)-plane onto the lines $\theta = 0$ and $\tau = 1$. Using this deformation in each section $x' \times I \times I'$, there results a deformation retraction of $X' \times I \times I'$ into $(X' \times I \times 1) \cup (X' \times 0 \times I')$, and $X' \times I \times 0$ is deformed over itself into $X' \times 0 \times 0$. Since the points of $X' \times I \times I'$ which are identified with points of X in X_f are in $X' \times I \times 1$ and these points are fixed under the deformation, it follows that this deformation induces a deformation of X_f which clearly has the required properties.

The first corollary follows from the theorem and the fact that homotopy equivalence of spaces is a symmetric and transitive relation. The second corollary follows from 9.6 and 9.9.

REMARK. One can prove the isomorphic character of T_{0*} directly. Since t is the identity map of X, t_* is the identity map of $H_p(X)$. It is obvious that $t'_0: X' \to X' \times I$ is a homotopy equivalence, hence $t'_{0*}: H_p(X') \approx H_p(X' \times I)$. Since T_{0*} is a homomorphism of the H.S. of f_0 into that of f_1, and t_*, t'_{0*} are isomorphisms, it follows from the "five lemma" [1, Ch. 1, 4.3] that $\bar{l}_{0*}$ is also an isomorphism.

THEOREM 10.4. *If the map $f: X' \to X$ is homotopic to a constant, then all products $u \underset{f}{\smile} v$ and $v \underset{f}{\frown} z$ which are defined are zero.*

PROOF. By 10.3, we can assume that f is a constant, and f can be factored into the composition of two maps $f_1 : X' \to P$, $f_2: P \to X$ where P is a single point. Since the groups of P are zero in dimensions $\neq 0$, all $\underset{f_2}{\smile}$ products are zero if of dimension $\neq 0$. Since f_2^* maps $H^0(X)$ onto $H^0(P)$, we have $H^0(P)/L^0(f_2, v) = 0$; so every 0-dimensional $\underset{f_2}{\smile}$ is zero. If $u \underset{f}{\smile} v$ is defined, then $u \smile v = 0$ and $f^*u = f_1^* f_2^* u = 0$. This implies $f_2^* u = 0$ since $H^p(P) = 0$ $(p \neq 0)$, and f_1^* maps $H^0(P)$ isomorphically into $H^0(X')$. Therefore $u \underset{f_2}{\smile} v$ is defined. Since it is zero, it follows from 9.7 that $u \underset{f}{\smile} v = 0$.

If $v \underset{f}{\frown} z$ is defined, then $f_* z = f_{2*} f_{1*} z = 0$. Since the groups of P are zero save for $H_0(P)$, and f_{2*} maps this group isomorphically into $H_0(X)$, it follows that $f_{1*} z = 0$. Hence $(f_2^* v) \underset{f_1}{\frown} z$ is defined. Since the groups of P vanish except in the dimension 0, $(f_2^* v) \underset{f_1}{\frown} z$ can be different from zero only if it has dimension 0 and v has dimension 0. But this requires z to have dimension -1, so $z = 0$. By 9.9, we have $v \underset{f}{\frown} z = f_{1\#}((f_2^* v) \underset{f_1}{\frown} z) = f_{1\#}(0) = 0$.

11. Application to sphere bundles

In this section it is assumed that X is a compact orientable n-manifold without boundary and that X' is an orientable sphere bundle with base space X and fibres of dimension $k > 0$. Since X' is locally a product space over X, it follows that X' is a compact manifold of dimension $n + k$. Since X is orientable and the bundle is orientable, it follows that X' is orientable as a manifold. Let $f: X' \to X$ be the natural projection of the bundle into its base space.

The structure of the mapping cylinder X_f is readily described. Since each fibre is a k-sphere and f maps it into a point, the portion of X_f corresponding to a single fibre is a $(k + 1)$-cell. Thus X_f is fibred into $(k + 1)$-cells and is a bundle over X with projection $f': X_f \to X$. It follows that X_f is itself an orientable $(n + k + 1)$-manifold having X' as a regular boundary.

Assume now that ${}_1H$, ${}_2H$ are cohomology theories paired to a third, H, by a cup product so that the Lefschetz-Poincaré duality theorem holds in (X_f, X') (see §7). Define ${}_1K^p(X)$, ${}_2K^q(X)$ to be the kernels of $f^*: {}_1H^p(X) \to {}_1H^p(X')$ and $f^*: {}_2H^q(X) \to {}_2H^q(X')$. Let the integers p, q satisfy

$$p + q = n + k + 1. \tag{11.1}$$

Since the composition of $i: X' \subset X_f$ and $f': X_f \to X$ is f, it follows that f'^* maps ${}_1K^p(X)$ and ${}_2K^q(X)$ isomorphically onto the kernels ${}_1K^p$ and ${}_2K^q$ of i^* (see §7). From 8.8 and 7.2, we obtain

THEOREM 11.2. *Under the above assumptions, the pairing $u \underset{f}{\smile} v$ of ${}_1K^p(X)$, ${}_2K^q(X)$ to $H^{n+k}(X')$ is completely orthogonal.*

Consider now the special case where both X and X' are themselves spheres. It has been shown [7] that, in this case, the fibres have dimension $n - 1$ and, therefore, dim $X' = 2n - 1$. Hopf [6] has given examples of this for $n = 2$, 4 and 8. Since the groups of an r-sphere are zero except in the dimension 0 and r, we have ${}_1K^p(X) = 0$ $(p \neq n)$, and ${}_1K^n(X) = {}_1H^n(X)$. Then 11.2 becomes

THEOREM 11.3. *If S, S' are spheres of dimensions n, $2n - 1$, and S' is a sphere bundle over S with projection $f: S' \to S$, then the pairing $u \underset{f}{\smile} v$ of ${}_1H^n(S)$, ${}_2H^n(S)$ to $H^{2n-1}(S)$ is completely orthogonal.*

In particular, suppose ${}_1H$, ${}_2H$ and H coincide and have integer coefficients; and the cup product is defined by the natural pairing of the coefficients. Since S, S' are spheres, they have no torsion. It follows that (S_f, S') has no torsion. Hence duality holds in (S_f, S') with integer coefficients. Then 11.3 specializes to

COROLLARY 11.4. *Using integer coefficients, let u be a generator of the cyclic group $H^n(S)$. Then $u \underset{f}{\smile} u$ is a generator of the cyclic group $H^{2n-1}(S)$.*

To obtain analogous results for the $\underset{f}{\frown}$ pairing we make the same assumptions on the cohomology theory ${}_1H$ and the homology theories ${}_2H$ and H as those preceding 7.5. Then 7.5 yields

THEOREM 11.5. *Let X be an orientable k-sphere bundle over the orientable n-manifold X, and $f: X' \to X$ the projection. Let z' be a generator of the cyclic group ${}_2H^{n+k}(X')$, and let ${}_1K^q(X)$ be the kernel of $f^*: {}_1H^q(X) \to {}_1H^q(X')$. Then the operation which sends $v \in {}_1K^q(X)$ into $v \underset{f}{\frown} z'$ is an isomorphism of ${}_1K^q(X)$ onto $H_r(X)/f_*H_r(X')$ $(r = n + k - q + 1)$.*

This theorem is a mild translation of the central theorem of Gysin's work on sphere bundles [4]. To obtain his form of the theorem one must eliminate cohomology as follows. Let z generate $H_n(X)$. The operation $v \to v \frown z$ maps ${}_1K^q(X)$ isomorphically onto a subgroup $K_{n-q}(X)$ of $H_{n-q}(X)$. Let $p = n - q$. Then Gysin's assertion is that $K_p(X)$ is isomorphic to $H_r(X)/f_*H_r(X')$ $(r = p + k + 1)$. Our description of $K_p(X)$ differs from Gysin's but can be proved equivalent.

EXAMPLE. The result 11.4 leads to the following example of a non-trivial functional product where all elements involved are of order 2. Let S^2 be a 2-sphere and let P^3 be the rotation group of S^2. Let $x_0 \in S^2$ be fixed, and define $h: P^3 \to S^2$ by $h(r) = r(x_0)$ for $r \in P^3$. Now P^3 is equivalent to real projective 3-space. Let S^3 be a 3-sphere, and $f: S^3 \to P^3$ the double covering. Then the composition $g = hf: S^3 \to S^2$ is the Hopf fibre mapping. Using integer coefficients throughout, let u_1 be a generator of $H^2(S^2)$, and let u be the non-zero element of $H^2(P^3)$ which is a cyclic group of order 2. Then $h^*u_1 = u$. By 11.4, $u_1 \underset{g}{\smile} u_1$ is a generator of $H^3(S^3)$. If t is the identity map of S^3, by 9.8, we have $t^{\#}(u_1 \underset{g}{\smile} u_1) = u \underset{f}{\smile} u$. Since f is the double covering, $t^{\#}$ is just reduction mod 2. Thus $u \underset{f}{\smile} u$ is the non-zero element of $H^3(S^3)/f^*H^3(P^3)$.

12. Extension of functional products to the relative case

It was observed in §3, that, if $(X; A, B)$ is a triad and $u \in {}_1H^p(X, A)$, $v \in {}_2H^q(X, B)$, then $u \smile v$ is an element of $H^r(X, A \cup B)$ for $r = p + q$. Let $f: (X'; A', B') \to (X; A, B)$ be a map of one triad in another, and suppose $u \smile v = 0$, and $f^*u = 0$ in ${}_1H^p(X', A')$. It is to be expected that a product

$u \underset{f}{\smile} v$ can be defined as an element of $H^{r-1}(X', A' \cup B')/L^{r-1}(f, v)$ where

$$L^{r-1}(f, v) = f^*H^{r-1}(X, A \cup B) + H^{p-1}(X', A') \smile f^*v.$$

This is the case and we will indicate briefly the procedure.

Denote by X_f, A_f, B_f the mapping cylinders of the maps $X' \to X$, $A' \to A$ and $B' \to B$ that f defines. We can regard A_f, B_f as subspaces of X_f in a natural way. The diagram of maps shown in Fig. 6 is the

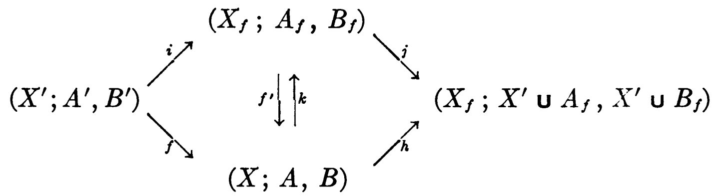

Fig. 6

generalization of the mapping cylinder of Fig. 4. The maps h, i, j, k are inclusions and f' is the collapsing map of §8. Just as in §8, we have

12.1. $f = f'i$ and $h = jk$.

12.2. The pair of maps f', k form a homotopy equivalence of the two triads.

As a corollary:

12.3. The homology and cohomology groups of (X, A), (X, B) and $(X, A \cup B)$ are isomorphic, respectively, to the groups of (X_f, A_f), (X_f, B_f) and $(X_f, A_f \cup B_f)$ under the homomorphisms induced by f and k.

Consider now the inclusion

$$l: (X'; A', B') \subset (X' \cup A_f \cup B_f; A_f, B_f).$$

12.4. The homology and cohomology groups of (X', A'), (X', B') and $(X', A' \cup B')$ are isomorphic, respectively, to the groups of $(X' \cup A_f, A_f)$, $(X' \cup B_f, B_f)$ and $(X' \cup A_f \cup B_f, A_f \cup B_f)$ under the homomorphisms induced by l.

We prove the first case, the others are similar. Delete from the pair $(X' \cup A_f, A_f)$ the points of A_f for which the τ-parameter exceeds 1/2 (including all points of A). By 2.7, this excision induces isomorphisms of all groups. The excised pair is retracted into (X', A') by the deformation obtained by allowing the τ-interval [0, 1/2] to contract into 0. Thus the excised pair and (X', A') are homotopically equivalent, so their groups are isomorphic.

Now let $v \in {}_2H^q(X, B)$ be fixed, and let $f'^*v = v' \in {}_2H^q(X_f, B_f)$. If $u \in {}_1H^p(X, A)$ satisfies $u \smile v = 0$ and $f^*u = 0$, it follows from 12.1 and 12.3 that $u' \smile v' = 0$ and $i^*u' = 0$.

The generalization of Fig. 2 is shown in Fig. 7.

$$\begin{array}{ccccccc}
{}_1H^{p-1}(X', A') & \xrightarrow{\delta} & {}_1H^p(X_f, X' \cup A_f) & \xrightarrow{j^*} & {}_1H^p(X_f, A_f) & \xrightarrow{i^*} & {}_1H^p(X', A') \\
\downarrow{\smile v'} & & \downarrow{\smile v'} & & \downarrow{\smile v'} & & \downarrow{\smile v'} \\
H^{r-1}(X', A' \cup B') & \xrightarrow{\delta} & H^r(X_f, X' \cup A_f \cup B_f) & \xrightarrow{j^*} & H^r(X_f, A_f \cup B_f) & \xrightarrow{i^*} & H^r(X', A' \cup B')
\end{array}$$

Fig. 7

The homomorphisms i^*, j^* are induced by inclusion maps. The operation δ on the upper line is the composition $\delta' m^* l^{*-1}$:

$$ {}_1H^{p-1}(X', A') \xleftarrow{l^*} {}_1H^{p-1}(X' \cup A_f, A_f) \xrightarrow{m^*} {}_1H^{p-1}(X' \cup A_f) \xrightarrow{\delta'} {}_1H^p(X_f, X' \cup A_f) $$

where m is the indicated inclusion map, δ' is the ordinary coboundary, and l^{*-1} exists by 12.4. This δ is called the *coboundary operator* of the ordered triad $(X_f ; X', A_f)$. The upper line of Fig. 7 is called the *cohomology sequence* of $(X_f ; X', A_f)$. As proved in [1, Ch. 1, 14.4], this sequence is exact. Similarly, the lower line of Fig. 7 is the cohomology sequence of the ordered triad $(X_f ; X', A_f \cup B_f)$, and is also exact.

Commutativity in the middle and right squares of Fig. 7 follow readily from 3.4. Commutativity in the left square is not so obvious. This is proved in [1, Ch. 15] for the singular cohomology theory for general spaces, and for the Čech cohomology theory on compact Hansdorff spaces. It can be deduced for triangulable spaces from the properties listed in §3.

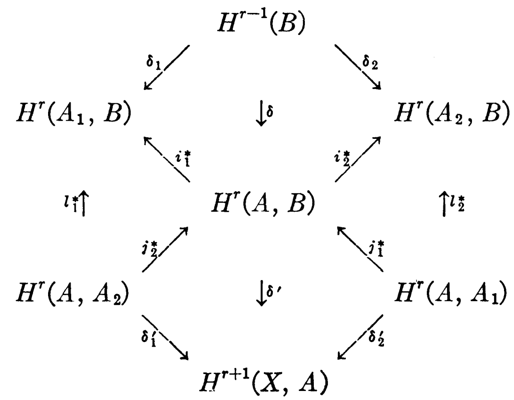

Fig. 8

Having established that $\smile v'$ is a homomorphism of one exact sequence into another, the product $u' \underset{i}{\smile} v'$ is defined (§4) and is an element of $H^{r-1}(X', A' \cup B')/L^{r-1}(i, v')$. It follows from 12.1 and 12.3 that $L^{r-1}(i, v') = L^{r-1}(f, v)$. Thus we define $u \underset{f}{\smile} v$ by

$$ u \underset{f}{\smile} v = u' \underset{i}{\smile} v'. $$

The results of §6 concerning bilinearity, and the results of §§9, 10, concerning invariance under transformations and homotopy, carry over to this generalized product. The details are left to the reader.

13. Relations with coboundary

We shall obtain formulas for $\delta(u \underset{f}{\smile} v)$ analogous to the formulas 3.2′, 3.3′. An essential step is provided by a general proposition which we consider first.

Let $(X; A_1, A_2)$ be a triad, and define

$$ A = A_1 \cup A_2, \qquad B = A_1 \cap A_2 . $$

In Fig. 8 we have an associated diagram of groups and homomorphisms. The maps i, j, l are all inclusions. The coboundary operations issuing from $H^{r-1}(B)$ are those of the indicated pairs, and $\delta_1', \delta_2', \delta'$ are those of the indicated triples. We shall assume moreover that $(X; A_1, A_2)$ is a *proper* triad, i.e. that the excision maps l_1, l_2 induce isomorphisms onto. Then the coboundary operators of the ordered triads $(X; A_1, A_2)$ and $(X; A_2, A_1)$ are defined:

$$\bar{\delta}_\alpha = \delta'_\alpha l_\alpha^{*-1} \qquad (\alpha = 1, 2).$$

THEOREM 13.1. *Under the above assumptions, for each $w \in H^{r-1}(B)$ we have*

$$\bar{\delta}_1\delta_1 w = -\bar{\delta}_2\delta_2 w.$$

This is a corollary of the purely group theoretic "hexagonal lemma" see [1, Ch. 1, 15.1]. It is only necessary to verify that (1) commutativity holds in each triangle of Fig. 8, (2) $\delta'\delta = 0$, and (3) the image of j_α^* is the kernel of i_α^* $(\alpha = 1, 2)$. Commutativity is obvious. To prove (2) observe that δ' is a composition of j^*: $H^r(A, B) \to H^r(A)$ followed by an ordinary coboundary. Since $j^*\delta = 0$ by exactness, (2) follows. The fact that the C.S. of the triple (A, A_α, B) is exact embodies (3).

THEOREM 13.2. *Let f: $(X', A') \to (X, A)$, and let $f_1 : A' \to A$ be the map defined by f. If $u \in {}_1H^p(A)$ and $v \in {}_2H^q(X)$ are such that $u \smile v = 0$ and $f_1^* u = 0$, then both $u \underset{f_1}{\smile} v$ and $(\delta u) \underset{f}{\smile} v$ are defined. Furthermore δ', the coboundary operator of (X', A'), maps $L^{r-1}(f_1, v)$ into $L^r(f, v)$ $(r = p + q)$ thereby inducing a homomorphism*

$$\delta'^{\#} : H^{r-1}(A')/L^{r-1}(f_1, v) \to H^r(X', A')/L^r(f, v),$$

and

$$\delta'^{\#}(u \underset{f_1}{\smile} v) = -(\delta u) \underset{f}{\smile} v.$$

PROOF. By 3.2′, $u \smile v = 0$ implies $\delta u \smile v = 0$, and $f^*\delta u = \delta' f_1^* u = 0$. So both products are defined. Since $\delta' f_1^* = f^*\delta$, and $\delta'(s \smile f^*v) = \delta' s \smile f^*v$, it follows that δ' maps $L^{r-1}(f_1, v)$ into $L^r(f, v)$. Figure 9 shows the groups and homomorphisms involved in constructing the two products.

$$\begin{array}{ccccccc}
H^{r-1}(A') & \xrightarrow{\delta_1} & H^r(A_f, A') & \xleftarrow{\smile v'} & {}_1H^p(A_f, A') & \xrightarrow{j_1^*} & {}_1H^p(A_f) \\
\downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta_3} & & \downarrow{\scriptstyle \delta_3} & & \downarrow{\scriptstyle \delta} \\
H^r(X', A') & \xrightarrow{\delta_2} & H^{r+1}(X_f, X' \cup A_f) & \xleftarrow{\smile v'} & {}_1H^{p+1}(X_f, X' \cup A_f) & \xrightarrow{j^*} & {}_1H^{p+1}(X_f)
\end{array}$$

Fig. 9

All coboundary operators have been shown to exist save for δ_3, the coboundary operator of the ordered triad $(X_f ; A_f, X')$. To prove its existence, one must show that the groups of (A_f, A') and $(X' \cup A_f, X')$ are isomorphic under the inclusion map. But this follows from the fact that X' is a deformation retract of the closed neighborhood in $X' \cup A_f$ consisting of points of τ-value $\leqq 1/2$. Thus full excision holds in $(X' \cup A_f, X')$ (see [1, Ch. 1, 12.2]).

Let $u' = f'^* u$ and $v' = f'^* v$. Then a representative of $u \underset{f_1}{\smile} v$ is an element w of $H^{r-1}(A')$ such that a $u_1' \in H^p(A_f, A')$ exists with the properties $\delta_1 w = u_1' \smile v'$

and $j_1^* u_1' = u'$. Now $\delta u' = f'^* \delta u$ and $j^* \delta_3 u_1 = \delta j_1^* u_1$ follow from 2.4. The commutativity relation $\delta_3(u_1' \smile v') = \delta_3 u_1' \smile v'$ is proved in [1, Ch. 15]. However anti-commutativity holds in the left square of Fig. 9. This follows from 13.1 applied to the triad $(X_f ; X', A_f)$. It follows that $-\delta' w$ lies in the coset $\delta u \underset{f}{\smile} v$.

THEOREM 13.3. *Let f: $(X', A') \to (X, A)$, and let $f_1 : A' \to A$ be the map defined by f. If $u \in {}_1H^p(X)$ and $v \in {}_2H^q(A)$ are such that $u \smile v = 0$ and $f^* u = 0$, then $u \underset{f_1}{\smile} v$ and $u \underset{f}{\smile} (\delta v)$ are both defined. If, in addition, $f_1^* v = 0$, then δ' induces a homomorphism*

$$\delta'^{\#} : H^{r-1}(A')/f_1^* H^{r-1}(A) \to H^r(X', A')/f^* H^r(X, A)$$

and

$$\delta'^{\#} (u \underset{f_1}{\smile} v) = -(-1)^p \, u \underset{f}{\smile} (\delta v).$$

The proof is similar to that of 13.2. The condition $f_1^* v = 0$ disposes of the term $H^{p-1}(A') \smile f_1^* v$ which may not behave well under δ'.

14. Review of squaring operations

Let ${}_1G$ and G be abelian groups and suppose a pairing of ${}_1G$ with itself to G is given which is commutative (i.e. a bilinear operation $g_1 \cdot g_2 \in G$ is defined for $g_1, g_2 \in {}_1G$ and $g_1 \cdot g_2 = g_2 \cdot g_1$). Let G_2 denote the factor group of G by its subgroup of elements divisible by 2: $G_2 = G/2G$. The pairing $g_1 \cdot g_2$ followed by the natural homomorphism ξ: $G \to G_2$ yields a self-pairing $\xi(g_1 \cdot g_2)$ of ${}_1G$ to G_2.

Now let ${}_1H$, H, H_2 be cohomology theories having the coefficient groups ${}_1G$, G, G_2 respectively. A set of *squaring operations* is a collection of homomorphisms, one for each pair (X, A) and each pair of integers p, α:

$$\mathrm{Sq}^\alpha : {}_1H^p(X, A) \to H^{p+\alpha}(X, A) \qquad \alpha \text{ odd},$$

$$\mathrm{Sq}^\alpha : {}_1H^p(X, A) \to H_2^{p+\alpha}(X, A) \qquad \alpha \text{ even}.$$

These homomorphisms satisfy the following conditions

14.1. If f: $(X', A') \to (X, A)$, then $f^* \mathrm{Sq}^\alpha = \mathrm{Sq}^\alpha f^*$ for all p, α.

14.2. If δ is the coboundary operator of (X, A), then $\delta \mathrm{Sq}^\alpha u = \mathrm{Sq}^\alpha \delta u$ for each $u \in {}_1H^p(A)$.

14.3. $\mathrm{Sq}^\alpha = 0$ if $\alpha < 0$.

14.4. If $\alpha > p$, and $u \in {}_1H^p(X, A)$, then $\mathrm{Sq}^\alpha u = 0$.

14.5. If $u \in {}_1H^p(X, A)$, then $\mathrm{Sq}^p u = u \smile u$ if p is odd, and $\mathrm{Sq}^p u = \xi(u \smile u)$ if p is even (ξ is reduction mod 2).

The existence of Sq^α corresponding to a prescribed self-pairing of ${}_1G$ to G was proved in [8]. The operation denoted by Sq_i in [8] is related to the present Sq^α by $\mathrm{Sq}^\alpha = \mathrm{Sq}_i$ where $\alpha = p - i$. The squaring operations were derived from a cochain product $u \underset{i}{\smile} v$, and $\mathrm{Sq}_i u = u \underset{i}{\smile} u$. It was shown that $\mathrm{Sq}^\alpha u$ is always of order 2 even for an odd α. It is not known if the conditions 14.1 to 14.5 are sufficient to characterize the operations Sq^α.

15. The functional squaring operations

The operations Sq^α lead to operations $\smile_{f\alpha}$ just as cup products lead to $\smile_f$-products. Let f: $(X', A') \to (X, A)$, and let ${}_1K^p(f, \alpha)$ denote the subgroup of elements $u \in {}_1H^p(X, A)$ such that

$$\mathrm{Sq}^\alpha u = 0, \qquad f^*u = 0. \tag{15.1}$$

Adopting the notations of §12 for the mapping cylinder, let $u' = f'^*u$. Then, by 14.1, $\mathrm{Sq}^\alpha u' = 0$ and $i^*u' = 0$.

Figure 10 shows the homomorphism Sq^α of the C.S. of the ordered triad $(X_f\,;\, X', A_f)$ based on ${}_1H$ into the same based on H. In this case α is odd.

$$\begin{array}{ccccccc}
{}_1H^{p-1}(X', A') & \xrightarrow{\delta} & {}_1H^p(X_f, X' \cup A_f) & \xrightarrow{j_*} & {}_1H^p(X_f, A_f) & \xrightarrow{i_*} & {}_1H^p(X', A') \\
\downarrow{\scriptstyle Sq^\alpha} & & \downarrow{\scriptstyle Sq^\alpha} & & \downarrow{\scriptstyle Sq^\alpha} & & \downarrow{\scriptstyle Sq^\alpha} \\
H^{p+\alpha-1}(X', A') & \xrightarrow{\delta} & H^{p+\alpha}(X_f\,;\, X' \cup A_f) & \xrightarrow{j_*} & H^{p+\alpha}(X_f, A_f) & \xrightarrow{i_*} & H^{p+\alpha}(X', A')
\end{array}$$

Fig. 10

When α is even, replace H by H_2 in the lower line of Fig. 10. Commutativity in the middle and right square holds by 14.1. Since δ decomposes into $\delta' m^* l^{*-1}$ as in Fig. 7, commutativity in the left square follows from 14.1 and 14.2.

Since u' belongs to the intersection of the kernels of i^* and Sq^α, the operation of §4 can be applied to u' yielding a coset of $L^{p+\alpha-1}(f, \alpha)$ in $H^{p+\alpha-1}(X', A')$ which is denoted by $\smile_f^\alpha u$. Thus

$$\smile_f^\alpha u \in H^{p+\alpha-1}(X', A')/L^{p+\alpha-1}(f, \alpha)$$

where

$$L^{p+\alpha-1}(f, \alpha) = f^*H^{p+\alpha-1}(X, A) + \mathrm{Sq}^\alpha\{{}_1H^{p-1}(X', A')\}.$$

This holds when α is odd; when α is even, replace H by H_2. From 4.3, we have

15.2. $\smile_f^\alpha u$ is linear in u.

By 14.3, we have

15.3. If $\alpha < 0$, then $\smile_f^\alpha u = 0$.

By 14.4, we have

15.4. If $u \in {}_1H^p(X, A)$ and $\alpha > p$, then $\smile_f^\alpha u = 0$.

By 14.5, we have

15.5. If $u \in {}_1H^p(X, A)$, then $\smile_f^p u = u \smile_f u$ if p is odd, and $\smile_f^p u = \xi(u \smile_f u)$ if p is even.

Note that, in this case, the term $\mathrm{Sq}^p\{{}_1H^{p-1}(X', A')\}$ is zero by 14.4 so that $\smile_f^p u$ and $u \smile_f u$ lie in the same group.

THEOREM 15.6. *Let* f: $(X', A') \to (X, A)$, *and* g: $(Y', B') \to (Y, B)$; *and let* $T: f \to g$ *be a transformation (see §9). Then* t^* *maps* ${}_1K^p(g, \alpha)$ *into* ${}_1K^p(f, \alpha)$, *and* t'^* *induces a homomorphism*

$$t'^{\#} : H^r(Y', B')/L^r(g, \alpha) \to H^r(X', B')/L^r(f, \alpha), \qquad r = p + \alpha - 1.$$

Finally, for each $u \in {}_1K^p(g, \alpha)$,

$$t'^{\#} \smile_g^\alpha u = \smile_f^\alpha t^*u.$$

The proof is similar to that of 9.6. The transformation T induces a map of the MC(f) into that of g, and thereby induces a homomorphism of the diagram of Fig. 10 for (Y, B), g into the same for $(X, A), f$. All commutativity relations are readily proved, and 4.5 applies.

As corollaries of 15.6 we have the analogs of 9.7 and 9.8:

COROLLARY 15.7. *Let* $t'\colon (X', A') \to (Y', B')$, *and* $g\colon (Y', B') \to (Y, B)$. *Then* ${}_1K^p(g, \alpha) \subset {}_1K^p(gt', \alpha)$, *and*

$$t'^{\#} \underset{g}{\smile}{}^{\alpha} u = (\underset{gt'}{\smile})^{\alpha} u, \qquad u \in {}_1K^p(g, \alpha).$$

COROLLARY 15.8. *Let* $f\colon (X', A') \to (X, A)$, *and* $t\colon (X, A) \to (Y, B)$. *Then* t^* *maps* ${}_1K^p(tf, \alpha)$ *into* ${}_1K^p(f, \alpha)$, *and*

$$t'^{\#} (\underset{tf}{\smile})^{\alpha} u = \underset{f}{\smile}{}^{\alpha} t^* u \qquad u \in {}_1K^p(tf, \alpha)$$

where t' *is the identity map of* (X', A').

THEOREM 15.9. *If* $f_0, f_1 : (X', A') \to (X, A)$, *and* $f_0 \simeq f_1$, *then, if either* $\underset{f_0}{\smile}{}^{\alpha} u$ *or* $\underset{f_1}{\smile}{}^{\alpha} u$ *is defined, both are defined and they are equal.*

This is also a corollary of 15.6. It is only necessary to observe that the proof of 10.1 shows that the map $\bar{t}_\theta\colon (X_{f_\theta}; X', A_{f_\theta}) \to (X_f; X' \times I, A_f)$ is a homotopy equivalence of the two triads.

COROLLARY 15.10. *If f is homotopic to a constant map, then* $\underset{f}{\smile}{}^{\alpha} u = 0$ *for all* α *and* u.

The proof is similar to that of 10.4.

THEOREM 15.11. *Let* $f\colon (X', A') \to (X, A)$, *and let* $f_1 : A' \to A$ *be the map defined by f. Then δ maps* ${}_1K^p(f_1, \alpha)$ *into* ${}_1K^{p+1}(f, \alpha)$, *and δ' induces a homomorphism*

$$\delta'^{\#} : H^r(A')/L^r(f_1, \alpha) \to H^{r+1}(X', A')/L^{r+1}(f, \alpha), \qquad r = p + \alpha - 1$$

and

$$\delta'^{\#} \underset{f_1}{\smile}{}^{\alpha} u = -\underset{f}{\smile}{}^{\alpha} \delta u \qquad u \in {}_1K^p(f_1, \alpha).$$

The proof is based on the diagram obtained from Fig. 9 by replacing $\smile v'$ by Sq^α and setting $r = p + \alpha$. Commutativity in the right square holds as before. Commutativity in the middle square is proved by expanding it into three squares according to the definition of δ_3 as a composition of three homomorphisms, and then applying 14.1 and 14.2. In the left square, anti-commutativity holds by 13.1.

16. Computation of products using cochains

We shall give formulas for the functional products and squares using cochains.

For any space X, let $S(X)$ denote the total singular complex of X [see 1, Ch. 7]. For any pair (X, A) and any singular cohomology theory H, let $C^p(X, A)$ denote the group of cochains of $S(X)$ which are zero on $S(A)$ (used in defining $H^p(X, A)$). Let $Z^p(X, A)$ be the subgroup of cocycles, and $B^p(X, A)$ the subgroup of coboundaries of $C^{p-1}(X, A)$. Then $H^p(X, A) = Z^p(X, A)/B^p(X, A)$. If $f\colon (X', A') \to (X, A)$, let $f^{\#} : C^p(X, A) \to C^p(X', A')$ be the cochain mapping induced by f.

Suppose, as in §12, that $f: (X'; A', B') \to (X; A, B)$, $u \in {}_1H^p(X, A)$ and $v \in {}_2H^q(X, B)$ are such that $f^*u = 0$ and $u \smile v = 0$. Choose representative cocycles $u_1 \in {}_1Z^p(X, A)$, $v_1 \in {}_2Z^p(X, B)$ of u, v. Since $u \smile v = 0$, there exists a cochain $b \in C^{r-1}(X, A \cup B)$ $(r = p + q)$ such that $\delta b = u_1 \smile v_1$. Since $f^*u = 0$, there exists a cochain $a \in {}_1C^{p-1}(X', A')$ such that $\delta a = f^{\#}u_1$. Define w_1 by

$$w_1 = f^{\#}b - a \smile f^{\#}v_1. \tag{1}$$

THEOREM 16.1. *The* $(p + q - 1)$*-cochain* w_1*, defined in* (1)*, is a cocycle of* $S(X')$ *and is zero on* $S(A' \cup B')$*. Its cohomology class* w *is an element of the coset* $u \underset{f}{\curlyvee} v$ *of* $L^{r-1}(f, v)$.

PROOF. Applying standard coboundary relations, we have

$$\delta w_1 = \delta f^{\#}b - \delta(a \smile f^{\#}v_1) = f^{\#}\delta b - (\delta a) \smile f^{\#}v_1$$
$$= f^{\#}(u_1 \smile v_1) - f^{\#}u_1 \smile f^{\#}v_1 = 0.$$

Thus, w_1 is a cocycle.

Suppose $a_1 \in {}_1C^{p-1}(X', A')$ and $\delta a_1 = f^{\#}u_1$. Define w_1' as in (1) using a_1 instead of a. Then $a_1 - a \in {}_1Z^{p-1}(X', A')$, and $w_1 - w_1' = (a_1 - a) \smile f^{\#}v_1$. Therefore, altering a alters w_1 by a cocycle representing an element of ${}_1H^{p-1}(X', A') \smile f^*v$. Since this group is contained in $L^{r-1}(f, v)$, it suffices to prove 16.1 for any suitable choice of a.

For each step of the construction of $u \underset{f}{\curlyvee} v$ (see §12) we choose representative cochains as follows. Clearly

$$u_2 = f'^{\#}u_1 \in {}_1Z^p(X_f, A_f), \qquad v_2 = f'^{\#}v_1 \in {}_2Z^q(X_f, A_f)$$

are cocycles representing f'^*u and f'^*v. In addition,

$$u_2 \smile v_2 = \delta f'^{\#}b.$$

Consider now the inclusion maps

$$l: (X'; A', B') \subset (X' \cup A_f; A_f, B_f), \quad k: (X' \cup A_f; A_f, B_f) \subset (X_f; A_f, B_f).$$

Clearly $f'kl = f$. Since $f^*u = 0$, and the excision map l induces isomorphisms, it follows that $k^*f'^*u = 0$. Hence $k^{\#}u_2 = \delta_1 a'$ for some $a' \in {}_1C^{p-1}(X' \cup A_f, A_f)$ where δ_1 denotes coboundary in $S(X' \cup A_f)$. Extend a' to a cochain $a'' \in {}_1C^{p-1}(X_f, A_f)$ by giving a'' the value zero on any singular simplex of $S(X_f)$ which is not in $S(X' \cup A_f)$. Then $k^{\#}a'' = a'$. It follows that $k^{\#}(u_2 - \delta a'') = 0$. Therefore $u_2 - \delta a'' \in {}_1Z^p(X_f, X' \cup A_f)$, and, if $u' \in {}_1H^p(X_f, X' \cup A_f)$ is its cohomology class, then $j^*u' = f'^*u$. Observe also that

$$\delta l^{\#}a' = l^{\#}\delta_1 a' = l^{\#}k^{\#}u_2 = f^{\#}u_1.$$

Define $a = l^{\#}a'$. As proved above, we can suppose the w' of (1) is defined using this a. Then

$$l^{\#}k^{\#}(f'^{\#}b - a'' \smile f'^{\#}v_1) = f^{\#}b - a \smile f^{\#}v_1 = w_1, \tag{2}$$

and

$$\delta(f'^{\#}b - a'' \smile f'^{\#}v_1) = (u_2 - \delta a'') \smile f'^{\#}v_1. \tag{3}$$

Recall now the definition of δ: $H^{r-1}(X', A' \cup B') \to H^r(X_f, X' \cup A_f \cup B_f)$. If $w \in H^{r-1}(X', A' \cup B')$, we choose a representative cocycle w_1, then we extend w_1 to a cochain $z \in C^{r-1}(X_f, A_f \cup B_f)$ and then form $\delta z \in Z^r(X_f, X' \cup A_f \cup B_f)$. By (2), $z = f'^{\#} b - a'' \smile f'^{\#} v'$ is such an extension; and by (3), the cohomology class of δz is $u' \smile f'^* v$ where $j^* u' = f'^* v$. Hence $\delta w = u' \smile f'^* v$. This proves 16.1.

THEOREM 16.2. *If $(X'; A', B')$ and $(X; A, B)$ are simplicial complexes, and the map f is simplicial, and w' in (1) is a simplicial cochain constructed from simplicial cochains u', v', a, b, then the cohomology class of w' is likewise a representative of $u \underset{f}{\smile} v$.*

PROOF. If X is the space of a simplicial complex K, then $K \subset S(X)$ and any cochain on $S(X)$ defines one on K. This transformation ϕ: $C^p(X) \to C^p(K)$ is known to be a cochain equivalence and induces an isomorphism $H^p(X) \approx H^p(K)$. If u_1, v_1, a, b are as in (1), then

$$\phi w_1 = f^{\#} \phi b - (\phi a) \smile f^{\#} \phi v_1$$

and ϕw_1 also represents $u \underset{f}{\smile} v$. Thus 16.2 holds for simplicial cochains which are ϕ-images. To complete the proof, one need only show that the cohomology class mod $L^{r-1}(f, v)$ of w_1 is independent of the choice of representatives u_1, v_1, a, b as simplicial cochains.

If u_1, v_1 are fixed, one can vary the choice of a and b only by cocyles. From the form of (1), it follows that the class of w varies by elements of ${}_1H^{p-1}(X', A') \smile f^* v$ and $f^* H^{r-1}(X, A \cup B)$ which are in $L^{r-1}(f, v)$.

Suppose selections u_1, v_1, a, b are made, and the choice u_1 is altered by a coboundary: $u_1' = u_1 + \delta s$. Let $b' = b + s \smile v_1$ and $a' = a + f^{\#} s$. Then the corresponding w_1' is

$$\begin{aligned} w_1' &= f^{\#} b' - a' \smile f^{\#} v_1 = f^{\#}(b + s \smile v_1) - (a + f^{\#} s) \smile f^{\#} v_1 \\ &= f^{\#} b + f^{\#} s - f^{\#} v_1 - a \smile f^{\#} v_1 - f^{\#} s \smile f^{\#} v_1 = w_1 . \end{aligned}$$

Thus, altering the choice of u_1 does not alter w_1 if the a and b are correspondingly altered.

Suppose selections u_1, v_1, a, b are made, and v_1 is altered by a coboundary: $v_1' = v_1 + \delta t$. Let $b' = b + (-1)^p u_1 \smile t$. Then $\delta b' = u_1 \smile v_1'$, and the corresponding w_1' becomes

$$w_1' = f^{\#} b' - a \smile f^{\#} v_1' = w_1 + (-1)^p \delta(a \smile f^{\#} t).$$

Thus w_1 is altered by a coboundary.

Now any choice u_1, v_1, a, b can be carried into any other by simple alterations of the type just considered. Since the class mod $L^{r-1}(f, v)$ of w_1 does not change under simple alterations, it will not change under any. This proves 16.2.

There is a corresponding cochain construction for the functional squaring operations. Suppose f: $(X', A') \to (X, A)$ and that $u \in {}_1H^p(X, A)$ satisfies 15.1. Let $u_1 \in {}_1Z^p(X, A)$ be a representative of u. Let $i = p - \alpha$. Then there exists a $b \in C^{p+\alpha-1}(X, A)$ such that $\delta b = u_1 \smile_i u_1$. This is taken mod 2 when α is even. Similarly there exists an $a \in {}_1C^{p-1}(X', A')$ such that $\delta a = f^{\#} u_1$.

Define w_1 by

$$w_1 = f^{\#}b - a \cup_{i-1} a - a \cup_i \delta a. \tag{4}$$

THEOREM 16.3. *The* $(p + \alpha - 1)$*-cochain* w_1, *defined in* (4), *is a cocycle of* $S(X')$ *and is zero on* $S(A')$. *Its cohomology class* w *is an element of the coset* $\underset{f}{\cup}^{\alpha} u$ *of* $L^{p+\alpha-1}(f, \alpha)$. *When* α *is even this holds* mod 2. *The same is true using simplicial cochains when* (X', A'), (X, A) *are simplicial pairs and* f *is simplicial.*

PROOF. By [8, formula 6.3], we have

$$\begin{aligned} \delta w_1 &= f^{\#}\delta b - \delta(a \cup_i \delta a + a \cup_{i-1} a) \\ &= f^{\#}(u_1 \cup_i u_1) - \delta a \cup_i \delta a + \epsilon(a \cup_{i-2} a + a \cup_{i-1} \delta a) \end{aligned}$$

where $\epsilon = 0$ for an odd α, and $\epsilon = \pm 2$ for an even α. Since $f^{\#}(u_1 \cup_i u_1) = \delta a \cup_i \delta a$, we have $\delta w_1 = 0$ for α odd, and $\delta w_1 \equiv 0 \pmod 2$ for α even.

Suppose a is varied by adding a cocycle $z \in {}_1Z^{p-1}(X', A')$. The new w_1 is

$$w_1' = f^{\#}b - (a + z) \cup_i f^{\#}u_1 - (a + z) \cup_{i-1} (a + z).$$

Applying the bilinearity of these products and the coboundary formula [6, 5.1] we obtain

$$w_1 - w_1' = z \cup_{i-1} z - (-1)^p \delta(z \cup_i a) + [(-1)^\alpha + 1]z \cup_{i-1} a.$$

The last term is zero or zero mod 2 according as α is odd or even. It follows that the cohomology class of w_1 has been altered by an element of $\mathrm{Sq}^\alpha\{{}_1H^{p-1}(X', A')\}$. Thus it suffices to prove 16.3 for any convenient choice of a.

For each step of the construction of $\underset{f}{\cup}^{\alpha} u$, we choose representative cocycles. Let $u_2 = f'^{\#}u_1$. Then $u_2 \cup_i u_2 = \delta f'^{\#}b$. Define a', a'' and a as in the proof of 16.1. Let $u_2' = u_2 - \delta a''$. Then $u_2' \in {}_1Z^p(X_f, X' \cup A_f)$, and $j^*u' = f'^*u$ where u' is the class of u_2'. Define w_2 by

$$w_2 = f'^{\#}b + a'' \cup_{i-1} a'' + a'' \cup_i \delta a'' - (-1)^p \delta a'' \cup_{i+1} u_2 .$$

Applying the coboundary formula [8, 5.1], we obtain

$$\begin{aligned} \delta w_2 &= (u_1 - \delta a'') \cup_i (u_1 - \delta a'') \\ &\quad -[(-1)^i + (-1)^p](a'' \cup_{i-2} a'' + a'' \cup_{i-1} \delta a'' - (-1)^p \delta a'' \cup_i u_2). \end{aligned} \tag{5}$$

The lower line is zero (zero mod 2) if α is odd (even). It follows that w_3 defined by

$$w_3 = l^{\#}k^{\#}w_2 = f^{\#}b + a \cup_{i-1} a + a \cup_i \delta a - (-1)^p \delta a \cup_{i+1} \delta a \tag{6}$$

is a cocycle (cocycle mod 2) on $S(X')$ and is zero on $S(A')$. If w is its cohomology class, we have, by (5), that $\delta w = \mathrm{Sq}^\alpha u'$. Thus w belongs to the coset $\underset{f}{\cup}^{\alpha} u$. It remains to show that w_1 of (4) is cohomologous to w_3. But $w_1 - w_3 = (-1)^p \delta a \cup_{i+1} \delta a - 2(a \cup_{i-1} a + a \cup_i \delta a)$, and, by [8, 6.3], this coincides with

$$(-1)^p \delta(a \cup_i a + a \cup_{i+1} \delta a)$$

when α is odd, and is congruent to it mod 2 when α is even. Thus $w_1 \sim w_3$, and the proof that w_1 represents $\underset{f}{\smile}^{\alpha} u$ is complete.

In the simplicial case, it is clear that (4) gives the proper cocycle provided a and b are ϕ-images (see proof of 16.2). To complete the proof it is only necessary to show that a change in the choice of u_1, a, b alters the cohomology class of w_1 only by elements of $L^{p+\alpha-1}(f, \alpha)$. This is similar to the proof of 16.2, and is omitted.

The cochain form for $\underset{f}{\smile}^{\alpha} u$ enables us to establish

THEOREM 16.4. *If* $\underset{f}{\smile}^{\alpha} u$ *is defined, then* $2 \underset{f}{\smile}^{\alpha} u = 0$.

PROOF. If α is even, the result is trivial since the coefficients of $\underset{f}{\smile}^{\alpha} u$ are reduced mod 2. Let α be odd. Consider first the case of the singular theory. Select cochains u_1, a, b and define w_1 as in (4). Since $\alpha = p - i$ is odd, we have, by [8, 6.3], that

$$2(f^{\#} b - a \smile_{i-1} a - a \smile_i \delta a)$$
$$= f^{\#}(2b - (-1)^p u \smile_{i+1} u) + (-1)^p \delta(a \smile_i a + a \smile_{i+1} \delta a).$$

By [8, 6.2], $2b - (-1)^p u \smile_{i+1} u$ is a cocycle. Thus $2w_1$ is the sum of a coboundary and the image of a cocycle. Hence its cohomology class is in $L^{p+\alpha-1}(f, \alpha)$. This proves 16.4 for the singular theory. In particular it holds on complexes. This implies that it holds for the Čech theory since it holds on each approximating complex.

17. The Hopf invariant

Let $f: S' \to S$ be a map of a $(2n - 1)$-sphere S' into an n-sphere S $(n > 1)$. Using cohomology groups based on integer coefficients, let u and u' be generators of $H^n(S)$, $H^{2n-1}(S')$ respectively. Since $L^{2n-1}(f, u) = 0$ (see 8.7), we have, for some integer $\gamma(f)$, that

$$u \underset{f}{\smile} u = \gamma(f) u'. \tag{1}$$

In view of 16.1 and the discussion in the introduction, we may refer to $\gamma(f)$ as the Hopf invariant of f (there is a difference in sign which is irrelevant since the sign of $\gamma(f)$ depends on the choice of u).

The known properties of the Hopf invariant are readily deduced from the definition (1). Its invariance under a homotopy of f follows from the invariance of $u \underset{f}{\smile} u$. In particular, if f is inessential, then, by 10.4, $\gamma(f) = 0$.

If n is odd, by 6.2, $u \underset{f}{\smile} u = -u \underset{f}{\smile} u$; hence $\gamma(f)$ is zero.

If f is the projection of a representation of S' as a sphere bundle over S, then, by 11.4, $\gamma(f) = \pm 1$. By the preceding result, this can happen only if n is even.

If S_1' is a $(2n - 1)$-sphere, and $g: S_1' \to S'$, then, by 9.7, $g^*(u \underset{f}{\smile} u) = u \underset{fg}{\smile} u$. Therefore $\gamma(fg) = \gamma(f) \cdot \text{degree}(g)$.

If S_1 is an n-sphere, and $h: S \to S_1$, and u_1 is a generator of $H^n(S_1)$, then, by 9.8,

$$u_1 \underset{hf}{\smile} u_1 = (h^* u_1) \underset{f}{\smile} (h^* u_1) = (u \underset{f}{\smile} u) d^2$$

where $d = \text{degree}(h)$. Therefore $\gamma(hf) = \gamma(f) d^2$.

Suppose f_1, f_2: $S' \to S$ are two maps of a 3-sphere into a two sphere, and $\gamma(f_1) = \gamma(f_2)$. Let $f_0 : S' \to S$ be the Hopf fibre mapping. As shown by Pontrjagin,[5] the map $f_i (i = 1, 2)$ can be factored into a map $g_i : S' \to S'$ followed by f_0. Choose u' so that $\gamma(f_0) = 1$. Since $\gamma(f_0 g_i) =$ degree (g_i), it follows that g_1 and g_2 have equal degrees; hence they are homotopic. This implies that f_1 and f_2 are homotopic. Thus, in the case $n = 2$, $\gamma(f)$ is characteristic of the homotopy class.

18. A generalization of the Hopf invariant

Let f: $S' \to S$ be a map of an $(n + \alpha - 1)$-sphere into an n-sphere. The *suspension* (= Einhängung) of f is a map g: $\hat{S}' \to \hat{S}$ of an $(n + \alpha)$-sphere into an $(n + 1)$-sphere obtained by regarding S, S' as equators of $\hat{S}$, $\hat{S}'$ and extending f as follows. Each $x \in S$ determines a longitudinal semi-circle C_x. Then g is the map which carries each $C'_{x'}$ isometrically onto $C_{f(x')}$.

Let E_+, E_- denote the two closed hemispheres into which S divides $\hat{S}$. For any cohomology theory, the following homomorphisms are isomorphisms onto $(n > 0)$:

$$H^n(S) \xrightarrow{\delta} H^{n+1}(E_+, S) \xleftarrow{l^*} H^{n+1}(\hat{S}, E_-) \xrightarrow{j^*} H^{n+1}(\hat{S}). \tag{18.1}$$

The maps j, l are the indicated inclusions. For the proof see [1, Ch. 3, 3.2]. If $u \in H^n(S)$, we will call $j^* l^{*-1} \delta u$ the *suspension* of u.

THEOREM 18.2. *Let f: $S' \to S$ be a map of an $(n + \alpha - 1)$-sphere $(\alpha \geqq 2)$ into an n-sphere, and let $u \in {}_1H^n(S)$. If g: $S' \to S$ is the suspension of f, and if $v \in {}_1H^{n+1}(S)$ is the suspension of u, then $\underset{g}{\smile}^{\alpha} v$ is the suspension of $-\underset{f}{\smile}^{\alpha} u$.*

The proof is based on the diagram of Fig. 11.

$$\begin{array}{ccccccc}
{}_1H^n(S) & \xrightarrow{\delta} & {}_1H^{n+1}(E_+, S) & \xleftarrow{l^*} & {}_1H^{n+1}(\hat{S}, E_-) & \xrightarrow{j^*} & {}_1H^{n+1}(\hat{S}) \\
\downarrow \smile_{f}^{\alpha} & & \downarrow \smile_{g_1}^{\alpha} & & \downarrow \smile_{g_2}^{\alpha} & & \downarrow \smile_{g}^{\alpha} \\
H^{r-1}(S') & \xrightarrow{\delta} & H^r(E'_+, S') & \xleftarrow{l'^*} & H^r(\hat{S}', E'_-) & \xrightarrow{j'^*} & H^r(\hat{S}')
\end{array}$$

Fig. 11

Here $r = n + \alpha$, and g_1, g_2 are maps defined by g. Since $\alpha > 1$, we have $r > n + 1$; hence the groups L^r, L^{r-1} vanish, and the groups ${}_1K^{n+1}$, ${}_1K^n$ are the entire cohomology groups. Thus $\underset{f}{\smile}^{\alpha}$, $\underset{g_1}{\smile}^{\alpha}$, $\cdots$ have the indicated domains and ranges. Since $gj = j'g_2$, and $g_1 l = l'g_2$, commutativity holds in the right and

[5] *A classification of mappings of a 3-complex into a 2-sphere*, Rec. Math. [Mat. Sbornik] N.S. 9 (51) (1941) Theorem 1. The following is a simpler proof. Let $\phi: E \to S'$ be a representation of S' as a 3-cell E with its boundary $\dot{E}$ pinched to a point. By the covering homotopy theorem [7], the map $f_1\phi$ can be factored into $\phi_1 : E \to S'$ followed by f_0. Then $\phi_1(\dot{E})$ lies on a single fibre S^1. Since $\pi_2(S^1) = 0$, $\phi_1 \mid \dot{E}$ can be contracted on S^1 to a point. Extend this to a homotopy of ϕ_1 into a map ψ. Then ψ can be factored into ϕ followed by a map g: $S' \to S'$. Then $f_0 g$ is homotopic to f_1. A second covering homotopy deforms g into a map g_1 such that $f_0 g_1 = f_1$.

middle squares by 15.6. In the left square, anti-commutativity holds by 15.11. This implies 18.2.

COROLLARY 18.3. *If* $\smile_{f^\alpha}^\alpha u$ *is non-zero, so also is* $\smile_{g^\alpha}^\alpha v$.

We may now generalize the Hopf invariant as follows. Let S, S' be spheres of dimensions n and $n + \alpha - 1$ where $\alpha > 1$. Using cohomology groups based on integer coefficients, let u be a generator of $H^n(S)$. Let u' be the non-zero element of $H_2^{n+\alpha-1}(S')$ based on integers mod 2 as coefficients. Let the pairing of coefficients be ordinary multiplication reduced mod 2. If f is a map $f: S' \to S$, then $\smile_f^\alpha u$ is an element of $H_2^{n+\alpha-1}(S')$ since $L^{n+\alpha-1}(f, \alpha) = 0$ (see §15). Hence, for some integer $\gamma_\alpha(f) = 0$ or 1 we have

$$\smile_f^\alpha u = \gamma_\alpha(f)u'. \tag{18.4}$$

We shall call $\gamma_\alpha(f)$ *the generalized Hopf invariant of* f.

By 15.9 and 15.10, we have

18.5. *If* f *is homotopic to* g, *then* $\gamma_\alpha(f) = \gamma_\alpha(g)$. *In particular, if* f *is inessential, then* $\gamma_\alpha(f) = 0$.

By 15.5, we have

18.6. *If* f *is a map of a* $(2n - 1)$*-sphere into an* n*-sphere, then* $\gamma_n(f) \equiv \gamma(f) \bmod 2$.

By 18.2, we have

18.7. *If* f *is a map of an* $(n + \alpha - 1)$*-sphere into an* n*-sphere, and* $\hat{f}$ *is the suspension of* f, *then* $\gamma_\alpha(f) = \gamma_\alpha(\hat{f})$.

18.8. *If* α *is odd, then* $\gamma_\alpha(f) = 0$.

PROOF. Since α is odd, we can define $\smile_f^\alpha u$ in the group $H^{n+\alpha-1}(S')$ without reducing mod 2. If we can prove this element to be zero, then reducing mod 2 will show that 18.4 is zero. Now

$$\delta: H^{n+\alpha-1}(S') \approx H^{n+\alpha}(S_f, S')$$

since the cohomology groups of S, and hence of S_f, vanish in the dimensions $n + \alpha$ and $n + \alpha - 1$. It suffices, therefore, to prove $\delta\smile_f^\alpha u = 0$. By definition (§15), $\delta\smile_f^\alpha u = \mathrm{Sq}^\alpha u_1$ where $u_1 \in H^n(S_f, S')$ satisfies $j^*u_1 = f'^*u$. By [8, 6.10], $\mathrm{Sq}^\alpha u_1$ is of order 2 in $H^{n+\alpha}(S_f, S')$. But this group is infinite cyclic, so $\mathrm{Sq}^\alpha u_1 = \delta\smile_f^\alpha u = 0$.

18.9. *If* S' *is an* $(n + \alpha - 1)$*-sphere,* S *and* S_1 *are* n*-spheres,* $f: S' \to S$, *and* $h: S \to S_1$ *has degree* d, *then*

$$\gamma_\alpha(hf) = d \cdot \gamma_\alpha(f) \qquad \bmod 2.$$

PROOF. Let u_1 be a generator of $H^n(S_1)$. Then $h^*u_1 = du$. By 15.8 and 15.2,

$$\gamma_\alpha(hf)u' = \smile_{hf}^\alpha u_1 = \smile_f^\alpha(h^*u_1) = \smile_f^\alpha(du) = d\smile_f^\alpha u = d\gamma_\alpha(f)u'.$$

18.10. *If* S_1', S' *are* $(n + \alpha - 1)$*-spheres,* S *an* n *sphere.* $f\cdot S' \to S$, *and* g: $S_1' \to S'$ *has degree* d, *then*

$$\gamma_\alpha(fg) = d \cdot \gamma_\alpha(f) \qquad \bmod 2.$$

This follows directly from 15.7.

18.11. *If f is a map of a $(2n - 1)$-sphere into an n-sphere, and f' is any map obtained from f by a series of suspensions, then* $\gamma_n(f') \equiv \gamma(f) \bmod 2$.

This follows immediately from 18.6 and 18.7.

18.12. *If f_1, f_2 are two maps of an $(n + 1)$-sphere into an n-sphere, then f_1 is homotopic to f_2 if and only* $\gamma_2(f_1) = \gamma_2(f_2)$.

Proof. Half of the statement follows from 18.5. To prove the second half, we use the result of Freudenthal [3] that there are just two homotopy classes of maps of an $(n + 1)$-sphere into an n-sphere: the class of maps homotopic to a constant, and the class of maps homotopic to the result of a succession of suspensions of the Hopf map of the 3-sphere on the 2-sphere. In the first case $\gamma_2(f) = 0$. In the second case, by 18.11, $\gamma_2(f) = 1$.

18.13. *If f is a map of an $(n + \alpha - 1)$-sphere into a n-sphere, and $\alpha > n$, then* $\gamma_\alpha(f) = 0$.

This follows from 15.4.

Note that these results include a part of the results of Freudenthal [3]. If f is a map of a $(2n - 1)$-sphere into an n-sphere, and the result of suspending f a number of times yields an inessential map, then 18.11 asserts that f has an even Hopf invariant. In particular the result of a series of suspensions of any one of the Hopf maps $S^3 \rightarrow S^2$, $S^7 \rightarrow S^4$ and $S^{15} \rightarrow S^8$ yields an essential map.

It is to be observed that the results of Freudenthal, even for a map $S^{n+1} \rightarrow S^n$, do not provide an effectively calculable method of deciding the homotopy class of the map. If the map is simplicial, then $\gamma_\alpha(f)$ is calculable using 16, (4).

19. The Whitehead product

Let S_1, S_2 be spheres of positive dimensions p, q. Let $x_1 \in S_1$, $x_2 \in S_2$ be reference points. In the product space $S_1 \times S_2$, let $X = (S_1 \times x_2) \cup (x_1 \times S_2)$. Then X consists of two copies of S_1, S_2 with a point in common. The projections of $S_1 \times S_2$ into each factor define maps

$$h_1 : X \rightarrow S_1, \qquad h_2 : X \rightarrow S_2 .$$

J. H. C. Whitehead, in a study on homotopy groups [11], constructed a map of a $(p + q - 1)$-sphere S onto X as follows. Let E_1, E_2 be cells of dimensions p, q. For $\alpha = 1, 2$, choose a map f_α: $E_\alpha \rightarrow S_\alpha$ which maps $E_\alpha - \dot{E}_\alpha$ on $S_\alpha - x_\alpha$ topologically and carries $\dot{E}_\alpha$ into x_α. Let $S = (E_1 \times \dot{E}_2) \cup (\dot{E}_1 \times E_2)$ Then S is a $(p + q - 1)$-sphere. The product mapping $f_1 \times f_2 : E_1 \times E_2 \rightarrow S_1 \times S_2$ carries S into X. Let f: $S \rightarrow X$ denote this map. Whitehead showed that f is essential, and generates an infinite cyclic subgroup of $\pi_{p+q-1}(X)$. This is also a consequence of the following

Theorem 19.1. *Using cohomology groups based on integer coefficients, let u, v generate the groups $H^p(S_1)$, $H^q(S_2)$ respectively. Then $(h_1^* u) \smile_f (h_2^* v)$ is a generator of* $H^{p+q-1}(S)$.

Proof. We shall show that the mapping cylinder X_f is a $(p + q)$-manifold with regular boundary S. The conclusion will follow then from 7.3. Choose a $(p + q)$-cell E in the interior of $E_1 \times E_2$ in such a way that there is a homeomorphism h of $S \times I$ onto $E_1 \times E_2$ minus the interior of E, and h maps $S \times 1$ onto S and $S \times 0$ onto E. Then the composition of h and $f_1 \times f_2$ maps $S \times 1$ onto X in the manner of f and maps $S \times I - S \times 1$ topologically into $S_1 \times S_2$. Thus X_f is homeomorphic to the manifold obtained by deleting an open $(p + q)$-cell from $S_1 \times S_2$.

20. Functional products of exterior differential forms

Suppose X, X' are differentiable manifolds and f: $X \to X'$ is differentiable. According to the Whitney formulation of the deRham theorem [13], the cohomology classes of exact exterior differential forms of degree p of a manifold are in $1 - 1$ correspondence with the elements of the p^{th} cohomology group of the manifold based on real numbers as coefficients. Under this isomorphism, addition is preserved, and products of exterior forms correspond to cup products of cocycles. It is clear therefore that the functional products of cocycles must correspond to an operation on exterior forms which we now describe.

Let U, V be exact exterior forms of degrees p and q, and suppose $U \cdot V = \delta B$ is an outer derivative. Under f, the form U has an image form, $f^{\#} U$ in X'. Suppose $f^{\#} U = \delta A$ is also an outer derivative. Then the form

$$W = f^{\#} B - A \cdot f^{\#} V \tag{1}$$

is an exact form of degree $p + q - 1$. Its cohomology class may not be a unique function of U and V; however it is unique modulo images of exact forms of X, and products of exact $(p - 1)$-forms of X' with images of exact q-forms of X.

It is clear that, if u, v, a, b are cochains corresponding to the forms U, V, A, B, then $w = f^{\#} b - a \smile f^{\#} v$ corresponds to the form W. By 16.1, the cocycle w rerpresents the product $u \smile_f v$. Thus the operation (1) corresponds to the functional product. All of the properties of $u \smile_f v$ now carry over to properties of (1).

If one attempts to carry over the operation $\smile_f^{\alpha} u$ in a similar way by using the formula 16, (4), there results only a trivial operation on exterior forms. The reason for this is that $\mathrm{Sq}^{\alpha} u$ is always of order 2. If real coefficients are used, it is always zero.

In a recent paper J. H. C. Whitehead [Proc. Nat. Acad. Sci. 33 (1947), 117–123] gave an integral formula for the Hopf invariant of a differentiable map f: $S^3 \to S^2$. This can be derived from the preceding considerations as follows. Let U be an exact form of degree 2 on S^2 which corresponds to a generator u of $H^2(S^2)$ based on integral coefficients. For example, the element of area of S^2 divided by 4π is such a form since its integral over S^2 is 1. Since any exact form on S^3 of degree 2 is a derived form, there exists a form A on S^3 of degree 1 such that $\delta A = f^{\#} U$. Define W by (1) $(B = 0)$. Then W corresponds to $u \smile_f u = \gamma(f) u'$ (see §17,(1)). Hence the integral of W over S^3 is the Hopf invariant $\gamma(f)$.

PRINCETON UNIVERSITY

BIBLIOGRAPHY

1. S. EILENBERG-N. STEENROD, Foundations of algebraic topology, to appear.
2. R. H. FOX, *On homotopy type and deformation retracts*, Ann. of Math. 44 (1943), 40–50.

3. H. Freudenthal, *Über die Klassen der Sphärenabbildungen*, Compositio Math. 5 (1937), 299-314.
4. W. Gysin, *Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten*, Comm. Math. Helv. 14 (1942), 61-122.
5. H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Ann. 104 (1931), 639-665.
6. ———, *Über die Abbildungen von Sphären auf Sphären niedriger Dimension*, Fund. Math. 25 (1935), 427-440.
7. W. Hurewicz-N. Steenrod, *Homotopy relations in fibre spaces*, Proc. Nat. Acad. Sci. 27 (1941), 60-64.
8. N. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. 48 (1947), 290-320.
9. ———, *Cohomology invariants of mappings*, Proc. Nat. Acad. Sci. 33 (1947), 124-128.
10. J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. 45 (1939), 243-327.
11. ———, *On adding relations to homotopy groups*, Ann. of Math. 42 (1941), 409-428.
12. H. Whitney, *On products in a complex*, ibid 39 (1938), 397-432.
13. ———, *Topological properties of differentiable manifolds*, Bull. Amer. Math. Soc. 43 (1937), 785-805.