ANNALS OF MATHEMATICS
Vol. 56, No. 1, July, 1952
Printed in U.S.A.

REDUCED POWERS OF COHOMOLOGY CLASSES[1]

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(Received April 18, 1951)
(Revised March 20, 1952)

1. Introduction

We shall present a set of new operations which interrelate the elements of the various dimensional cohomology groups of a space. They are topologically invariant, and provide sharper methods for distinguishing topological types and homotopy types of spaces. They are generalizations of the squaring operations introduced in [4].

Let $H^q(K;G)$ denote the q^{th} cohomology group of a complex K with coefficient group G. Let integers q, i be given, and let G, G' be fixed coefficient groups. Then a *cohomology operation*, relative to (q,i,G,G'), is a mapping

$$\phi: H^q(K;G) \rightarrow H^{q+i}(K;G')$$

defined for each K and such that $f^*\phi = \phi f^*$ for any map f of one complex into another.

For any integer $p > 0$, the p^{th} power of a cohomology class in the sense of cup products is a cohomology operation. The new operations are closely related to the powers. Let Π_p be the symmetric group on p elements and Γ_p its group ring with integer coefficients. A 0-*sequence* in Γ_p is a sequence $\alpha_1, \alpha_2, \cdots$ such that $\alpha_i\alpha_{i-1} = 0$ $(i = 2, 3, \cdots)$, and the sum of the coefficients of α_1 is zero. Corresponding to each such sequence and any p-linear function from G to G', we construct a sequence of cohomology operations

$$\mathcal{P}^p_i: H^q(K;G) \rightarrow H^{pq-i}(K;G'_n), \qquad \text{all } q,$$

where G'_n is G' reduced mod n for some n depending on q and α_i.

If we set e = identity of π_p, g = the p fold cyclic permutation, and

$$\alpha_{2j-1} = g - e, \qquad \alpha_{2j} = \textstyle\sum_{k=0}^{p-1} g^k, \qquad j = 1, 2, \cdots$$

we obtain a 0-sequence. The corresponding $\mathcal{P}^p_i$ are called *cyclic reduced powers*. When $p = 2$, they are the squaring operations.

In the present paper we present the theory for a general 0-sequence. The special properties of the cyclic reduced powers will be given in a subsequent paper.

The importance of cohomology operations to the study of homotopy groups is seen as follows. Let $f: S^n \rightarrow S^q$ be a map of an n-sphere into a q-sphere $(n > q)$. Form a complex K by adjoining to S^q an $(n + 1)$-cell by the map f of its bound-

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[1] Preliminary results were announced at the International Congress at Harvard. A major part of the investigation was made while the author was a John Simon Guggenheim Memorial Fellow.

ary into S^q. Then H^q and H^{n+1} are the only non-trivial cohomology groups of K. If f is inessential, K has the homotopy type of $S^q \vee S^{n+1}$ (two spheres with a point in common). Since S^{n+1} is a retract of $S^q \vee S^{n+1}$, it follows that any cohomology operation $\phi: H^q(K) \to H^{n+1}(K)$ must be zero. Hence, if some ϕ is non-zero, the map f must be essential.

The above method will be used in the subsequent paper to show that $\pi_{n+3}(S^n)$ admits a homomorphism onto a cyclic group of order 3 when $n \geqq 3$. This agrees with recent results of Serre [3]. The results of Serre are of course much more extensive. However it should be emphasized that the reduced power operations are effectively computable; and any result describing the structure of a homotopy group should be buttressed by an effective method for deciding which element of the homotopy group is represented by a given map.

The method we will use to construct the new operations has its origin in a procedure devised by Lefschetz for constructing cup products [1; pp. 173–190]. He used the product complex $K \times K$, the cross product of cocycles, and chain transformations approximating the diagonal map of K into $K \times K$. The clue to our generalization is the observation that the diagonal map is invariant under the permutation of the factors but the chain transformation is not. We then apply the Smith-Richardson technique [2] to obtain measures of the impossibility of obtaining a symmetric chain transformation. This leads directly to the squaring operations. Replacing $K \times K$ by p-fold products yields the general reduced powers.

2. Preliminaries

We shall deal with geometric cell complexes K which are closure finite. The closure $\bar{\sigma}$ of any cell σ is required to be a subcomplex of K; hence also $\dot{\sigma} = \bar{\sigma} - \sigma$. We write $\sigma < \tau$ to indicate that σ is a face of τ. The group $C_q(K)$ of q-chains is the free abelian group generated by the q-cells (i.e. finite chains with integer coefficients). If $q < 0$, $C_q(K) = 0$. The boundary operator is denoted by $\partial: C_q(K) \to C_{q-1}(K)$. The Kronecker index, $\operatorname{In}(c)$, of a 0-chain c is the sum of its coefficients. As usual $\partial\partial c = 0$ for each $c \in C_q(K)$, and $\operatorname{In}(\partial c) = 0$ for $c \in C_1(K)$. We denote by $Z_q(K)$, $B_q(K)$ and $H_q(K)$ the groups q-cycles, q-boundaries and q-dimensional homology classes respectively. A complex is *acyclic* if each q-cycle, $q > 0$, is a boundary, and each 0-cycle of index zero is a boundary. *We shall consider henceforth only those cell complexes K such that $\bar{\sigma}$ is acyclic for each σ in K.*

For any abelian coefficient group G, the group $C^q(K;G)$ of q-cochains is the group of homomorphisms of $C_q(K)$ into G (i.e. infinite cochains). The value of $u \in C^q(K;G)$ on $c \in C_q(K)$ is denoted by $u \cdot c$. Given a homomorphism $\phi: C_q(K) \to C_r(K')$ of one chain group into another, then the homomorphism $\phi: C^r(K';G) \to C^q(K;G)$ defined by

$$(2.1) \qquad (\phi u) \cdot c = u \cdot (\phi c), \qquad u \in C^r(K';G),\ c \in C_q(K),$$

is called the *dual* of ϕ. In particular, the coboundary $\delta: C^q(K;G) \to C^{q+1}(K;G)$ is the dual of $\partial: C_{q+1}(K) \to C_q(K)$.

If L is a subcomplex of K, then $C^q(K,L,G)$ denotes the subgroup of q-cochains which are zero on cells of L. Clearly δ maps $C^q(K,L;G)$ into $C^{q+1}(K,L;G)$. We designate by $Z^q(K,L;G)$, $B^q(K,L;G)$ and $H^q(K,L;G)$ the q^{th} cocycle, coboundary and cohomology group of K mod L respectively.

A *chain transformation* ϕ of (K,L) into (K',L') is a sequence of homomorphisms $\phi\colon C_q(K) \to C_q(K')$ (all q) such that

$$\phi C_q(L) \subset C_q(L'), \tag{2.2}$$

$$\operatorname{In}(\phi c) = \operatorname{In}(c), \qquad c \in C_0(K), \tag{2.3}$$

$$\partial\phi c = \phi\partial c, \qquad c \in C_q(K). \tag{2.4}$$

Then the dual of ϕ satisfies $\phi C^q(K',L';G) \subset C^q(K,L;G)$, and $\delta\phi = \phi\delta$. It thereby induces a homomorphism

$$\phi^*\colon H^q(K',L';G) \to H^q(K, L;G). \tag{2.5}$$

If ϕ, ψ are two chain transformations of (K,L) into (K',L'), then a *chain homotopy* D of ϕ into ψ is a sequence of homomorphisms $D\colon C_q(K) \to C_{q+1}(K')$ such that

$$DC_q(L) \subset C_{q+1}(L) \tag{2.6}$$

$$\partial Dc + D\partial c = \psi c - \phi c, \qquad c \in C_q(K). \tag{2.7}$$

Then the duals of D, ϕ, ψ satisfy $DC^q(K',L';G) \subset C^{q-1}(K,L;G)$ and

$$D\delta u + \delta Du = \psi u - \phi u, \qquad u \in C^q(K',L';G). \tag{2.8}$$

Taking u to be a cocycle, this gives $\phi u \sim \psi u$; hence $\phi^* = \psi^*$.

A *carrier* C from (K,L) to (K',L') is a function which assigns to each cell σ of K a non-vacuous subcomplex $C(\sigma)$ of K' such that $\sigma \in L$ implies $C(\sigma) \subset L'$, and $\sigma < \tau$ implies $C(\sigma) \subset C(\tau)$. The carrier is *acyclic* if $C(\sigma)$ is acyclic for every σ. We say that C *carries* the chain transformation ϕ [homotopy D] if, for each $\sigma \in K$, $\phi\sigma$ [$D\sigma$] is a chain on $C(\sigma)$.

2.9 LEMMA. *If C is an acyclic carrier $(K,L) \to (K',L')$, then C carries some chain transformation ϕ; and, if ϕ, ψ are two chain transformations carried by C, then C carries a chain homotopy D of ϕ into ψ.*

For each vertex v of K, let ϕv be a 0-chain of index 1 on $C(v)$, e.g. a vertex. Extend ϕ linearly over $C_0(K)$ by setting $\phi(\sum a_i v_i) = \sum a_i\phi v_i$. For any 1-cell σ with $\partial\sigma = \tau_1 - \tau_2$, we have $\phi\partial\sigma = \phi\tau_1 - \phi\tau_2$ is a 0-cycle on $C(\sigma)$ of index 0. As $C(\sigma)$ is acyclic, it contains a 1-chain $\phi\sigma$ such that $\partial\phi\sigma = \phi\partial\sigma$. Extend ϕ over $C_1(K)$ linearly.

Suppose ϕ constructed for chains of dimensions $<q$. Let σ be a q-cell. For any $(q - 1)$-face τ of σ, $\phi\tau$ is defined and lies on $C(\tau) \subset C(\sigma)$. Thus $\phi\partial\sigma$ is a chain on $C(\sigma)$. It is a cycle since $\partial\phi\partial\sigma = \phi\partial\partial\sigma = 0$. As $C(\sigma)$ is acyclic, it contains a q-chain $\phi\sigma$ such that $\partial\phi\sigma = \phi\partial\sigma$. Extend ϕ over $C_q(K)$ linearly. This completes the inductive construction of ϕ.

The construction of D proceeds similarly. For any vertex σ, $\psi\sigma - \phi\sigma$ is a

0-cycle of index 0 on $C(\sigma)$; hence it bounds a 1-chain $D\sigma$ on $C(\sigma)$. Extend D over $C_0(K)$ linearly. Assume D constructed for chains of dimensions $<q$. For any q-cell σ, the chain $z = \psi\sigma - \phi\sigma - D\partial\sigma$ is defined and lies on $C(\sigma)$. If we apply 2.7 with $c = \partial\sigma$, it follows that z is a cycle. As $C(\sigma)$ is acyclic, we may choose a chain $D\sigma$ on $C(\sigma)$ with boundary z. Extend D over $C_q(K)$ linearly. This completes the inductive construction of D and the proof of the lemma.

2.10. LEMMA. *If the acyclic carrier C satisfies* $\dim C(\sigma) = \dim \sigma$ *for each* σ, *then the chain transformation carried by C is unique.*

Suppose C carries ϕ and ψ. By 2.9, C carries a chain homotopy D of ϕ into ψ. But $D\sigma$ is a $(q + 1)$-chain on $C(\sigma)$ having dimension q. Thus $D\sigma = 0$ for each σ. Then 2.7 gives $\phi = \psi$.

Let f be a map $(K,L) \to (K',L')$, i.e. a continuous function $|K| \to |K'|$ such that $f(|L|) \subset |L'|$. A carrier C from (K,L) to (K',L') is a *carrier* for f if $f(\sigma) \subset C(\sigma)$ for each cell σ. The *minimal* carrier for f is obtained by defining $C(\sigma)$ to be the least subcomplex of K' containing $f(\sigma)$. The map f is called *proper* if its minimal carrier is acyclic.

2.11. LEMMA. *If f is a proper map, then, for any acyclic carrier C of f, and any chain transformation ϕ carried by C, the induced homomorphism ϕ^* is independent of the choice of C and ϕ.*

Let C' be the minimal carrier, and let ψ be a chain transformation carried by C' which exists by 2.9. Then C carries ϕ, ψ. By 2.9, C carries a chain homotopy of ϕ into ψ. By 2.8, $\phi^* = \psi^*$. Any other choice of C, ϕ would give the same result; so the lemma is proved.

When f is proper, we shall denote by $f_\#$ a chain transformation which has a common acyclic carrier with f. We call $f_\#$ an *algebraic approximation* to f. Its dual is denoted by $f^\#$. The homomorphisms of cohomology groups induced by $f^\#$ are denoted by f^*. By 2.11, f^* depends only on f.

3. Operator Complexes

Let K, K' be cell complexes. *An operation of degree i from K to K'* is defined to be a sequence of homomorphisms

$$D_i\colon C_q(K) \to C_{q+i}(K') \tag{3.1}$$

defined for all q. Although the integer i is unrestricted, it will be positive or zero in applications. Let O_i denote the set of all operations of degree i. It forms an additive group with addition defined by $(D_i + D'_i)c = D_ic + D'_ic$. We define the *boundary* operator

$$\omega\colon O_i \to O_{i-1} \tag{3.2}$$

by the rule

$$(\omega D_i)c = \partial D_i c + (-1)^{i+1} D_i \partial c, \qquad c \in C_q(K). \tag{3.3}$$

It is easily seen that $\omega\omega = 0$. Hence the sequence of groups O_i and the homomorphisms ω form a chain complex (in the algebraic sense of W. Mayer). It is called the *operator complex* from K to K', and is denoted by $O(K,K')$.[2]

If D_i is an i-cycle, i.e. $\omega D_i = 0$, then 3.3 states that D_i carries cycles into cycles, boundaries into boundaries, and thereby induces homomorphisms $H_q(K) \to H_{q+i}(K')$ for each q. If D_i, D'_i are i-cycles, and $\omega E_{i+1} = D'_i - D_i$ so that $D_i \sim D'_i$ in $O(K,K')$, then 3.3 states that $\partial E_{i+1}z = D'_i z - D_i z$ for each cycle z of K. Thus $D_i z \sim D'_i z$ in K', and the homomorphisms of homology groups induced by D_i, D'_i coincide.

Let D_0 be a 0-cycle. We shall say that D_0 has an *index* if there is an integer k such that $\text{In}(D_0 c) = k\text{In}(c)$ for each $c \in C_0(K)$; and k is called the index of D_0. Thus

$$\text{(3.4)} \qquad \text{In}(D_0 c) = \text{In}(D_0)\text{In}(c), \qquad c \in C_0(K).$$

If K is connected, every 0-cycle has an index. Let v be a vertex of K, and $k = \text{In}(D_0 v)$. For any $c \in C_0(K)$, the connectedness of K implies the existence of a $d \in C_1(K)$ such that $\partial d = c - \text{In}(c)v$. If we apply D_0 to both sides of this relation, use $\partial D_0 = D_0\partial$ (D_0 is a 0-cycle), take the index of both sides, and use $\text{In}\partial = 0$, we obtain 3.4.

It is clear that any 0-cycle D_0 of index 1 is a chain transformation (2.3, 2.4).

In 3.3, let $i = 1$ and $q = 0$. Then $\partial c = 0$. Using $\text{In}\partial = 0$, it follows that $\text{In}((\omega D_1)c) = 0$. Thus ωD_1 has an index, and it is zero.

Let C be a carrier $K \to K'$. The *operator complex of* C, denoted by $O(C)$, consists of those elements D_i of $O(K,K')$ such that, (i) $D_i\sigma$ is a chain of $C(\sigma)$ for each cell σ, (ii) $D_i = 0$ for $i < 0$, and (iii) when $i = 0$, D_0 is a 0-cycle having an index. Clearly $D_i \in O(C)$ implies $\omega D_i \in O(C)$. Hence $O(C)$ is a chain complex. The basic result of this section is the following.

3.5. Lemma. *Let C be an acyclic carrier from K to K'. Then the operator complex $O(C)$ contains 0-cycles of index 1, and $O(C)$ is acyclic.*

The existence of a 0-cycle of index 1 is given by 2.9.

Now let D be an i-cycle of $O(C)$, and if $i = 0$, let $\text{In}(D) = 0$. We shall construct an $(i + 1)$-chain E of $O(C)$ such that $\omega E = D$. If σ is a vertex, then $\omega D = 0$ implies that $D\sigma$ is an i-cycle on $C(\sigma)$. If $i = 0$, we have in addition that $\text{In}(D\sigma) = 0$. As $C(\sigma)$ is acyclic, there is a chain $E\sigma$ on $C(\sigma)$ such that $\partial E\sigma = D\sigma$. As σ is a vertex, $\partial\sigma = 0$; hence by 3.3, $(\omega E)\sigma = D\sigma$. For any 0-chain $c = \sum a_i\sigma_i$, set $Ec = \sum a_i E\sigma_i$. Then $(\omega E)c = Dc$.

Suppose, inductively, that Ec is defined for r-chains c, $r < q$, so that $(\omega E)c = Dc$, and $E\tau$ is a chain on $C(\tau)$ for each r-cell τ. Let σ be a q-cell. Then the chain $z = D\sigma + (-1)^{i+1}E\partial\sigma$ is defined and lies on $C(\sigma)$. It is a cycle, for

$$\begin{aligned} \partial z &= \partial D\sigma + (-1)^{i+1}\partial E\partial\sigma \\ &= (\omega D + (-1)^{i}D\partial)\sigma + (-1)^{i+1}(\omega E + (-1)^{i+1}E\partial)\partial\sigma \\ &= (-1)^{i}D\partial\sigma + (-1)^{i+1}(\omega E)\partial\sigma = 0. \end{aligned}$$

[2] This complex has been considered by Lefschetz [1; Ch. V]. It is the product $K \times K'^{*}$ of K with the dual of K'. The correspondence is obtained by assigning to $\sigma \times \tau'^{*}$ the operation D such that $D\sigma = \tau'$ and D is zero on all other cells.

Hence we may choose a chain $E\sigma$ on $C(\sigma)$ such that $\partial E\sigma = z$. Then $(\omega E)\sigma = D\sigma$. For each q-chain $c = \sum a_i\sigma_i$, we set $Ec = \sum a_i E\sigma_i$. It follows that $(\omega E)c = Dc$, and the inductive construction is complete.

4. Product Complexes

If K_1, K_2 are cell complexes, then their product $K_1 \times K_2$ has as cells the products of the cells of K_1 and K_2. Given incidence numbers $[\sigma_1 : \tau_1]$ in K_1 and $[\sigma_2 : \tau_2]$ in K_2, we define incidences in $K_1 \times K_2$ by

$$(4.1) \qquad \begin{aligned} [\sigma_1 \times \sigma_2 : \tau_1 \times \sigma_2] &= [\sigma_1 : \tau_1] \\ [\sigma_1 \times \sigma_2 : \sigma_1 \times \tau_2] &= (-1)^q[\sigma_2 : \tau_2], \qquad q = \dim \sigma_1, \end{aligned}$$

and all other incidence numbers are zero. The products $\sigma \times \tau$ of the generators of the chain groups define a bilinear pairing of $C_q(K_1)$ and $C_r(K_2)$ to $C_{q+r}(K_1 \times K_2)$ called the *cross product*. Then 4.1 is equivalent to

$$(4.2) \qquad \partial(c_1 \times c_2) = (\partial c_1) \times c_2 + (-1)^q c_1 \times (\partial c_2)$$

where $c_1 \in C_q(K_1)$ and $c_2 \in C_r(K_2)$.

The product of relative complexes is defined by

$$(K_1,L_1) \times (K_2,L_2) = (K_1 \times K_2,\ K_1 \times L_2 \cup L_1 \times K_2).$$

Abbreviate this by (K,L). Assume that the coefficient groups G_1, G_2 are paired to a third G_3. Define the *cross product* pairing of $C^q(K_1,L_1;G_1)$, $C^r(K_2,L_2;G_2)$ to $C^{q+r}(K,L;G_3)$ by

$$(4.3) \qquad (u_1 \times u_2)\cdot(\sigma_1 \times \sigma_2) = (u_1\cdot\sigma_1)(u_2\cdot\sigma_2).$$

From this and 4.2, we obtain

$$(4.4) \qquad \delta(u \times v) = (\delta u) \times v + (-1)^q u \times (\delta v).$$

It follows that the product of cocycles is a cocycle, and its cohomology class depends only on the classes of its factors. We obtain thus a *cross product* pairing of $H^q(K_1,L_1;G_1)$, $H^r(K_2,L_2;G_2)$ to $H^{q+r}(K,L;G_3)$.

Let $g(x_1, x_2) = (x_2, x_1)$; then g is a map

$$g: K_1 \times K_2 \to K_2 \times K_1.$$

It has the carrier $C(\sigma_1 \times \sigma_2) = \bar\sigma_2 \times \bar\sigma_1$ which is acyclic since the product of acyclic spaces is acyclic. If we set

$$(4.5) \qquad g_\#(\sigma_1 \times \sigma_2) = (-1)^{qr}\sigma_2 \times \sigma_1, \qquad q = \dim \sigma_1,\ r = \dim \sigma_2,$$

then, using 4.2, we have $\partial g_\# = g_\#\partial$ and $\mathrm{In}\, g_\# (v_1 \times v_2) = 1$ where v_1, v_2 are vertices. Thus, by 2.10, $g_\#$ is the unique algebraic approximation to (g,C).

Let a pairing G_2, G_1 to G_3 be defined in terms of the pairing G_1, G_2 to G_3 by $ba = ab$ for $a \in G_1$, $b \in G_2$. Then $u_2 \times u_1$ is defined for $u_1 \in C^q(K_1,L_1;G_1)$,

$u_2 \in C^r(K_2,L_2;G_2)$. Using 4.3 and the definition 2.1 of the dual $g^\#$ of $g_\#$, we have

$$g^\#(u_2 \times u_1)\cdot(\sigma_1 \times \sigma_2) = (u_2 \times u_1)\cdot g_\#(\sigma_1 \times \sigma_2) = (-1)^{qr} u_2 \times u_1 \cdot \sigma_2 \times \sigma_1$$
$$= (-1)^{qr}(u_2\cdot\sigma_2)(u_1\cdot\sigma_1) = (-1)^{qr}(u_1\cdot\sigma_1)(u_2\cdot\sigma_2).$$

It follows that

$$g^\#(u_2 \times u_1) = (-1)^{qr} u_1 \times u_2 . \tag{4.6}$$

In case $(K_1,L_1) = (K_2,L_2)$, then the product complexes based on the two orders of the factors coincide; and $g^\#$ becomes an automorphism of the group of cochains. If also $G_1 = G_2$, then 4.6 is ambiguous; because the cross product on the left is based on a different pairing from that on the right. However, if the initial pairing of $G_1 = G_2$ with itself to G_3 is commutative, then the two cross products coincide, and 4.6 is unambiguous.[3]

The p-fold product complex

$$(K',L') = \textstyle\prod_{i=1}^{p}(K_i,L_i)$$

is defined by an obvious induction. Henceforth we shall omit the cross symbol and write $c_1 \cdots c_p$ and $u_1 \cdots u_p$ for the products of chains c_i and cochains u_i of K_i. If $q_1 = \dim c_i$ and $r_i = \dim u_i$, then an induction based on 4.2, 4.4 gives

$$\partial(c_1 \cdots c_p) = \textstyle\sum_{i=1}^{p} (-1)^{q_1+\cdots+q_{i-1}} c_1 \cdots c_{i-1}(\partial c_i)c_{i+1} \cdots c_p . \tag{4.7}$$

$$\delta(u_1 \cdots u_p) = \textstyle\sum_{i=1}^{p} (-1)^{r_1+\cdots+r_{i-1}} u_1 \cdots u_{i-1}(\delta u_i)u_{i+1} \cdots u_p . \tag{4.8}$$

Our principal concern is with the p-fold product where all the factors coincide with a fixed (K,L). In this case the product is denoted by $(K,L)^p$ and is called the p-fold *power* of (K,L). Let g be the map of $(K,L)^p$ onto itself obtained by interchanging the i^{th} and $(i+1)^{\text{st}}$ factors. The same reasoning as for 4.5 gives

$$g_\#(\sigma_1 \cdots \sigma_p) = (-1)^{q_i q_{i+1}} \sigma_1 \cdots \sigma_{i-1}\sigma_{i+1}\sigma_i\sigma_{i+2} \cdots \sigma_p \tag{4.9}$$

where $q_i = \dim \sigma_i$.

To have p-fold products of cochains, we assume given two coefficient groups G, G' and a p-linear function $b(a_1, \cdots, a_p) \in G'$ for $a_1, \cdots, a_p$ in G. For example, if R is a commutative ring, we may take $G = G' = R$ and let the p-linear function be the product $a_1 \cdots a_p$. If $u_1, \cdots, u_p$ are cochains of (K,L) with coefficients in G, we define

$$(u_1 \cdots u_p)\cdot(\sigma_1 \cdots \sigma_p) = b(u_1\cdot\sigma_1, \cdots, u_p\cdot\sigma_p).$$

This product of cochains is a p-linear function whose values are cochains of $(K,L)^p$ with coefficients in G'. We assume henceforth that $b(a_1, \cdots, a_p)$ is

[3] If the pairing G_1 to G_3 is skew symmetric then 4.6 remains valid if the sign of the right hand member is changed.

commutative, i.e. its value is unchanged by any permutation of the arguments. Then 4.9 and the same reasoning as for 4.6 give

$$g^{\#}(u_1 \cdots u_p) = (-1)^{r_i r_{i+1}} u_1 \cdots u_{i-1} u_{i+1} u_i u_{i+2} \cdots u_p \tag{4.10}$$

where $r_j = \dim u_j$.

Let Π_p denote the full permutation group on p letters. The identity element of Π_p is denoted by e. We may regard Π_p as the permutation group of the factors of $(K,L)^p$ for any (K,L). Each $g \in \Pi_p$ determines a unique $g_{\#}$ which, for each q, is an automorphism of $C_q(K^p)$ giving thereby a representation of Π_p as a group of automorphisms. The cochain operations $g^{\#}$ are automorphisms of $C^q((K,L)^p;G')$. They provide an *anti-representation* of Π_p ; for, by 2.1,

$$\begin{aligned}((g_1g_2)^{\#} u)\cdot c &= u\cdot(g_1g_2)_{\#} c = u\cdot g_{1\#} g_{2\#} c \\ &= (g_1^{\#} u)\cdot(g_{2\#} c) = (g_2^{\#} g_1^{\#} u)\cdot c.\end{aligned}$$

Henceforth we shall write g for both $g_{\#}$ and $g^{\#}$, the context will prevent ambiguity.

Let Γ_p denote the group ring over the integers of Π_p. Elements of Γ will be written as linear combinations $\sum x_i g_i$ of $g_i \in \Pi$ with integer coefficients x_i. Let s denote the ring homomorphism of Γ into the integers obtained by mapping each element of Π into 1. If $\alpha = \sum x_i g_i$, then $s(\alpha) = \sum x_i$. We call $s(\alpha)$ the *index* of α. Let t denote the ring homomorphism of Γ into the integers obtained by mapping each even permutation into 1 and each odd permutation into -1. We call $t(\alpha)$ the *coindex* of α.

The representation of Π as operators on chains of K^p extends to a representation of Γ as endomorphisms of the chains. If σ is a vertex of K^p, it is clear that $\text{In}(\alpha\sigma) = s(\alpha)$; hence, for any $c \in C_0(K^p)$, we have

$$\text{In}(\alpha c) = s(\alpha)\,\text{In}(c). \tag{4.11}$$

Similarly Γ has an anti-representation as endomorphisms of cochains. If $u \in C^q(K,L;G)$ let u^p denote the cross product of p factors u. If $g \in \Pi$ is the simple interchange of two adjacent factors, then 4.10 gives

$$gu^p = (-1)^q u^p. \tag{4.12}$$

Then, for any $\alpha \in \Gamma$,

$$\alpha u^p = \begin{cases} s(\alpha)u^p \text{ } \textit{if } \dim u \textit{ is even}, \\ t(\alpha)u^p \text{ } \textit{if } \dim u \textit{ is odd}. \end{cases} \tag{4.13}$$

5. The Fundamental Construction

As the construction to be given is carried out under general circumstances, we shall describe first the basic idea in the simplest case.

Let $d: K \to K \times K$ be the diagonal map $d(x) = (x,x)$. It is not cellular; but has the minimal carrier $C(\sigma) = \bar{\sigma} \times \bar{\sigma}$ which is acyclic. Hence we may choose a chain transformation D_0 with carrier C. Lefschetz [1; Ch. V] defines cup products in K by

$$(u \smile v) \cdot \sigma = (u \times v) \cdot D_0 \sigma.$$

The permutation group Π of the factors consists of two elements e and g. Let $\Delta = g - e$, $\Sigma = g + e$ in the group ring Γ of Π. Now d is invariant under Π; but the algebraic approximation D_0 to d may not be. In fact, by taking K to be a 1-cell, it is easy to see that D_0 cannot be invariant. However gD_0 and D_0 are two chain transformations carried by C. Hence C carries a chain homotopy D_1 of D_0 into gD_0. In the language of §3,

$$\omega D_1 = gD_0 - D_0 = \Delta D_0 .$$

Now ωD_1 has the symmetry property expressed by $\Sigma \omega D_1 = 0$. In general $\Sigma D_i \neq 0$; but ΣD_1 is a 1-cycle in $O(C)$ since $\omega \Sigma D_1 = \Sigma \omega D_1 = 0$. As $O(C)$ is acyclic, we may choose D_2 in $O(C)$ such that $\omega D_2 = \Sigma D_1$. Then ωD_2 has the symmetry $\Delta \omega D_2 = 0$. Hence ΔD_2 is a 2-cycle in $O(C)$. Then there exists a D_3 in $O(C)$ such that $\omega D_3 = \Delta D_2$. We define in this way an infinite sequence $\{D_i\}$ in $O(C)$ such that

$$\omega D_{2i} = \Sigma D_{2i-1}, \qquad \omega D_{2i-1} = \Delta D_{2i-2} .$$

Thus the impossibility of approximating d by an invariant D_0 gives rise to an infinite sequence of operations $\{D_i\}$ from K to $K \times K$. Following the idea of Lefschetz, we define cup-i products in K by

$$(u \smile_i v) \cdot \sigma = (u \times v) \cdot D_i \sigma.$$

These products provide a new definition of the squaring operations given in [4].

We turn now to the construction in the general case. Generalizing slightly the situation of §4, we shall suppose that Π is a group of cellular mappings of a complex K', i.e. each g in Π is a homeomorphism of $| K' |$ which maps cells onto cells. Then the unique chain transformation assigned to each g provides a representation of the group ring Γ of Π as a ring of endomorphisms of the chain groups of K'. The *index* $s(\alpha)$, $\alpha \in \Gamma$, is defined as in §4.

Consider now the operator complex $O(K, K')$. If D is a chain of $O(K, K')$ and $\alpha \in \Gamma$, define αD in $O(K, K')$ by

$$(5.1) \qquad (\alpha D)c = \alpha(Dc), \qquad c \in C_q(K).$$

In this way Γ is represented as a ring of endomorphisms of $O(K, K')$. Since $\partial \alpha c = \alpha \partial c$ for chains c of K', it follows from 3.3 that

$$(5.2) \qquad \omega \alpha D = \alpha \omega D, \qquad \alpha \in \Gamma, D \in O(K, K').$$

A carrier C from K to K' is said to be *invariant* under Π if, for each $\sigma \in K$ and $g \in \Pi$, g maps $C(\sigma)$ onto itself. Then, if D is a chain of $O(C)$, it follows that αD is a chain of $O(C)$. In this way Γ is a ring of endomorphisms of $O(C)$.

5.3. DEFINITION. A *0-sequence* in Γ is an infinite sequence $\alpha_1, \alpha_2, \cdots$ of elements of Γ such that

$$s(\alpha_1) = 0, \text{ and } \alpha_{i+1}\alpha_i = 0 \qquad \text{for } i = 1, 2, \cdots . \tag{5.4}$$

REMARK. It is to be noted that, if we define $\alpha_0 \in \Gamma$ to be the sum of all elements of Π, each with coefficient 1, then $\alpha\alpha_0 = s(\alpha)\alpha_0$ for any $\alpha \in \Gamma$. Thus the condition $s(\alpha_1) = 0$ in 5.4 can be replaced by $\alpha_1\alpha_0 = 0$ in accord with the remaining conditions.

5.5. LEMMA. *Let C be an acyclic carrier from K to K' invariant under the group Π operating in K', and let $\{\alpha_i\}$ be a 0-sequence in the group ring Γ of Π. Then there exists a sequence $\{D_i\}$ where D_i is an i-chain of $O(C)$, $i = 0, 1, \cdots$, such that*

$$\operatorname{In}(D_0) = 1, \text{ and } \omega D_i = \alpha_i D_{i-1} \qquad \text{for } i = 1, 2, \cdots . \tag{5.6}$$

If $\{D_i\}$, $\{D'_i\}$ are two sequences satisfying 5.6, then there is a sequence $\{E_i\}$ where E_i is an i-chain of $O(C)$, $i = 0, 1, \cdots$, such that $E_0 = 0$, and

$$\begin{aligned} \omega E_1 &= D'_0 - D_0 \\ \omega E_{i+1} &= D'_i - D_i - \alpha_i E_i , \qquad i = 1, 2, \cdots . \end{aligned} \tag{5.7}$$

The existence of a D_0 of index 1 is given by 3.5. By 4.11, 3.4. we have

$$\operatorname{In}(\alpha_1 D_0) = s(\alpha_1) \operatorname{In}(D_0) = 0.$$

Then 3.5 states that D_1 exists in $O(C)$ such that $\omega D_1 = \alpha_1 D_0$. Suppose inductively that $D_0, \cdots, D_k$ have been found in $O(C)$ satisfying 5.6. Then

$$\omega\alpha_{k+1}D_k = \alpha_{k+1}\omega D_k = \alpha_{k+1}\alpha_k D_{k-1} = 0, \qquad \text{by 5.2, 5.4.}$$

Hence $\alpha_{k+1}D_k$ is a k-cycle of $O(C)$. By 3.5, $O(C)$ contains a $(k + 1)$-chain D_{k+1} such that $\omega D_{k+1} = \alpha_{k+1}D_k$. This completes the construction of the $\{D_i\}$.

To prove the second part of the lemma, we set $E_0 = 0$. Since $\operatorname{In}(D'_0 - D_0) = 0$, 3.5 provides an E_1 satisfying 5.7. Suppose $E_0, \cdots, E_k$ have been found in $O(C)$ satisfying 5.7. Then

$$\begin{aligned} \omega(D'_k - D_k - \alpha_k E_k) &= \omega D'_k - \omega D_k - \alpha_k \omega E_k \\ &= \alpha_k D'_{k-1} - \alpha_k D_{k-1} - \alpha_k(D'_{k-1} - D_{k-1} - \alpha_{k-1}E_{k-1}) \\ &= 0 \text{ by 5.4.} \end{aligned}$$

By 3.5, the cycle $D'_k - D_k - \alpha_k E_k$ is the boundary of some chain E_{k+1} in $O(C)$. This completes the proof.

6. Reduced Powers of Cocyles

The preceding result is now applied to the case where K' is the p-fold product K^p of K, $\Pi = \Pi_p$ is the permutation group of the factors (§4), and C is the minimal carrier for the *diagonal* map $d\colon K \to K^p$ given by $d(x) = (x, \cdots, x)$.

Then $C(\sigma) = \bar{\sigma}^p$ is acyclic. Let $\{\alpha_i\}$ be any 0-sequence in Γ, and let

(6.1) $$D_i: C_q(K) \rightarrow C_{q+i}(K^p), \qquad i, q = 0, 1, \cdots,$$

be a sequence in $O(C)$ satisfying 5.6.

Let L be a subcomplex of K. Clearly we may regard C as a carrier from (K,L) to (K^p, L^p). Passing to cochains with coefficients in G', each D_i has a dual which we denote by the same symbol:

(6.2) $$D_i: C^r(K^p,L^p;G') \rightarrow C^{r-i}(K,L;G').$$

If the relations 5.6 are expanded using 3.3, we obtain

$$\partial D_i c + (-1)^{i+1} D_i \partial c = \alpha_i D_{i-1} c.$$

The dual relation to this is obtained by evaluating $w \in C^r(K^p,L^p;G')$ on each term and applying the definition of the dual of ∂, D_i, α_i. This gives

(6.3) $$D_i \delta w + (-1)^{i+1} \delta D_i w = D_{i-1} \alpha_i w.$$

When w is a cocycle, we have

(6.4) $$\delta D_i w = (-1)^{i+1} D_{i-1} \alpha_i w.$$

Now let $u \in C^q(K,L;G)$ be a q-cocycle, and set $w = u^p \in C^{pq}((K,L)^p;G')$ (p-fold products are defined in §4 in terms of a p-linear commutative function $G \rightarrow G'$). Using 4.13, we obtain

(6.5) $$\delta D_i u^p = \begin{cases} (-1)^{i+1} s(\alpha_i) D_{i-1} u^p, & q \text{ even}, \\ (-1)^{i+1} t(\alpha_i) D_{i-1} u^p, & q \text{ odd}. \end{cases}$$

It follows that $D_i u^p$ is a $(pq - i)$-cocycle mod $s(\alpha_i)$ when q is even, and mod $t(\alpha_i)$ when q is odd. It is a cocycle of K which is zero on L. It is called the *i-fold reduction of the* p^{th} *power of* u, or briefly, the (p, i)-power of u. Naturally this concept depends on the portion $\alpha_1, \cdots, \alpha_i$ of the chosen 0-sequence. In the next two sections we show that the cohomology class of $D_i u^p$ depends only on that of u, and that the resulting operations on cohomology groups are topologically invariant. They depend only on $\alpha_1, \cdots, \alpha_i$; and, in particular, do not depend on the choice of the sequence D_i.

Since s and t are ring homomorphisms, $\alpha_i \alpha_{i-1} = 0$ implies

(6.6) $$s(\alpha_i) s(\alpha_{i-1}) = 0 = t(\alpha_i) t(\alpha_{i-1}).$$

It follows from 6.5 that, for any i, either $D_i u^p$ or $D_{i-1} u^p$ is an absolute cocycle.

7. Invariance Under A Cohomology

Suppose $u, v \in C^q(K,L;G)$ are cohomologous cocycles, i.e.

(7.1) $$\delta a = u - v \quad \text{for some} \quad a \in C^{q-1}(K,L;G).$$

Define a_1 in $C^{pq-1}((K,L)^p;G')$ by

(7.2) $$a_1 = \sum_{k=0}^{p-1} (-1)^{qk} v^k a u^{p-k-1}.$$

Applying 3.8, we obtain

(7.3) $$u^p - v^p = \delta a_1 .$$

Thus $u \sim v$ implies $u^p \sim v^p$. We should like to infer from this that $D_i u^p \sim D_i v^p$. So we apply D_i to both sides of 7.3, and use 6.3 to obtain

(7.4) $$D_i u^p - D_i v^p = D_i \delta a_1 = D_{i-1} \alpha_i a_1 + (-1)^i \delta D_i a_1 .$$

If $\alpha_i a_1$ were zero mod $s(\alpha_i)$ or mod $t(\alpha_i)$ according as q is even or odd, then we would have the desired cohomology relation. As this need not be the case, a more involved argument is required. The clue to this argument is the observation that $\alpha_i a_1$ is a cocycle, and in fact is the coboundary of some cochain a_2. Then 6.3 can be applied to $D_{i-1}\delta a_2$ to give a further reduction of 7.4. Continuing this process p steps yields the desired result. The details follow.

Construct an algebraic cochain complex M as follows. The group $C^r(M)$ of r-cochains is zero if $r \neq q$ or $q - 1$; $C^q(M)$ is the *free* abelian group on two generators u, v; and $C^{q-1}(M)$ is the *free* group on one generator a. The coboundary δ in M is defined by 7.1. Then the cohomology group $H^r(M)$ is zero if $r \neq q$, and $H^q(M)$ is infinite cyclic.

Form now the p-fold tensor product

$$M^p = M \otimes M \otimes \cdots \otimes M.$$

The cochain groups of M^p are the free groups generated by p-fold products of a, u, v. The dimension of a product is the sum of the dimensions of the factors. Then $C^r(M^p) = 0$ if $r > pq$ or if $r < pq - q$. If $pq - p \leqq r \leqq pq$, then $C^r(M^p)$ is the free group whose base consists of p-fold products of a, u, v in which there are $pq - r$ factors a. The coboundary operator for M^p is defined by 4.8.

7.5. LEMMA. *If $r \neq pq$, then every cocycle of $C^r(M^p)$ is a coboundary.*

The assertion is trivial for $p = 1$. Proceeding by induction on p, we assume the lemma holds for $p - 1$ in place of p. Let z be an r-cocycle of M^p where $r \neq pq$. If z is written as a sum of products, and we collect those terms having first factors a, u, v respectively, then z takes the form

(7.6) $$z = aE + uF + vG$$

where $E \in C^{r-q+1}(M^{p-1})$ and $F, G \in C^{r-q}(M^{p-1})$. Then by 4.8,

$$\delta z = (u - v)E + (-1)^{q-1} a\delta E + (-1)^q (u\delta F + v\delta G)$$
$$= u(E + (-1)^q \delta F) - v(E - (-1)^q \delta G) - (-1)^q a\delta E.$$

Since $\delta z = 0$, and a, u, v are distinct generators of M, it follows that each term of the last expression is zero. Hence

$$E = (-1)^{q-1}\delta F = (-1)^q \delta G.$$

Therefore $F + G$ is a cocycle of $C^{r-q}(M^{p-1})$, and the inductive hypothesis provides a cochain $L \in C^{r-q-1}(M^{p-1})$ such that $\delta L = F + G$. It follows quickly that

$$\delta(aF + (-1)^q vL) = z,$$

and the lemma is proved.

We return now to the hypotheses of 7.1, and suppose p, q and i are fixed integers. Define $n = n(q, \alpha_i)$ by

$$n = \begin{cases} s(\alpha_i), & q \text{ even}, \\ t(\alpha_i), & q \text{ odd}. \end{cases} \tag{7.7}$$

Then $D_i u^p$, $D_i v^p$ are cocycles mod n.

7.8. LEMMA. *If a, u, v satisfy 7.1, then there exist cochains a_k, $b_k \in C^{pq-k}((K,L)^p; G')$, $k = 1, \cdots, p$, each of which is a sum of products of a, u, v, and such that*

$$\begin{aligned} \delta a_1 &= u^p - v^p \\ \delta a_{k+1} &= \alpha_{i-k+1} a_k + (-1)^k n b_k, \qquad k = 1, \cdots, p-1 \\ 0 &= \alpha_{i-p+1} a_p + (-1)^p n b_p . \end{aligned} \tag{7.9}$$

We prove first the existence of a_k, b_k in $C^{pq-k}(M^p)$ satisfying 7.9. The operations of Π_p and Γ_p in M^p are defined by 4.10.

Suppose first that $n = 0$. Define a_1 by 7.4. Then

$$\delta \alpha_i a_1 = \alpha_i \delta a_1 = \alpha_i(u^p - v^p) = n(u^p - v^p) = 0.$$

By 7.5, the cocycle $\alpha_i a_1$ is the coboundary of some cochain a_2 of M^p. Suppose $a_1, \cdots, a_l$ in M^p have been found so as to satisfy 7.9. Then

$$\delta \alpha_{i-l+1} a_l = \alpha_{i-l+1} \delta a_l = \alpha_{i-l+1}(\alpha_{i-l+2} a_{l-1}) = 0.$$

This last step requires the observation that the α's form a 0-sequence in Γ, and the operations of Γ in M^p form an anti-representation. By 7.5, the cocycle $\alpha_{i-l+1} a_l$ bounds a cochain a_{l+1} in M^p. The construction continues until a_p has been found. The preceding argument shows that $\alpha_{i-p+1} a_p$ is a cocycle of $C^{pq-p}(M^p)$. As a^p generates this group, $\delta a^p \neq 0$, and M^p is free, we must have $\alpha_{i-p+1} a_p = 0$.

Now suppose $n \neq 0$. By 6.6, if q is even, $s(\alpha_{i-1}) = 0$, and, if q is odd, $t(\alpha_{i-1}) = 0$. Therefore, in either case, the preceding argument applies with i replaced by $i - 1$ and provides elements $b_k \in C^{pq-k}(M^p)$ satisfying

$$\delta b_1 = u^p - v^p, \qquad \delta b_{k+1} = \alpha_{i-k} b_k, \qquad 0 = \alpha_{i-p} b_p .$$

Now define a_1 by 7.2. Then 7.3 holds and

$$\delta(\alpha_i a_1 - n b_1) = \alpha_i(u^p - v^p) - n(u^p - v^p) = 0.$$

By 7.5, the cocycle $\alpha_i a_1 - n b_1$ bounds a cochain a_2 in M^p. Suppose $a_1, \cdots, a_l$ in M^p satisfy 7.9. Then

$$\begin{aligned} &\delta(\alpha_{i-l+1} a_l + (-1)^l n b_l) \\ &\qquad = \alpha_{i-l+1}(\alpha_{i-l+2} a_{l-1} + (-1)^{l-1} n b_{l-1}) + (-1)^l n \alpha_{i-l+1} b_{l-1} = 0. \end{aligned}$$

It follows that a_{l+1} can be chosen properly. Continuing until a_p has been found, the same argument shows that $\alpha_{i-p+1}a_p + (-1)^p nb_p$ is a cocycle of $C^{pq-p}(M^p)$, and must therefore be zero.

Now map M^p into the cochain groups of $(K,L)^p$ by letting a, u, v in M correspond to the cochains of (K,L) represented by the same symbols. This homomorphism commutes with δ and the operations of Γ. Then the images of a_k , b_k fulfill the conditions of the lemma.

7.10. THEOREM. *If* a_k , b_k *are chosen as in* 7.8, *then*

$$D_i u^p - D_i v^p = \delta \textstyle\sum_{k=1}^{p} (-1)^{i-k+1} D_{i-k+1} a_k + n \textstyle\sum_{k=1}^{p} (-1)^{k+1} D_{i-k} b_k .$$

Apply D_{i-k} to both sides of 7.9, expand $D_{i-k}\delta$ using 6.3, and rearrange terms to obtain

$$D_{i-k-1}\alpha_{i-k}a_{k+1} - D_{i-k}\alpha_{i-k+1}a_k + (-1)^{i-k}\delta D_{i-k}a_{k+1} + (-1)^{k+1} nD_{i-k}b_k = 0.$$

If these equations are summed for $k = 1, \cdots, p$, the initial two terms cancel in pairs save for $D_{i-1}\alpha_i a_1$ and $D_{i-p-1}\alpha_{i-p}a_{p+1}$. However $a_{p+1} = 0$, and the term $D_{i-1}\alpha_i a_1$ is eliminated by using 7.4. The resulting equation is 7.10.

An obvious consequence of 7.10 is the

COROLLARY. $u \sim v$ *implies* $D_i u^p \sim D_i v^p$ mod n.

Thus we can make the following definition.

7.11. DEFINITION. If $w \in H^q(K,L;G)$ and u is a representative cocycle of w, the cohomology class of $D_i u^p$ mod n is denoted by $\mathcal{P}_i^p w$ and is called the *i-fold reduction of the* p^{th} *power of* w. Thus $\mathcal{P}_i^p$ is a set of transformations

$$\mathcal{P}_i^p: H^q(K,L;G) \to H^{pq-i}(K,L;G'_n), \qquad q = 0, 1, \cdots$$

where G'_n denotes G' reduced mod n, and $n = n(q, \alpha_i)$ is given by 7.7.

It should be kept in mind that $\mathcal{P}_i^p$ is not generally a homomorphism.

8. Topological Invariance

8.1. THEOREM. *Let* f *be a proper map* $(K,L) \to (K',L')$. *For a fixed* p *and a fixed* 0*-sequence* $\{\alpha_i\}$ *in* Γ_p , *let operations* $\{\mathcal{P}_i^p\}$ *be defined on the cohomology groups of* (K,L) *and* (K',L') *based on choices* $\{D_i\}$, $\{D'_i\}$, *respectively, of chain homomorphisms satisfying* 5.6. *Then, for any* $w \in H^q(K',L';G)$, *we have*

$$f^*\mathcal{P}_i^p w = \mathcal{P}_i^p f^* w. \tag{8.2}$$

By hypothesis, the minimal carrier C for f is acyclic. Hence C carries a chain transformation $f_\#$. Define F: $(K,L)^p \to (K',L')^p$ by

$$F(x_1, \cdots, x_p) = (f(x_1), \cdots, f(x_p)).$$

Then

$$C(\sigma_1 \cdots \sigma_p) = C(\sigma_1) \cdots C(\sigma_p)$$

is an acyclic carrier for F. Define $F_\#$ with this carrier by

$$F_\#(\sigma_1 \cdots \sigma_p) = (f_\#\sigma_1) \cdots (f_\#\sigma_p), \qquad \sigma_i \in K.$$

Then its dual satisfies

(8.3) $$F^{\#}(u_1 \cdots u_p) = (f^{\#}u_1) \cdots (f^{\#}u_p),$$

where each u_i is a cochain of (K',L') with coefficients in G.

Let f': $K \to K'^p$ be the composition of f and the diagonal map $K' \to K'^p$. Then C' defined by $C'(\sigma) = C(\sigma)^p$ is an acyclic carrier for f'. It is invariant under the permutation of the factors of K'^p. Consider the homomorphisms

$$F_{\#}D_i\,,\ D'_i f_{\#} : C_q(K) \to C_{q+i}(K'^p).$$

Clearly, for any cell σ of K, $F_{\#}\, D_i\sigma$ and $D'_i f_{\#}\, \sigma$ are chains on $C'(\sigma)$; hence $F_{\#}\, D^i$ and $D'_i f_{\#}$ are chains of $O(C')$. It is easy to verify that the conditions 5.6 hold with $\{D_i\}$ replaced by $\{F_{\#}\, D_i\}$ and also by $\{D'_i f_{\#}\}$. Therefore the second half of Lemma 5.5 asserts the existence of a sequence of homomorphisms $\{E_i\}$, where E_i is an i-chain of $O(C')$, such that

$$\partial E_{i+1} + (-1)^i E_{i+1}\partial = F_{\#}\, D_i - D'_i f_{\#} - \alpha_i E_i\,.$$

Passing to duals and denoting the dual of E_i by the same symbol, we obtain

$$E_{i+1}\delta + (-1)^i \delta E_{i+1} = D_i F^{\#} - f^{\#}\, D'_i - E_i \alpha_i\,.$$

If u is a q-cocycle of K' which is zero on L', we obtain, using 4.13,

$$(-1)^i \delta E_{i+1} u^p = D_i F^{\#} u^p - f^{\#} D'_i u^p - \begin{cases} s(\alpha_i) E_i u^p, & q \text{ even'} \\ t(\alpha_i) E_i u^p, & q \text{ odd}^{\cdot} \end{cases}$$

But $F^{\#} u^p = (f^{\#} u)^p$ by 8.3; and $E_{i+1} u^p$ is zero on L. Therefore $D_i(f^{\#} u)^p \sim f^{\#} D'_i u^p$ in (K,L) mod $s(\alpha_i)$ or $t(\alpha_i)$. This completes the proof.

If, in 8.1, we take $(K,L) = (K',L')$ and f to be the identity, then f^* is the identity and we have

8.4. Corollary. *The reduced power operations $\mathcal{P}^p_i$ are independent of the choice of the sequence $\{D_i\}$ satisfying 5.6. They depend therefore only on the 0-sequence $\{\alpha_i\}$ in Γ_p. In particular $\mathcal{P}^p_i$ depends only on the portion $\alpha_1, \cdots, \alpha_i$.*

Consider now the case where f in 8.1 is unrestricted. Let (K'_1,L'_1) be a simplicial subdivision of (K',L'). The inverse images under f of the open stars of vertices of K'_1 form a covering U of K by open sets. If K is locally finite, it is well known that there is a sufficiently fine subdivision (K_1,L_1) of (K,L) such that the stars of vertices of K_1 form a covering which refines U. We obtain thus a decomposition of f into three maps

$$(K,L) \xrightarrow{h} (K_1,L_1) \xrightarrow{g} (K'_1,L'_1) \xrightarrow{k} (K',L')$$

where

$$h(x) = x, \qquad g(x) = f(x), \qquad k(x') = x'.$$

The map h is proper since any subdivision of a cell is acyclic. Also k is proper since each cell of K'_1 lies in a least cell of K'. Finally g is proper since, for any cell σ of K_1, $g(\sigma)$ lies in the star of some vertex v of K'_1; hence the minimal complex containing $g(\sigma)$ is contractible on itself to v. It follows that $\mathcal{P}_i^p$ commutes with h^*, g^*, k^*. Since $f = kgh$ we must have $f^* = h^*g^*k^*$; and this implies that $\mathcal{P}_i^p$ commutes with f^*:

8.5. THEOREM. *If we restrict consideration to locally finite complexes, then* 8.2 *holds for arbitrary maps* f.

This result includes topological invariance; for, if f is a homeomorphism, f^* is a set of isomorphisms, and 8.2 shows that the operations $\mathcal{P}_i^p$ correspond under these isomorphisms.

This restriction to locally-finite complexes may not be necessary if one always uses the weak topology in K in the sense of J. H. C. Whitehead [6]. This is the topology in which a set is closed if and only if its intersection with each closed cell is a closed set of the cell. With this topology it seems likely that any open covering admits a refinement by the stars of vertices of some subdivision. Once this is done the above argument applies.

9. General Properties

The results of this section apply to the operations $\{\mathcal{P}_i^p\}$ based on any 0-sequence $\{\alpha_i\}$ in Γ_p.

In defining $\{\mathcal{P}_i^p\}$, we have slighted somewhat the case $i = 0$. Since D_0 is a chain transformation, $\partial D_0 = D_0\partial$; and the dual D_0 satisfies $D_0\delta = \delta D_0$. Hence, for any cocycle u of (K,L), we have $\delta D_0 u^p = 0$. Thus $D_0 u^p$ is a cocycle without reduction of the coefficients. In the proof of invariance under a cohomology, 7.4 reduces to

$$D_0 u^p - D_0 v^p = D_0 \delta a_1 = \delta D_0 a_1 .$$

So $D_0 u^p \sim D_0 v^p$. The proof of topological invariance holds with the modification $E_0 = 0$ so that $D_0(f^{\#}u)^p \sim f^{\#} D_0' u^p$ without reduction of the coefficients. Thus $\mathcal{P}_0^p$ is an invariant operation sending $H^q(K,L;G)$ into $H^{pq}(K,L;G')$. Clearly it is the same for all 0-sequences.

9.1. THEOREM. $\mathcal{P}_0^p u$ *is the* p^{th} *power of* u *in the sense of cup products.*

The case $p = 2$ is trivial since Lefschetz [1; p. 178] defines the cup product $u \smile v$ of two cochains by choosing a chain approximation D_0 to the diagonal map $K \to K^2$, and setting

$$(u \smile v)\cdot\sigma = (u \times v)\cdot D_0\sigma.$$

Then $D_0 u^2 = u \smile u$. The case $p = 3$ is included in an argument given by Lefschetz [1; p. 182] to prove the associative law $(u \smile v) \smile w \sim u \smile (v \smile w)$ where u, v, w are cocycles. The general case is omitted since it is obtained by an induction in which the central argument occurs already when $p = 3$.

9.2. THEOREM. *If* $\dim u = q$, *then* $\mathcal{P}_i^p u = 0$ *for all* $i > pq - q$.

Let σ be a cell of dimension $pq - i$ where $i > pq - q$. Then $\dim \sigma < q$ and $\dim \sigma^p < pq$. Since $D_i\sigma$ is a chain of dimension pq lying on $\bar{\sigma}^p$ whose dimension is $<pq$, we must have $D_i\sigma = 0$. Then, for any q-cocycle u, $D_i u^p \cdot \sigma = u^p \cdot D_i\sigma = 0$. As this holds for every $(pq - i)$-cell, we must have $D_i u^p = 0$.

9.3. THEOREM. *There is an integer m depending only on p, q and* $\alpha_1, \cdots, \alpha_{pq-q}$ *such that for any cochain* $u \in C^q(K,L;G)$ *and q-cell* σ *of K*

$$(D_{pq-q}u^p)\cdot\sigma = m(u\cdot\sigma)^p \bmod n$$

where $n = s(\alpha_{pq-q})$ *for q even and* $n = t(\alpha_{pq-q})$ *for q odd.*

Let $k = pq - q$. For any q-cell σ, $D_k\sigma$ is a pq-chain on $\bar\sigma^p$ whose only pq-cell is σ^p. Hence $D_k\sigma = m\sigma^p$ for some m. Then

$$(D_ku^p)\cdot\sigma = u^p\cdot D_k\sigma = mu^p\cdot\sigma^p = m(u\cdot\sigma)^p.$$

It suffices to show that m reduced mod n is independent of σ. Let Z be the ring of integers. Then $H^q(\bar\sigma,\dot\sigma;Z)$ is an infinite cyclic group generated by v. Using standard ring operations in Z, $\mathcal{P}_k^p v$ is defined and lies in $H^q(\bar\sigma,\dot\sigma;Z_n)$. From $D_k\sigma = m\sigma^p$, we obtain $\mathcal{P}_k^p v = mv \bmod n$. Now let $\bar\tau$ be a complex consisting of q-cell τ and its boundary $\dot\tau$, and let f: $(\bar\tau,\dot\tau) \to (\bar\sigma,\dot\sigma)$ map $\bar\tau - \dot\tau$ topologically onto $\bar\sigma - \dot\sigma$ with degree 1. Using 8.5, we obtain $\mathcal{P}_k^p w = mw \bmod n$ where $w = f^*v$ generates $H^q(\bar\tau,\dot\tau;Z)$. Since this relation on w is independent of σ, it follows that m mod n is independent of σ.

9.4 THEOREM. *For any integer k,* $\mathcal{P}_i^p(ku) = k^p\mathcal{P}_i^p u$.

This follows immediately from $(ku)^p = k^pu^p$.

If R is any commutative ring, we can always take $G = G' = R$ and let the p-linear function from G to G' be the ordinary product of p elements of R.

9.5. THEOREM. *If Z is the ring of integers, then, for any* $u \in H^1(K,L;Z)$, *we have* $\mathcal{P}_i^p u = 0$ *for* $i \neq p - 1$.

Consider first the special case where $K = S^1$ is a 1-sphere, $L = x_0$ is a vertex, and $u = u_0$ is a generator of $H^1(S^1,x_0;Z)$. Since $q \neq 1$ implies $H^q(S^1,x_0) = 0$ for any coefficient group, we must have $\mathcal{P}_i^p u_0 = 0$ for $i \neq p - 1$.

In the general case, we apply the theorem of Hopf which says that there is a map f: $(K,L) \to (S^1,x_0)$ such that f^*u_0 is the given u. By 8.5, we have

$$\mathcal{P}_i^p u = \mathcal{P}_i^p f^*u_0 = f^*\mathcal{P}_i^p u_0 = 0 \qquad \text{if } i \neq p - 1.$$

9.6. THEOREM. *Let Z be the ring of integers. For any* $q > 0$, *there is an integer m such that, for any* $u \in H^q(K,L;Z)$,

$$\mathcal{P}_{pq-q}^p u = mu \bmod n \tag{9.7}$$

where $n = s(\alpha_{pq-q})$ *for q even, and* $n = t(\alpha_{pq-q})$ *for q odd. In addition,*

$$\mathcal{P}_{pq-q-1}^p u = 0. \tag{9.8}$$

Let S^q denote the q-sphere, let $x_0 \in S^q$, and let u_0 generate $H^q(S^q,x_0;Z)$. Since $H^q(S^q,x_0;Z_n)$ is generated by u_0 mod n, 9.7 holds for some m with $u = u_0$. This determines m. As $H^{q+1}(S^q,x_0) = 0$ for any coefficients, 9.8 is valid for u_0.

Now let (K,L) be $(q + 1)$-dimensional, and $u \in H^q(K,L;Z)$. The Hopf theorem asserts the existence of a map f: $(K,L) \to (S^q,x_0)$ such that $f^*u_0 = u$. Since 9.7 and 9.8 hold for u_0, and f^* commutes with $\mathcal{P}_i^p$, it follows that they hold for u.

Let (K,L) be arbitrary, and $u \in H^q(K,L;Z)$. Let (K',L') be the $(q+1)$-skeleton of (K,L), and let $g\colon (K',L') \to (K,L)$ be the inclusion map. Then $g^*\colon H^q(K,L) \approx H^q(K',L')$, and the kernel of $g^*\colon H^{q+1}(K,L) \to H^{q+1}(K',L')$ is zero for any coefficient group. The preceding case asserts that 9.7 and 9.8 are valid for g^*u in place of u. Since g^* commutes with $\mathcal{P}_i^p$, and its kernels are zero in the dimensions q, $q+1$, it follows that 9.7 and 9.8 are valid for u. This completes the proof.

It is evident that the m of 9.7 coincides with that of 9.3.

9.9. Theorem. *Let Z be the ring of integers. If i is odd, and $u \in H^2(K,L;Z)$, then $\mathcal{P}_i^p u = 0$. If $i = 2k$ is even, then there is an integer m such that, for any $u \in H^2(K,L;Z)$,*

$$(9.10) \qquad \mathcal{P}_{2k}^p u = mu^{p-k} \bmod s(\alpha_{2k})$$

where u^{p-k} is the $(p-k)^{\text{th}}$ power in the sense of cup products.

Let M^{2l} denote the complex projective space of $2l$ real dimensions, and let x_0 be a point of M^{2l}. It is well known [5; p. 136] that $H^r(M^{2l},x_0) = 0$ for r odd and any coefficients, and $H^r(M^{2l},x_0;Z) \approx Z$ for r even and $\leqq 2l$. Furthermore if u_0 generates $H^2(M^{2l},x_0;Z)$, then u_0^{p-k} generates $H^{2p-2k}(M^{2l},x_0;Z)$. Setting $n = s(\alpha_{2k})$, and $Z_n = Z \bmod n$, it follows that u_0^{p-k}, reduced mod n, generates $H^{2p-2k}(M^{2l},x_0;Z_n)$. When i is odd, $\mathcal{P}_i^p u_0$ has odd dimension, and is therefore zero. When $i = 2k$, choose $l = p - k$. Then, for some integer m, 9.10 holds for $u = u_0$. This determines m.

Consider now the general case. Let $M^2 \subset M^{2l}$ be a complex projective line containing x_0. It is a 2-sphere. Let (K',L') be the 3-skeleton of (K,L), and g' the inclusion map $(K',L') \subset (K,L)$. Then $g'^*\colon H^2(K,L) \approx H^2(K',L')$ for any coefficients. By the Hopf theorem, there is a map $f'\colon (K',L') \to (M^2,x_0)$ such that

$$(9.11) \qquad f'^*u_0 = g'^*u.$$

It is well known [5; p. 108] that the homotopy groups $\pi_j(M^{2l}) = 0$ for $2 < j \leqq 2l$. It follows that we may extend f' to a map f'' of the $(2l+1)$-skeleton $(K'',L'') \to (M^2,x_0)$. Let g'' be the inclusion map $(K'',L'') \subset (K,L)$. From 9.11, we have

$$(9.12) \qquad f''^*u_0 = g''^*u.$$

If i is odd, we choose l so that $2l + 1 \geqq 2p - i$, and we apply $\mathcal{P}_i^p$ to 9.12. Using 8.5, and $\mathcal{P}_i^p u_0 = 0$, we obtain $g''\mathcal{P}_i^p u = 0$. Since $g''^*\colon H^{2p-i}(K,L) \to H^{2p-i}(K'',L'')$ has the kernel zero, it follows that $\mathcal{P}_i^p u = 0$.

When $i = 2k$, we have $l = p - k$ as above. Hence $2l + 1 = 2p - 2k + 1$. Therefore $g''^*\colon H^{2p-2k}(K,L) \approx H^{2p-2k}(K'',L'')$ for any coefficient group. From 9.10, we have

$$\begin{aligned} g''^*\mathcal{P}_{2k}^p u &= \mathcal{P}_{2k}^p g''^*u = \mathcal{P}_{2k}^p f''^*u_0 = f''^*\mathcal{P}_{2k}^p u_0 \\ &= f''^*mu_0^{p-k} = m(f''^*u_0)^{p-k} = m(g''^*u)^{p-k} = g''^*mu^{p-k}. \end{aligned}$$

As the kernel of g''^* is zero, the proof is complete.

10. Factorization

The aim of this section is to show that 0-sequences with certain properties lead to trivial $\mathcal{P}_i^p$.

10.1. LEMMA. *Suppose* $\omega D_i = \alpha_i D_{i-1}$ *as in* 5.6. *If, for some integer* $k > 0$, *there are chains* E_k, E_{k+1} *in* $O(C)$ *such that*

$$\omega E_{k+1} = D_k - \alpha_k E_k, \tag{10.2}$$

then $\mathcal{P}_i^p = 0$ *for* $i \geqq k$.

The existence of chains E_i in $O(C)$, $i = k, k+1, \cdots$, such that

$$\omega E_{i+1} = D_i - \alpha_i E_i \tag{10.3}$$

is established inductively. Suppose $E_k, \cdots, E_l$ are given, then

$$\omega(D_l - \alpha_l E_l) = \alpha_l D_{l-1} - \alpha_l(D_{l-1} - \alpha_{l-1}E_{l-1}) = 0.$$

Since $O(C)$ is acyclic, we may choose E_{l+1} so as to satisfy 10.3 with $i = l$.

Expanding the left side of 10.3 according to 3.3, and passing to duals, we obtain

$$E_{i+1}\delta + (-1)^i \delta E_{i+1} = D_i - E_i\alpha_i. \qquad i \geqq k.$$

If u is a q-cocycle of (K,L), we obtain, by 4.13,

$$(-1)^i \delta E_{i+1} u^p = D_i u^p - \begin{cases} s(\alpha_i) E_i u^p, & q \text{ even}, \\ t(\alpha_i) E_i u^p, & q \text{ odd}. \end{cases}$$

Therefore $D_i u^p \sim 0 \bmod s(\alpha_i)$ or $t(\alpha_i)$.

10.4. THEOREM. *If the* 0-*sequence* $\{\alpha_i\}$ *has the property* $\alpha_1\beta = 0$ *for some* $\beta \in \Gamma$ *such that* $s(\beta) = m \neq 0$, *then, for any* (K,L), q *and* $u \in H^q(K,L;G)$, *we have* $m\mathcal{P}_i^p u = 0$ *for* $i \geqq 1$.

Let e denote the unit of Γ. Then $s(me - \beta) = 0$. Therefore $(me - \beta)D_0$ has index 0. As $O(C)$ is acyclic, 3.5 provides an $E_1 \in O(C)$ such that $\omega E_1 = mD_0 - \beta D_0$. Then

$$\omega(mD_1 - \alpha_1 E_1) = m\alpha_1 D_0 - \alpha_1(mD_0 - \beta D_0) = 0.$$

By 3.5, there is an E_2 in $O(C)$ such that $\omega E_2 = mD_1 - \alpha_1 E_1$. Thus 10.1 applies with $k = 1$ and with $\{D_i\}$ replaced by $\{mD_i\}$.

10.5. COROLLARY. *If* $\alpha_1\beta = 0$ *for some* β *with* $s(\beta) = 1$, *then* $\mathcal{P}_i^p = 0$ *for* $i \geqq 1$.

10.6. EXAMPLE. Let $p = 3$, let the permutation g interchange the first and second factors, and let g' interchange the second and third. Set

$$\alpha = g - g' + gg' - g'g \in \Gamma.$$

Then $s(\alpha) = 0$. Also $\alpha\alpha = 0$. Setting $\alpha_i = \alpha$ for $i = 1, 2, \cdots$, we obtain a 0-sequence. If

$$\beta = g' + gg'g - gg',$$

we have $s(\beta) = 1$ and $\alpha\beta = 0$. By 10.5, the 0-sequence gives trivial operations.

10.7. THEOREM. *If* $\{\alpha_i\}$ *has the property that* α_1 *is contained in the group ring* *of a subgroup* Π' *of* Π *of order* m, *then, for any* (K,L), q *and* $u \in H^q(K,L;G)$, *we have* $m\mathcal{P}_i^p u = 0$ *for* $i \geqq 1$. *Taking* $\Pi' = \Pi$, *these relations hold for* $m = p!$. *Thus, in all cases,* $\mathcal{P}_i^p u$ *has a finite order dividing* $p!$.

Let β be the sum of all elements of Π' each with coefficient 1. Then $s(\beta) = m$. If $g \in \Pi'$, it is clear that $g\beta = \beta$. From this it follows that $\alpha\beta = s(\alpha)\beta$ for any α in the group ring of Π'. Therefore $s(\alpha_1) = 0$ implies $\alpha_1\beta = 0$. The conclusion of the theorem follows now from 10.4.

10.8. Theorem. *If* $\{\alpha_i\}$ *has the property that, for some* $k \geqq 1$ *and some integer* m, $m\alpha_k$ *can be factored* $m\alpha_k = \beta\gamma$ *so that* $\alpha_{k+1}\beta = 0$ *and* $\gamma\alpha_{k-1} = 0$ (*or* $s(\gamma) = 0$ *when* $k = 1$), *then, for any* (K,L), q *and* $u \in H^q(K,L;G)$, *we have* $m\mathcal{P}_i^p u = 0$ *for all* $i > k$.

If $k = 1$, $s(\gamma) = 0$ implies $\operatorname{In}(\gamma D_0) = 0$. If $k > 1$, then $\omega\gamma D_{k-1} = \gamma\alpha_{k-1}D_{k-2} = 0$. In either case, the acyclicity of $O(C)$ ensures the existence of a chain E_k such that $\omega E_k = \gamma D_{k-1}$. Then

$$\omega(mD_k - \beta E_k) = m\alpha_k D_{k-1} - \beta\gamma D_{k-1} = 0.$$

Hence $O(C)$ contains E_{k+1} such that

$$\omega E_{k+1} - mD_k - \beta E_k .$$

Then

$$\omega(mD_{k+1} - \alpha_{k+1}E_{k+1}) = m\alpha_{k+1}D_k - \alpha_{k+1}(mD_k - \beta E_k) = 0.$$

Hence $O(C)$ contains E_{k+2} such that

$$\omega E_{k+2} = mD_{k+1} - \alpha_{k+1}E_{k+1} .$$

The conclusion follows now by applying 10.1 to the sequence mD_i .

Remark. If we extend the 0-sequence $\{\alpha_i\}$ by adjoining α_0 = the sum of the elements of Π (see remark following 5.3), then 10.4 can be included in 10.8. For, $s(\beta) = m$ is equivalent to $m\alpha_0 = \beta\alpha_0$ which is the analog for $k = 0$ of the factorization in 10.8.

10.9. Theorem. *If the* 0-*sequence* $\{\alpha_i\}$ *in* Γ_p *lies in the subring* Γ_{p-1} *corresponding to the permutation group* Π_{p-1} *of the first* $p - 1$ *factors of* K^p, *and if* $\{\mathcal{P}_i^p\}$ $\{\mathcal{P}_i^{p-1}\}$ *are the operations corresponding to* $\{\alpha_i\} \subset \Gamma_p$, $\{\alpha_i\} \subset \Gamma_{p-1}$ *respectively, then, for any* (K,L), q *and* $u \in H^q(K,L;R)$ (R = *ring*), *we have*

$$\mathcal{P}_i^p u = (\mathcal{P}_i^{p-1}u) \smile u$$

where $\smile$ *denotes the cup product.*

Choose operations $D_i : K \to K^{p-1}$ corresponding to $\{\alpha_i\} \subset \Gamma_{p-1}$. Let I denote the identity transformation of the chains of K. Let $D_i \times I$ in $O(K \times K, K^p)$ be defined by $(D_i \times I)(\sigma \times \tau) = (D_i\sigma) \times \tau$. A simple calculation shows that

$$\omega(D_i \times I) = (\omega D_i) \times I = (\alpha_i D_{i-1}) \times I.$$

Choose also an algebraic approximation $d_\#$ to the diagonal map $K \to K \times K$. It is clear that the compositions $(D_i \times I)d_\#$ are in $O(C)$ where C is the minimal carrier for the diagonal map $K \to K^p$. Since $\partial d_\# = d_\#\partial$, we obtain

$$\omega((D_i \times I)d_\#) = (\omega(D_i \times I))d_\# = ((\alpha_i D_{i-1}) \times I)d_\# = \alpha_i(D_{i-1} \times I)d_\#.$$

This last step is valid by virtue of the identification of $\alpha_i \in \Gamma_{p-1}$ with an element of Γ_p. Thus $(D_i \times I)d_\#$ may be used to define $\mathcal{P}_i^p$. Doing so, the dual cochain operation is $d_\#(D_i \times I)$; and applying it to u^p, where u is a cocycle of (K,L), we obtain

$$d^\#(D_i \times I)u^p = d^\#((D_i u^{p-1}) \times u).$$

Lefschetz's definition of the cup product is $v \smile u = d^\#(u \times v)$. From this the conclusion of the theorem is immediate.

11. Equivalence of 0-Sequences

We say that $\alpha \in \Gamma$ is *regular* if it has a unique inverse $\alpha^{-1} \in \Gamma: \alpha\alpha^{-1} = e = \alpha^{-1}\alpha$. It is well known that, if α has a right or left inverse β in Γ, then α is regular and $\beta = \alpha^{-1}$. Two 0-sequences $\{\alpha_i\}$, $\{\beta_i\}$ in Γ are called *equivalent* if there is a sequence of regular elements $\gamma_0, \gamma_1, \cdots$ in Γ such that

$$\beta_i = \gamma_i \alpha_i \gamma_{i-1}^{-1}, \qquad i = 1, 2, \cdots .$$

11.1 THEOREM. *The operations $\mathcal{P}_i^p$ based on equivalent 0-sequences differ at most in sign.*

Suppose $\omega D_i = \alpha_i D_{i-1}$ and $\gamma_0, \gamma_1, \cdots$ are regular elements of Γ. Then

$$\omega\gamma_i D_i = \gamma_i \alpha_i \gamma_{i-1}^{-1} \gamma_{i-1} D_{i-1} = \beta_i \gamma_{i-1} D_{i-1} .$$

Thus $\{\gamma_i D_i\}$ satisfies 5.6 for the 0-sequence $\{\beta_i\}$. Since $\gamma_i\gamma_i^{-1} = e$ implies $s(\gamma_i) = \pm 1$ and $t(\gamma_i) = \pm 1$, it follows that the operation $D_i\gamma_i$ dual to $\gamma_i D_i$ when applied to the cocycle u^p gives $D_i\gamma_i u^p = \pm D_i u^p$.

12. Remarks

We have not as yet given a single example of an operation $\mathcal{P}_i^p$ which is both non-zero and which differs significantly from standard operations. The results of §10 show that the 0-sequence must be selected with care if $\mathcal{P}_i^p$ is not to be identically zero. The results of §9 show that, regardless of the choice of the 0-sequence, the operations $\mathcal{P}_i^p$ on 1 and 2-dimensional classes will be zero or will coincide with standard operations. In a subsequent paper we will show that the cyclic reduced powers (see §1) are generally non-trivial.

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