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THE CLASSIFICATION OF SPHERE BUNDLES

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1. Introduction

Whitney has introduced [13] the general notion of one space A being a fibre bundle over a second space B. It is a topological condition on a function f mapping A onto B which insures that f shall have a high degree of smoothness. It is the topological counterpart of the analytic requirement, in case A and B are differentiable manifolds, that f shall have a Jacobian of maximum rank at every point. Fibre bundles should prove of importance since they arise in many connections. The factor spaces of a Lie group form a lattice of fibre bundles (see Theorem 1). The spaces of tensors over a manifold are fibre bundles [9].

Among the simpler bundles are those for which the fibres (inverse images of points of B) are k-spheres. One of the main problems with which Whitney has concerned himself is the classification of the k-sphere bundles over a given space B.

In the present paper, this problem is reduced to a familiar problem of topology. A factor space M_l^k of the rotation group of a $(k + l + 1)$-sphere is selected, and it is proved that the equivalence classes of k-sphere bundles over a complex B are in 1-1 correspondence with the homotopy classes of maps of B in M_l^k.

In addition, the homotopy groups of M_l^k are computed for dimensions $\leqq 6$. This leads to a complete solution of the classification problem when B is a sphere of dimension $\leqq 6$.

Sections 2 and 3 contain definitions and discussion of fibre bundles and related concepts. Sections 4, 5 and 6 contain the statements of the principal results without proofs. Sections 7 and 8 are concerned with showing that covering homotopies exist in a fibre bundle. With this mechanism and its consequences at hand, the proofs of the main results are given in the remaining sections.

2. Fibre bundles

The concept of fibre bundle is somewhat complicated. There must be given three topological spaces. The first of these, denoted by A, is called the *fibre bundle*. The second, B, is called the *base space*; and the third, F, is called the *fibre*. There must likewise be given a fixed map ψ of A onto B; it is called the *projection*. (It will appear from subsequent conditions that, for each point $b \in B$, $\psi^{-1}(b)$ is homeomorphic to F.) There must also be given a fixed group G of homeomorphisms of F, so that F is a space with a geometry. Finally, there is given a family $\{N\}$ of neighborhoods covering B, and, for each N, a function ϕ_N mapping the product space $N \times F$ homeomorphically onto $\psi^{-1}(N)$ so as to satisfy two conditions. The first of these is

$$\psi\phi_N(b, y) = b \qquad \text{for } b \in N, y \in F.$$

For the second, define $\phi_{N,b}$ ($b \in N$) to be the map $F \rightarrow \psi^{-1}(b)$ given by $\phi_{N,b}(y) = \phi_N(b, y)$; then, for any $b \in B$ and neighborhoods N, N' containing b, we must have

$$\phi_{N,b}^{-1}\phi_{N',b} \in G.$$

The functions ϕ are called the *coordinate functions.*

The entire structure can be pictured in A somewhat as follows. The space A is decomposed continuously into disjoint closed sets denoted generically by F and referred to as fibres. Each F possesses a geometry G. A sufficiently small neighborhood of an F can be resolved into a product space in such a way that the fibres F' neighboring F appear as parallel sections, and under the parallel displacement $F \rightarrow F'$ we have in addition $G \rightarrow G'$. The space B is the space whose points are the fibres of A, and ψ is the natural map $A \rightarrow B$ which attaches to each point of A the fibre containing it.

If the group G consists of the identity alone, then the functions ϕ_N can be amalgamated into a single function which represents A as the product space $B \times F$, and A is called the *product bundle.*

If F is a linear space and G is the full linear group, A is called a *bundle of linear spaces.* This is the case of importance to differential geometry since the tangent space of a differentiable manifold M is such a bundle. In addition each manifold of tensors of a specified type over M is a bundle of linear spaces with the base space M [see 9].

If F is a k-dimensional sphere (k-sphere) and G is the full orthogonal group, A is called a *k-sphere-bundle.* In case G is the group of rotations (i.e. the orthogonal transformations preserving orientation), A is called an *orientable* k-sphere-bundle.

If the differentiable manifold M has a Riemann metric, then, by selecting the unit sphere in the tangent space of each point, one obtains a sphere-bundle over M. A similar operation is possible in the tensor spaces of each point.

By a *change of coordinates* in the neighborhood N, we shall mean the substitution of a function ϕ'_N for ϕ_N such that the fibre bundle conditions still hold, and, in addition, for $b \in N$, we have $\phi_{N,b}^{-1}\phi'_{N,b} \in G$. If λ is a continuous map $N \rightarrow G$, we obtain a change of coordinates if we set $\phi'_{N,b} = \phi_{N,b}\,\lambda(b)$. In general we do not wish to distinguish between a bundle and one obtained from it by coordinate transformations. This leads to the notion of equivalence of bundles.

If A, A' are two bundles over the same base space B with the same fibre F and group G, they are said to be *equivalent* if there exists a homeomorphism τ of A' onto A which, for each b in B, maps the fibre $\psi'^{-1}(b)$ of A' onto the fibre $\psi^{-1}(b)$ of A in such a fashion that, for any neighborhoods N, N' containing b and attached to A, A', respectively, we have

$$\phi_{N,b}^{-1}\tau\phi'_{N',b} \in G. \tag{E}$$

Clearly any number of coordinate changes in A merely replaces A by an equivalent bundle.

If, for example, G is the group of all homeomorphisms, the condition E is unnecessary.

The classification problem for fibre bundles is the following: *For fixed B, F and G, exhibit one and only one A from each equivalence class.*

Now let B, F be fixed and let G, G' be two groups of homeomorphisms of F such that $G \subset G'$. If A is a fibre bundle over B with fibre F and group G, then the same coordinate functions ϕ represent A as a fibre bundle over B with fibre F and group G'. This new bundle A' is distinct from A since the groups of the fibres are larger. The transformation $A \rightarrow A'$ carries equivalent bundles into such, and is therefore an operation on equivalence classes. It is to be expected in general that this correspondence of classes is a many-to-one map and need not be a mapping *onto*.

Since the identity homeomorphism of F is contained in every G, the product bundle $B \times F$ determines an equivalence class of bundles relative to B, F and G. A bundle of this class will be called a *product bundle* with little danger of confusion.

It is sometimes convenient to consider an equivalence of two bundles not possessing the same base space and fibre. Let A, B, F, G be a bundle and A', B', F', G' a second. We shall say that they are *equivalent* (in the unrestricted sense) if there exists a homeomorphism τ of A' onto A which carries fibres into fibres, inducing thereby a homeomorphism τ of B' onto B, and is such that, if $\tau(b') = b$, $b \in N$ and $b' \in N'$, then the homeomorphism $\phi_{N,b}^{-1}\tau\phi'_{N',b'}$ of F' onto F must carry the group G' onto G.

For A, A' to be equivalent, it is necessary that B, B' be homeomorphic, and that F, F' be homeomorphic in such a way that G' corresponds to G. Assuming these conditions satisfied, we cannot specify these correspondences in advance unless we wish to consider the restricted equivalence defined earlier.

On the other hand, two bundles A, A' with the same B, F, G may be equivalent in the unrestricted sense but not in the restricted sense. Of course, in such a case, τ would of necessity induce a non-trivial transformation of B, or of F or of both. Naturally there are fewer equivalence classes in the unrestricted sense.

In our main problem equivalence will be used only in the restricted sense.

3. Analogy with group extensions

We digress for a moment to remark the analogy between a fibre bundle and a group extension. In the group case F and G coalesce into the group G, B is likewise a group, and A is an extension of G by B (i.e. G is an invariant subgroup of A and $A/G = B$).

The equivalence of two extensions parallels the equivalence of two bundles; in the first case τ is isomorphic, in the second, homeomorphic.

The problem of classifying bundles with fixed B, F and G is the analog of determining all extensions of the group G by the group B.

We shall see that these ideas come together quite naturally when A, B and G are topological groups.

4. Summary of principal results

If R is a topological group and K is a closed subgroup, the symbol R/K is used to denote the set of left cosets of K in R. The *natural map* $R \to R/K$ attaches to each r in R the coset rK. A set of cosets is an *open set* of R/K if their point set sum is an open set of R. With this definition of open set, R/K is a topological space, and the natural map is continuous [see 8, Ch. 3]. If the elements of a coset of K are all multiplied on the left by a fixed element of R, the resulting set is also a left coset of K. In this way R is realized as a group of homeomorphisms of R/K. Of course R may not operate effectively in R/K; the effective group is R/K' where K' is the intersection of all subgroups conjugate to K.

It is well-known that, if R is a Lie group, then R/K is an analytic manifold, and the natural map has Jacobian of maximum rank at every point. Even more, we have

THEOREM 1. *If R is a Lie group; and H, K are closed subgroups with $H \subset K$, then, with respect to the natural map, R/H is a fibre bundle over R/K with fibre $F = K/H$ subject to the group $G = K$ of left translations of K/H.*

Now let R be the group of rotations (i.e. the orthogonal maps preserving orientation) of a sphere S of dimension $k + l + 1$. In S we select a fixed great k-sphere S^k. Let S^l be the great sphere of S orthogonal to S^k. A rotation of S^k (S^l) becomes a rotation of S if we let it act as the identity on S^l (S^k). In this way the groups of rotations $R(S^k)$, $R(S^l)$ of S^k, S^l, respectively, are subgroups of R. Since they commute with one another, we may regard their direct product

$$\tilde{R}(S^k, S^l) = R(S^k) \times R(S^l)$$

as a subgroup of R. It is contained in the group $R(S^k, S^l)$ of all elements of R which map S^k on itself. The latter is larger since it contains the rotations of S which reverse orientations of both S^k and S^l. Since any two such differ by an element of $\tilde{R}(S^k, S^l)$, it follows that $R(S^k, S^l)$ has just two components and $\tilde{R}(S^k, S^l)$ is the component of the identity. We shall make extensive use of the left coset spaces

$$M_l^k = R/R(S^k, S^l), \qquad \tilde{M}_l^k = R/\tilde{R}(S^k, S^l).$$

From Theorem 1, we obtain

$\tilde{M}_l^k$ is a two-fold covering of M_l^k.

Two rotations r, r' lie in the same left coset of $R(S^k, S^l)$ if and only if $r(S^k) = r'(S^k)$. Since any great k-sphere of S is the image $r(S^k)$ for a suitable r, the correspondence $r \to r(S^k)$ between the great k-spheres and the left cosets of $R(S^k, S^l)$ is 1-1. Therefore

M_l^k is the space whose points are the great k-spheres of a $(k + l + 1)$-sphere.[1]

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[1] A simple duality shows that the bundle R over M_l^k is equivalent to the bundle R over M_k^l. In particular M_l^k and M_k^l are homeomorphic. We might therefore regard M_l^k as the space of great l-spheres of a $(k + l + 1)$-sphere. We adhere to the first interpretation throughout the paper.

By a similar argument, we obtain

$\tilde{M}_l^k$ is the space whose points are the oriented great k-spheres of a $(k + l + 1)$-sphere.

We might equally well regard M_l^k as the space of $(k + 1)$-planes passing through the origin of a $(k + l + 2)$-space.

It is worth noting that M_l^0 is projective $(l + 1)$-space and $\tilde{M}_l^0$ is an $(l + 1)$-sphere.

DEFINITION. To a map g of a topological space B into M_l^k we attach a space $A(g)$ as follows: $A(g)$ is the subset of $B \times S^{k+l+1}$ consisting of those pairs (b, s) such that s is a point of the k-sphere corresponding to $g(b)$. Define $\psi(b, s) = b$, so that ψ maps $A(g)$ onto B.

It is clear that $\psi^{-1}(b)$ in $A(g)$ is the k-sphere $b \times g(b)$. The main results hinging on the function $A(g)$ are embodied in the following theorems.[2]

THEOREM 2. *For any map g of B in M_l^k, the space $A(g)$ is a k-sphere bundle over B with projection ψ.*

THEOREM 3. *If B is compact and g, g' are homotopic maps of B in M_l^k, then the bundles $A(g)$, $A(g')$ are equivalent.*

THEOREM 4. *If B is a complex,[3] A a k-sphere bundle over B, and $l \geqq \dim B - 1$, then there is a map g of B in M_l^k such that $A(g)$ is equivalent to A.*

THEOREM 5. *If B is a complex,[3] $l \geqq \dim B$, and g, g' are two maps of B in M_l^k such that $A(g)$, $A(g')$ are equivalent, then g, g' are homotopic.*

As an immediate consequence of Theorems 2 to 5, we have

COROLLARY. *The problem of classifying the k-sphere bundles over a complex B is equivalent to the problem of enumerating the homotopy classes of maps of B in M_l^k for any $l \geqq \dim B$.*

5. The homotopy structure of M_l^k

It should be observed that, if g maps B into a point of M_l^k, then $A(g)$ is a product bundle $B \times S^k$. From this and Theorem 3, it follows that inessential maps of B in M_l^k determine product bundles. As a corollary, if B is contractible on itself to a point, then any sphere bundle over B is a product bundle.

To obtain bundles which are not product bundles one must therefore study spaces B which are not contractible. The simplest such are the spheres of various dimensions. This leads to the problem of classifying the maps of spheres in M_l^k, and this in turn to the determination of the homotopy groups $\pi_i(M_l^k)$. Furthermore an inspection of existing results on the number of homotopy classes of maps of one space in another [5] reveals that the homotopy groups of the latter space play a fundamental role. Therefore our immediate objective is to obtain such information as we can about the groups $\pi_i(M_l^k)$.

Much is already known concerning the homotopy groups of the orthogonal groups [2; 12]. Since M_l^k is intimately related to several such groups, the obvious procedure is to examine this connection more closely. This leads to the following results.

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[2] Theorems 2 and 4 have been stated by Whitney in a different but equivalent form (see [15, §8]).

[3] The complex B may be infinite but is required to be locally finite.

THEOREM 6. *If k or l is positive,*[4] *the fundamental group of M_l^k is cyclic of period 2. If $R(S^k)$ is the group of rotations of S^k, and if it is considered as a subgroup of R, then there is a natural isomorphism*

$$\pi_i(M_l^k) \cong \pi_{i-1}(R(S^k)) \qquad 2 \leqq i \leqq l.$$

The restriction $i \leqq l$ is of no significance in the classification problem since l may be chosen as large as convenient.

Consider now the classification problem for B = i-sphere. An element of $\pi_i(M_l^k)$ is a class of maps of B in M_l^k which map a fixed reference point b_0 of B into another such point x_0 of M_l^k, and any two maps of the class are homotopic within the class. If C_i denotes the set of homotopy classes of maps B in M_l^k, then each element of $\pi_i(M_l^k)$ is contained in some element of C_i; so that a map $\pi_i \to C_i$ is at hand. Since M_l^k is arcwise connected, any map of B in M_l^k is homotopic to one mapping b_0 into x_0. Thus C_i is the image of π_i.

The algebraic background of the correspondence $\pi_i \to C_i$ has been given by Eilenberg [5, pp. 63–64] as follows. A closed path beginning and ending at x_0 determines an automorphism of π_i. The effect of the automorphism on any i-sphere with b_0 mapped into x_0 is obtained by deforming the sphere in M_l^k so that b_0 describes the closed path. The automorphism depends only on the homotopy class of the closed path. Indeed, the fundamental group $\pi_1(M_l^k)$ appears as a group of automorphisms of $\pi_i(M_l^k)$. The principal result that concerns us is the following:

Two elements of $\pi_i(M_l^k)$ belong to the same homotopy class of maps of an i-sphere in M_l^k if and only if there is an element of $\pi_1(M_l^k)$ mapping one into the other.

Whenever each element of π_1 induces the identity automorphism in π_i the space is said to be *simple* in the dimension i. In this case the map $\pi_i \to C_i$ is 1-1.

We are able to establish

THEOREM 7. *If k and l are even, then M_l^k is simple in all dimensions. In any case M_l^k is simple in all dimensions $\leqq$ Min (k, l). If $l \geqq 2$, then M_l^1 is not simple in the dimension 2; for $\pi_2(M_l^1)$ is infinite cyclic and the non-trivial element α of $\pi_1(M_l^1)$ reverses sign in $\pi_2(M_l^1)$.*

We have not been able to decide the question of simplicity in the remaining cases except for trivial cases where $\pi_i = 0$ or is cyclic of period 2.

6. Sphere bundles over spheres

Combining the results of the preceding section with the known facts [2; 12] about the groups $\pi_{i-1}(R(S^k))$, we can obtain numerous special results concerning the number of k-sphere bundles over a sphere B of dimension $\leqq 6$. We state some of these in this section.

Since $\pi_1(M_l^k)$ is cyclic of period 2, we have

I. *The k-sphere bundles over S^1 are of two types: the product bundle, and the generalized Klein bottle.*

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[4] The case $k = l = 0$ is trivial since M_0^0 is a 1-sphere.

The term *Klein bottle* is justified by the fact that, if S^k is carried around S^1, it comes back with orientation reversed. This is proved by noting that the nontrivial element α of $\pi_1(M_l^k)$ is covered by a path in R joining the identity to an element of $R(S^k, S^l)$ which reverses the orientation of S^k.

Since $\pi_2(M_l^1) \cong \pi_1(R(S^1))$ is infinite cyclic,

II. *There are an infinity of* 1-*sphere bundles over* S^2.

Since the automorphism α is a reversal of sign, an element of $\pi_2(M_l^1)$ defines the same 1-sphere bundle as does its negative. We can thus set up a natural correspondence between the integers $n = 0, 1, 2, \cdots$ and the 1-sphere bundles over S^2. Let A_n denote the bundle corresponding to n. It is clear the A_0 is the product bundle.

It is well known that, if S^3 = group of unit quaternions, and S^1 = subgroup of complex numbers of absolute value 1, then S^3/S^1 is homeomorphic to S^2. In this way S^3 is a 1-sphere bundle over S^2 (see Theorem 1). It is a reasonable conjecture that this is the bundle A_1.

If G_m is the subgroup of S^1 generated by exp $(2\pi i/m)$, by Theorem 1, S^3/G_m is a bundle over $S^3/S^1 = S^2$. The fibre is S^1/G_m which is again a 1-sphere. In this way S^3/G_m is a 1-sphere bundle over S^2. It is reasonable to conjecture that it is the bundle A_m. Note that S^3/G_m is the lens space corresponding to the pair of integers $(m, 1)$.

Since $\pi_1(R(S^k))$ is cyclic of period 2 for $k \geqq 2$, we have

III. *If* $k \geqq 2$, *there is just one k-sphere bundle over* S^2 *other than the product bundle.*

Since $\pi_2(R(S^k)) = 0$ for any k, we have

IV. *A k-sphere bundle over* S^3 *is always a product bundle.*

Since $\pi_{n-1}(R(S^1)) = 0$ for $n \geqq 3$, we have

V. *If* $n \geqq 3$, *a* 1-*sphere bundle over* S^n *is always a product bundle.*

Since $\pi_3(R(S^k))$ contains an infinity of elements for $k > 1$, we have

VI. *If* $k > 1$, *there are an infinity of k-sphere bundles over* S^4.

Since $\pi_4(R(S^k)) = 0$ for $k \geqq 5$, and $\pi_5(R(S^k)) = 0$ for $k \neq 5$, we have

VII. *A k-sphere bundle over* S^5 (S^6) *is always a product bundle if* $k \geqq 5 (k \neq 5)$.

One may obtain additional information by examining the results of [2; 12].

7. Factor spaces of groups

Since we must deal extensively with the factor spaces of the rotation group R, a few general remarks about factor spaces are in order.

DEFINITION. Let R be a topological group and K a closed subgroup. A function ϕ defined in a neighborhood N of the point K of the left coset space R/K with values in R is called a *slicing function* for K if (1) ϕ is continuous, (2) $\phi(b)$ is an element of b for each b in N, (3) $\phi(K)$ is the identity element of R. We shall say that ϕ is *invariant* under the subgroup H of R if $hKh^{-1} = K$ and $\phi(hbh^{-1}) = h\phi(b)h^{-1}$ for each b in N and h in H.

A slicing function is a continuous cross-section through the identity of the family of left cosets of K neighboring K. It is invariant if the cross-section is mapped on itself when conjugated by an element of H. In practice, H will be a subgroup of K.

THEOREM 8. *If K is a closed subgroup of the Lie group R, then there exists a slicing function ϕ for K. If H is a compact subgroup of R such that $hKh^{-1} = K$ for each h in H, then ϕ may be chosen invariant under H.*

PROOF. Choose a system of canonical coordinates of the first kind for a neighborhood of the identity e of R [see 8, p. 187]. A sufficiently small neighborhood of e in K is a linear subspace in these coordinates [8, p. 196, 203]. By means of a linear transformation, it can be arranged that the subspace K is defined by equations $x^1 = x^2 = \cdots = x^r = 0$ where $(x) = (x^1, \cdots, x^m)$ are the coordinates of the element x of R. If the elements x, y are near enough to e, then the coordinates of $z = xy$ are differentiable functions of those of x and y: $z^i = f^i(x, y)$. The system of equations

$$\text{(A)} \qquad f^i(x^{-1}, y) = 0 \qquad (i = 1, \cdots, r), \qquad y^j = 0 \qquad (j = r+1, \cdots, m)$$

reduces to the system $y^i = 0$ $(i = 1, \cdots, m)$ when $x = e$. Thus the Jacobian of (A) in the y^is is non-singular for x, y near e. It follows from the implicit function theorem that there exists a system of continuous functions $\phi^i(x)$, defined for x in a neighborhood of e, such that (A) becomes an identity in x when y^i is replaced by $\phi^i(x)$. The system (A) states that y lies in the intersection of the left coset xK with the coordinate subspace complementary to K. The uniqueness part of the implicit function theorem assures us that there is just one such intersection near to e for x near enough to e. Therefore, for such x's, $\phi(x)$ is constant along the coset xK. Thus ϕ is a function of the coset alone and is therefore the desired slicing function.

To prove the second part of the theorem, observe first that the operation of conjugation of R by a fixed element of R is a linear transformation of the canonical coordinates of the first kind [8, p. 280]. Since the subgroup H is compact, there is a linear transformation to new coordinates in which conjugation by an element of H is an orthogonal transformation.[5] Since K is invariant under H, the linear subspace orthogonal to K is likewise invariant. These new coordinates are canonical, and may be chosen so that K is a coordinate subspace. Using these coordinates, we define ϕ as above. Then, for h in H, both $\phi(hxh^{-1})$ and $h\phi(x)h^{-1}$ lie in the intersection of the left coset $(hxh^{-1})K$ and the coordinate plane orthogonal to K. Since this intersection consists of just one point near e when x is near e, we have $\phi(hxh^{-1}) = h\phi(x)h^{-1}$ as desired.

Theorem 1 is an immediate consequence of Theorem 8 and the following

THEOREM 1′. *If R is a topological group, K a closed subgroup of R which admits a slicing function ϕ, and H a closed subgroup of K, then, with respect to the natural map, R/H is a fibre bundle over R/K with fibre $F = K/H$ subject to the group $G = K$ of left translations of K/H.*

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[5] In the rotation group R with which we are mainly concerned, the Cayley coordinates have this property [11, p. 56].

PROOF. Let ψ denote the natural map $R/H \to R/K$. Let N denote the neighborhood of the point K of R/K in which ϕ is defined. The left translation of N by r in R is denoted by N_r. The totality of these translations covers R/K. For each N_r, we define a map ϕ_r of $N_r \times K/H$ into R/H as follows

$$\phi_r(b, c) = [r\phi(r^{-1}b)]c, \qquad b \in N_r, \ c \in K/H.$$

Since b is in N_r, $r^{-1}b$ is in N, so that $\phi(r^{-1}b)$ is defined. Since $K/H \subset R/H$, $c \in K/H$ is an element of R/H, and $\phi_r(b, c)$ is the left translation of c by the element $r\phi(r^{-1}b)$ of R. It is clear that ϕ_r is continuous.

Since $\phi(r^{-1}b)$ is a point of the coset $r^{-1}b$, $r\phi(r^{-1}b)$ is a point of the coset b. If c be regarded as a subset of K, it follows that $\phi_r(b, c)$ is a subset of b; and therefore $\psi\phi_r(b, c) = b$.

For an element a in $\psi^{-1}(N_r)$, let $\zeta_r(a)$ be the left translation of a by $[r\phi(r^{-1}\psi(a))]^{-1}$. It follows that $\zeta_r\phi_r(b, c) = c$, and $\phi_r(\psi(a), \zeta_r(a)) = a$. The existence of the continuous functions ψ and ζ_r proves that ϕ_r is a homeomorphism of $N_r \times K/H$ onto $\psi^{-1}(N_r)$.

Finally if b is in both N_r and $N_{r'}$, the map $\phi_{r,b}^{-1}\phi_{r',b}$ of K/H on itself is given by

$$c' = [r'\phi(r'^{-1}b)]^{-1}[r\phi(r^{-1}b)]c$$

which is clearly a left translation of K/H by an element of K.

8. Covering homotopies

Suppose ψ is a continuous map of A into B, f a continuous map of X into A, and $h(x, t)$, $x \in X$, $0 \leqq t \leqq 1$, a homotopy in B of the map ψf of X into B (i.e. $h(x, 0) = \psi f(x)$). A homotopy $g(x, t)$ of f in A is said to *cover* the homotopy $h(x, t)$ if $\psi g(x, t) = h(x, t)$ for all x, t. Clearly, if a homotopy $g(x, t)$ of f be given, then $g(x, t)$ covers the homotopy $\psi g(x, t)$. Of more significance is the matter of constructing a $g(x, t)$ for a given $h(x, t)$.

The covering homotopy g of h is said to be *stationary* with h, if, whenever $h(x_0, t)$ is constant in a t-interval $t' \leqq t \leqq t''$, then $g(x_0, t)$ is likewise constant in that interval (i.e. whenever x_0 remains at rest under h, it is likewise at rest under g).

In a recent paper by Hurewicz and the author [7] the question of the existence of covering homotopies is considered in some detail. The property "A is a fibre space over B relative to ψ" is introduced, and it is proved that, in the presence of this property, there exists a g covering a given h which is stationary with h.

The notions of fibre space and fibre bundle do not coincide, there are fibre spaces that are not fibre bundles. The fibres of a fibre space need not be pairwise homeomorphic, although they do belong to the same homotopy type. Whether every fibre bundle is a fibre space is not yet determined. What concerns us here is to establish the existence of covering homotopies for fibre bundles, for we shall make extensive use of the consequences.

THEOREM 9[6]. *If A is a fibre bundle over B, X a compact space, f a continuous*

[6] The essential content of this theorem has been stated by Ehresmann and Feldbau [4].

map $X \to A$, and $h(x, t)$ a homotopy of the map ψf of X in B, then there exists a covering homotopy $g(x, t)$ which is stationary with $h(x, t)$.

PROOF. For each b in B we select a pair of neighborhoods U, V of b such that $\bar{U} \subset V$ and $\bar{V}$ is contained in some coordinate neighborhood N. Since $X \times I$ is compact (I is the unit t-interval), there exists a $\delta > 0$ such that, for any point x of X and interval $I' \subset I$ of length $<\delta$, the image of $x \times I'$ under h is contained wholly in some member of the family $\{U\}$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of I such that $t_{i+1} - t_i < \delta$. We shall suppose that $g(x, t)$ has been constructed for all x, t such that $t \leqq t_k$, and proceed to show that g may be extended as required over the interval I': $t_k < t \leqq t_{k+1}$.

Since, for each x, there exists a U containing the image of $x \times I'$ under h, it follows by continuity that there is a neighborhood W of x such that h maps $W \times I'$ into some U. Since X is compact, we may select a finite covering by these neighborhoods: $W_1, \cdots, W_m$. Denote the U, V pair such that $W_i \times I' \to U$ by U_i, V_i.

By the Urysohn lemma, there exist a continuous, real-valued function $u_i(b)$ defined on B such that $u_i = 1$ for b in $\bar{U}_i$, $u_i = 0$ for b outside V_i, and $0 \leqq u_i \leqq 1$. Define

$$\tau_j(x) = t_k + (t_{k+1} - t_k) \operatorname{Max}_{i \leqq j} u_i(h(x, t_k)).$$

Clearly τ_j is continuous, $t_k \leqq \tau_j \leqq t_{k+1}$, $\tau_j(x) \leqq \tau_{j+1}(x)$, and $\tau_m(x) = t_{k+1}$. Define $\tau_0(x) = t_k$. Then $g(x, t)$ is defined for $t \leqq \tau_0(x)$. We shall suppose, inductively, that $g(x, t)$ has been properly constructed for $t \leqq \tau_j(x)$, and show that it may be extended to $t \leqq \tau_{j+1}(x)$.

Denote by T the set of pairs (x, t) such that $\tau_j(x) < t \leqq \tau_{j+1}(x)$. It is over this set that we must extend $g(x, t)$. It follows from the definition of the functions τ that h maps the closure $\bar{T}$ of T into $\bar{V}_{j+1}$ which is contained in some coordinate neighborhood N with coordinate function ϕ. Since ϕ is a homeomorphism of $N \times F$ with $\psi^{-1}(N)$, there exists a map ζ of $\psi^{-1}(N)$ into F such that $\zeta\phi(b, c) = c$ for each c in F. Since $(x, t) \in \bar{T}$ implies $\psi g(x, \tau_j(x)) = h(x, \tau_j(x))$ is in N, the function $\zeta g(x, \tau_j(x))$ is defined. We can therefore define

$$g(x, t) = \phi(h(x, t); \zeta g(x, \tau_j(x))) \qquad \text{for} \quad (x, t) \quad \text{in} \quad T.$$

It follows from the properties of ϕ that $\psi g(x, t) = h(x, t)$, and that g is stationary with h. To see that g has been continuously extended, we note that the expression for g is defined and continuous for (x, t) in $\bar{T}$, and, for (x, t) in $\bar{T} - T$ it agrees with the values of g already assigned.

This completes the general step in the secondary induction. Since $\tau_m(x) = t_{k+1}$, the completion of the secondary induction establishes the general step of the primary induction. This completes the proof.

As an immediate consequence of Theorem 9 and a theorem of R. H. Fox[7], we have

[7] Fibre spaces II, Bull. Amer. Math. Soc. 49 (1943).

THEOREM 10. *If A is a compact fibre bundle over B, and B is an absolute neighborhood retract, then A is a fibre space over B.*

In the case of factor spaces of groups we can establish directly the fibre space nature of the fibre bundles.

THEOREM 11. *Let R be a topological group and H, K closed subgroups with $H \subset K$. If H admits a slicing function, and K admits a slicing function invariant under H, then R/H is a fibre space over R/K with the natural map as projection.*

PROOF. Let $\phi(b)$ be the slicing function for K which is invariant under H. Let η be the natural map $R \to R/H$, and ψ the natural map $R/H \to R/K$. For any pair (a, b) such that a is in R/H and b is in the left translation to $\psi(a)$ of the domain N of definition of ϕ, we define the slicing function

$$\phi(a, b) = \eta(r\phi(r^{-1}b))$$

where r has been chosen arbitrarily in $\eta^{-1}(a)$. That ϕ is independent of the choice of r in $\eta^{-1}(a)$ follows from the invariance of ϕ under H; for

$$(rh)\phi((rh)^{-1}b) = r\phi(r^{-1}bh^{-1})h = r\phi(r^{-1}b)h.$$

To prove that ϕ is continuous in (a, b), it suffices to note that, since H admits a slicing function, we may choose r in $\eta^{-1}(a)$ as a continuous function of a neighboring a particular choice r_0 in $\eta^{-1}(a_0)$.

COROLLARY. *If R is a Lie group, K a closed subgroup, and H a compact subgroup of K, then R/H is a fibre space over R/K with the natural map as projection.*

9. Proofs of Theorems 6 and 7

We first prove that $\tilde{M}_l^k$ is simply-connected. Let a_0 be the identity element of R, and let b_0 be its projection in $\tilde{M}_l^k$. Let $g(x)$ be a continuous map of the interval $0 \leqq x \leqq 1$ in $\tilde{M}_l^k$ with $g(0) = g(1) = b_0$. If it is not already the case, one can obtain by a simple homotopy that $g(x) = b_0$ for $\frac{1}{2} \leqq x \leqq 1$. Since g may be regarded as a homotopy of the image of a_0, it follows from the existence of a covering homotopy that there is a function $f(x)$ defined for $0 \leqq x \leqq \frac{1}{2}$ with values in R such that $f(0) = a_0$, and the projection of $f(x)$ is $g(x)$. This last implies that $f(\frac{1}{2})$ is in $\tilde{R}(S^k, S^l)$ which, being connected, admits a path $f(x)$, $\frac{1}{2} \leqq x \leqq 1$, from $f(\frac{1}{2})$ to a_0. Then the projection of $f(x)$ is $g(x)$ for $0 \leqq x \leqq 1$. By assumption k or l is positive, say l. It is proved in [7, Th. 5] that a closed path f in R is contractible into $R(S^l)$. The image in $\tilde{M}_l^k$ of this contraction of f provides the desired contraction of g into b_0.

Since $\tilde{M}_l^k$ is simply-connected and is a two-fold covering of M_l^k, it follows that the fundamental group of M_l^k is cyclic of period 2. The non-trivial element of this group is denoted by α.

Our procedure in proving Theorems 6 and 7 is to exhibit a set of four natural isomorphisms. Their combination will be the isomorphism of Theorem 6. Under these isomorphisms the automorphism α of $\pi_i(M_l^k)$ is transformed into an automorphism of each of the groups, likewise denoted by α. In each case, the automorphism will be exhibited in a new form.

The first of these isomorphisms is

(A) $$\pi_i(M_l^k) \cong \pi_i(\tilde{M}_l^k) \qquad i \geqq 2$$

that always exists [7, Cor. 3] between the groups of a space and its covering space, and is induced by the covering map $\xi: \tilde{M}_l^k \rightarrow M_l^k$.

It is well known that the fundamental group of a space X is isomorphic in a natural way to the group of covering transformations of its universal covering space $\tilde{X}$. Since $\tilde{X}$ is simple in all dimensions, a homeomorphism of $\tilde{X}$ on itself induces an automorphism of $\pi_i(\tilde{X})$. In this way $\pi_1(X)$ appears as a group of automorphisms of $\pi_i(\tilde{X})$. Thus $\pi_1(X)$ operates in both $\pi_i(X)$ and $\pi_i(\tilde{X})$. Eilenberg has shown [6, p. 171] that these operations commute with the natural isomorphism $\pi_i(X) \cong \pi_i(\tilde{X})$.

The covering transformation α of $\tilde{M}_l^k$ is obtainable as follows. Let r_0 be any rotation of S which maps S^k on itself with orientation reversed. It will then map S^l on itself with orientation reversed. Therefore r_0 lies in the component of $R(S^k, S^l)$ other than the component of the identity $\tilde{R}(S^k, S^l)$. If a left coset of $R(S^k, S^l)$ be multiplied on the *right* by r_0, the two components of the coset are interchanged. As an operation on the left cosets of $\tilde{R}(S^k, S^l)$, it is precisely the covering transformation α.

Left translation of $\tilde{M}_l^k$ by any r in R is homotopic to the identity; for R is arcwise connected, and a path in R from r to a_0 provides the 1-parameter family of left translations. (Note that right translation of $\tilde{M}_l^k$ by r has no meaning unless r is in $R(S^k, S^l)$; for otherwise the image of a left coset of $\tilde{R}(S^k, S^l)$ is not a left coset.) Therefore right translation by r_0 followed by left translation by r_0^{-1} (i.e. conjugation by r_0) is a map T of $\tilde{M}_l^k$ homotopic to the covering transformation α. It induces, therefore, the automorphism α in $\pi_i(\tilde{M}_l^k)$. It has the advantage that it leaves fixed the point b_0; for $r_0^{-1}\tilde{R}(S^k, S^l)r_0 = \tilde{R}(S^k, S^l)$. We choose b_0 as the base point for $\pi_i(\tilde{M}_l^k)$.

Suppose, for the moment, that both k and l are even. In this case, the dimension of S is odd so that the antipodal transformation of S is in R and may be chosen as r_0. Since r_0 commutes with each element of R (r_0 is represented by the scalar matrix -1 of order $k + l + 2$), it follows that conjugation by r_0 is the identity in $\tilde{M}_l^k$. This proves the first part of Theorem 7.

We introduce now a space of importance in subsequent sections as well as the present one. As in §4, $R(S^l)$ is a subgroup of R, and we define

$$N_l^k = R/R(S^l)$$

using left cosets. It is easy to see that two elements r, r' of R lie in the same left coset of $R(S^l)$ if and only if they coincide on S^k (i.e. $r(y) = r'(y)$ for each y in S^k). On the other hand, any orthogonal map of S^k in S can by extended to all of S so as to be a rotation of S. In this way *N_l^k is the space of all orthogonal maps of S^k in S.* It will be so regarded in the sequel.

Since $R(S^l) \subset \tilde{R}(S^k, S^l)$, we have the natural maps

$$\eta: N_l^k \rightarrow \tilde{M}_l^k, \qquad \zeta: R \rightarrow N_l^k.$$

By Theorem 1, each space is a fibre bundle over its image with the natural map as projection.

We note that ζ maps $R(S^k)$ homeomorphically onto a subset $R^*(S^k)$ which is the fibre of N_l^k over the point b_0. The point $n_0 = \zeta(a_0)$ is taken as base point for homotopy groups in N_l^k.

Since $R^*(S^k)$ is the fibre over b_0, the projection η induces, by [7, Th. 2], a natural isomorphism

$$\text{(B)} \qquad \pi_i(\tilde{M}_l^k) \cong \pi_i(N_l^k, R^*(S^k)).$$

The element r_0, chosen as above, has the property $r_0^{-1}R(S^l)r_0 = R(S^l)$. Therefore conjugation of R by r_0 permutes the left cosets of $R(S^l)$, and yields thus a transformation T' of N_l^k. Since conjugation in R preserves the inclusion relations among the subsets of R, we have $\eta T' = T\eta$. It is also clear that $T'(R^*(S^k)) = R^*(S^k)$, and $T'(n_0) = n_0$. We have thus proved that the automorphism α of $\pi_i(N_l^k, R^*(S^k))$ is that induced by T'.

An element of $\pi_i(N_l^k, R^*(S^k))$ is represented by a map of an i-cell in N_l^k with boundary mapped into $R^*(S^k)$ and a fixed reference point of the boundary into n_0. The correspondence between the i-cell and its boundary has been shown [12, Th. 1] to induce a homomorphism

$$\text{(C)} \qquad \pi_i(N_l^k, R^*(S^k)) \to \pi_{i-1}(R^*(S^k)).$$

Since the image under T' of the boundary of a cell is the boundary of its image, the operations of T' in the two groups of (C) commute with the homomorphism.

We shall prove that (C) is an isomorphism if $i \leqq l$. As a first step, we prove

LEMMA. *$\pi_i(N_l^k) = 0$ if $i \leqq l$.*

Since R is a fibre bundle over N_l^k with fibre $R(S^l)$, we have by [7, Th. 2] that the projection ζ induces an isomorphism

$$\pi_i(N_l^k) \cong \pi_i(R, R(S^l)).$$

In [7, proof of Th. 5], it is shown that, if $i \leqq l$, the image of an i-cell in R with boundary mapped into $R(S^l)$ is contractible into $R(S^l)$ leaving its boundary fixed. But this is equivalent to $\pi_i(R, R(S^l)) = 0, i \leqq l$, and the lemma is proved.

An element u of the kernel of the homomorphism (C) is represented by a map of an i-cell with the property that its boundary is contractible in $R^*(S^k)$ to n_0. If this homotopy of the boundary be extended to the cell, we find that u is represented by a map of an i-cell with boundary mapped into n_0. By the lemma, if $i \leqq l$, the image of the i-cell may be contracted into n_0 leaving the boundary fixed. Therefore, the kernel of the homomorphism (C) is zero if $i \leqq l$.

An element v of $\pi_{i-1}(R^*(S^k))$ may be represented by a map of an $(i - 1)$-sphere in $R^*(S^k)$. If $i - 1 \leqq l$, we have, by the lemma, that the map may be extended continuously over an i-cell with the given $(i - 1)$-sphere as boundary. We obtain in this way an element u of $\pi_i(N_l^k, R^*(S^k))$ whose image under (C) is v. Therefore (C) is an isomorphism for $i \leqq l$ and a homomorphism *onto* for $i = l + 1$.

The final isomorphism

$$\text{(D)} \qquad \pi_{i-1}(R^*(S^k)) \cong \pi_{i-1}(R(S^k))$$

is that induced by ζ which, as observed before, is a homeomorphism of $R(S^k)$ onto $R^*(S^k)$. The combination of the isomorphisms (A), (B), (C) and (D) is the isomorphism stated in Theorem 6.

It is clear that the transformation T' of $R^*(S^k)$ corresponds under ζ to the operation T'' of conjugating $R(S^k)$ by r_0. Since each element of $R(S^k)$ acts as the identity on S^l, it follows that T'' is completely determined by the transformation r_0 restricted to S^k. Thus we have

THEOREM 7'. *Under the isomorphism of Theorem* 6, *the automorphism* α *of* $\pi_i(M_l^k)$ *is transformed into the automorphism of* $\pi_{i-1}(R(S^k))$ *induced by conjugating* $R(S^k)$ *by any orientation reversing orthogonal map of* S^k.

Choose a fixed great $(k-1)$-sphere S^{k-1} on S^k. As before, we may regard $R(S^{k-1})$ as a subgroup of $R(S^k)$. Choose the orthogonal map of Theorem 7' to be the reflection of S^k through the k-plane of S^{k-1}. Since S^{k-1} remains fixed, $R(S^{k-1})$ remains fixed under the conjugation of $R(S^k)$. In [7, Th. 5], it is shown that the identity map of $R(S^{k-1})$ in $R(S^k)$ induces an isomorphism between $\pi_{i-1}(R(S^{k-1}))$ and $\pi_{i-1}R(S^k)$ for $i \leqq k-1$ and a homomorphism onto if $i = k$. The last two statements imply that, if $i \leqq k$, the automorphism of $R(S^k)$ induces the identity automorphism of $\pi_{i-1}(R(S^k))$. This completes the proof that M_l^k is simple in dimensions $i \leqq k, l$.

Consider now the case $k = 1$, $i = 2$ and $l \geqq 2$. In this case S^k and $R(S^k)$ are both circles. It is easily verified that the conjugation of a rotation by a reflection reverses the sense of the rotation. Therefore the automorphism α of $\pi_1(R(S^k))$ is the reversal of sign. Thus, by Theorem 7', M_l^1 is not simple in the dimension 2.

REMARK: Eilenberg has shown [6, p. 175] that the projective space M_0^k of dimension $k+1$ is simple or not in the dimension $k+1$ according as $k+1$ is odd or even.

10. Proof of Theorem 2

Since the space N_l^k of §9 is a fibre bundle over M_l^k, we may choose a family $\{U\}$ of coordinate neighborhoods covering M_l^k, and a family $\{\phi_U\}$ of coordinate functions. The fibre of this bundle is $R(S^k, S^l)/R(S^l)$. Choose a fixed element u_0 of the fibre and define

$$\zeta_U(x) = \phi_U(x, u_0) \qquad \text{for } x \in U.$$

The function ζ_U provides a continuous cross-section of the fibres over U.

Corresponding to a map g of B in M_l^k, we define the family $\{N\}$ of coordinate neighborhoods in B to be the inverse images in B of the neighborhoods $\{U\}$ of M_l^k. We have seen in §9 that each element n of N_l^k is an orthogonal map of S^k into S. Let us agree to denote the image of the point y of S^k under the map n by $n \circ y$. We can then define the coordinate functions for $A(g)$ by

$$\phi_N(b, y) = (b, \zeta_U(g(b)) \circ y), \qquad b \in N = g^{-1}(U), \qquad y \in S^k.$$

Since $\zeta_U(g(b))$ maps S^k on the k-sphere $g(b)$, it follows that ϕ_N maps $N \times S^k$ on the part of $A(g)$ lying over N in such a way that $\psi\phi_N(b, y) = b$ If $g(b)$ is in two neighborhoods U, U', then $\zeta_U(g(b))$ and $\zeta_{U'}(g(b))$ map S^k orthogonally into the k-sphere $g(b)$. One followed by the inverse of the other is surely an orthogonal map of S^k on itself.

11. Proof of Theorem 3

For each point x of M_l^k, let X denote the great k-sphere of S corresponding to x as indicated in §4. In particular, let x_0 be the point of M_l^k corresponding to the reference sphere S^k.

By Theorem 8, there exists a slicing function ϕ for $R(S^k, S^l)$ invariant with respect to $R(S^k, S^l)$. It is defined in a neighborhood N of x_0. If $r \in R(S^k, S^l)$, then $rN = rNr^{-1} = N$ because of the invariance of ϕ. Therefore the left translation rN of N to the point $x = rx_0$ (r arbitrary) depends only on the left coset of $R(S^k, S^l)$ to which r belongs (i.e. on x alone). We denote this neighborhood of x by $N(x)$. These neighborhoods cover M_l^k.

For any pair x_1, x_2 in M_l^k such that $x_2 \in N(x_1)$, we choose an r_1 in R such that $r_1(X_1) = S^k$, and define

$$r(x_1, x_2) = r_1^{-1}\phi(r_1x_2)r_1.$$

It is easy to see that $r(x_1, x_2)$ is a rotation of S carrying X_1 into X_2. It appears to depend on the choice of r_1. This is not the case; for any other choice may be written r_0r_1 for $r_0 \in R(S^k, S^l)$ and we have

$$(r_0r_1)^{-1}\phi((r_0r_1)x_2)(r_0r_1) = r_1^{-1}\phi(r_1x_2r_0)r_1 = r_1^{-1}\phi(r_1x_2)r_1,$$

because of the invariance of ϕ and the fact that x_2 is a left coset of $R(S^k, S^l)$. Finally $r(x_1, x_2)$ is continuous simultaneously in x_1, x_2. We may choose r_1 as a continuous function of x_1 neighboring a particular choice because R is a fibre bundle over M_l^k. If r_1 is replaced by this function in the definition of $r(x_1, x_2)$, the continuity of the latter becomes apparent.

Suppose now that g, g' are two maps of B in M_l^k such that

$$r(b) = r(g(b), g'(b))$$

is defined for every b. A map τ of $B \times S$ on itself is then defined by $\tau(b, y) = (b, y')$ where y' is the image of $y \in S$ under $r(b)$. It follows immediately that τ, restricted to $A(g)$, provides an equivalence between the bundles $A(g)$ and $A(g')$.

Finally, let $g(b, t)$, b in B, $0 \leqq t \leqq 1$, be a homotopy in M_l^k of the map $g(b) = g(b, 0)$. Since B is compact, a familiar uniformity argument shows that there exists a $\delta > 0$ such that $|t - t'| < \delta$ implies $g(b, t')$ is in $N(g, b, t))$ for any b in B. Introduce now a subdivision of the t-interval of mesh $<\delta$. As shown in the preceding paragraph, the bundles over B corresponding to successive subdivision points are equivalent. Since equivalence is a transitive relation, Theorem 3 is proved.

REMARK: If B is not compact, but the homotopy $g(b, t)$ is *uniform* in the sense described above, the conclusion will still hold.

12. Proof of Theorem 4

A simplex of B is denoted by σ, its closure by $\bar{\sigma}$. We shall suppose that the subdivision of B is so fine that each $\bar{\sigma}$ lies wholly within some coordinate neighborhood of the bundle A. We choose one such neighborhood for each σ and denote it by $N(\sigma)$. The corresponding coordinate function is denoted by ϕ_σ.

Our procedure will be to construct stepwise over the simplexes of B, in the order of their dimension, a system of functions λ_σ from which we shall be able to deduce the existence of a map g and an equivalence τ between A and $A(g)$.

The function λ_σ is to be a continuous map of $\bar{\sigma}$ in the space N_l^k of §9 so that $\lambda_\sigma(b)$ for each $b \in \bar{\sigma}$ is an orthogonal map of S^k into S. These functions are to have the following basic property:

(P) If σ' is a face of σ and b is in $\bar{\sigma}'$, then the orthogonal map $\phi_{\sigma' b}^{-1}\phi_{\sigma b}$ of S^k on itself followed by the map $\lambda_{\sigma'}(b)$ of S^k into S is the map $\lambda_\sigma(b)$ of S^k into S.

We define λ_σ arbitrarily over the vertices σ of B. Suppose, inductively, that λ_σ has been defined for each σ of dimension $<i$ so that (P) holds. For an i-simplex σ and a point b of its boundary, we define $\lambda_\sigma(b)$ by choosing a face σ' of σ such that $b \in \bar{\sigma}'$ and letting $\lambda_\sigma(b)$ be the orthogonal map $\phi_{\sigma' b}^{-1}\phi_{\sigma b}$ of S^k on itself followed by $\lambda_{\sigma'}(b)$. By (P), $\lambda_\sigma(b)$ is independent of the choice of $\bar{\sigma}' \supset b$. Since $\lambda_\sigma(b)$ is continuous over each closed face of σ, it is continuous over the boundary of σ. Since $l \geqq \dim B - 1$, we have $i - 1 \leqq l$. Therefore, by the lemma of §9, λ_σ may be extended continuously over σ. Let this be done for each i-simplex. By the construction, (P) holds. Thus the general step in the induction is complete.

If η denotes the natural map $N_l^k \to M_l^k$, we notice that, if $b \in \bar{\sigma}' \subset \bar{\sigma}$, then

$$\eta\lambda_\sigma(b) = \eta\lambda_{\sigma'}(b) = g(b),$$

for $\lambda_\sigma(b)$ and $\lambda_{\sigma'}(b)$ map S^k into the same great k-sphere of S. Since g is continuous over each closed $\bar{\sigma}$, it is continuous over B and defines thereby a bundle $A(g)$.

Denote the projection $A \to B$ by ψ. For any $a \in A$, we choose a σ such that $b = \psi(a) \in \bar{\sigma}$, and denote by $h(a)$ the image of a in S under the map $\phi_{\sigma b}$ followed by $\lambda_\sigma(b)$. By the property (P), $h(a)$ is independent of the choice of σ. It is a continuous function since it is continuous over each of the closed sets $\psi^{-1}(\sigma)$ which cover A. Since both $\phi_{\sigma b}$ and $\lambda_\sigma(b)$ are 1-1 maps, $h(a)$ is 1-1 for a restricted to $\psi^{-1}(b)$.

We define the map τ of A in $B \times S$ by

$$\tau(a) = (\psi(a), h(a)).$$

From the stated properties of h, it follows immediately that τ defines an equivalence between A and $A(g)$.

13. Proof of Theorem 5

Let g_0, g_1 be two maps of a complex B in M^k such that $A(g_0)$, $A(g_1)$ are equivalent bundles. Let I be the t-interval $0 \leqq t \leqq 1$, let $B' = B \times I$, and $B_t = B \times t$. Define $g'(b, t) = g_0(b)$. In this way a bundle $A(g')$ over B' is obtained. If we identify B with B_0, the bundle $A(g_0)$ becomes identified with the part of $A(g')$ over B_0; because $g'(b, 0) = g_0(b)$.

Let $A(g_1')$ be the part of $A(g')$ over B_1. The map of $B_1 \times S$ onto $B_0 \times S$ defined by $((b, 1), y) \rightarrow ((b, 0), y)$ provides an equivalence between $A(g_1')$ and $A(g_0)$. Since the latter is equivalent to $A(g_1)$, the two equivalences provide an equivalence map τ of $A(g_1')$ onto $A(g_1)$.

Let $g(b, 0) = g_0(b)$, and $g(b, 1) = g_1(b)$. Our problem is to extend $g(b, t)$ over all of B'. To do this, we shall use the method of §12 with B' in place of B and $A(g')$ in place of A. In the present case, however, the function g is already specified on the subcomplex $B_0 + B_1$. Therefore, in order to apply §12, it is necessary to define first the functions λ_σ for σ in $B_0 + B_1$ so as to satisfy (P), and, in addition, $\eta\lambda_\sigma(b') = g(b')$ for each $b' \in \bar{\sigma}$. Once this is accomplished, the general step in the extension will follow as in §12 because $l \geqq \dim B \geqq \dim B' - 1$.

The essential point in obtaining the λ's for σ in $B_0 + B_1$ is that the function τ of §12 (defined there in terms of the λ's) is already given, in the present case, in the portion of $A(g')$ over $B_0 + B_1$. On $A(g_0)$, τ is the identity, on $A(g_1')$, it is the given equivalence with $A(g_1)$. We can use the τ to define the λ's.

As in §12, we suppose B' subdivided so fine that each closed cell lies wholly in a coordinate neighborhood for $A(g')$. We choose one such for each cell σ and denote the corresponding coordinate function by ϕ_σ. Map $B' \times S$ onto S by $\zeta(b', y) = y$. For σ in $B_0 + B_1$ and b' in $\bar{\sigma}$, define

$$\lambda_\sigma(b') = \zeta\tau\phi_{\sigma b'}.$$

In words, $\lambda_\sigma(b')$ is the orthogonal map of S^k into S obtained as the composition of the map $\phi_{\sigma b'}$ of S^k on the fibre over b' in $A(g')$, the equivalence map τ of this fibre into $B' \times S$, and finally the map ζ of this last image into S. One verifies immediately that (P) holds and $\eta\lambda_\sigma(b') = g(b')$. This completes the proof.

14. General extension theorem

Experience has shown that in most homotopy classification problems there is a basic extension theorem from which all results follow. We state here a single extension theorem which implies both Theorem 4 and Theorem 5.

THEOREM 12. *Let A be a k-sphere bundle over the complex B, let B' be a closed subcomplex of B, let g' be a map of B' in M_l^k, and let τ' be an equivalence map of the part of A over B' onto $A(g')$. If $l \geqq \dim B - 1$, then there exists an extension g of g' to all of B such that A and $A(g)$ are equivalent. Moreover this last equivalence may be chosen to be an extension of τ'.*

The proof can be found in sections 12 and 13.

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