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Semicontinuity of the singularity spectrum

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Introduction

To each isolated hypersurface singularity one can associate a sequence of μ rational numbers, called its *spectrum*. Here μ is the Milnor number of the singularity. (cf. [M]). The spectrum gathers the information about the eigenvalues of the monodromy operator of f and about the Hodge filtration on the vanishing cohomology (see (2.1) for a precise definition).

The idea behind the concept of spectrum is the following. In [S1] a mixed Hodge structure has been constructed on the vanishing cohomology of any isolated hypersurface singularity. It appears that many important invariants of the singularity can be expressed in terms of the Hodge filtration alone, so it may be enlightening to forget about the weight filtration. In this way the spectrum has arisen from the "characteristic pairs" of [S1]. For quasi-homogeneous singularities the spectrum numbers occur as eigenvalues of the residue of the Gauss-Manin connection on the lattice $\mathscr{H}''$ (see [Ma, Ex. (6, 7)]). Arnol'd was the first to discover the importance of the spectrum for deformation theory ([A]). He conjectured that the spectrum behaves semicontinuously under deformation of the singularity, in a certain sense. A stronger version of his conjecture has been proved recently by Varchenko [V2] for the case of deformations of low weight of quasi-homogeneous singularities: for such deformations, any open interval $(a, a+1)$ is a so-called semicontinuity domain (see (2.2)). This result covers almost all cases one meets in practice.

In this paper we prove a slightly weaker statement than Varchenko's for arbitrary deformations of isolated hypersurface singularities: any half open interval $(a, a+1]$ is a semicontinuity domain. This still implies a positive answer to Arnol'd's original question. The case $a \leqq -1$ has been proved by Varchenko in [V3], see also [V4].

Our proof depends heavily on Varchenko's idea to use the relation between the spectra of a function f and the function $f+z^q$, where z is a new variable. The extra argument we put in consists of recent results about limit mixed Hodge structures for families of projective varieties over a disc whose general fibre is not necessarily smooth ([EZ, GNP, B and SZ]). As byproducts we obtain semicontinuity results for Hodge numbers of complete intersection singularities, in particular for the geometric genus (treated before by Elkik) and a new proof that the spectrum is constant under deformations with constant Milnor number.

The spectrum is related to asymptotic integrals which can be associated to the critical point of f. For $0<|t|<\eta\ll\varepsilon\ll 1$ we let $X_t=f^{-1}(t)\cap B_\varepsilon$. If ω is a holomorphic $(n+1)$-form on a neighborhood of 0 in $\mathbb{C}^{n+1}$ and $\gamma(t)$ is a continuously varying homology class of dimension n on X_t, then the function

$$I(t)=\int_{\gamma(t)}\frac{\omega}{df}$$

admits an asymptotic expansion as t tends to zero:

$$I(t)=\sum_{\alpha,q} C^{\omega,\gamma}_{\alpha,q}\, t^\alpha(\log t)^q/q!$$

such that $q\in\mathbb{Z}$, $0\leqq q\leqq n$, $\alpha\in\mathbb{Q}$, $\alpha>-1$ and $\exp(-2\pi i\alpha)$ is an eigenvalue of the monodromy operator. The complex singularity index $\beta_{\mathbb{C}}$ is given by

$$\begin{aligned}\beta_{\mathbb{C}}-1&=\min\{\alpha\,|\,\exists\,\omega,\gamma,q \text{ with } C^{\omega,\gamma}_{\alpha,q}\neq 0\}\\&=\inf\{\alpha\,|\,\forall\,\omega,\gamma\colon \lim_{t\to 0} t^{-\alpha}I(t)=0\}.\end{aligned}$$

It has been conjectured by Malgrange [Ma] that $\beta_{\mathbb{C}}$ is semicontinuous under deformations of f. Varchenko's construction of the asymptotic Hodge filtration [V6] implies that $\beta_{\mathbb{C}}-1$ is equal to the smallest spectrum number, so its semicontinuity follows from the semicontinuity of the spectrum.

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1. Limit mixed Hodge structures

(1.1) The limit behaviour of a variation of Hodge structure over a punctured disc was analyzed by Schmid [Sc]. He shows that the limit object is a limit mixed Hodge structure. The Hodge filtration of this can be considered as the limit of the Hodge filtrations for the various parameter values t, as t tends to 0. The weight filtration is the socalled monodromy weight filtration, depending on the Jordan normal form of the monodromy. In case the variation of Hodge structure comes from geometry this limit mixed Hodge structure has been described more or less explicitly by the author [S1].

A family of singular varieties over a disc gives rise to a variation of mixed Hodge structure over a small punctured disc, over which one can stratify the map. As there is no general theory of limits of mixed Hodge structures available (see however [SZ]) one restricts to the geometric case. The methods of [S1] have been generalized by various authors to the singular (or/and non-proper) case. The idea is, to make a version of the formalism of vanishing cycles in terms of (cohomological) mixed Hodge complexes. This also applies to the cohomology of Milnor fibres.

Let X and U be reduced analytic spaces such that U is smooth. Let $f: X \to U$ be a flat projective mapping. We say that f is *locally trivial* if there exists a semi-simplicial resolution $\varepsilon: X_\bullet \to X$ of X such that all X_i are smooth and the induced mappings $f_i = f \circ \varepsilon_i: X_i \to U$ are projective submersions. Using the results of [GNP], say, it is easy to see that for every projective mapping there exists a partition of the base into locally closed submanifolds such that the restriction of the mapping to the inverse image of each stratum is locally trivial.

(1.2) **Lemma.** *Let $f: X \to U$ be a locally trivial projective map. Then the functions*

$$u \mapsto \dim_{\mathbb{C}} Gr_F^p H^m(X_u, \mathbb{C})$$

are locally constant on U. (Here X_u is the fibre over u and F is the Hodge filtration on its cohomology.)

Proof. It suffices to treat the case $\dim(U)=1$, for which we refer to [B, Théorème (2.6)].

We fix the following notations. We let S denote the unit disc in the complex plane and $f: Y \to S$ a flat projective map, where Y is a complex space of dimension $n+1$. We assume that f is locally trivial over the punctured disc S^*. Let S_∞ be a universal covering of S^* and let $Y_\infty = Y \times_S S_\infty$. Let $i: Y_0 \to Y$ and $k: Y_\infty \to Y$ denote the obvious maps (we let $Y_t = f^{-1}(t)$). Then Y_∞ is homotopy equivalent with each fibre $Y_t, t \neq 0$.

(1.3) **Theorem.** *There exist filtrations $W_\bullet$, $M_\bullet$ and $F^\bullet$ on each cohomology group $H^m(Y_\infty)$ with the following properties:*

a) *$W_\bullet$ and $M_\bullet$ are increasing, $F^\bullet$ is decreasing, $M_\bullet$ and $F^\bullet$ define a mixed Hodge structure on $H^m(Y_\infty)$;*

b) *Let $r \in \mathbb{Z}$. Then $Gr_r^W = W_r/W_{r-1}$, together with the filtrations induced on it by $M_\bullet$ and $F^\bullet$, coincides with the limit mixed Hodge structure (in Schmid's sense) of the variation of Hodge structure $Gr_r^W R^m f_* \mathbb{Q}_{Y|S^*}$;*

c) *Let T denote the monodromy operator on $H^m(Y_\infty)$ and T_s, T_u its semisimple and unipotent parts. Then T_s preserves the filtrations W, M and F, and $N = \log T_u$ satisfies: $N(W_i) \subset W_i$, $N(F^p) \subset F^{p-1}$ and $N(M_k) \subset M_{k-2}$.*

d) *The filtrations W, M and F exist already on the level of a cohomological mixed Hodge complex $\mathscr{A}^\bullet$ with the property that $\mathbf{H}^m(\mathscr{A}^\bullet) = H^m(Y_\infty)$ and which admits a morphism $\mathscr{K}^\bullet_{Y_0} \to \mathscr{A}^\bullet$ of cohomological mixed Hodge complexes. Here $\mathscr{K}^\bullet_{Y_0}$ is a cohomological mixed Hodge complex defining the mixed Hodge structure on the cohomology of Y_0.*

Proof. See [EZ II, Prop. 2.1; GNP Chap. 9; B; SZ Chap. 5].

(1.4) The sheaf of vanishing cycles $R\Phi$ is the cone of the natural morphism $\mathbb{C}_{Y_0} \to i^{-1} \mathbf{R} k_* \mathbb{C}_{Y_\infty}$. It gives a long exact sequence of hypercohomology groups

$$\dots \to H^m(Y_0) \to H^m(Y_\infty) \to \mathbf{H}^m(R\Phi) \to H^{m+1}(Y_0) \to \dots.$$

Because $R\Phi$ is quasi-isomorphic to the cone of the morphism $\mathscr{K}^\bullet_{Y_0} \to \mathscr{A}^\bullet$, one may give it the filtrations of the mixed cone (see [EZ]). In this way the sequence above becomes a long exact sequence of mixed Hodge structures.

Suppose that Y_0 has only isolated singularities. Then around each of its singular points x_i we take a small ball B_i and put $X_i = B_i \cap Y_t$ for t small enough (but unequal to 0). Then we may interpret $\mathbf{H}^m(R\Phi)$ as $\bigoplus_i \tilde{H}^m(X_i)$ where $\tilde{H}$ means reduced cohomology.

Because the construction of $R\Phi$ with its Hodge and weight filtrations is of local nature on Y_0, we may apply it in other than projective situations, e.g. Stein mappings for which the fibres all have at most isolated singularities and for which the critical set lies finitely over the base space. In particular the mixed Hodge structures on the hypercohomology groups $\mathbf{H}^m(R\Phi)$ do not depend on the global family but only on the map germs at the critical points (and on the choice of a local parameter on the disc).

(1.5) Let us now consider the case of a more-parameter family: we take a flat projective map $F\colon Y \to U$ between a complex space Y and an irreducible and reduced complex space U, whose fibers have dimension n and have at most isolated singularities. Then we can apply the previous construction as follows. We take a holomorphic mapping of the disc S to U and pull back the family. The resulting mixed Hodge structure will depend very much on the choice of the arc. In particular the weight filtration will vary with the position of the image with respect to a stratification of U as in the following lemma. However, the numerical invariants of the Hodge filtration behave much better.

(1.6) **Lemma.** *Let $F\colon Y \to U$ be a flat projective mapping. Then there exists a complex analytic stratification (S_j) of U such that the functions $u \mapsto \dim Gr_F^p H^m(Y_u)$ are constant on each S_j for all $p, m \in \mathbb{N}$.*

Proof. By Lemma (1.2) it suffices to take a stratification (S_j) of U such that the mappings $(Y \times_U S_j)_{\mathrm{red}} \to S_j$ induced by F are locally trivial. It has already been remarked that this stratification exists. □

(1.7) In the next lemma we consider some stratification, satisfying the properties of Lemma (1.6). We choose arcs $h\colon S \to U$ which map the punctured disc S^* into one stratum. We let $R\Phi_h$ denote the sheaf of vanishing cycles associated to the family $Y \times_U S$ over S, induced by h.

(1.8) **Lemma.** *The number $\sum_{i=0}^{2n} (-1)^{n+i} \dim Gr_F^p \mathbf{H}^i(R\Phi_h) = \chi_p(h)$ (p fixed) does not depend on h but only on the two strata which intersect $h(S)$.*

Proof. Let the family over S induced by h have a general fibre Y_v and let Y_∞ be constructed from it as in (1.1). According to Theorem (1.3) one has for every p and m:

$$\dim Gr_F^p H^m(Y_v) = \sum_k \dim Gr_F^p Gr_W^k H^m(Y_v) = \sum_k \dim Gr_F^p Gr_k^W H^m(Y_\infty) = \dim Gr_F^p H^m(Y_\infty).$$

Because morphisms of mixed Hodge structures are strictly compatible with the Hodge filtrations, the sequence

$$\dots \to Gr_F^p H^m(Y_u) \to Gr_F^p H^m(Y_\infty) \to Gr_F^p \mathbf{H}^m(R\Phi_h) \to Gr_F^p H^{m+1}(Y_u) \to \dots$$

with $u = h(0)$, is also exact. Taking alternating sums of dimensions we obtain

$$\chi_p(h) = \sum_{i=0}^{2n} (-1)^{n+i} [\dim Gr_F^p H^i(Y_v) - \dim Gr_F^p H^i(Y_u)].$$

This expression depends only on the strata of u and v by (1.6).

(1.9) **Definition.** Assume that the projective family $F: Y \to U$ is flat, its fibers have only isolated singular points and U is irreducible. Then we define functions $s_p: U \to \mathbb{Z}$ by $s_p(u) = \chi_p(h)$ where we choose the arc h such that $h(0) = u$ and h maps the punctured disc to the top dimensional stratum of U.

By Lemma (1.8) the functions s_p are analytically constructible on U.

(1.10) **Definition.** The map $F: Y \to U$ is said to have only *spherical* isolated singularities if for any arc h as in (1.7) one has $\mathbf{H}^i(R\Phi_h) = 0$ for $i \neq n = \dim Y - \dim U$.

Examples. By [G, Lemma 3.2] this is the case if the fibers of F have only isolated complete intersection singularities, and by [GS, Th. 2] if the fibers of F are normal surfaces and its generic fiber is smooth.

(1.11) **Theorem.** *Let $F: Y \to U$ be a flat projective map which has only spherical isolated singularities and such that U is irreducible. Then the functions s_p are upper semi-continuous on U.*

Proof. Because each s_p is analytically constructible, it suffices to prove the following: if $u \in U$ lies in the closure of the stratum of $v \in U$, then $s_p(u) \geqq s_p(v)$.

By the curve selection lemma there exists a holomorphic map germ $h: (S, 0) \to (U, u)$ mapping the punctured disc to the stratum of v. Then clearly the computation in (1.8) shows that

$$\begin{aligned} s_p(u) - s_p(v) &= \sum_{i=0}^{2n} (-1)^{n+i} \dim Gr_F^p \mathbf{H}^i(R\Phi_h) \\ &= \dim Gr_F^p \mathbf{H}^n(R\Phi_h) \geqq 0. \end{aligned}$$

(1.12) *Remark.* In the theorem above the hypothesis can be weakened to $Gr_F^p \mathbf{H}^i(R\Phi_h) = 0$ for $i \neq n$ (p fixed). This is always true for $p = n$: the case $i < n$ can be proved via the fact that the complex $Gr_F^n R\Phi_h$ has zero cohomology sheaves in degrees smaller than n, and the case $i > n$ follows from the fact that the "Milnor fiber" of h is a Stein space of dimension n. In particular the function s_n is always semi-continuous. If the generic fiber of F is smooth, the value of s_n at $u \in U$ is the sum of the geometric genera of the singular points in the fibre Y_u (cf. [S2, Prop. 2.12]). Its semicontinuity has been proven by R. Elkik (see [E, Th.1]).

(1.13) *Remark*. In the situation of Theorem (1.11), suppose that one has an automorphism g of Y with $Fg=F$. Then g acts on all terms of the exact sequence in (1.8) and the statement of (1.11) remains true if one replaces every space by its subspace on which g acts with a given eigenvalue λ. In this way $s_p = \sum_\lambda s_p^\lambda$ where we sum over all eigenvalues λ of g; each function s_p^λ is itself semicontinuous on U.

2. Spectra of isolated hypersurface singularities

(2.1) Each isolated hypersurface singularity (X_0, x_0) has a privileged class of smoothings, namely those for which the total space is smooth. These are just the germs $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ where f is a defining function for X_0. Let μ be the Milnor number of f and let $F^\bullet$ denote the Hodge filtration on the vanishing cohomology group $\mathbf{H}^n(R\Phi_f)$ of f. As stated in Theorem (1.3), the semisimple part T_s of the monodromy preserves $F^\bullet$.

The *spectrum* of f is defined as the unordered μ-tuple of rational numbers $(a_1, \ldots, a_\mu)$ with the following property: the frequency of the number a in the spectrum is equal to the dimension of the eigenspace of T_s acting on F^p/F^{p+1} with the eigenvalue $\exp(-2\pi i a)$, where we take $p=[n-a]$.

(2.2) If $F: X \to U$ is a good representative of a deformation of an isolated hypersurface singularity, we let Σ_u denote the union of all spectra of the critical points in the fiber $F^{-1}(u)$. Here we take the union "with multiplicities": the frequency of a in Σ_u is the sum of its frequencies in the spectra of all critical points with value u.

A subset A of $\mathbb{R}$ is called a *semicontinuity domain* (for deformations of isolated hypersurface singularities) if for every good representative as above, the function which associates to $u \in U$ the sum of the frequencies of the elements of A in Σ_u, is upper semicontinuous on U.

Example. According to (2.1) the function s_p from (1.9), applied to the case of (a globalization of) a smoothing of an isolated hypersurface singularity, just counts the number of points in Σ_u which lie in the interval $(n-p-1, n-p]$. Hence every such interval is a semicontinuity domain.

(2.3) V.I. Arnol'd [A] has conjectured that each half line $(-\infty, t]$ $(t \in \mathbb{R})$ is a semicontinuity domain. A.N. Varchenko [V2] has verified that for deformations of low weight of quasi-homogeneous isolated hypersurface singularities every open interval $(t, t+1)$ is a semicontinuity domain. Using the main ideas of his proof plus Theorem (1.11) we will prove a somewhat weaker statement in the general case, which still verifies Arnol'd's conjecture:

(2.4) **Theorem.** *Every half open interval* $(t, t+1]$ *is a semicontinuity domain for deformations of isolated hypersurface singularities.*

Before we give the proof, we show that every deformation of an isolated complete intersection singularity can be globalized. This is a well-known result, which shows that Theorem (1.11) can indeed be applied in the proof of (2.4).

(2.5) **Lemma.** *Let* $F: (X, x_0) \to (U, 0)$ *be a deformation of an isolated complete intersection singularity* (X_0, x_0). *Then after possibly shrinking* X *and* U *there exists a flat projective map* $\bar{F}: Y \to U$ *and an open embedding of* X *in* Y *such that* $F = \bar{F}_{|X}$ *and* $\bar{F}$ *has no critical points outside* X.

Proof. It suffices to check this for a versal deformation of (X_0, x_0). Hence we may assume that F is a polynomial mapping from $\mathbb{C}^{n+k}, 0$ to $\mathbb{C}^k, 0$ which is flat and finitely determined. Choose an integer s such that F is $(s-1)$-determined and such that its components $F_1, \ldots, F_k$ are of degree smaller than s. Consider the family of polynomial mappings $F+f$ where the components $f_1, \ldots, f_k$ of f are homogeneous polynomials in $z_1, \ldots, z_{n+k}$ of degree s. By Sard's theorem, there exists a dense open subset V of the space of all such mappings, such that for $F+f \in V$:

1) $F+f$ is equivalent to F as a germ at 0;

2) the closure of $(F+f)^{-1}(0)$ in $\mathbb{P}^{n+k}(\mathbb{C})$ has 0 as its only singular point.

We choose f such that $F+f \in V$ and let

$$G_i(t, z_0, \ldots, z_{n+k}) = z_0^s F_i(z_1/z_0, \ldots, z_{n+k}/z_0) + f_i - t_i z_0^s$$

for $i=1, \ldots, k$. Let B_r be the open ball in $\mathbb{C}^k$ with center 0 and radius r and let $Y = \{(t, z) \in B_r \times \mathbb{P}^{n+k} \mid G_i(t, z) = 0,\ i=1, \ldots, k\}$. Then for r sufficiently small, the projection map $\bar{F}: Y \to B_r$ satisfies all our requirements.

(2.6) **Corollary.** *The numbers* $\dim Gr_F^p \mathbf{H}^n(R\Phi)$, *which are defined for any 1-parameter smoothing of an isolated singularity, in the case of complete intersections do not depend on the choice of a smoothing. Hence they define invariants of the singularity.*

(2.7) *Proof of Theorem* (2.4). The main idea is due to Varchenko (see [V2, 3d]). If $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ has an isolated singularity and z is a new variable, then the spectrum of the function $f + z^q$ ($q \in \mathbb{N}$) consists of the numbers $a + k/q$ where a runs over the spectrum of f and $k = 1, \ldots, q-1$. Moreover the automorphism g of $\mathbb{C}^{n+2}$ defined by $(x, z) \mapsto (x, \exp(2\pi i/q) z)$ preserves the fibers of $f + z^q$ and hence acts on its vanishing cohomology, preserving the Hodge filtration. The number of spectrum points of f in the interval $(n-p-1-k/q,\ n-p-k/q]$ is equal to

$$\dim \{y \in Gr_F^p \mathbf{H}^{n+1}(R\Phi_{f+z^q}) \mid g^*(y) = \exp(2\pi i k/q) \cdot y\}.$$

Let $t \in \mathbb{R}$ and let $F: X \to U$ be a good representative for a deformation of (X_0, x_0), with smooth general fiber. Let f_u be the defining function for X_u. Then for any $q \in \mathbb{N}$ we may consider the family

$$\mathbb{C}^{n+1} \times \mathbb{C} \times U \to \mathbb{C} \times U$$

$$(x, z, u) \mapsto (f_u(x) + z^q, u)$$

with the automorphism g induced by the substitution $z \to \exp(2\pi i/q) \cdot z$. The fact that the function s_p^λ for this family, with $\lambda = \exp(2\pi i k/q)$, is semicontinuous on $\mathbb{C} \times U$ (see (1.12)) implies, that the interval $(n-p-1-k/q,\ n-p-k/q]$ is a semicontinuity domain for deformations of (X_0, x_0). Hence every interval $(a, a+1]$ with $a \in \mathbb{Q}$ is a semicontinuity domain. This implies the general case since spectrum numbers are rational.

(2.8) As a trivial consequence of Theorem (2.4) we recover a result of Varchenko [V1]:

Theorem. *The spectrum is constant in a deformation of isolated hypersurface singularities with constant Milnor number.*

Proof. If the Milnor number does not vary in a family, then each interval $(a, a+1]$ contains a constant number of spectrum points. This is only possible if the spectrum remains constant.

(2.9) For complete intersections we obtain the analogous result:

Theorem. *In a deformation of isolated complete intersection singularities with constant Milnor number the functions s_p are constant.*

Proof. This is a direct consequence of the semicontinuity of the s_p and the fact that their sum, the Milnor number, is constant.

(2.10) *Remark.* For surfaces the number of spectrum points in the interval $(0, 1)$ is equal to the number μ_- which denotes the maximal dimension of a negative definite subspace of the intersection form on the vanishing homology. This can be proved as follows. By a result of Durfee [Du] $\mu - \mu_- = \mu_+ + \mu_0 = 2p_g$. Moreover p_g is the number of spectrum points in $(-1, 0]$ or, by symmetry, in $[1, 2)$. As all spectrum numbers lie inside $(-1, 2)$, the result easily follows.

The semicontinuity of the number μ_- is an easy consequence of the fact, that, if a function g deforms into f, the vanishing homology of f embeds isometrically in the vanishing homology of g. Substituting this argument in the proof of Theorem (2.4) one obtains a result of Varchenko [V3, V4]: for isolated plane curve singularities every open interval $(a, a+1)$ is a semicontinuity domain.

We are not able to prove this in higher dimension. Finally Theorem (2.4) implies

(2.11) **Theorem.** *In any deformation of isolated hypersurface singularities the complex singularity index (the smallest spectrum number plus one) is lower semicontinuous.*

This was proved by Varchenko in the case of quasihomogeneous singularities for deformations of low weight and in the case that the complex singularity index is not bigger than 1 [V5].

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