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On the Mixed Hodge Structure on the Cohomology of the Milnor Fibre

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Introduction

In [18], the second author introduced a mixed Hodge structure on the cohomology of the Milnor fibre of an isolated hypersurface singularity. (For the definition of a mixed Hodge structure, cf. [17, Sect. 3.4].) The weight filtration is essentially the "monodromy weight filtration" [18, Sect. 4] and thus simplifies the Jordan normal form of the unipotent part of the monodromy. The significance of the Hodge filtration, however, is not so clear, and its description seems difficult to use. This paper gives another description of the Hodge filtration which we hope is easier to understand and apply. Its definition does not use resolution of singularities. Instead it relies on the theory of holonomic $\mathcal{D}$-modules in one variable with regular singularities.

This paper arose from conversations at the 1980 Arbeitstagung in Bonn, where Varchenko's conjectured Hodge filtration [22] had been discussed in Brieskorn's seminar. We have since learned that Varchenko obtained similar results to those in this paper in the summer and autumn of 1980 [23–25].

In Sect. 1 the Milnor fibration is embedded in a family of smooth projective hypersurfaces. The Hodge theory of a smooth projective hypersurface is explained in the fashion of Brylinski [2] in Sect. 2. In Sect. 3 we recall the description of the Gauss-Manin system from [8]. Our new formula for the Hodge filtration is explained in Sects. 4, 5. In Sect. 6 we prove that it gives the same result as [18]. We prove a "Thom-Sebastiani" formula for the Hodge filtration in Sect. 7 which leads to a proof of conjecture (5.4) of [18]. In Sect. 8 we show how to prove Varchenko's result about the Jordan normal forms of multiplication by f in the Jacobian ring of f and the logarithm of the unipotent monodormy of f with our method. In the last chapter the mixed Hodge structure is calculated for two examples:

$$x^p + y^q + z^r + axyz, p^{-1} + q^{-1} + r^{-1} < 1, a \neq 0$$

and

$$ax^5 + y^6 + x^4 y, \quad a \in \mathbb{C}.$$

Lastly we would like to point out that the formula for the Hodge filtration given here fits together very well with what Brylinski conjectures should be the Hodge filtration on an arbitrary holonomic $\mathscr{D}$-module with regular singularities.

1. The Milnor Fibration

Suppose f is a holomorphic function defined on some open neighborhood of 0 in $\mathbb{C}^{n+1}$ with $f(0)=0$. Assume that 0 is an isolated critical point of f. By making a holomorphic change of coordinates, if necessary, we can arrange that f is a polynomial of arbitrary large degree d, such that

(i) the closure Y_0 of $f^{-1}(0)$ in $\mathbb{P}^{n+1}(\mathbb{C})$ has only 0 as a singular point, and

(ii) for $|t|$ sufficiently small, $t \neq 0$, the fibre Y_t = closure of $f^{-1}(t)$ in $\mathbb{P}^{n+1}(\mathbb{C})$ is smooth [1, Sect. 1.1].

If we choose a sufficiently small open ball B around 0 in $\mathbb{C}^{n+1}$, then there exists a small open disc S around 0 in $\mathbb{C}$ such that

$$f: f^{-1}(S')\cap B \to S'$$

is a C^∞ fibre bundle, where $S'=S\setminus\{0\}$. Let $X=f^{-1}(S)\cap B$, $X'=f^{-1}(S')\cap B$. Without loss of generality we may assume that Y_t is smooth for $t\in S'$. Let

$$F(z_0,\ldots,z_{n+1})=z_{n+1}^d f(z_0/z_{n+1},\ldots,z_n/z_{n+1}).$$

Define $Y\subset \mathbb{P}^{n+1}(\mathbb{C})\times S$ by

$$Y=\{(z,t)|F(z)-tz_{n+1}^d=0\}=\{(z,t)|z\in Y_t\}.$$

Let $\pi: Y\to S$ be the projection onto the second factor and $Y'=\pi^{-1}(S')$. Then $\pi: Y' \to S'$ is also a C^∞ fibre bundle, and we can regard $f: X'\to S'$ as being embedded fibre-wise in $\pi: Y'\to S'$. Let $X_t=f^{-1}(t)$, $t\in S$.

For any $t\in S'$ we have an action of $\pi_1(S',t)$ on $H^n(X_t,\mathbb{C})$ and $H^n(Y_t,\mathbb{C})$. A closed path which starts at t and travels once around 0 in a counterclockwise direction represents a generator of $\pi_1(S',t)$. Thus it determines automorphisms τ of $H^n(Y_t,\mathbb{C})$ and σ of $H^n(X_t,\mathbb{C})$, called the monodromy of π, respectively, of f.

We have natural inclusions $i: X\to Y$, $i_t: X_t\to Y_t$ and $r_t: Y_t\to Y$ for $t\in S'$. These give us the following commutative diagram [18, Sect. 4] whose rows are exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^n(Y) & \xrightarrow{r_t^*} & H^n(Y_t) & \xrightarrow{i_t^*} & H^n(X_t) \\
& \| & & \downarrow \tau & & \downarrow \sigma \\
0 \longrightarrow & H^n(Y) & \xrightarrow{r_t^*} & H^n(Y_t) & \xrightarrow{i_t^*} & H^n(X_t)
\end{array}$$

Let $H^n_{X'}=R^n f_* \mathbb{C}_{X'}=\bigcup_{t\in S'} H^n(X_t)$ and $H^n_{Y'}=R^n\pi_*\mathbb{C}_{Y'}$. Then $H^n_{X'}$ and $H^n_{Y'}$ carry the structure of flat complex vector bundles on S'. Thus they have canonical flat connections, both of which will be denoted by ∂_t and called the Gauss-Manin connection. The inclusion $i: X\to Y$ determines a horizontal map $i^*: H^n_{Y'}\to H^n_{X'}$. Let $\mathcal{O}_{S'}(H^n_{Y'})$ and $\mathcal{O}_{S'}(H^n_{X'})$ denote the corresponding sheaves of germs of holomorphic sections.

We are interested in extensions of $\mathcal{O}_{S'}(H^n_{X'})$ and $\mathcal{O}_{S'}(H^n_{Y'})$ to locally free sheaves of $\mathcal{O}_S$-modules. Such extensions always exist, and all possible extensions are divided into meromorphy classes. Two extensions $\mathscr{L}$ and $\mathscr{L}'$ are said to belong to the same meromorphy class if there exist $a, b \in \mathbb{Z}$ such that

$$t^a \mathscr{L}' \subset \mathscr{L} \subset t^b \mathscr{L}'.$$

From geometry one obtains several extensions of $\mathcal{O}_{S'}(H^n_{X'})$ and $\mathcal{O}_{S'}(H^n_{Y'})$ which all belong to the same meromorphy class [1, 4, 3, 13, 17, etc.]. This class of lattices is characterised by the property that the Gauss-Manin connection has regular singularities with respect to them [3, Chap. II, Théorème 1.19]. For example, if $\bar{Y} \to Y$ is a resolution of π with singular fibre E, a divisor with normal crossings on $\bar{Y}$, and $\bar{Y} \xrightarrow{\bar{\pi}} S$ is the induced map, the sheaf $\mathbf{R}^n \bar{\pi}_* \Omega^{\cdot}_{\bar{Y}/S}(\log E)$ is an extension of $\mathcal{O}_{S'}(H^n_{Y'})$, which is characterised by the properties

(i) ∂_t has a simple pole on it, and

(ii) the residue $\mathrm{Res}_0(\partial_t)$ has its eigenvalues in the interval $[0, 1)$.

For our purposes it is more convenient to work with the locally free extension $\mathscr{L}_Y$ of $\mathcal{O}_{S'}(H^n_{Y'})$, satisfying

(i) ∂_t has a simple pole on $\mathscr{L}_Y$, and

(ii) $\mathrm{Res}_0(\partial_t)$ has its eigenvalues on $\mathscr{L}_Y / t\mathscr{L}_Y$ in the interval $(-1, 0]$.

The analogous extension of $\mathcal{O}_{S'}(H^n_{X'})$ will be denoted by $\mathscr{L}_X$. The restriction map i^* extends to a horizontal morphism

$$i^* : \mathscr{L}_Y \to \mathscr{L}_X .$$

As in [18, (2.12)] one extends σ and τ to the stalks $\mathscr{L}_{Y,0}$ and $\mathscr{L}_{X,0}$. It is on $\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}$ that our Hodge filtration will first appear. It will measure the behaviour of a section of $\mathscr{L}_X$ as t tends to 0. In Sect. 5 we shall explain how $\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}$ is isomorphic to the cohomology of the "canonical" fibre of the Milnor fibration. As Pham showed to us, the best way to describe our filtration is to use the formalism of $\mathscr{D}$-modules.

2. Hodge Theory of Smooth Hypersurfaces

In this section we recall the description of the cohomology of a smooth projective hypersurface in terms of "couches multiples" following Brylinski [2, Sect. 3]. For a complex manifold X we let $\mathscr{D}_X$ denote the sheaf of germs of holomorphic differential operators on X. Suppose that V is a complex submanifold of codimension one in an $(n+1)$-dimensional complex manifold W. Let

$$B_{[V]W} = \mathscr{H}^1_V(\mathcal{O}_W) = \mathcal{O}_W(*V)/\mathcal{O}_W ,$$

where $\mathcal{O}_W(*V)$ is the sheaf of germs of meromorphic functions on W with only poles along V. The sheaf $\mathscr{D}_W$ acts on $\mathcal{O}_W(*V)$ and hence $B_{[V]W}$ is a $\mathscr{D}_W$-module. If $s = s_0, s_1, \ldots, s_n$ are local coordinates on W such that V is given by $s = 0$, and $\delta(s)$ is the class of s^{-1} in $B_{[V]W}$, then

$$B_{[V]W} = \mathscr{D}_W \cdot \delta(s) = \mathscr{D}_W \Big/ \left(\mathscr{D}_W \cdot s + \sum_{j=1}^{n} \mathscr{D}_W \cdot \partial_{s_j} \right).$$

A local section of $B_{[V]W}$ can then be expressed as

$$x = \sum_j g_j \partial_s^j \delta(s)$$

with g_j local sections of $\mathcal{O}_W$. Define the de Rham complex of $B_{[V]W}$ by

$$\mathrm{DR}(B_{[V]W}) = \Omega_W^{\cdot} \otimes_{\mathcal{O}_W} B_{[V]W}$$

with differential

$$\mathrm{d}(\omega \otimes P\delta(s)) = \mathrm{d}\omega \otimes P\delta(s) + \sum_{j=0}^{n} \mathrm{d}s_j \wedge \omega \otimes \partial_{s_j} P\delta(s)$$

[8, p. 138] for $\omega \in \Omega_W^{\cdot}$, $P \in \mathcal{D}_W$. Let $i: V \to W$ be the inclusion mapping. Following [8, p. 139], define a map

$$a: i_* \Omega_V^{\cdot}[-1] \to \mathrm{DR}(B_{[V]W})$$

by

$$a(\omega) = (\omega' \wedge \mathrm{d}s) \otimes \delta(s), \ \omega \in \Omega_V^k, k \geqq 0,$$

where $\omega' \in \Omega_W^k$ and $i^*(\omega') = \omega$. Then a is well defined and, as Pham shows, it is a quasi-isomorphism.

Now $B_{[V]W}$ has a natural filtration given as follows. We put $B_{[V]W}^{(j)} = 0$ if $j < 0$ and we let $B_{[V]W}^{(j)} = \mathcal{O}_W((j+1)V)/\mathcal{O}_W$ if $j \geqq 0$. We give $\Omega_V^{\cdot}[-1]$ the trivial filtration ("filtration bête"): $F^k(\Omega_V^{\cdot}[-1])$ is the complex with zero on the m^{th} place for $m < k$ and Ω_V^{m-1} if $m \geqq k$. Remark that a is compatible with these filtrations.

(2.1) **Lemma.** *The map a is a filtered quasi-isomorphism.*

Proof. We have the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Omega_W^{\cdot} & \longrightarrow & \Omega_W^{\cdot}(\log V) & \xrightarrow{\text{res}} & i_*\Omega_V^{\cdot}[-1] & \longrightarrow 0 \\ & \| & & \downarrow b & & \downarrow a & \\ 0 \longrightarrow & \Omega_W^{\cdot} & \longrightarrow & \Omega_W^{\cdot}(*V) & \xrightarrow{c} & \mathrm{DR}(B_{[V]W}) & \longrightarrow 0. \end{array}$$

Here $\Omega_W^{\cdot}(\log V)$ is the sheaf of germs of holomorphic forms ω on $W - V$ for which ω and $\mathrm{d}\omega$ have simple poles along V, and b is the inclusion map. The map c is the canonical map from $\Omega_W^{\cdot}(*V)$ to $\Omega_W^{\cdot}(*V)/\Omega_W^{\cdot}$ and is expressed in terms of $\delta(s)$ by

$$c\left(\frac{\omega}{s^{k+1}}\right) = \frac{(-1)^{k+1}}{k!} \omega \otimes \partial_s^k \delta(s), \quad \omega \in \Omega_W^j, \quad j, k \geqq 0.$$

If we filter $\Omega_W^{\cdot}(*V)$ by pole order the way $\mathrm{DR}(B_{[V]W})$ was filtered above, then c will be filtration-preserving. The induced filtration on $\Omega_W^{\cdot}(\log V)$ is the trivial filtration, and b becomes a filtered quasi-isomorphism [3, p. 80]. So a is a filtered quasi-isomorphism too.

Thus

$$\mathbf{H}^{\cdot}(W, \mathrm{DR}(B_{[V]W})) = \mathbf{H}^{\cdot -1}(V, \Omega_V^{\cdot}) = H^{\cdot -1}(V, \mathbb{C}).$$

If V is a projective variety the induced filtration on $\mathbf{H}^{\cdot}(W, \mathrm{DR}(B_{[V]W}))$ corresponds to the Hodge filtration on V, with indices shifted by 1.

3. Gauss-Manin Systems

We now return to the families of hypersurfaces $f: X \to S$ and $\pi: Y \to S$. The sheaves $\mathcal{O}_{S'}(H_{X'}^n)$ and $\mathcal{O}_{S'}(H_{Y'}^n)$ with their Gauss-Manin connections are $\mathcal{D}_{S'}$-modules. In this section we shall recapitulate the definition of the Gauss-Manin systems of f and π given in [8, p. 153ff.]. These are $\mathscr{D}_S$-modules which extend $\mathcal{O}_{S'}(H^n_{X'})$ and $\mathcal{O}_{S'}(H^n_{Y'})$ over S. We shall also see how the Hodge filtration on $\mathcal{O}_{S'}(H^n_{Y'})$ extends.

The graphs of f and of π in $X\times S$ and $Y\times S$ respectively are smooth hypersurfaces. So we have

$$B_{[\pi]Y\times S}=\mathscr{H}^1_{[\mathrm{graph}\,\pi]}(\mathcal{O}_{Y\times S})=\mathcal{O}_{Y\times S}[(\pi-t)^{-1}]/\mathcal{O}_{Y\times S}$$

and

$$\begin{aligned}B_{[f]X\times S}&=\mathscr{H}^1_{[\mathrm{graph}\,f]}(\mathcal{O}_{X\times S})=\mathcal{O}_{X\times S}[(f-t)^{-1}]/\mathcal{O}_{X\times S}\\&=\mathscr{D}_{X\times S}\delta(f-t)=\mathscr{D}_{X\times S}\Big/\left(\mathscr{D}_{X\times S}(f-t)+\sum_{i=0}^{n}\mathscr{D}_{X\times S}(\partial_i+f_i\partial_t)\right),\end{aligned}$$

where $\partial_i=\partial/\partial x_i$ and $f_i=\partial f/\partial x_i$ [8, p. 129].

Now form the relative de Rham complexes of these two modules, e.g.

$$\mathrm{DR}_{\mathrm{rel}}(B_{[f]X\times S})=\Omega^{\cdot}_{X\times S/S}\otimes_{\mathcal{O}_{X\times S}}B_{[f]X\times S}$$

with differential

$$\mathbf{d}(\omega\otimes P\delta(f-t))=\mathrm{d}\omega\otimes P\delta(f-t)+\sum_{j=0}^{n}\mathrm{d}x_j\wedge\omega\otimes\partial_jP\delta(f-t)$$

for ω, P local sections of $\Omega^k_{X\times S/S}$, $k\geqq 0$, and $\mathscr{D}_{X\times S}$ respectively [8, p. 145]. This complex can be thought of as the complex of relative forms on $X\times S$ with poles along the graph of f modulo forms without poles. Clearly it is a complex of $\mathscr{D}_S$-modules.

The Gauss-Manin systems $\mathscr{H}_X$ and $\mathscr{H}_Y$ of f and π respectively are defined by

$$\mathscr{H}_X=\mathbf{R}^{n+1}p_{X_*}\mathrm{DR}_{\mathrm{rel}}(B_{[f]X\times S}),\ \mathscr{H}_Y=\mathbf{R}^{n+1}p_{Y_*}\mathrm{DR}_{\mathrm{rel}}(B_{[\pi]Y\times S}),$$

where $p_X: X\times S\to S$ and $p_Y: Y\times S\to S$ are the projections.

We can simplify this description more. Let $j=\mathrm{id}_Y\times\pi$ be the inclusion of Y as the graph of π in $Y\times S$. As the complex $\mathrm{DR}_{\mathrm{rel}}(B_{[\pi]Y\times S})$ is supported on the graph of π, one obtains

$$\mathscr{H}_Y=\mathbf{R}^{n+1}\pi_*(j^{-1}\mathrm{DR}_{\mathrm{rel}}(B_{[\pi]Y\times S}))$$

and similarly for $\mathscr{H}_X$. Now it follows from [8, p. 159] that

$$j^{-1}\mathrm{DR}_{\mathrm{rel}}(B_{[\pi]Y\times S})\cong\Omega^{\cdot}_Y[D]$$

(polynomials in the indeterminate D with coefficients in the complex $\Omega^{\cdot}_Y$) as $\mathscr{D}_S$-complexes. Here the differential $\mathbf{d}$ in the complex $\Omega^{\cdot}_Y[D]$ is given by $\mathbf{d}(\sum_i\omega_iD^i)=\sum_i\mathrm{d}\omega_iD^i-\sum_i\mathrm{d}f\wedge\omega_iD^{i+1}$ and the isomorphism is given by the correspondence

$$\omega\cdot D^i\to\partial_t^i\left(\left[\frac{\omega}{\pi-t}\right]\right)=\left[\frac{i!\,\omega}{(\pi-t)^{i+1}}\right].$$

The $\pi^{-1}\mathscr{D}_S$-action on $\Omega^{\cdot}_Y[D]$ is given by

$$\begin{cases}\partial_t\cdot\omega D^i=\omega D^{i+1},\\ t\cdot\omega D^i=f\omega D^i-i\omega D^{i-1}.\end{cases}$$

As a consequence

$$\mathscr{H}_Y = \mathbf{R}^{n+1}\pi_* \Omega_Y^\cdot[D],$$
$$\mathscr{H}_X = \mathbf{R}^{n+1} f_* \Omega_X^\cdot[D].$$

We define a filtration F on $\Omega_Y^\cdot[D]$ (the *Hodge filtration*) by

$$F^{p-k-1}\Omega_Y^p[D] = \bigoplus_{i=0}^{k} \Omega_Y^p \cdot D^i.$$

Then $F^\cdot$ is a decreasing filtration by subcomplexes and ∂_t maps F^p to F^{p-1}. Moreover $\Omega_Y^\cdot[D] = \lim F^p$ (inductive limit for $p \to -\infty$). By restriction we obtain the Hodge filtration for $\Omega_X^\cdot[D]$. We let

(3.1)
$$F^p\mathscr{H}_Y = \text{image of } \mathbf{R}^{n+1}\pi_*(F^p\Omega_Y^\cdot[D]) \text{ in } \mathscr{H}_Y,$$
$$F^p\mathscr{H}_X = \text{image of } \mathbf{R}^{n+1} f_*(F^p\Omega_X^\cdot[D]) \text{ in } \mathscr{H}_X.$$

(3.2) **Lemma.** $\mathscr{H}_Y = \lim \mathbf{R}^{n+1}\pi_*(F^p\Omega_Y^\cdot[D])$.

Proof. Because π is a proper mapping, formation of $\mathbf{R}^i\pi_*$ commutes with inductive limits (compare R. Godement: Théorie des Faisceaux, p. 194. Paris: Hermann 1964).

Corollary. *Let V be a Stein open subset of S and let $\mathscr{U}$ be a suitable Stein covering of $\pi^{-1}(V)$. Then if $C^\cdot(\mathscr{U}, \Omega_Y^\cdot[D])$ denotes the single complex of $\mathscr{D}_S(V)$-modules, obtained from the Čech double complex of $\mathscr{U}$ with values in $\Omega_Y^\cdot[D]$, then*

$$\mathscr{H}_Y(V) = H^{n+1}(C^\cdot(\mathscr{U}, \Omega_Y^\cdot[D])).$$

Indeed, because $F^k\Omega_Y^p[D]$ is a coherent $\mathscr{O}_Y$-module for all k, p, we get by Cartan's Theorem B that

$$H^q(U, F^k\Omega_Y^p[D]) = 0, \quad (q>0)$$

for all $U \subset Y$ open and Stein. Hence for each k:

$$\Gamma(V, \mathbf{R}^{n+1}\pi_*(F^k\Omega_Y^\cdot[D])) = H^{n+1}(C^\cdot(\mathscr{U}, F^k\Omega_Y^\cdot[D])).$$

If each U in the nerve of $\mathscr{U}$ has only a finite number of connected components, then for these U one has

$$\lim \Gamma(U, F^k\Omega_Y^p[D]) = \Gamma(U, \Omega_Y^p[D])$$

(on each connected component the order of a section of $\Omega_Y^p[D]$ is bounded by analytic continuation) so we can take inductive limits to get

$$\mathscr{H}_Y(V) = \lim \Gamma(V, \mathbf{R}^{n+1}\pi_*(F^k\Omega_Y^\cdot[D])) = \lim H^{n+1}(C^\cdot(\mathscr{U}, F^k\Omega_Y^\cdot[D]))$$
$$= H^{n+1}(\lim C^\cdot(\mathscr{U}, F^k\Omega_Y^\cdot[D])) = H^{n+1}(C^\cdot(\mathscr{U}, \Omega_Y^\cdot[D]))$$

as required.

Observe that Cartan's Theorem B does not hold in general for quasi-coherent sheaves (this was pointed out to us by M. Saito). As a consequence the reasoning in [8, p. 143] is not correct. Hence we need a different argument to get a similar result for $\mathscr{H}_X$:

(3.3) **Lemma.** $\mathscr{H}_X \cong f_*(\Omega_X^{n+1}[D])/\mathrm{d}f_*(\Omega_X^n[D])$.

Proof. Write $K^\cdot = \Omega_X^\cdot[D]$. We have the first spectral sequence of hypercohomology

$$E_1^{pq} = R^q f_*(K^p) \Rightarrow \mathbf{R}^{p+q} f_*(K^\cdot).$$

If Cartan's Theorem B would hold for K^p, then E_1^{pq} would be zero for $q>0$ and the desired isomorphism would follow easily from this. As this is not true, we proceed in the following way. As $E_1^{pq}=0$ for $p>n+1$ we obtain a natural map (edge homomorphism)

$$E_2^{n+1,0} = \mathscr{H}^{n+1}(f_* K^\cdot) \to \mathbf{R}^{n+1} f_*(K^\cdot) = E_\infty^{n+1}.$$

The cohomology sheaves of the complex $K^\cdot$ satisfy:

$$\mathscr{H}^i(K^\cdot) = 0 \quad \text{for} \quad i \neq 1, n+1;$$

$$\mathscr{H}^1(K^\cdot) = f^{-1}\mathcal{O}_S \quad \text{and} \quad \mathscr{H}^{n+1}(K^\cdot) \text{ has support in the singularity } x$$

(see [8, pp. 159–161]). The same properties hold for the subcomplexes $F^iK^\cdot$ for $i<0$. Let $Q^\cdot = K^\cdot/F^0K^\cdot$ Then it follows that $\mathscr{H}^i(Q^\cdot)=0$ for $i \neq n+1$ and $\mathscr{H}^{n+1}(Q^\cdot)$ has support in x. These properties also hold for its subcomplexes $F^iQ^\cdot = F^iK^\cdot/F^0K^\cdot$ $(i<0)$. Hence $\mathbf{R}^i f_*(Q^\cdot)=0$ for $i \neq n+1$ and $\mathbf{R}^{n+1} f_*(Q^\cdot)$ has support in x with stalk $H^{n+1}(Q_x^\cdot)$, as follows from the second spectral sequence of hypercohomology. Again the same holds for $F^iQ^\cdot$, $i<0$. The coherence of $F^0K^p (p \geqq 0)$ implies that the sequence

$$0 \longrightarrow f_*(F^0K^\cdot) \longrightarrow f_*(K^\cdot) \longrightarrow f_*Q^\cdot \longrightarrow 0$$

is exact. We obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathscr{H}^{n+1}(f_*F^0K^\cdot) & \longrightarrow & \mathscr{H}^{n+1}(f_*K^\cdot) & \longrightarrow & \mathscr{H}^{n+1}(f_*Q^\cdot) & \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 \longrightarrow & \mathbf{R}^{n+1} f_*(F^0K^\cdot) & \longrightarrow & \mathbf{R}^{n+1} f_*(K^\cdot) & \longrightarrow & \mathbf{R}^{n+1} f_*(Q^\cdot) & \longrightarrow 0.
\end{array}$$

The map α is an isomorphism because of the coherence of all F^0K^p. So to prove that β is an isomorphism it suffices to show that γ is an isomorphism.

Because each F^iQ^p is locally free, $f_*Q^\cdot = \lim f_*F^iQ^\cdot$. Both source and target of γ have support in $0 \in S$ and

$$\begin{aligned}
\mathscr{H}^{n+1}(f_*Q^\cdot)_0 &= \mathscr{H}^{n+1}(\lim f_*F^iQ^\cdot)_0 = \lim \mathscr{H}^{n+1}(f_*F^iQ^\cdot)_0 \\
&= \lim(\mathbf{R}^{n+1} f_*F^iQ^\cdot)_0 = \lim H^{n+1}(F^iQ_x^\cdot) \\
&= H^{n+1}(\lim F^iQ_x^\cdot) = H^{n+1}(Q_x^\cdot) = (\mathbf{R}^{n+1} f_*Q^\cdot)_0 .
\end{aligned}$$

This concludes the proof.

Let $H' = \mathrm{d}f \wedge (f_*\Omega_X^n)/\mathrm{d}f \wedge \mathrm{d}(f_*\Omega_X^{n-1}) \subset H'' = f_*\Omega_X^{n+1}/\mathrm{d}f \wedge \mathrm{d}(f_*\Omega_X^{n-1})$. These are modules studied by Brieskorn [1]. By a result of Sebastiani [28] H' and H'' are locally free $\mathcal{O}_S$-modules of rank μ, the Milnor number of f. Moreover H''/H' has dimension μ.

The map $\omega \mapsto [\omega]$ embeds H'' into $\mathscr{H}_X$ [8, p. 160] with image $F^n\mathscr{H}_X$. With this identification we have that $\partial_t H' = H''$.

(3.4) **Lemma.** *$F^p\mathscr{H}_X=\partial_t^{n-p}F^n\mathscr{H}_X$ for $p\leqq n$.*

Proof. This is clear for $p=n$. We prove by induction on k that $\partial_t F^{n-k}\mathscr{H}_X=F^{n-k-1}\mathscr{H}_X$. The inclusions $\subset$ are obvious. Take $\eta\in F^{n-1}\mathscr{H}_X$: then $\eta=\eta_1 D^2+\eta_0\cdot D$ with $\eta_0\in H''$. Then $\eta_0 D=\partial_t(\omega_0 D)$ for some $\omega_0\in H'$ so $\eta=\partial_t\omega$ with $\omega=\omega_0 D+\eta_1 D\in F^n\mathscr{H}_X$. This settles the case $k=0$. The induction step is similar.

(3.5) **Proposition.** *$\mathscr{H}_X$ is a coherent $\mathscr{D}_S$-module and ∂_t is invertible on it.*

Proof. The submodules $F^p\mathscr{H}_X$ form a good filtration of $\mathscr{H}_X$, as

$$\mathscr{D}_S^{(r)}F^p\mathscr{H}_X=F^{p-r}\mathscr{H}_X$$

and each $F^p\mathscr{H}_X=\partial_t^{n-p}H''$ is a locally free $\mathscr{O}_S$-module. Moreover each quotient $F^p/F^{p+1}(p<n)$ is an isomorphic copy of H''/H' via the mapping ∂_t^{n-p}, so the associated graded module $\mathrm{gr}_F\mathscr{H}_X$ is a coherent $\mathrm{gr}\mathscr{D}_S$-module. By [8, Ex. 8.2.0, p. 81] this implies that $\mathscr{H}_X$ is $\mathscr{D}_S$-coherent. The invertibility of ∂_t follows easily from the fact that the map $\partial_t: H'\to H''$ is bijective.

Recall [8, 11.7] that a germ of $\mathscr{D}_S$-module at 0 is called *regular singular* if it is generated as a $\mathscr{D}_S$-module by a coherent $\mathscr{O}_S$-submodule which is stable under $t\partial_t$. Brieskorn has shown that $H''[t^{-1}]$ with its obvious $\mathscr{D}_S$-structure is regular singular [1]. Moreover $H''[t^{-1}]$ can be identified with the localisation $\mathscr{H}_{X,(t)}$ of $\mathscr{H}_X$ at 0:

$$\mathscr{H}_{X,(t)}=\mathscr{H}_X\otimes_{\mathscr{O}_S}\mathscr{O}_S[t^{-1}]$$

with the action $\partial_t\cdot(a\otimes b)=(\partial_t a)\otimes b+a\otimes\dfrac{\mathrm{d}b}{\mathrm{d}t}$.

Therefore $\mathscr{H}_{X,(t)}$ is regular singular. This implies that $\mathscr{H}_X$ is itself regular singular [8, 11.7.3].

Next we show that the restrictions of $\mathscr{H}_X$ and $\mathscr{H}_Y$ to the punctured disc S' coincide with the sheaves $\mathscr{O}_{S'}(H^n_{X'})$ and $\mathscr{O}_{S'}(H^n_{Y'})$ respectively. Observe that

$$\mathscr{O}_{S'}(H^n_{X'})=\mathbf{R}^n f_*\mathbb{C}_{X'}\otimes_{\mathbb{C}_{S'}}\mathscr{O}_{S'}=R^n f_*(f^{-1}\mathscr{O}_{S'}),$$

where we denote by f also its restriction to X'. Because f has no critical point on X', the complex $\Omega^{\cdot}_{X'}[D]$ is quasi-isomorphic to $f^{-1}\mathscr{O}_{S'}[-1]$ (i.e. the complex which has $f^{-1}\mathscr{O}_{S'}$ at place 1 and zero elsewhere), hence

$$\mathscr{H}_{X|S'}=\mathbf{R}^{n+1}f_*\Omega^{\cdot}_{X'}[D]=R^n f_*(f^{-1}\mathscr{O}_{S'})$$

as required. The same argument works for $\mathscr{H}_Y$.

We have two Hodge filtrations on $\mathscr{H}_{Y|S'}$: one from the filtration F of (3.1) and the other from the Hodge filtrations on the fibres of π: $F^pH^n(Y_t,\mathbb{C})=\bigoplus_{r\geqq p}H^{r,n-r}(Y_t)$; this provides a filtration of $\mathscr{O}_{S'}(H^n_{Y'})$ by holomorphic subbundles. Again these two filtrations coincide. This can be proved as follows. First observe that the Hodge filtration on $H^n(Y_t)$ is obtained from the trivial filtration of $\Omega^{\cdot}_{Y_t}$ (see Sect. 2). A relative version of this tells us that $F^p\mathscr{O}_{S'}(H^n_{Y'})=\mathbf{R}^n\pi_*(\sigma_{\geqq p}\Omega^{\cdot}_{Y'/S'})$ where $\sigma_{\geqq}$ denotes the trivial filtration. Cup product with $\mathrm{d}\pi$ produces a morphism of complexes

$$\Omega^{\cdot}_{Y'/S'}\to\Omega^{\cdot}_{Y'}[D][1]$$

such that $\sigma_{\geqq p}\Omega^{\cdot}_{Y'/S'}$ is mapped to $F^p\Omega^{\cdot+1}_Y[D]$. If we grade each side for its filtration we obtain the map

$$\Omega^p_{Y'/S'} \to \bigoplus_{r>0} \Omega^{p+r}_{Y'} \cdot D^{r-1}$$

and the differential in the right-hand side is reduced to cup product with $-d\pi$. This implies that we have a filtered quasi-isomorphism between $(\Omega^{\cdot}_{Y'/S'}, \sigma_{\geqq})$ and $(\Omega^{\cdot}_{Y'}[D][1], F)$. Hence $\mathbf{R}^n\pi_*(\sigma_{\geqq p}\Omega^{\cdot}_{Y'/S'}) = \mathbf{R}^{n+1}\pi_*(F^p\Omega^{\cdot}_{Y'}[D]) = F^p\mathscr{H}_{Y|S'}$.

(3.6) **Lemma.** *$\mathscr{H}_Y$ is $\mathscr{D}_S$-coherent.*

Proof. The trivial filtration on $\Omega^{\cdot}_Y[D]$ is a filtration by $\pi^{-1}\mathscr{D}_S$-subcomplexes, hence the first spectral sequence of hypercohomology

$$E_1^{pq} = R^q\pi_*\Omega^p_Y[D] \Rightarrow \mathbf{R}^{p+q}\pi_*\Omega^{\cdot}_Y[D]$$

is a spectral sequence of $\mathscr{D}_S$-modules. The E_1^{pq}'s are $\mathscr{D}_S$-coherent, as

$$R^q\pi_*\Omega^p_Y[D] = R^q\pi_*\Omega^p_Y \otimes_{\mathscr{O}_S} \mathscr{D}_S$$

and each $R^q\pi_*\Omega^p_Y$ is $\mathscr{O}_S$-coherent because π is proper.

(3.7) **Proposition.** *The restriction mapping $i^*: \mathscr{O}_{S'}(H^n_{Y'}) \to \mathscr{O}_{S'}(H^n_{X'})$ extends in a natural way to a homomorphism of $\mathscr{D}_S$-modules*

$$i^*: \mathscr{H}_Y \to \mathscr{H}_X$$

whose kernel and cokernel are $\mathscr{O}_S$-coherent.

Proof. The existence of i^* follows immediately from the description of $\mathscr{H}_Y$ and $\mathscr{H}_X$ with Čech double complexes; in fact, for any complex $I^{\cdot}$ of sheaves on Y one has a morphism of double complexes

$$C^{\cdot}(\mathscr{U}_Y, I^{\cdot}) \to C^{\cdot}(\mathscr{U}_X, I^{\cdot})$$

which induces the restriction mapping on the hypercohomology groups, and is compatible with additional structures on $I^{\cdot}$ like a $\mathscr{D}_S$-action. On S' the mapping i^* comes from a horizontal mapping between $H^n_{Y'}$ and $H^n_{X'}$, so $\ker(i^*)$ and coker (i^*) are coherent $\mathscr{O}_S$-modules on S'. As they are $\mathscr{D}_S$-coherent, it suffices to show that their stalks at $0 \in S$ are $\mathscr{O}_{S,0}$-modules of finite type.

By the proper base change theorem

$$\mathscr{H}_{Y,0} = \mathbf{H}^{n+1}(Y_0, \Omega^{\cdot}_Y[D]_{|Y_0}),$$

From the second spectral sequence of hypercohomology

$$E_2^{pq} = H^p(Y_0, \mathscr{H}^q(I^{\cdot})) \Rightarrow \mathbf{H}^{p+q}(Y_0, I^{\cdot})$$

($I^{\cdot}$ a complex of sheaves on Y_0, $\mathscr{H}^{\cdot}$ stands for cohomology sheaf) we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^n(Y_0, \mathscr{O}_{S,0}) & \longrightarrow & \mathscr{H}_{Y,0} & \longrightarrow & \mathscr{H}_{X,0} & \longrightarrow & H^{n+1}(Y_0, \mathscr{O}_{S,0}) \\
\downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \sim} & & \downarrow \\
H^n(X_0, \mathscr{O}_{S,0}) & \longrightarrow & \mathscr{H}_{X,0} & \xrightarrow{\sim} & \mathscr{H}_{X,0} & \longrightarrow & H^{n+1}(X_0, \mathscr{O}_{S,0})
\end{array}$$

because $\mathscr{H}^i(\Omega_Y^\cdot[D])=0$ for $i\neq 1, n+1$ and $\mathscr{H}^1(\Omega_Y^\cdot[D])_{|Y_0}$ is the constant sheaf with fibre $\mathcal{O}_{S,0}$. This shows that i^* can be considered as an edge homomorphism in the spectral sequence. The stalk of $\ker(i^*)$ at 0 is then a quotient of $H^n(Y_0, \mathcal{O}_{S,0}) = H^n(Y_0, \mathbb{C})\otimes_{\mathbb{C}}\mathcal{O}_{S,0}$ which is a free $\mathcal{O}_{S,0}$-module of finite rank. Hence $\ker(i^*)$ is of finite type. Similarly, $\operatorname{coker}(i^*)$ is an $\mathcal{O}_{S,0}$-submodule of a free $\mathcal{O}_{S,0}$-module of finite rank, hence it is also free of finite type.

(3.8) **Corollary.** *$\mathscr{H}_Y$ is regular singular.*

4. The Canonical Lattice

In this section we show how to embed the canonical extension $\mathscr{L}_X$ of $\mathcal{O}_{S'}(H^n_{X'})$ (cf. Sect. 1) in the Gauss-Manin system $\mathscr{H}_X$. Let us first summarize the facts we know about $\mathscr{H}_X$:

(i) $\mathscr{H}_X$ is a coherent $\mathscr{D}_S$-module and its restriction to S' is a locally free $\mathcal{O}_{S'}$-module of rank μ, the Milnor number of f;

(ii) $\mathscr{H}_X$ is regular singular at 0;

(iii) the operator ∂_t is invertible on $\mathscr{H}_{X,0}$.

It happens to be the case that such $\mathscr{D}_S$-modules have been classified. For $a\in\mathbb{C}$ and $q\in\mathbb{N}$ let

$$\mathscr{M}^{a,q}=\mathscr{D}_S/\mathscr{D}_S\cdot(t\partial_t-a)^q.$$

Then by [8, p. 107] there exist $a_j\in\mathbb{C}$ and $q_j\in\mathbb{N}$, $j=1,\dots,k$ with $a_j\neq -1, -2,\dots$ and an isomorphism of $\mathscr{D}_S$-modules

$$\bigoplus_{j=1}^{k}\mathscr{M}^{a_j,q_j}\xrightarrow{\sim}\mathscr{H}_X. \tag{4.1}$$

The numbers a_j which occur have the property that $\exp(-2\pi i a_j)$ is an eigenvalue of the monodromy operator σ with a Jordan block of size q_j. By the monodromy theorem the a_j are rational numbers. In fact we can arrange that $-1<a_j\leqq 0$.

For $a\in\mathbb{C}$ we define

$$C_a=\bigcup_{r>0}\operatorname{Ker}(t\partial_t-a)^r\subset\mathscr{H}_{X,0}.$$

(4.2) **Lemma.** *For every a the space C_a has finite dimension and $C_a=0$ if $\exp(-2\pi i a)$ is not an eigenvalue of σ.*

Proof. It clearly suffices to show these properties for each of the summands in the left-hand side of (4.1) separately. Hence fix $b\in\mathbb{Q}$, $b\neq -1, -2,\dots$ and $q\in\mathbb{N}$. Let $\mathscr{M}=\mathscr{M}_0^{b,q}$ with the canonical generator u corresponding to the class of 1.

Put $B=\mathbb{C}\cdot u\oplus\mathbb{C}\cdot Nu\oplus\dots\oplus\mathbb{C}\cdot N^{q-1}u$ where $N=t\partial_t-b$. One easily checks that the natural mapping from B to $\mathscr{M}$ is injective. For $k\in\mathbb{Z}$ we let $\mathscr{M}_k=\mathcal{O}_{S,0}\cdot\partial_t^{-k}B\subset\mathscr{M}$. It is easy to prove that $\mathscr{M}=\cup\mathscr{M}_k$, $\cap\mathscr{M}_k=0$, $\mathscr{M}_i\subset\mathscr{M}_j$ if and only if $i\geqq j$, $\mathscr{M}_s=\partial_t^{k-s}M_k$. Moreover $\mathscr{M}_0=B\oplus\mathscr{M}_1$ so $\mathscr{M}_k=\partial_t^{-k}B\oplus\mathscr{M}_{k+1}$ for all k. The operator N acts on each $\mathscr{M}_k$ and on $\mathscr{M}_k/\mathscr{M}_{k+1}\cong\partial_t^{-k}B$ it acts with eigenvalue $-k$; more precisely: $(N+k)^q=0$ on $\mathscr{M}_k/\mathscr{M}_{k+1}$. We determine the intersection of C_a with $\mathscr{M}$. Let $v\in C_a\cap\mathscr{M}$, $v\neq 0$. Then $(t\partial_t-a)^r v=0$ for some $r>0$. Choose k such that $v\in\mathscr{M}_k$ but $v\notin\mathscr{M}_{k+1}$. Because N acts on $\mathscr{M}_k/\mathscr{M}_{k+1}$ with eigenvalue $-k$ and

$(N+b-a)^r v=0$ we have that $a=b+k$. A similar argument shows that $C_{b+k} \cap \mathscr{M}=\partial_t^{-k}B$. The monodromy operator of $\mathscr{M}$ acts with eigenvalue $\exp(-2\pi i b)$ so $b-a\in\mathbb{Z}$ if and only if $\exp(-2\pi i a)$ is an eigenvalue of the same operator.

(4.3) *Definition.* A filtration V on $\mathscr{H}_{X,0}$ is defined as follows: we let $V_a\mathscr{H}_{X,0}$ be the $\mathscr{O}_{S,0}$-submodule generated by all subspaces C_b with $b\geqq a$ and we let $V_{>a}\mathscr{H}_{X,0}$ be the submodule generated by all C_b with $b>a$.

It follows immediately from Lemma (4.2) that for every a:

$$V_a\mathscr{H}_{X,0}=C_a\oplus V_{>a}\mathscr{H}_{X,0}.$$

Remark that $V_a\mathscr{H}_{X,0}$ is a free $\mathscr{O}_{S,0}$-module for $a>-1$, because the $\mathscr{O}_S$-torsion part is contained in $\sum_{a\leqq -1} C_a$.

(4.4) *Definition.* We let $\mathscr{L}_X\subset\mathscr{H}_X$ be the $\mathscr{O}_S$-submodule which coincides with $\mathscr{H}_X$ on S' and with $\mathscr{L}_{X,0}=V_{>-1}\mathscr{H}_{X,0}$.

Then $t\mathscr{L}_{X,0}=\partial_t^{-1}\mathscr{L}_{X,0}=V_{>0}\mathscr{H}_{X,0}$ and $\mathscr{L}_X$ is a free $\mathscr{O}_S$-module of rank μ. Hence $\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}=\bigoplus_{-1<a\leqq 0} C_a$ and $t\partial_t$ acts on this space with eigenvalues $a\in(-1,0]$. Thus $\mathscr{L}_X$ is indeed the canonical extension of $\mathscr{O}_{S'}(H^n_{X'})$.

The filtration V induces also a filtration on $\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}$, which we denote by the same symbol. It is on

$$\mathrm{Gr}_V(\mathscr{L}_{X,0}/t\mathscr{L}_{X,0})=\bigoplus_{-1<a\leqq 0} C_a$$

that our Hodge filtration will first appear. Recall that $C_a\cong V_a/V_{>a}$. We let

(4.5) $$F^pC_a=[F^p\cap V_a+V_{>a}]/V_{>a}=\text{image of } \partial_t^{n-p}H''\cap V_a \text{ in } C_a$$

and

$$F^p(\mathscr{L}_{X,0}/t\mathscr{L}_{X,0})=\bigoplus_{-1<a\leqq 0} F^pC_a.$$

Here $F^\cdot$ is the filtration on $\mathscr{H}_{X,0}$ from (3.3).

Remark. The filtration V as defined above is the same as the order filtration associated to the "microlocal asymptotical expansion", considered by Pham [9, p. 274].

Example. Let $f\in\mathbb{C}[z_0,\dots,z_n]$ be quasihomogeneous with weights $w_0,\dots,w_n$, i.e. f is a linear combination of monomials $z_0^{m_0}\dots z_n^{m_n}$ with $\sum_{i=0}^{n} w_im_i=1$. Suppose that f has an isolated singularity at 0 and let $\{z^m|m\in A\}$ be a monomial basis for the Artinian ring $\mathbb{C}[z_0,\dots,z_n]/(\partial f/\partial z_0,\dots,\partial f/\partial z_n)$. Put $\omega_m=z^m dz_0\wedge\dots\wedge dz_n/(f-t)$. Then $\{\omega_m|m\in A\}$ is an $\mathscr{O}_S$-basis of H''. Moreover

$$t\partial_t\omega_m=(a(m)-1)\,\omega_m,$$

where $a(m)=\sum_{i=0}^{n} w_i(m_i+1)$, hence $\omega_m\in C_{a(m)-1}$. Let $k(m)=-[-a(m)]$ and $\eta_m=\partial_t^{k(m)}\omega_m$. Then $\mathscr{L}_{X,0}$ is generated by all η_m, $m\in A$. The resulting Hodge filtration on $\bigoplus_{m\in A}\mathbb{C}\cdot\eta_m\cong\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}$ is the same as the one given in [19].

(4.6) *Definition.* We let $\mathscr{L}_Y \subset \mathscr{H}_Y$ be given by $\mathscr{L}_Y = (i^*)^{-1}\mathscr{L}_X$, where $i^*: \mathscr{H}_Y \to \mathscr{H}_X$ is as in (3.7). Because $\ker(i^*)$ is $\mathcal{O}_S$-coherent, $\mathscr{L}_Y$ is then indeed isomorphic to the canonical extension of $\mathcal{O}_{S'}(H^n_{Y'})$.

5. The Canonical Milnor Fibre

In this section we explain how our filtration F appears on the cohomology of the canonical fibre X_∞ of the Milnor fibration.

Let $e: U \to S'$ be a universal covering of S'. Set $X_\infty = X \times_{S'} U$. Then X_∞ is the total space of a differentiable fibre bundle over the contractible space U; so for any $s \in S'$ we have a diffeomorphism $X_\infty \xrightarrow{\sim} X_s \times U$ and for all $u \in U$ the natural inclusion

$$j_u : X_{e(u)} \to X_\infty, j_u(x) = (x, u)$$

is a homotopy equivalence.

In this way the space $H^n(X_\infty, \mathbb{C})$ is naturally isomorphic to the space of multivalued horizontal sections of $H^n_{X'}$ over S', i.e. to the space of constant sections of $e^*H^n_{X'}$ over U. Write the monodromy σ as $\sigma = \sigma_s\sigma_u = \sigma_u\sigma_s$ with σ_s semi-simple and σ_u unipotent. Let $N = \log\sigma_u$. There is a natural action of σ on $H^n(X_\infty, \mathbb{C})$. So let $H^n(X_\infty, \mathbb{C})_a$, $-1 < a \leq 0$, be the eigenspace of σ_s for the eigenvalue $\exp(-2\pi i a)$. Thus

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1 < a \leq 0} H^n(X_\infty, \mathbb{C})_a .$$

We define an isomorphism

$$\Phi : \mathscr{L}_{X,0}/t\mathscr{L}_{X,0} \to H^n(X_\infty, \mathbb{C})$$

by mapping the summand C_a to the summand $H^n(X_\infty, \mathbb{C})_a$ via the map

$$v \mapsto t^{-a} \exp(N \log t / 2\pi i) v .$$

To show that C_a is mapped indeed to $H^n(X_\infty, \mathbb{C})_a$, use the fact that $Nv = -2\pi i(t\partial_t - a)v$ for $v \in C_a$ [3, p. 53]. The inverse mapping is given by

$$w \mapsto t^a \exp(-N \log t / 2\pi i) w .$$

Define the filtration F on $H^n(X_\infty, \mathbb{C})$ by

$$F^p H^n(X_\infty, \mathbb{C}) = \Phi F^p(\mathscr{L}_{X,0}/t\mathscr{L}_{X,0}) .$$

We define the space Y_∞ in a similar way, and interpret its cohomology $H^n(Y_\infty, \mathbb{C})$ as the space of multivalued horizontal sections of $\mathcal{O}_{S'}(H^n_{Y'})$. Again we have the restriction map

$$i^* : H^n(Y_\infty, \mathbb{C}) \to H^n(X_\infty, \mathbb{C}) .$$

On $H^n(Y_\infty, \mathbb{C})$ we have the limit Hodge filtration $F^{\cdot}_S$ of Schmid [15], and on $H^n(X_\infty, \mathbb{C})$ the Hodge filtration $F^{\cdot}_{St}$ defined by the second author [18]. It is our purpose to show that $F^{\cdot}_{St} = F^{\cdot}$, but we will use $F^{\cdot}_S$ to prove this.

We first recapitulate the construction of $F^{\cdot}_S$. For $u \in U$, we identify $H^n(Y_{e(u)})$ with $H^n(Y_\infty)$ via the homotopy equivalence $y \mapsto (y, u)$ (this amounts to evaluation of horizontal sections of $e^*H^n_{Y'}$ at $u \in U$). The holomorphically varying Hodge filtration on the fibres of $\pi: Y' \to S'$ gives us a holomorphic mapping from U to a suitable flag manifold of $H^n(Y_\infty, \mathbb{C})$: $u \mapsto F^{\cdot}_{e(u)}$. Let us take for U the subset of $\mathbb{C}$ given by $\mathrm{Im}(u) > c$ for some $c \in \mathbb{R}$ and $e(u) = \exp(2\pi i u)$. Clearly under our identifications $F^{\cdot}_{e(u+1)} = \tau^{-1} F^{\cdot}_{e(u)}$ where τ is the monodromy operator on $H^n(Y_\infty, \mathbb{C})$. By abuse of notation we also let N denote $\log \tau_u$. According to Schmid's results, the limit $\lim_{\mathrm{Im}\, u \to \infty} \exp(Nu) F^{\cdot}_{e(u)}$ exists: this is the limit Hodge filtration $F^{\cdot}_S$.

It was observed by the first author [16] that the map i^* is surjective provided that the degree of f is sufficiently large. As a given isolated hypersurface singularity can always be represented by a polynomial germ of arbitrary large degree, it may be assumed that i^* is surjective. But then $F^{\cdot}_{St}$ is just the image of $F^{\cdot}_S$ under i^*, due to the fact that i^* is a morphism of mixed Hodge structures and hence strictly compatible with the Hodge filtrations.

An isomorphism

$$\Psi: \mathscr{L}_{Y,0}/t\mathscr{L}_{Y,0} \to H^n(Y_\infty, \mathbb{C})$$

is defined in a similar way as Φ, and such that $i^* \circ \Psi = \Phi \circ i^*$. The space F^p_S certainly contains $\Psi(y + t\mathscr{L}_{Y,0})$ for $y \in F^p\mathscr{H}_{Y,0} \cap \mathscr{L}_{Y,0}$. Hence to prove the inclusion

$$(5.1) \qquad F^p H^n(X_\infty, \mathbb{C}) \subset F^p_{St}$$

it suffices to show that, if $d = \deg f$ is sufficiently large, one has the equality

$$(5.2) \qquad F^p \mathscr{H}_X = i^* F^p \mathscr{H}_Y.$$

For, let us assume this and let $[x] \in F^p C_a$, $-1 < a \leqq 0$, be represented by $x \in F^p\mathscr{H}_{X,0} \cap V_a$. Then we may choose $y \in F^p\mathscr{H}_{Y,0}$ with $i^*(y) = x$. As $x \in \mathscr{L}_{X,0}$, $y \in \mathscr{L}_{Y,0} \cap F^p\mathscr{H}_{Y,0}$ so $\Psi(y + t\mathscr{L}_{Y,0}) \in F^p_S$. Hence $\Phi([x]) = \Phi([i^*y]) = i^*\Psi(y + t\mathscr{L}_{Y,0}) \in i^*F^p_S \subset F^p_{St}$. We make (5.2) precise as follows:

(5.3) **Lemma.** *Suppose that $d = \deg f$ is so big that there exist $Q_1, \ldots, Q_\mu \in \mathscr{O}_{S,0}[z_0, \ldots, z_n]$ of degree $\leqq d - n - 2$ such that the forms $\omega_i = Q_i dz_0 \wedge \ldots \wedge dz_n/(f - t)$, $i \in 1, \ldots, \mu$ generate $F^n\mathscr{H}_{X,0}$ as an $\mathscr{O}_{S,0}$-module. Then $F^p\mathscr{H}_{X,0} = i^*F^p\mathscr{H}_{Y,0}$.* ▫

Proof. Let $x \in F^p\mathscr{H}_{X,0}$. Then $x = \partial_t^{n-p} x'$ for some $x' \in F^n\mathscr{H}_{X,0}$. Write $x' = \sum_{i=1}^{\mu} g_i \omega_i$ with $g_i \in \mathscr{O}_{S,0}$, $i = 1, \ldots, \mu$. As $\deg Q_j \leqq d - n - 2$ the forms ω_i extend to forms η_i on $\mathbb{P}^{n+1} \times S$ with a pole of order 1 along Y, so $x' = i^*(y')$ with $y' = \sum_{i=1} g_i \eta_i \in F^n\mathscr{H}_{Y,0}$. Then $x = \partial_t^{n-p} x' = \partial_t^{n-p}(i^* y') = i^* \partial_t^{n-p}(y') \in i^* F^p\mathscr{H}_{Y,0}$. We conclude that $F^p\mathscr{H}_{X,0} \subset i^*F^p\mathscr{H}_{Y,0}$. The converse inclusion follows from (3.3) and the proof of (3.7).

The converse of (5.1) will be proved in the next section.

6.

In this section we will prove that

$$F^p H^n(X_\infty, \mathbb{C}) = F^p_{St}$$

for all $p \in \mathbb{Z}$. Observe that this trivially holds for $p > n$: in that case $F^p = 0$ as $F^p_{St} = 0$ and $F^p \subset F^p_{St}$. Note however, that $F^0_{St} = H^n(X_\infty, \mathbb{C})$ but that the equality $F^0 = H^n(X_\infty, \mathbb{C})$ is not at all obvious (and yet unproven); it is equivalent to the statement that $\mathscr{L}_{X,0} \subset \partial_t^n H''$.

Our proof is similar to Varchenko's proof that his asymptotic Hodge filtration induces the same filtration on $\mathrm{Gr}_W H^n(X_\infty, \mathbb{C})$ where W is the weight filtration [see the proof of (6.5)]. The basic tool is the following

(6.1) **Lemma.** *For $b \in Q$ let us define $S(b)$ to be the image of $F^n \mathscr{H}_{X,0} \cap V_b$ under the canonical map $V_b \to V_b/V_{>b} = C_b$, and let $d(b) = \dim S(b) - \dim S(b-1)$. (This is a non-negative number as $\partial_t^{-1} : S(b-1) \to S(b)$ is injective.) Then*

$$\sum_b b d(b) \leqq (n-1)\mu/2 .$$

Proof. See [25, Lemma 1.4].

(6.2) **Lemma.** *With notations as in (6.1), write $b = a + n - p$ where $p \in \mathbb{Z}$, $-1 < a \leqq 0$. Then $d(b) = \dim \mathrm{Gr}_F^p C_a$.*

Proof. The space $S(b)$ is mapped isomorphically to $F^p C_a$ by ∂_t^{n-p}. Hence $d(b) = \dim F^p C_a - \dim F^{p+1} C_a = \dim \mathrm{Gr}_F^p C_a$.

Write $\mu = \mu' + \mu''$ with $\mu' = \dim C_0$, $\mu'' = \sum_{-1<a<0} \dim C_a$.

(6.3) **Lemma.** $\sum_{p=0}^{n} \dim F^p \geqq (n+3)\mu'/2 + (n+2)\mu''/2$.

Proof.

$$\begin{aligned}
\sum_{p=0}^{n} \dim F^p &= \sum_{-1<a\leqq 0} \sum_{p=0}^{n} \sum_{q=p}^{n} \dim \mathrm{Gr}_F^q C_a \\
&= \sum_{-1<a\leqq 0} \sum_{p=0}^{n} \sum_{q=0}^{p} \dim \mathrm{Gr}_F^p C_a \\
&= \sum_{-1<a\leqq 0} \sum_{p=0}^{n} (p+1) \dim \mathrm{Gr}_F^p C_a \\
&\geqq \sum_{-1<a\leqq 0} \sum_{p\leqq n} (p+1) \dim \mathrm{Gr}_F^p C_a \\
&= \sum_{-1<a\leqq 0} \sum_{p\leqq n} (n+1+a) \dim \mathrm{Gr}_F^p C_a \\
&\quad - \sum_{-1<a\leqq 0} \sum_{p\leqq n} (a+n-p) \dim \mathrm{Gr}_F^p C_a \\
&= (n+1)\mu + \sum_{-1<a<0} a \dim C_a - \sum_b b d(b)
\end{aligned}$$

[use complex conjugation on $H^n(X_\infty, \mathbb{C})$ to see that $\dim C_a = \dim C_{-1-a}$ for $-1 < a < 0$ so $\sum_{-1<a<0} a \dim C_a = \frac{1}{2} \sum_{-1<a<0} (a - 1 - a) \dim C_a = -\mu''/2$]

$$\geqq (n+1)\mu - \mu''/2 - (n-1)\mu/2 = (n+3)\mu'/2 + (n+2)\mu''/2 .$$

(6.4) Our next step will be to show that equality holds in (6.3) if we replace $F^\cdot$ by $F^\cdot_{\mathrm{St}}$. Then we may conclude that $\dim F^p = \dim F^p_{\mathrm{St}}$ for $p = 0, \ldots, n$, so $F^p = F^p_{\mathrm{St}}$ for $p = 0, 1, \ldots, n$. As $F^0_{\mathrm{St}} = H^n(X_\infty, \mathbb{C})$ and $F^\cdot$ is a decreasing filtration, this implies that in fact $F^p = F^p_{\mathrm{St}}$ for all $p \in \mathbb{Z}$.

The desired equality is based on a symmetry between the Hodge numbers of the mixed Hodge structure $(H^n(X_\infty), F_{\mathrm{St}}, W)$. With the help of the fact that the map $i^* : H^n(Y_\infty) \to H^n(X_\infty)$ is surjective this admits an easy proof, so we will give it here.

(6.5) The main facts we need are the following. Consider the map $N = \log \tau_u$ on $H^n(Y_\infty)$ where τ_u is the unipotent part of the monodromy. Then $N^{n+1} = 0$ hence there exists a unique filtration

$$(0) \subset W_0 \subset W_1 \subset \ldots \subset W_{2n-1} \subset W_{2n} = H^n(Y_\infty)$$

with the properties that N maps W_i to W_{i-2} and the induced mappings $N^r: W_{n+r}/W_{n+r-1} \to W_{n-r}/W_{n-r-1}$ are isomorphisms. The filtration W is called the weight filtration of N with center n. Moreover N maps F_S^p to F_S^{p-1} and the triple $(H^n(Y_\infty), F_S, W)$ is a mixed Hodge structure. As N shifts the Hodge filtration by -1 and the weight filtration by -2 it is a morphism of mixed Hodge structures of type $(-1, -1)$. The semisimple part τ_s of the monodromy preserves both F_S and W.

Consequently we get a splitting of the mixed Hodge structure:

$$H^n(Y_\infty) = H^n(Y_\infty)_1 \oplus H^n(Y_\infty)_{\neq 1},$$

where the subscript refers to the eigenvalues of the action of τ_s on the corresponding subspace. Consider the exact sequence

$$0 \to H^n(Y_0) \xrightarrow{\text{sp}} H^n(Y_\infty) \xrightarrow{i^*} H^n(X_\infty) \to 0.$$

This is an exact sequence of mixed Hodge structures [18, p. 543] and by the invariant cycle theorem the image of sp coincides with $\ker(\tau - I) = \ker(N) \cap H^n(Y_\infty)_1$ [17, (5.12)].

Thus we get a complete description of the induced weight filtration on $H^n(X_\infty)$ and its relation to the induced morphism N:

- on $H^n(X_\infty)_{\neq 1} = H^n(Y_\infty)_{\neq 1}$, W is the weight filtration of N centered at n as before;
- however, on $H^n(X_\infty)_1 = H^n(Y_\infty)_1/\ker(N)_1$, W is the weight filtration of N centered at $n+1$. The proof of this miracle is an easy exercise.

The Hodge numbers of $(H^n(X_\infty), F_{St}, W)$ are given by

$$h^{pq} = \dim \operatorname{Gr}^p_{F_{St}} \operatorname{Gr}^W_{p+q} H^n(X_\infty);$$

we have an obvious decomposition $h^{pq} = h_1^{pq} + h_{\neq 1}^{pq}$. Moreover $\dim \operatorname{Gr}^p_{F_{St}} = \sum_q h^{pq}$.

We have a double symmetry between these Hodge numbers:

- due to complex conjugation on $\operatorname{Gr}^W_{p+q}$ we get $h_1^{pq} = h_1^{qp}$ and $h_{\neq 1}^{pq} = h_{\neq 1}^{qp}$;
- due to the isomorphism $N^{p+q-n}: \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q} \xrightarrow{\sim} \operatorname{Gr}^{n-q}_F \operatorname{Gr}^W_{2n-p-q}$ $(p+q \geqq n)$ on the eigenvalue $\neq 1$ part we obtain:

$$h_{\neq 1}^{pq} = h_{\neq 1}^{n-q, n-p}$$

and similarly (taking care of the shift in indices)

$$h_1^{pq} = h_1^{n+1-q, n+1-p}.$$

(6.6) **Lemma.** $\sum_{p=0}^{n} \dim F_{St}^p = (n+3)\mu'/2 + (n+2)\mu''/2.$

Proof. The computation is done in [25, Sect. 3.1]. This finishes the proof that our Hodge filtration coincides with F_{St}.

7. Monodromy and Multiplication by f

In this section we apply our main result to give a simple proof of a result of Varchenko [24] (cf. also [14]). We assume that f has an isolated singularity at $0\in\mathbb{C}^{n+1}$. We let

$$Q^f=\Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f\wedge\Omega^n_{\mathbb{C}^{n+1},0}\,,$$

a space which has dimension μ, the Milnor number of f, and which we identify with $F^n\mathscr{H}_{X,0}/\partial_t^{-1}F^n\mathscr{H}_{X,0}$ by the correspondence $[\omega]\to[\omega/(f-t)]$.

As an $\mathcal{O}_{S,0}$-subquotient module of $\mathscr{L}_{X,0}$, the latter module inherits the filtration V. It is given explicitly by

$$V_a(F^n\mathscr{H}_{X,0}/\partial_t^{-1}F^n\mathscr{H}_{X,0})=[V_a\cap F^n\mathscr{H}_{X,0}+\partial_t^{-1}F^n\mathscr{H}_{X,0}]/\partial_t^{-1}F^n\mathscr{H}_{X,0}\,.$$

We also denote by V the corresponding filtration of Q^f.

Remark that $f\cdot V_bQ^f\subset V_{b+1}Q^f$ as $f\omega=t\omega$ for $\omega\in F^n\mathscr{H}_{X,0}$ and $tV_b\subset V_{b+1}$ for all b. Hence multiplication by f induces a graded endomorphism of degree one of Gr^VQ^f, denoted by $\{f\}$.

(7.1) **Theorem** (Varchenko). *The maps* $\{f\}$ *and* $N=\log\sigma_u\in\mathrm{End}\,H^n(X_\infty,\mathbb{C})$ *have the same Jordan normal form.*

Proof. The map N is a morphism of mixed Hodge structures of type $(-1,-1)$. Hence all powers of N are strictly compatible with the filtration F (with the appropriate shift). This implies the existence of a splitting of the Hodge filtration, i.e. a graduation of $H^n(X_\infty,\mathbb{C})$ which has F as its associate filtration, such that N becomes a graded morphism of degree -1. In particular one concludes that N and its induced endomorphism Gr_FN of degree -1 of $\mathrm{Gr}_FH^n(X_\infty,\mathbb{C})$ have the same Jordan normal form.

We have a canonical isomorphism

$$\mathrm{Gr}_FH^n(X_\infty,\mathbb{C})=\bigoplus_{-1<a\leq 0}\mathrm{Gr}_FC_a$$

and the corresponding endomorphisms $N_{p,a}$: $\mathrm{Gr}^p_FC_a\to\mathrm{Gr}^{p-1}_FC_a$ are given by $N_{p,a}(x)=-2\pi i(t\partial_t-a)x\equiv-2\pi i\,t\partial_t\,x \bmod F^p$.

On the other hand it is immediately seen that for $b\in\mathbb{Q}$, $b=n-p+a$ with $p\in\mathbb{Z}$ and $-1<a\leq 0$, the map

$$\partial_t^{n-p}:V_b\cap F^n\mathscr{H}_{X,0}\to V_a/V_{>a}=C_a$$

induces an isomorphism from $\mathrm{Gr}^V_bQ^f$ to $\mathrm{Gr}^p_FC_a$, and the diagram

$$\begin{array}{ccc}\mathrm{Gr}^V_bQ^f & \xrightarrow{\{f\}} & \mathrm{Gr}^V_{b+1}Q^f\\ \cong\downarrow\partial_t^{n-p} & & \cong\downarrow\partial_t^{n-p+1}\\ \mathrm{Gr}^p_FC_a & \xrightarrow{N_{p,a}} & \mathrm{Gr}^{p-1}_FC_a\end{array}$$

commutes up to the factor $-2\pi i$. Hence $\{f\}$ and Gr_FN have the same Jordan type.

(7.2) *Remark*. Suppose that the function f is non-degenerate with respect to its Newton diagram [5]. Then one has the so-called Newton filtration on $\Omega^{n+1}_{\mathbb{C}^{n+1},0}$. It has been proved by Saito [12] that the filtration on Q^f, induced by this Newton filtration coincides with the V-filtration with indices shifted by 1.

(7.3) *The singularity spectrum.* For $b \in \mathbb{Q}$ we have the number $d(b) = \dim \mathrm{Gr}_F^p C_a = \dim \mathrm{Gr}_b^V Q^f$ where $p \in \mathbb{Z}$, $-1 < a \leq 0$ (6.2). The spectrum of the singularity f is the unordered sequence $\{b_1, \ldots, b_\mu\}$ in which the number b occurs with multiplicity $d(b)$. These numbers have first been considered in [18, (5.3)] in connection with the Thom-Sebastiani problem for the mixed Hodge structure (see the next section). Their importance for deformation theory has been first emphasized by Arnol'd [30], see also [31]. The most important properties of the spectrum are the following:

(i) *range:* $d(b) \neq 0$ implies that $-1 < b < n$;

(ii) *symmetry:* $d(n-1-b) = d(b)$; this can be deduced from the results in Sect. 6.

(iii) *additivity:* for polynomials $f \in \mathbb{C}[x_0, \ldots, x_n]$, $g \in \mathbb{C}[y_0, \ldots, y_m]$ with isolated singularities at 0 the spectrum of $f+g$ is obtained as $\{a_i + b_j + 1\}$ where a_i, b_j run over the spectrum of f and g respectively. This is due to Varchenko ([25, Theorem 7.3]) and is a consequence of the Thom-Sebastiani result which we prove in Sect. 8.

(iv) *semicontinuity:* refine the notation $d(b)$ as follows when f is variable too: let $d(b; f, 0)$ denote $d(b)$ for the germ of the function f at 0. Suppose that we have a deformation f_t of f_0 and critical points $x_1, \ldots, x_r$ of f_t which approach the critical point 0 of f_0 as t tends to 0 and which have the same critical value. Then one has for every $a \in \mathbb{R}$ the inequality

$$\sum_{i=1}^{r} \sum_{a < b \leq a+1} d(b; f_t, x_i) \leq \sum_{a < b \leq a+1} d(b; f_0, 0).$$

This has been proved by Varchenko for deformations of low weight of quasihomogeneous functions [27] and by the second author in general [20].

8.

The goal of this section is to prove a "Sebastiani-Thom" formula for the Hodge filtration.

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$, $g \in \mathbb{C}[y_0, \ldots, y_m]$, both with an isolated singularity at the origin. Denote by X_∞^f and X_∞^g the corresponding "canonical Milnor fibres" and σ_f, σ_g the monodromy operators.

Then $f+g \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ also has an isolated singularity at the origin, and X_∞^{f+g} has the homotopy type of the join of X_∞^f and X_∞^g. Hence

$$H^{n+m+1}(X_\infty^{f+g}, \mathbb{C}) \cong H^n(X_\infty^f, \mathbb{C}) \otimes_{\mathbb{C}} H^n(X_\infty^g, \mathbb{C}). \tag{8.1}$$

Moreover

$$\sigma_{f+g} = \sigma_f \otimes \sigma_g \quad \text{(cf. [29])}.$$

Our formula for the Hodge filtration is as follows.

(8.2) **Theorem.**

$$F^k H^{n+m+1}(X_\infty^{f+g}, \mathbb{C})_c = \bigoplus_{a+b=c} \sum_{i+j=k} F^i H^n(X_\infty^f, \mathbb{C})_a \otimes_{\mathbb{C}} F^j H^m(X_\infty^g, \mathbb{C})_b$$
$$\bigoplus_{a+b=c-1} \sum_{i+j=k-1} F^i H^n(X_\infty^f, \mathbb{C})_a \otimes_{\mathbb{C}} F^j H^m(X_\infty^g, \mathbb{C})_b.$$

The proof of this result uses the idea of Malgrange's proof of the Thom-Sebastiani result [6, p. 424]. Let

$$R=\mathbb{C}\{\{\partial_t^{-1}\}\}=\left\{\sum_{p=0}^{\infty} a_p\partial_t^{-p}\in\mathbb{C}[[\partial_t^{-1}]]\,\middle|\,\sum_{p=0}^{\infty} a_p t^p/p!\in\mathbb{C}\{t\}\right\}$$

be the ring of microdifferential operators with constant coefficients, and let $L=R[\partial_t]$ be its quotient field. Then according to [7], Proposition 2.5 H''_f is a free R-module of rank μ_f and $\mathscr{H}_{X_f,0}=H''_f\otimes_R L$. This is most conveniently checked by considering a decomposition of $\mathscr{H}_{X_f,0}$ as in (4.1):

(8.3) **Lemma.** *Let $b\in\mathbb{Q}$, $b\neq-1,-2,\ldots$ and let $q\in\mathbb{N}$. Let u_b be the class of* 1 *in $\mathscr{M}_0^{b,q}=\mathscr{D}_{S,0}/\mathscr{D}_{S,0}(t\partial_t-b)^{q+1}$. Then $\{(t\partial_t-b)^j u_b | j=0,\ldots,q\}$ is an L-basis of $\mathscr{M}_0^{b,q}$.*

Proof. First suppose $q=0$. Thus $t\partial_t u_b=bu_b$. It follows that

$$t^k\partial_t^r u_b=(b-r+k)\ldots(b-r+1)\partial_t^{r-k}u_b$$

for $r\geqq 0$, $k>0$. Any $w\in\mathscr{D}_{S,0}u_b$ may be written

$$w=\sum_{r=0}^{m} a_r(t)\partial_t^r u_b$$

with $a_0,\ldots,a_m\in\mathcal{O}_{S,0}$. If we write $a_r(t)=\sum_{k=0}^{\infty} a_{kr}t^k$, then substituting for $t^k\partial_t^r$, we obtain

$$w=\sum_{r=0}^{m}\left(\sum_{k=1}^{\infty} a_{kr}(b-r+k)\ldots(b-r+1)\partial_t^{r-k}u_b+a_{0r}\partial_t^r u_b\right).$$

Clearly $\sum_{k=r+1}^{\infty} a_{kr}(b-r+k)\ldots(b-r+1)\partial_t^{r-k}\in R$; so $\mathscr{M}_0^{b,0}=L\cdot u_b$. In general

$$\mathscr{M}_0^{b,q}/\mathscr{D}_{S,0}(t\partial_t-b)^q u_b=\mathscr{M}_0^{b,q-1}.$$

Arguing by induction, we find that $u_b,\ldots,(t\partial_t-b)^q u_b$ is a basis of $\mathscr{M}_0^{b,q}$.

For notational convenience, let us write $\mathscr{H}_f$, $\mathscr{L}_f$, V_b^f, C_b^f instead of $\mathscr{H}_{X_f,0}$, $\mathscr{L}_{X_f,0}$ etc.

(8.4) **Corollary.** *$\mathscr{H}_f$ is an L-vectorspace of dimension μ_f and each $V_b^f\mathscr{H}_f$ is a free R-submodule of* rank μ_f.

Notice that $\mathscr{M}_0^{a,q}\otimes_L\mathscr{M}_0^{b,r}$ becomes a $\mathscr{D}_{S,0}$-module if we let

(8.5) $$t(u\otimes v)=tu\otimes v+u\otimes tv,\ u\in\mathscr{M}_0^{a,q},\ v\in\mathscr{M}_0^{b,r}.$$

Thus

$$\partial_t t(u\otimes v)=\partial_t tu\otimes v+u\otimes\partial_t tv$$

and

(8.6) $$(\partial_t t-a-1-b-1)^k u\otimes v=\sum_{j=0}^{k}\binom{k}{j}(\partial_t t-a-1)^j u\otimes(\partial_t t-b-1)^{k-j}v.$$

Now (8.5) makes $\mathscr{H}_f\otimes_L\mathscr{H}_g$ into a $\mathscr{D}_{S,0}$-module too. We want to show that it is isomorphic to $\mathscr{H}_{f+g}$.

(8.7) **Lemma.** *There is a natural isomorphism*

$$\kappa: H''_f \otimes_R H''_g \to H''_{f+g}.$$

Proof. One defines κ as follows. Consider the map

$$\kappa_0: H''_f \times H''_g \to H''_{f+g}$$

given by $\kappa_0(x, y) = x \wedge y$. One checks immediately that κ_0 is well-defined and that $\kappa_0(\partial_t^{-1} x, y) = \kappa_0(x, \partial_t^{-1} y)$. Thus κ_0 is R-bilinear and defines a homomorphism as above. Computing modulo ∂_t^{-1}, one obtains the multiplication map

$$\mathbb{C}\{x\}/J_f \otimes_{\mathbb{C}} \mathbb{C}\{y\}/J_g \to \mathbb{C}\{x, y\}/J_{f+g}$$

which is bijective (J stands for the Jacobian ideal, i.e. the ideal generated by the partial derivatives of the corresponding function). So κ is an isomorphism. Moreover it is $\mathcal{O}_{S,0}$-linear.

It clearly extends to an isomorphism of $\mathcal{D}_{S,0}$-modules

$$\kappa: \mathcal{H}_f \otimes_L \mathcal{H}_g \to \mathcal{H}_{f+g}.$$

Our next step is the computation of C_c^{f+g} in terms of the spaces C_a^f and C_b^g. Observe that $C_a^f = \bigcup_{q>0} \ker(\partial_t t - a - 1)^q$ as $\partial_t t = t\partial_t + 1$. Using formula (8.6) one deduces that, if $x \in C_a^f$ and $y \in C_b^g$, then $x \otimes y \in C_{a+b+1}^{f+g}$. Using the relation $\partial_t C_a^f = C_{a-1}^f$ and a dimension count one obtains:

(8.8) **Lemma.** $C_c^{f+g} = \bigoplus_{-1<a\leqq 0} C_a^f \otimes C_{c-a-1}^g$.

Proof. The right-hand side injects into C_c^{f+g}; moreover one has

$$\sum_{-1<c\leqq 0} \dim C_c^{f+g} = \mu_{f+g} = \mu_f \cdot \mu_g$$

and

$$\sum_{-1<c\leqq 0} \sum_{-1<a\leqq 0} \dim C_a^f \cdot \dim C_{c-a-1}^g$$

$$= \sum_{-1<a\leqq 0} \dim C_a^f \cdot \sum_{-a-2<b\leqq -a-1} \dim C_b^g = \mu_f \cdot \mu_g$$

hence the lemma follows.

It is clear from the above that the isomorphism

$$\kappa: \mathcal{H}_f \otimes_L \mathcal{H}_g \to \mathcal{H}_{f+g}$$

does not map $\mathcal{L}_f \otimes_R \mathcal{L}_g$ isomorphically to $\mathcal{L}_{f+g}$. However, one can write $\mathcal{L}_f \otimes_R \mathcal{L}_g = \mathcal{A} \oplus \mathcal{B}$ where $\mathcal{A}$ is the R-submodule generated by

$$A = \bigoplus_{\substack{a+b>-1 \\ -1<a,b\leqq 0}} C_a^f \otimes C_b^g$$

and $\mathcal{B}$ is generated over R by

$$B = \bigoplus_{\substack{a+b\leqq -1 \\ -1<a,b\leqq 0}} C_a^f \otimes C_b^g.$$

Let us write $C^f = \bigoplus_{-1<a\leqq 0} C_a^f$ and use C^g, C^{f+g} in a similar way. Then the reasoning above shows that $C^f \otimes C^g = A \oplus B$ while $C^{f+g} = \partial_t \kappa(A) \oplus \kappa(B)$; moreover $\mathcal{L}_{f+g}$ is

the R-module generated by C^{f+g}. More generally $\kappa(V_a^f \otimes_R V_b^g) = V_{a+b+1}^{f+g}$. From this one deduces that

$$\kappa(F^i \mathscr{H}_f \otimes_R F^j \mathscr{H}_g) = F^{i+j+1} \mathscr{H}_{f+g}.$$

We now proceed to the proof of Theorem (8.2). Let

$$'F^k A = \bigoplus_{\substack{a+b>-1 \\ -1<a,b\leqq 0}} \sum_{i+j=k} F^i C_a^f \otimes F^j C_b^g \subset A$$

and

$$'F^k B = \bigoplus_{\substack{a+b\leqq -1 \\ -1<a,b\leqq 0}} \sum_{i+j=k-1} F^i C_a^f \otimes F^j C_b^g \subset B.$$

We write $'F^k = \partial_t \kappa('F^k A) \oplus \kappa('F^k B) \subset C^{f+g}$. To prove the theorem we must show that $'F^k = F^k C^{f+g}$ for all k. To achieve this, we first prove that $'F^k \subset F^k C^{f+g}$ and then that $\sum_{k=0}^{n+m+1} \dim 'F^k = \sum_{k=0}^{n+m+1} \dim F^k C^{f+g}$. (This is the same line of proof as Varchenko's in [25, Sect. 7].)

Take $x \in F^i C_a^f$, $y \in F^j C_b^g$, $-1 < a, b \leqq 0$. Suppose that $a+b > -1$. We must show that $\partial_t \kappa(x \otimes y) \in F^{i+j} C_{a+b}^{f+g}$. By the definition of $F^\cdot$, there exist $x' \in H_f''$, $y' \in H_g''$ such that $x - \partial_t^{n-i} x' \in V_{>a}^f$, $y - \partial_t^{m-j} y' \in V_{>b}^g$. Then

$$\partial_t \kappa(x \otimes y) - \partial_t^{n+m+1-i-j}(x' \otimes y') = \partial_t [\kappa((x - \partial_t^{n-i} x') \otimes y)$$
$$+ \kappa(\partial_t^{n-i} x' \otimes (y - \partial_t^{m-j} y'))] \in \partial_t [\kappa(V_{>a}^f \otimes V_b^g) + \kappa(V_a^f \otimes V_{>b}^g)] \subset \partial_t V_{>a+b+1}^{f+g} = V_{>a+b}^{f+g}.$$

Hence $\partial_t(x \otimes y) \in F^{i+j} C_{a+b}^{f+g}$. In a similar way one shows that, if $a+b \leqq -1$, then $\kappa(x \otimes y) \in F^{i+j+1} C_{a+b+1}^{f+g}$. As a consequence, $'F^k \subset F^k C^{f+g}$.

To prove the dimension statement, put $h_{f,a}^i = \dim \mathrm{Gr}_F^i C_a^f$ and similarly for $h_{g,b}^j$. Then due to the double symmetry between Hodge numbers (6.5) we obtain that

(8.9) $$h_{f,a}^i = h_{f,-1-a}^{n-i} \quad \text{if} \quad a \neq 0, \; h_{f,0}^i = h_{f,0}^{n+1-i}.$$

Thus

$$\sum_{k=0}^{n+m+1} \dim 'F^k = \sum_{k=0}^{n+m+1} (k+1) \dim 'F^k/'F^{k+1}$$
$$= \sum_{-1<a+b\leqq 0} \sum_{i,j} (i+j+1) h_{f,a}^i h_{g,b}^j + \sum_{-2<a+b\leqq 1} \sum_{i,j} (i+j+2) h_{f,a}^i h_{g,b}^j.$$

To use the symmetries of (8.9) we must distinguish several cases, according to whether a, b are zero or not etc. We obtain that our sum is equal to

$$\tfrac{1}{2}(n+m+4) \textstyle\sum' h_{f,a}^i h_{g,b}^j + \tfrac{1}{2}(n+m+3) \sum'' h_{f,a}^i h_{g,b}^j,$$

where in $\sum'$ we take the summation over all i, j, a, b with $a+b=0$ or -1 and in $\sum''$ we take the i, j, a, b with $a+b \neq 0, -1$. Thus

$$\textstyle\sum' h_{f,a}^i h_{g,b}^j = \mu_{f+g}' \quad \text{and} \quad \sum'' h_{f,a}^i h_{g,b}^j = \mu_{f+g}''.$$

So our desired equality follows from (6.6), applied to $f+g$.

Next we indicate, how one can prove Conjecture (5.4) from [18], using Theorem (8.2). We will formulate a more precise result.

Consider the category consisting of mixed Hodge structures together with an automorphism of finite order. We define the *join* of two such objects (U, γ_u) and (V, γ_v) as $(U * V, \gamma_u \otimes \gamma_v)$ where

(i) $(U * V)_{\mathbb{Z}} = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$.

(ii) For $a \in (-1, 0]$ write $U_{\mathbb{C}} = \bigoplus_{-1 < a \leqq 0} U_a$ where $U_a = \operatorname{Ker}(\gamma_u - e^{-2\pi i a})$. For $V_{\mathbb{C}}$ one has a similar decomposition. Define the filtration W on $(U * V)_{\mathbb{C}} = U_{\mathbb{C}} \otimes_{\mathbb{C}} V_{\mathbb{C}}$ by

$$W_k(U * V) = \bigoplus_{a,b} \sum_{i,j} W_i U_a \otimes W_j V_b$$

with summation over all i, j such that

$$i + j = k \quad \text{if} \quad a = b = 0$$
$$i + j = k - 2 \quad \text{if} \quad a + b = -1$$
$$i + j = k - 1 \text{ else}.$$

(iii) The Hodge filtration F on $(U * V)_{\mathbb{C}}$ is given by

$$F^p(U * V) = \bigoplus_{a,b} \sum_{k,l} F^k U_a \otimes_{\mathbb{C}} F^l V_b$$

with summation over k, l such that

$$k + l = p \quad \text{if} \quad a + b > -1,$$
$$k + l = p - 1 \quad \text{if} \quad a + b \leqq -1.$$

We omit the proof that $(U * V, W, F)$ is again a mixed Hodge structure: this is straightforward but tedious. The reader who likes such computations may amuse himself by checking the associativity of the join product.

(8.10) *Example.* Let V be the trivial Hodge structure on $\mathbb{Q}$, purely of type $(0, 0)$, $\gamma = -1$. Then $V * V = \mathbb{Q}(-1)$ with trivial action, purely of type $(1, 1)$. More generally: for any U one has $(U * V) * V = U(-1)$.

(8.11) **Theorem.** *Let f and g be as before. Let U_f be the pair consisting of the mixed Hodge structure on $H^n(X^f_\infty, \mathbb{C})$ together with the automorphism σ_s of finite order. Define U_g and U_{f+g} in a similar way. Then*

$$U_{f+g} \cong U_f * U_g.$$

Proof. By virtue of the Sebastiani-Thom theorem and Theorem (8.2) we only must check that the weight filtration is the right one. According to (6.5) this filtration is completely determined by $N_f = \log \sigma_u$. Let $L(N)$ denote the monodromy weight filtration of the nilpotent endomorphism N, centered at 0. Then the problem is to show that

$$L(N_{f+g})_k = \sum_{i+j=k} L(N_f)_i \otimes L(N_g)_j.$$

The Thom-Sebastiani result implies that

$$\sigma_u^f \otimes \sigma_u^g = \sigma_u^{f+g}$$

hence

$$N_{f+g} = N_f \otimes 1 + 1 \otimes N_g.$$

Choose representations ϱ and ϱ' of the Lie algebra $\mathfrak{sl}_2$ such that

$$\varrho\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=N_f, \quad \varrho'\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=N_g.$$

Let $H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\in\mathfrak{sl}_2$. Then the structure theory of representations of $\mathfrak{sl}_2$ shows that $L(N_f)_k$ is generated by the elements u such that $\varrho(H)u=cu$ and $c\leqq k$. In a similar way $L(N_g)$ and $L(N_{f+g})$ are determined by the actions of $\varrho'(H)$ and

$$\varrho''(H):=\varrho(H)\otimes 1+1\otimes\varrho'(H)$$

respectively.

Finally it is easily checked that the eigenspace of $\varrho''(H)$ for the eigenvalue k is generated by elements $u\otimes v$ which satisfy $\varrho(H)u=ru$, $\varrho'(H)v=sv$ with $r+s=k$.

From this the theorem follows. □

Let

$$h_a^{pq}(f)=\dim_{\mathbb{C}}\mathrm{Gr}_F^p\mathrm{Gr}_{p+q}^W H^n(X_\infty^f,\mathbb{C})_a,$$

be the *Hodge numbers* of f. A formal consequence of Theorem (8.11) is

(8.12) **Corollary.** $h_c^{p,q}(f+g)=\sum_{a,b}\sum_{i,j,r,s}h_a^{ij}(f)\,h_b^{rs}(g)$ *with summation over all* a,b,i,j,r,s *such that* $a+b-c\in\mathbb{Z}$ *and:*

$$a=c \quad\text{or}\quad b=c \Rightarrow i+r=p,\ j+s=q;$$

$$a\neq c \quad\text{and}\quad b\neq c;$$

$$c=0,\ a+b=-1 \Rightarrow i+r=p-1,\ j+s=q-1;$$

$$c\neq 0,\ a+b=c \Rightarrow i+r=p,\ j+s=q-1;$$

$$a+b=c-1 \Rightarrow i+r=p-1,\ j+s=q.$$

(8.13) *Remark.* A special case of Theorem (8.11) is the so-called inductive structure of the cohomology of Fermat varieties, discovered by Shioda [16] and applied by him to verify the Hodge conjecture for those. In fact, this inductive structure needs only the corresponding result for weighted homogeneous polynomials, which is already implicit in [19].

(8.14) *Remark.* It can easily be deduced from the proof of Theorem (8.2) that the map κ induces an isomorphism between $Q^f\otimes Q^g$ and Q^{f+g} and that under this identification

$$\mathrm{Gr}_c^V Q^{f+g}=\bigoplus_{a+b=c-1}\mathrm{Gr}_a^V Q^f\otimes\mathrm{Gr}_b^V Q^g.$$

In particular, the spectrum numbers of $f+g$ are of the form $a+b+1$, where a and b run over the spectrum numbers of f and g respectively.

9. Examples

(9.1) Let $f(x,y,z)=x^p+y^q+z^r+axyz$, $a\neq 0$, $p^{-1}+q^{-1}+r^{-1}<1$. Then f defines a cusp singularity of type $\mathrm{T}_{p,q,r}$. Following the method described in [13], one finds that a basis of $\mathscr{L}_f$ consists of the forms

$$\omega, t\partial_t\omega, \partial_t x\omega, \ldots, \partial_t x^{p-1}\omega,$$
$$\partial_t y\omega, \ldots, \partial_t y^{q-1}\omega, \partial_t z\omega, \ldots, \partial_t z^{r-1}\omega,$$

where $\omega = [\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z/(f-t)]$. Except for a few small values of p, q, and r, $t\partial_t$ has the following form on C^f:

$$t\partial_t(t\partial_t\omega)=0;\; t\partial_t(\partial_t x^j\omega)=\frac{j-p}{p}\,\partial_t x^j\omega \quad \text{for} \quad j=1,\ldots,p-1;$$

$$t\partial_t(\partial_t y^j\omega)=\frac{j-q}{q}\,\partial_t y^j\omega \quad \text{for} \quad j=1,\ldots,q-1;$$

$$t\partial_t(\partial_t z^j\omega)=\frac{j-r}{r}\,\partial_t z^j\omega \quad \text{for} \quad j=1,\ldots,r-1.$$

Therefore ω generates F^2C^f and $F^1C^f = C^f$. The definition of the weight filtration shows that $W_2 = W_3$ is generated by all base vectors different from ω and $W_4 = C^f$.

(9.2) Let $g(x,y) = ax^5 + y^6 + x^4y$, $a \in \mathbb{C}$. Let $\omega = [dx \wedge dy/(g-t)]$. Then the forms $\partial_t y^4\omega$, $\partial_t xy^3\omega$, $\partial_t x^2y^2\omega$, $\partial_t x^4\omega$, $\partial_t xy^4\omega$, $\partial_t x^2y^3\omega$, ω, $\partial_t x^5\omega$, $\partial_t x^2y^4\omega$, $y\omega$, $x\omega$, $\partial_t x^6\omega$, $y^2\omega$, $xy\omega$, $x^2\omega$, $y^3\omega$, $xy^2\omega$, $x^2y\omega$, $x^3\omega$ give a basis of $\mathscr{L}_g$ over R and of C^g over $\mathbb{C}$. With respect to this basis, $-24t\partial_t$ has the following matrix:

$$\begin{pmatrix}
23 & 0 & & & & & & & & & & & & & & & & & 0 \\
\cdot & 22 & 0 \\
\cdot & \cdot & 21 & 0 \\
\cdot & \cdot & \cdot & 19 & 0 \\
\cdot & \cdot & \cdot & \cdot & 18 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 17 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 15 & 0 & & & & & \emptyset \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 14 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 13 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 11 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 10 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 7 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{pmatrix}$$

The dots indicate entries which depend on the parameter a and may be non-zero. For $a=0$ they all vanish, because then f is quasi-homogeneous. It follows that F^1C^g is generated by

$$\omega, y\omega, x\omega, y^2\omega, xy\omega, x^2\omega, y^3\omega, xy^2\omega, x^2y\omega, x^3\omega.$$

Since the matrix above is diagonalizable all the basis elements lie in W_1 except for $x^3\omega$ which first appears in $W_2=C^g$.

This example illustrates the fact, that spectrum numbers remain constant under deformations with constant Milnor number [26]. It was also studied in [21, Sect. 11], in connection with another filtration which is discontinuous with respect to the parameter a and is related to the zeroes of the Bernstein polynomial.

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