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HOMOLOGY WITH LOCAL COEFFICIENTS

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1. Introduction

In a recent paper [16] the author has had occasion to introduce and use what he believed to be a new type of homology theory, and he named it *homology with local coefficients*. It proved to be the natural and full generalization of the Whitney notion of *locally isomorphic complexes* [18]. Whitney, in turn, credits the source of his idea to de Rham's *homology groups of the second kind in a non-orientable manifold* [13]. It has since come to the author's attention that homology with local coefficients is equivalent in a complex to Reidemeister's *Überdeckung* [10].

Since this new homology theory (which includes the old) seems to have such wide applicability, a complete review of the older theory is needed to determine to what extent and in what form its theorems generalize. The object of this paper is to make such a survey. The general conclusion is that all major parts of the older theory do extend to the new. In addition the newer theory fills in several gaps in the old. The most noteworthy of these is a full duality and intersection theory in a non-orientable manifold (§14).

For the sake of completeness, some of the results of Reidemeister have been included. The new approach and new definitions make for easier and more intuitive proofs. They lead also to results not obtained by Reidemeister. The most important is a proof of the topological invariance of all the homology groups obtained.[1] In addition developments are given of the subjects of multiplications of cycles and cocycles, chain mappings, continuous cycles, and Čech cycles.

Part I contains an abstract development of systems of local groups in a space entirely apart from their applications to homology. Any fibre bundle over a base space R [18] determines many such systems in R (one for each homology group, homotopy group, etc., of the fibre). These are invariants of the bundle. They should prove to be of some help in classifying fibre bundles.

Part II, which contains the extended homology theory, presupposes on the part of the reader a knowledge of the classical theory such as can be found in the books of Lefschetz [7] and Alexandroff-Hopf [1].

I. LOCAL GROUPS IN A SPACE

2. Notations

We shall be dealing throughout with an arcwise connected topological space R. For any point x of R, let F_x be the fundamental (Poincaré) group of R with

[1] It is not determined however whether or not the combinatorial invariant called "torsion" by Reidemeister [9] and its generalizations by Franz [4] and de Rham [14] are true topological invariants.

x as initial and terminal point. If A is a curve from x to y, the class of curves from x to y homotopic to A with end points fixed we shall denote by a symbol such as α_{xy}. Its inverse is denoted by α_{xy}^{-1} or α_{yx}. The elements of F_x are abbreviated α_x, β_x, etc. The class α_{xy} determines an isomorphism $F_x \rightarrow F_y$ (denoted by α_{xy}) defined by $\alpha_{xy}(\beta_x) = \alpha_{yx}\beta_x\alpha_{xy}$. In keeping with this notation, the product $\alpha_x\beta_x$ means the element of F_x obtained by traversing first a curve of the class α_x then one of β_x. As is well known, the combination of two isomorphisms $\beta_{yz}(\alpha_{xy}(\gamma_x))$ is the isomorphism $(\alpha_{xy}\beta_{yz})(\gamma_x)$.

3. Local groups

We shall say that we have *a system of local groups* (*rings*) *in the space R* if (1) for each point x, there is given a group (ring) G_x, (2) for each class of paths α_{xy}, there is given a group (ring) isomorphism $G_x \rightarrow G_y$ (denoted by α_{xy}), and (3) the result of the isomorphism α_{xy} followed by β_{yz} is the isomorphism corresponding to the path $\alpha_{xy}\beta_{yz}$.

It follows from the transitivity condition (3) that the identity path from x to x is the identity transformation in G_x. A further consequence is that the inverse of the isomorphism α_{xy} is α_{yx}. By (2), a closed path $\alpha_x \in F_x$ determines an automorphism of G_x. From (3) it follows that F_x is a group of automorphisms of G_x. The invariant subgroup of F_x acting as the identity on G_x is denoted F_x^1. Since, by (3),

$$\alpha_{xy}(\beta_x(\alpha_{yx}(g))) = (\alpha_{yx}\beta_x\alpha_{xy})(g), \qquad g \in G_y,$$

it follows that

$$\alpha_{xy}(\beta_x(g)) = [\alpha_{xy}(\beta_x)](\alpha_{xy}(g)), \qquad g \in G_x. \tag{3.1}$$

We shall say that the system $\{G_x\}$ is *simple* if every $F_x^1 = F_x$. If this happens for one x, it will be true for all. If $\{G_x\}$ is simple, the isomorphism α_{xy} is independent of the path from x to y. Choosing a fixed point o as origin, we find that each G_x is uniquely isomorphic to G_o. Thus the local system consists of one G_o and as many copies of G_o as there are points $x \neq 0$.

Two systems $\{G_x\}$, $\{H_x\}$ are said to be *isomorphic* if, for each x, there is an isomorphism ϕ_x of G_x onto H_x such that

$$\phi_y(\alpha_{xy}(g)) = \alpha_{xy}(\phi_x(g)), \qquad g \in G_x.$$

We shall deal only with properties of systems which are invariant under isomorphisms. In each case the proof of invariance is trivial and will be omitted.

It was proved in §2 that the collection $\{F_x\}$ is a system of local groups. It is simple if and only if it is abelian.

In some instances a system $\{G_x\}$ will consist of topological groups. The isomorphisms α_{xy} will then be continuous. In the following pages we shall omit continuity considerations whenever such are reasonably obvious.

We shall consistently attempt to reduce the study of a system to the study of what occurs at one point of R. As a first step we have

THEOREM 1. *If G is a group (ring), o a point of R, and ψ a homomorphism of F_o into the group of automorphisms of G, then there is one and only one system $\{G_x\}$ of local groups (rings) in R such that G_o is a copy of G and the operations of F_o in G_o are those determined by ψ.*

For each point $x \epsilon R$, choose a class of paths λ_{ox}, choosing λ_{oo} = identity. Let G_x be a group (ring) isomorphic to G. Associate this isomorphism with λ_{xo}. If α_{xy} is a path, we attach to it the isomorphism $G_x \to G_y$ defined by

$$\alpha_{xy}(g) = \lambda_{oy}[(\lambda_{ox}\alpha_{xy}\lambda_{yo})(\lambda_{xo}(g))].$$

The second operation is the automorphism of G attached to $\lambda_{ox}\alpha_{xy}\lambda_{yo} \epsilon F_o$ by ψ. Since ψ is a homomorphism, the transitivity condition (3) holds.

If $\{G'_x\}$ is a local system, and ϕ an isomorphism of G'_o into G_o such that $\alpha\phi = \phi\alpha$ for $\alpha \epsilon F_o$, map $G'_x \to G_x$ with the operation $\phi_x(g') = \lambda_{ox}(\phi(\lambda_{xo}(g')))$. It is easily verified that $\{\phi_x\}$ establishes an isomorphism between $\{G'_x\}$ and $\{G_x\}$. This proves the uniqueness and completes the theorem.

4. Automorphisms

Let $\{G_x\}$, $\{A_x\}$ be two systems of local groups, and suppose that A_x is a group of automorphisms of G_x in such a way that, for any α_{xy}, we have

$$\alpha_{xy}(a(g)) = \alpha_{xy}(a)(\alpha_{xy}(g)), \qquad a \epsilon A_x,\ g \epsilon G_x. \tag{4.1}$$

Then $\{A_x\}$ is called *a system of local automorphisms of* $\{G_x\}$. By (3.1), it follows that $\{F_x\}$ is such a system for any $\{G_x\}$.

Let A^1_x be the subgroup of A_x acting as the identity on G_x. By (4.1), $\{A^1_x\}$ is a system of local groups. It follows that, under the natural isomorphisms $A_x/A^1_x \to A_y/A^1_y$ induced by $A_x \to A_y$, $\{A_x/A^1_x\}$ is a system of local automorphisms of $\{G_x\}$.

If A is a group, and, for each x, A is a group of automorphisms of G_x such that

$$\alpha_{xy}(a(g)) = a(\alpha_{xy}(g)), \qquad a \epsilon A,\ g \epsilon G_x, \tag{4.2}$$

we shall call A *a group of uniform automorphisms of* $\{G_x\}$. If a system $\{A_x\}$ of local automorphisms is simple, then any one of its groups is, in a natural way, a group of uniform automorphisms of $\{G_x\}$. If F_x or F_x/F^1_x is abelian, we have such a group of uniform automorphisms for any $\{G_x\}$.

As in Theorem 1, complete knowledge of a system of automorphisms is obtainable from knowledge of what occurs at a single point.

THEOREM 2. *Let A be a group of automorphisms of G with only the identity acting as such in G (i.e. $A^1 = 1$). Let o be a point of R, and let F_o be represented as a group of automorphisms of G in such a way that the automorphism $\alpha(a(\alpha^{-1}(g)))$ of G is in A for every $\alpha \epsilon F_o$, $a \epsilon A$. Then there is one and only one system $\{A_x\}$ of local automorphisms of $\{G_x\}$ such that the collection (F_o, G_o, A_o) is isomorphic to (F_o, G, A). $\{A_x\}$ is simple and therefore A is a group of uniform automorphisms of $\{G_x\}$ if and only if the automorphisms of F_o and A in G commute.*

By Theorem 1, the system $\{G_x\}$ is completely determined. By assumption the automorphism $\alpha a \alpha^{-1}$ is a unique element of A. Thus each α determines an automorphism of A, and F_o is a group of such. By Theorem 1, the system $\{A_x\}$ of local groups is completely determined. For $a \epsilon A_x$, $g \epsilon G_x$, we choose a path α_{ox} and define

$$a(g) = \alpha_{ox}(\alpha_{xo}(a)(\alpha_{xo}(g))).$$

Clearly (4.1) will hold once we have proved the right side to be independent of α_{ox}. This is shown as follows.

$$\begin{aligned}\beta_{xo}(\alpha_{ox}(\alpha_{xo}(a)(\alpha_{xo}(g)))) &= (\alpha_{ox}\beta_{xo})(\alpha_{xo}(a)(\alpha_{xo}(g))) \\ &= [(\alpha_{ox}\beta_{xo})(\alpha_{xo}(a))(\alpha_{ox}\beta_{xo})^{-1}]((\alpha_{ox}\beta_{xo})(\alpha_{xo}(g))) \\ &= \beta_{xo}(a)(\beta_{xo}(g)).\end{aligned}$$

5. Operator rings

If G is an additive abelian group, the set H of all homomorphisms of G into itself forms a ring under the operations

$$(a + b)(g) = a(g) + b(g), \qquad (ab)(g) = a(b(g)), \qquad a, b \epsilon H, g \epsilon G.$$

A group A of automorphisms of G forms a multiplicative subgroup of H. It generates a subring A^* of H with unit = identity. If the symbol $a(g)$ be abbreviated by ag, this multiplication of $g \epsilon G$ by the scalar $a \epsilon A^*$ obeys the usual laws: $(a + b)g = ag + bg$, $(ab)g = a(bg)$, $1g = g$, $0g = 0$, $a(g + g') = ag + ag'$. One may therefore speak of linear combinations, linear independence, and bases in G relative to A.

We shall say that the system $\{A_x\}$ is *a system of operator rings for the abelian system* $\{G_x\}$ if, for each x, A_x is a ring of operators for G_x, and for each path α_{xy}, we have

$$\alpha_{xy}(ag) = \alpha_{xy}(a)\alpha_{xy}(g), \qquad a \epsilon A_x, g \epsilon G_x.$$

The analogue of Theorem 2 is proved with only slight modifications. If a system of operator rings is simple, we are led naturally to the concept of *a uniform operator ring for* $\{G_x\}$.

For any $\{G_x\}$, the system $\{F_x^*\}$ of group rings of $\{F_x\}$ is a system of operator rings. If F_x or F_x/F_x^1 is abelian, the group ring of F_o or factor ring thereof is a uniform operator ring for $\{G_x\}$.

6. Dual systems

Two abelian systems $\{G_x\}$, $\{H_x\}$ of local groups form a *pair* with respect to a third $\{K_x\}$ if, for each x, G_x and H_x form a pair with respect to K_x (i.e. a multiplication $gh = k$ is given which is linear in each factor) in such a way that, for each path α_{xy}, we have $\alpha_{xy}(gh) = \alpha_{xy}(g)\alpha_{xy}(h)$. Analogous to Theorem 1, we have

THEOREM 3. *Let G, H form a pair with respect to K, and let F_o be realized as a group of automorphisms of each of G, H, K in such a way that $\alpha(gh) = \alpha(g)\alpha(h)$ for $\alpha \epsilon F_o$, $g \epsilon G$, $h \epsilon H$. Then there is one and only one set of systems $\{G_x\}$, $\{H_x\}$,*

$\{K_x\}$ such that the first two form a pair with respect to the third, and G_o, H_o, K_o and the automorphisms F_o form a set isomorphic to the given G, H, K, F_o.

By Theorem 1, there are unique systems $\{G_x\}$, $\{H_x\}$, $\{K_x\}$ corresponding to G, H, K and the operations of F_o in these groups. In the construction of all three systems let us use the same collection of paths λ_{ox} (see proof of Theorem 1). Under the isomorphism of G_x, H_x, K_x with G, H, K attached to λ_{ox} the multiplication in the latter groups carries over into a multiplication in the former. The relation $\alpha_{xy}(gh) = \alpha_{xy}(g)\alpha_{xy}(h)$ for $g \in G_x$, $h \in H_x$ is proved by using the property $\alpha(g'h') = \alpha(g')\alpha(h')$ of the closed path $\alpha = \lambda_{ox}\alpha_{xy}\lambda_{yo}$ where g', h' correspond to g, h under λ_{ox}. Any other allowable product which agrees with the constructed product in G_o, H_o will likewise agree in G_x, H_x since both products are invariant under the translation along λ_{xo}.

Of particular interest is the case of character groups.

THEOREM 4. *If K = real numbers* mod 1, *G = a discrete, or compact, or locally compact separable group, and F_o is realized in two arbitrary ways as a group of automorphisms of K_o and G_o respectively, then F_o can be realized in one and only one way as a group of automorphisms of the character group H of G satisfying $\alpha(gh) = \alpha(g)\alpha(h)$. Thus to given systems $\{K_x\}$, $\{G_x\}$ is attached a unique system $\{H_x\}$ of character groups.*

The character $\alpha(h)$ is defined to be the one with the value $\alpha(\alpha^{-1}(g)h)$ on any g. The remainder of the theorem is readily verified.

It is to be noted that K admits but one non-trivial automorphism, namely: $k \to -k$. Thus there are as many character systems of a given system $\{G_x\}$ as there are factor groups of F_o of order 2. If it should be desirable to have a *unique* character system, it would be natural to choose $\{K_x\}$ to be simple.

7. Local groups under mappings

Let R, R' be two arcwise connected spaces and ϕ a continuous map $R' \to R$. Let F, F' be their fundamental groups relative to base points o, o' such that $\phi(o') = o$. If $\{G_x\}$ is a system of local groups in R, then ϕ *induces* in R' a system $\{G'_y\}$ as follows. The group G'_y is chosen isomorphic to G_x where $x = \phi(y)$. The isomorphism is denoted by ϕ. The path α_{yz} in R' maps G'_y isomorphically on G'_z according to the rule

$$\alpha_{yz}(g) = \phi^{-1}(\phi(\alpha_{yz})(\phi(g))), \qquad g \in G'_y. \tag{7.1}$$

The transitivity condition is immediately verified.

The existence of the induced system is apparent in view of Theorem 1 and the fact that the homomorphism $F' \to F$ realizes F' as a group of automorphisms of G_o.

The induced system has numerous properties which we list without proofs.

(a) If $\{G_x\}$ is simple, so is $\{G'_y\}$.

(b). $\{G'_y\}$ is simple if and only if F' is mapped by ϕ on the subgroup F^1 of F leaving G_o fixed.

(c). If $\{G_x\}$, $\{H_x\}$ are paired relative to $\{K_x\}$, then likewise the induced systems in R'.

(d). If $\{A_x\}$ is a system of local automorphisms or operator rings for $\{G_x\}$, then likewise the induced systems in R'.

(e). If A is a group of uniform automorphisms or a uniform operator ring for $\{G_x\}$, it is also one for the induced system.

It follows from (b) that, if R' is the covering space of R corresponding to the subgroup F^1 of F, the induced system is simple. Thus any system in R can be considered as the continuous image of a simple system in some covering space.

It is natural to inquire under what circumstances a given system in R' is induced by one in R. For this it is necessary and sufficient that (1) the kernel $\bar{F}'$ of the homomorphism $F' \rightarrow F$ shall act as the identity on $G'_{o'}$, and (2) the map of the subgroup $F'/\bar{F}'$ of F into the group of automorphisms of $G'_{o'}$ shall admit a homomorphic extension to F.

8. Special local groups

Because of their importance in the work of Reidemeister, we shall discuss certain local systems based on the fundamental group F of R.

Let F^1 be an invariant subgroup of F, and let $\mathcal{F} = F/F^1$. Let $\mathcal{G}$ be an abelian group. Let G be the set of functions from $\mathcal{F}$ to $\mathcal{G}$. If two such functions are added by adding functional values at each element of $\mathcal{F}$, G becomes an abelian group. A function $f \in G$ is called a *restricted* function if $f(\alpha) = 0$ except for a finite number of $\alpha \in \mathcal{F}$. The restricted functions form a subgroup G' of G. The structure of G (G') can be described as the unrestricted (restricted) direct sum of as many copies of $\mathcal{G}$ as there are elements of $\mathcal{F}$. If $\mathcal{G}$ = integers and $F^1 = 1$, then G' is the ordinary group ring of F.

If $\mathcal{G}$ is a ring, we define a multiplication of two functions $f, g \in G'$ by

$$f \times g(\alpha) = \textstyle\sum_\beta f(\alpha\beta^{-1})g(\beta), \qquad \alpha, \beta \in \mathcal{F}.$$

Since the functions in G' are restricted, the sum is finite; thus G' is a ring. If $\mathcal{G}$ = integers, this product is the usual one in the group ring.

The group F can be realized in three natural ways as a group of automorphisms of the groups G, G'. For any $\gamma \in F$, we define three operations on a function $f \in G$ (G') by

$$L_\gamma f(\alpha) = f(\gamma^{-1}\alpha), \qquad R_\gamma f(\alpha) = f(\alpha\gamma), \qquad T_\gamma f(\alpha) = f(\gamma^{-1}\alpha\gamma), \qquad \alpha \in \mathcal{F}.$$

Then $L_\gamma f$, $R_\gamma f$ and $T_\gamma f$ are in G (G'), and are called the *left translation*, *right translation*, and *transform* of f by γ, respectively. It is easy to verify that L_γ, R_γ, T_γ are automorphisms of G (G'), and that $L_\gamma L_\delta = L_{\gamma\delta}$, $R_\gamma R_\delta = R_{\gamma\delta}$, and $T_\gamma T_\delta = T_{\gamma\delta}$. The subgroup acting as the identity for both L and R is F^1.

In the case that $\mathcal{G}$ and therefore G' is a ring, the left and right translations are not *ring* automorphisms. However the transforms are: $T_\gamma(f \times g) = (T_\gamma f) \times (T_\gamma g)$.

It follows from Theorem 1 that, corresponding to each of the three ways that

F can act on G (G'), we have a system of local groups. These we denote by $\{G_x^L\}$, $\{G_x^R\}$, $\{G_x^T\}$, and similarly for G'. The last of these, $\{G_x'^T\}$, is a system of local rings, whenever $\mathcal{G}$ is a ring.

Since $\mathcal{G}$ is associative, $L_\gamma R_\beta f = R_\beta L_\gamma f$. Therefore, by Theorem 2, the left translations form a group of uniform automorphisms of $\{G_x^R\}$, and likewise for the right translations of $\{G_x^L\}$. Then, as in §5, the group ring of F is a ring of uniform operators for these systems.

Under the automorphism ϕ of G (G') defined by $\phi f(\alpha) = f(\alpha^{-1})$, we have $\phi L_\gamma f = R_\gamma \phi f$. Therefore, by means of ϕ, an isomorphism exists between $\{G_x^L\}$ and $\{G_x^R\}$.

In the work of Reidemeister on homotopy chains, $\mathcal{G}$ = integers, $F^1 = 1$, so that G' is the group ring of F. The coefficients of the homotopy chains belong to the local groups $\{G_x'^R\}$ (these are not rings), and the ring of uniform operators is likewise G' where left translations are used. This distinction between the two usages of the group ring of F is necessary for a comprehension of the subject of homotopy chains.

II. The combinatorial theory

9. Chains with local coefficients

Let $\{G_x\}$ be a system of local abelian groups in a space R which is decomposed into a cell complex[2] K. A q-cell of K is denoted by σ^q, incidence by $\sigma < \sigma'$, and incidence numbers by $[\sigma^{q-1}:\sigma^q]$. We suppose as usual that

$$\sum\nolimits_{\sigma^{q-1}} [\sigma^{q-2}:\sigma^{q-1}][\sigma^{q-1}:\sigma^q] = 0. \tag{9.1}$$

In each cell σ we choose a representative point $x(\sigma)$ and abbreviate the symbol $G_{x(\sigma)}$ by G_σ. A q-chain of K is a function[3] f attaching to each oriented q-cell σ an element $f(\sigma) \in G_\sigma$ with the property $f(-\sigma) = -f(\sigma)$. Chains are added by adding functional values. They then form a group isomorphic to the direct sum of the groups G_σ for all q-cells σ.

If $\sigma' < \sigma$, we may choose a path in the closure of σ joining $x(\sigma)$ to $x(\sigma')$ and obtain therefrom an isomorphism $G_\sigma \to G_{\sigma'}$ which is denoted by $h_{\sigma'\sigma}$. In order that $h_{\sigma'\sigma}$ shall be independent of the path, we postulate that the closure of each cell is simply connected. A second consequence of this and the transitivity condition is

$$h_{\sigma''\sigma'}h_{\sigma'\sigma} = h_{\sigma''\sigma} \tag{9.2}$$

[2] For the sake of simplicity we suppose K is finite and closed. The extension of the subsequent results to relative complexes, open complexes, and to the finite and infinite chains of locally-finite complexes will be obvious.

[3] We shall abide by the functional notation throughout. This will prove to be as convenient as the classical linear form notation. We abandon the latter since it has algebraic implications more prejudicial than suggestive in the present discussion.

By means of the isomorphisms h, we can define the boundary ∂f and coboundary δf of a q-chain f:

$$\begin{aligned} \partial f(\sigma^{q-1}) &= \textstyle\sum_{\sigma^q} [\sigma^{q-1}:\sigma^q] h_{\sigma^{q-1}\sigma^q}(f(\sigma^q)), \\ \delta f(\sigma^{q+1}) &= \textstyle\sum_{\sigma^q} [\sigma^q:\sigma^{q+1}] h^{-1}_{\sigma^q\sigma^{q+1}}(f(\sigma^q)). \end{aligned} \tag{9.3}$$

The sums extend over the q-cells which (a) have σ^{q-1} as a face, (b) are faces of σ^{q+1}. These new functions are $q - 1$ and $(q + 1)$-chains respectively. It follows from (9.1) and (9.2) that $\partial\partial f = 0$, $\delta\delta f = 0$. Therefore cycles, bounding cycles, homology and cohomology groups can be defined as usual.

A special convention for 0-cycles is necessary. We shall agree that any 0-chain is a 0-cycle. Note that the Kronecker index (i.e. the sum of the coefficients of a 0-chain) has no meaning unless the system $\{G_x\}$ is simple.[4]

Finally it is to be remarked that we obtain a completely isomorphic situation if we choose new representatives $y(\sigma)$ in each σ. The isomorphism is established by means of a path $x(\sigma)$ to $y(\sigma)$ in σ for each σ.

10. Automorphisms of chains

Let A be a group of *uniform* automorphisms or a ring of *uniform* operators of $\{G_x\}$. For any chain f and $a \in A$, define $(af)(\sigma) = af(\sigma)$ for each σ. Then af is a chain, and A appears as a group of automorphisms or a ring of operators of the group of q-chains for each q.

Since the operations of A commute with translations of the G_x along paths, it follows that the operations of A on the chains commute with ∂ and δ. Therefore A appears as an automorphism group or ring of operators of the groups of cycles, cocycles, boundaries, coboundaries, and consequently of the homology and cohomology groups.

11. Multiplication of chains

It should be noted that a cycle with local coefficients is locally a cycle in the ordinary sense; for, in any simply connected open set U (e.g. the star of a vertex), isomorphisms of the local groups can be set up with a fixed group (using paths in U) in such a way as to transform each chain with local coefficients into an ordinary chain mod $(K - U)$ so that the boundary relations are preserved. It follows that any operation on ordinary chains which is a sum of local operations can be carried over to chains with local coefficients. We have seen that this is true of the boundary operator. We shall see that this is also true of products of cocycles, products of cycles and cocycles, and intersections of cycles. The linking number like the Kronecker index is not of this category.

[4] Classical homology with a single coefficient group G is isomorphic to homology with coefficients in the simple system of local groups determined by G.

In a comprehensive paper of Whitney on products [17], it is proved that, corresponding to cells σ_i^p, σ_j^q, σ_k^{p+q} in K, there is an integer ${}^{pq}\Gamma_k^{ij}$ such that

(Γ_1) if σ_i^p, σ_j^q are not both faces of σ_k^{p+q}, then ${}^{pq}\Gamma_k^{ij} = 0$,
(Γ_2) for all p, q, i, j, k,

$$\sum_m [\sigma_m^{p+q}:\sigma_k^{p+q+1}]^{pq}\Gamma_m^{ij} = \sum_m [\sigma_i^p:\sigma_m^{p+1}]^{p+1,q}\Gamma_k^{mj} + (-1)^p\sum_m [\sigma_j^q:\sigma_m^{q+1}]^{p,q+1}\Gamma_k^{im},$$

(Γ_3) for all q and j, $\sum_i {}^{0q}\Gamma_j^{ij} = 1$.
Although the last two conditions appear not to be local in nature, they are so by virtue of the first.

Let $\{G_x\}$ be a system of local rings with units. If f^p and f^q are p and q-chains respectively with coefficients in $\{G_x\}$, we define their *cup* product to be the $(p + q)$-chain

$$f^p \cup f^q(\sigma_k^{p+q}) = \sum_{ij} {}^{pq}\Gamma_k^{ij}[h_{k,i}^{p+q,p}(f^p(\sigma_i^p))][h_{k,j}^{p+q,q}(f^q(\sigma_j^q))]. \tag{11.1}$$

Here we heve abbreviated the isomorphism of the local group of σ_i^p on that of σ_k^{p+q} by $h_{k,i}^{p+q,p}$. The sum extends over the faces σ_i^p, σ_j^q of σ_k^{p+q}. The relations

(P_1) $f^p \cup f^q$ is zero on any σ^{p+q} which has not both a face in f^p and a face in f^q,
(P_2) $\delta(f^p \cup f^q) = \delta f^p \cup f^q + (-1)^p f^p \cup \delta f^q$,
(P_3) $I \cup f^q = f^q$,

follow from the relations (Γ), the transitivity of the h's (9.2), and the preservation of the products in G_x under an h. In (P_3), I is the 0-cocycle which attaches to each vertex V the unit of G_V. Since h is a ring isomorphism, it preserves the unit; therefore $\delta I = 0$. It follows just as in Whitney [17] that a product is definable for the cohomology classes with the usual properties. The associative law and the commutation rule for the special products in a simplicial complex are given local proofs. Once invariance under subdivision has been established (see §16), the same laws will hold in a general complex.

For the *cap* product, we shall suppose that $\{G_x\}$, $\{H_x\}$ are paired with respect to $\{L_x\}$. Given a q-cochain f^q (coef. $\{G_x\}$) and a $(p + q)$-chain g^{p+q} (coef. $\{H_x\}$), we define their cap product to be the p-chain

$$f^q \cap g^{p+q}(\sigma_m^p) = \sum_{jk} {}^{pq}\Gamma_k^{mj}[h_{m,j}^{p,q}(f^q(\sigma_j^q))][h_{m,k}^{p,p+q}(g^{p+q}(\sigma_k^{p+q}))] \tag{11.2}$$

with coefficients in $\{L_x\}$. The sum extends over those j, k for which σ_m^p, σ_j^q are faces of σ_k^{p+q}. Just as before, we obtain

(Q_1) $f^q \cap g^{p+q}$ is zero on any σ^p which with no σ^q of f^q is a face of a σ^{p+q} of g^{p+q},

(Q_2) $\partial(f^q \cap g^{p+q}) = (-1)^p\delta f^q \cap g^{p+q} + f^q \cap \partial g^{p+q}$.

In order to obtain an analogue of (Γ_3), we shall take the special case where $\{L_x\} = \{H_x\}$ and $\{G_x\}$ is a system of operator rings for $\{H_x\}$ (see §5). Then the 0-cocycle I is defined, and we have (Q_3) $I \cap g^p = g^p$.

Under the same assumption, it can be shown that

$$(f^q \cup f^r) \cap g^{p+q+r} \sim f^q \cap (f^r \cap g^{p+q+r}),$$

for cocycles f^q, f^r and a cycle g^{p+q+r}.

The uniqueness of the products in the following sense can be proved. Suppose ${}^{pq}\bar{\Gamma}_k^{ij}$ is another set of Γ's for which the relations (Γ) hold. Corresponding to the two sets of Γ's is the Whitney operation $\wedge$ (Theorem 8, [17]). Let ${}^{pq}\Delta_k^{ij}$ be the coefficient of σ_k^{p+1} in the product $\sigma_i^q \wedge \sigma_j^{p+q}$. Define $f^q \wedge g^{p+q}$ with equations analogous to (11.2) using Δ in place of Γ. Then the Whitney relations (R) (loc. cit.) with chains in place of cells can be proved to hold. It follows that the two sets of Γ's determine the same products among the homology and cohomology classes.

12. Duality

Let L be the group of real numbers mod 1, and let $\{L_x\}$ be the corresponding simple system in K. Let $\{G_x\}$ and $\{H_x\}$ be character systems of one another with respect to $\{L_x\}$. Since $\{L_x\}$ is simple, a 0-cycle f^0 with coefficients in $\{L_x\}$ may be regarded as a 0-cycle with coefficients in L. It possesses therefore a Kronecker index (= the sum of its coefficients) which we denote by (f^0). As usual $(f^0) = 0$ is equivalent to $f^0 \sim 0$. The scalar product of a q-cochain f^q coef. $\{G_x\}$ and a q-chain g^q coef. $\{H_x\}$ is defined to be the Kronecker index of their cap product:

$$f^q \cdot g^q = (f^q \cap g^q) \qquad \text{in } L. \tag{12.1}$$

If f^q and g^q are zero except on a single σ, then, by Γ_3, $f^q \cdot g^q$ is the product $f^q(\sigma)g^q(\sigma)$. Therefore, due to the linearity, the scalar product of arbitrary f^q, g^q is the sum of products of corresponding coefficients. It follows that the groups of q-cochains and q-chains are character groups of one another.

For any q-cochain f^q and $(q + 1)$-chain g^{q+1} we obtain from Q_2 that

$$\delta f^q \cdot g^{p+1} = f^q \cdot \partial g^{q+1}. \tag{12.2}$$

It follows now in the usual way (see Whitney [17]) that the q^{th} cohomology group coef. $\{G_x\}$ and the q^{th} homology group coef. $\{H_x\}$ are character groups with the scalar product as the multiplication.

13. Intersection in an orientable manifold

Let K be an orientable simplicial n-manifold and let K^* be its dual. Denote by $\mathfrak{D}\sigma$ the cell of K^* dual to the oriented simplex σ of K relative to a fixed choice of the fundamental n-cycle Z^n with integer coefficients. Let $\{G_x\}$ be a system of local groups to be used as coefficients in both K and K^*. Let the coefficient groups of σ and $\mathfrak{D}\sigma$ be the group G_x where x is their common point. Then for any chain f of K (cochain f^* of K^*), the equation

$$f^*(\mathfrak{D}\sigma) = f(\sigma) \tag{13.1}$$

defines its *dual* cochain (chain) of K^* (K) of the complementary dimension. This isomorphism between the two groups of chains has, as usual, the property

$$(13.2) \qquad (\partial f)^* = (-1)^q \delta f^*, \qquad q = \text{dimension of } f.$$

It follows that *the* q^{th} *homology group coef.* $\{G_x\}$ *and the* $(n - q)^{\text{th}}$ *cohomology group coef.* $\{G_x\}$ *are isomorphic.*

In case $\{G_x\}$ is a system of local rings, we have as in §11 a multiplication defined for the cochains of K^*. The isomorphisms just established between the chains of K and cochains of K^* enable us to carry over the product in K^* into an intersection in K. We define the *intersection* of two chains f_1, f_2 of K to be the dual of the cup product of their duals:

$$(13.3) \qquad f_1 \circ f_2 = (f_1^* \cup f_2^*)^*.$$

(Compare Whitney [17], p. 422, formula (19.9)). It follows that *a system of local rings in an orientable manifold determines an intersection ring of cycles isomorphic to the ring of cocycles (same coefficients) under the operation of dual.*

As is well known, the ring of integers is a ring of operators for any group G which commute with any automorphism of G. The ring of integers is therefore a uniform operator ring (§5) for any $\{G_x\}$. The equation (11.1) may be interpreted as defining the cup product of a cochain f^p with (simple) integer coefficients and a cochain f^q with local coefficients $\{G_x\}$. The relations (P) still hold, and these together with the associative law lead to the conclusion that *the cohomology ring with simple integer coefficients constitutes a ring of operators for the cohomology groups coef.* $\{G_x\}$.

The dual of this last result is that *the intersection ring of an orientable manifold with simple integer coefficients is a ring of operators for the homology groups coef.* $\{G_x\}$.

14. Intersection in a non-orientable manifold

In a non-orientable manifold K there is no n-cycle with simple integer coefficients. One cannot therefore determine the orientation of $\mathfrak{D}\sigma$ uniformly over K so that (13.2) holds. A customary device is to use integers mod 2 as coefficients so as to restore the basic n-cycle and escape orientation difficulties. The resulting duality and intersection theory is a bit weak due to the inadequacy of the coefficients. A more ingenious device has been used by de Rham [13]. We shall see that a suitable use of local coefficients permits a full development of De Rham's notion and leads to a complete and satisfying duality and intersection theory in a non-orientable manifold.

Since K is non-orientable, the elements of its fundamental group F divide into two classes according as they do or do not preserve orientation. Those which do form an invariant subgroup F^1 of index 2. Let T be the group of integers. For each integer $t \in T$ and $\alpha \in F$, let $\alpha(t)$ be $+t$ or $-t$ according as α is or is not in F^1. In this way F is a group of automorphisms of T. Let $\{T'_x\}$ be the corresponding system of local groups given by Theorem 1. We shall say that chains with coefficients in $\{T'_x\}$ have *twisted integer coefficients.*

Let G be an abelian group and let F be represented as a group of automorphisms of G. Let $\bar{G}$ be the direct sum of two copies of G (i.e. the group of pairs (g_1, g_2)). Identify G with the subgroup of elements of the form $(g, 0)$, and call G the real part of $\bar{G}$. The subgroup G' of elements of the form $(0, g)$ we call the imaginary part of $\bar{G}$. The product of (g_1, g_2) with the complex number $a + ib$ (a, b are integers) is defined by

$$(14.1) \qquad (a + ib)(g_1, g_2) = (ag_1 - bg_2, ag_2 + bg_1).$$

It follows immediately that the complex integers form a ring of operators for the group $\bar{G}$, and that each element of $\bar{G}$ can be written uniquely in the form $g_1 + ig_2$ (g_1, g_2 real). If we set

$$(14.2) \qquad \alpha(g_1 + ig_2) = \begin{cases} \alpha(g_1) + i\alpha(g_2) & \text{if } \alpha \in F^1, \\ \alpha(g_1) - i\alpha(g_2) & \text{if } \alpha \notin F^1, \end{cases}$$

the automorphisms of F in G are extended to $\bar{G}$. For any complex integer $a + ib$, define $\alpha(a + ib) = a \pm ib$ according as α is or is not in F^1. Let $\bar{T}$ be the ring of complex integers, and let $\{\bar{T}_x\}$ be the system of local rings corresponding to $\bar{T}$ and these automorphisms. It follows that $\{\bar{T}_x\}$ is a system of operator rings for the system $\{\bar{G}_x\}$ corresponding to $\bar{G}$ and the automorphisms (14.2) (see §5). We refer to $\{\bar{G}_x\}$ as the *complex extension of* $\{G_x\}$.

If G is a ring, and the elements of F are ring automorphisms, we define a product in $\bar{G}$ in the usual way: $(g_1, g_2)(g_1', g_2') = (g_1g_1' - g_2g_2', g_2g_1' + g_1g_2')$. The equations (14.2) define ring automorphisms of $\bar{G}$. Furthermore the elements of $\bar{T}$ associate and commute with the multiplications in $\bar{G}$. Thus $\{\bar{G}_x\}$ is a system of local rings with $\{\bar{T}_x\}$ as a system of operator rings.

We shall use $\{\bar{G}_x\}$ and $\{\bar{T}_x\}$ as coefficients for chains of K and cochains of K^*. The group of n-cycles of K coef. $\{\bar{T}_x\}$ is infinite cyclic. A generator is constructed as follows. Choose an oriented n-cell σ and let $Z^n(\sigma) = \pm i$ in $\bar{T}_\sigma$ (because of $\alpha(i) = \pm i$, the sign of i has only a local significance). If $|\sigma'|$ is any other n-cell, choose a path α of n-cells from σ to $|\sigma'|$ (successive cells having an $(n - 1)$-face in common). The path α determines an orientation σ' of $|\sigma'|$ concordant with that of σ. Now define $Z^n(\sigma') = \alpha(Z^n(\sigma))$. In words: the orientation and coefficient of σ' are determined by translating along a path α both the orientation and coefficient of σ. Translating along a second path β will either produce the same orientation and the same coefficient or reverse the sign of both according as $\alpha\beta^{-1}$ is or is not in F^1. Thus the chain Z^n is independent of the paths chosen in its construction. It is a cycle since, in any simply connected domain, it is a cycle. The usual argument shows that any n-cycle of K coef. $\{\bar{T}_x\}$ is an integral multiple of Z^n. Thus the fundamental n-cycle of K has pure imaginary coefficients (or equally well, twisted integer coefficients).

If ϵ is an oriented simply-connected neighborhood in K, $Z^n(\epsilon)$ will be the coefficient $Z^n(\sigma)$ of an n-cell σ in ϵ oriented concordantly with ϵ. Corresponding to a q-cell $\sigma \in K$ and an oriented neighborhood ϵ of σ, there is in the usual way a unique orientation of the dual cell $\mathfrak{D}_\epsilon\sigma$ in K^*. If f is any q-chain of K coef. $\{\bar{G}_x\}$, we define its dual to be the $(n - q)$-cochain f^* in K^* coef. $\{\bar{G}_x\}$ defined by

$$f^*(\mathfrak{D}_\epsilon\sigma) = -f(\sigma)Z^n(\epsilon). \tag{14.3}$$

It is to be understood that $f(\sigma)$ is in $\bar{G}_x$ and $Z^n(\epsilon)$ is in $\bar{T}_x$ where x is the point common to σ and $\mathfrak{D}_\epsilon\sigma$. Since reversing the orientation of ϵ changes the sign of both sides of (14.3), f^* is independent of the choices of the ϵ's. The dual of a chain f^* in K^* is given by

$$f^{**}(\sigma) = f^*(\mathfrak{D}_\epsilon\sigma)Z^n(\epsilon). \tag{14.4}$$

Since $Z^n(\epsilon)Z^n(\epsilon) = -1$, we have that *any chain is the dual of its dual.* The dual of Z^n is the unit 0-cocycle I. The dual of a real (imaginary) chain is imaginary (real). The formula (13.2) now follows for the dual in a non-orientable manifold; for it is a statement of local properties, and (14.3), (14.4) differ from (13.1) locally by a constant factor. As in §13, it follows that *the* q^{th} *homology group* coef. $\{\bar{G}_x\}$ *and the* $(n - q)^{\text{th}}$ *cohomology group* coef. $\{\bar{G}_x\}$ *are isomorphic.* Using (13.3) to define the intersection, we obtain a homology ring isomorphic to the cohomology ring whenever the coef. $\{\bar{G}_x\}$ form a system of local rings. In any case, the homology ring coef. $\{\bar{T}_x\}$ forms a ring of operators for the homology groups coef. $\{\bar{G}_x\}$.

Since $\bar{G}$ is the direct sum of its real and imaginary part, any chain is uniquely a sum of a real chain and an imaginary chain. The operations ∂ and δ preserve the property of being real or imaginary. Therefore the homology and cohomology groups decompose into direct sums of their real and imaginary parts. Since passing to the dual interchanges real and imaginary, we obtain the following results. *The* q^{th} *homology group* coef. $\{G_x\}$ (coef. $\{G'_x\}$) *is isomorphic to the* $(n - q)^{\text{th}}$ *cohomology group* coef. $\{G'_x\}$ (coef. $\{G_x\}$). *The homology classes* coef. $\{G'_x\}$ *(i.e. the imaginary ones) form a ring isomorphic to the cohomology ring* coef. $\{G_x\}$. *The intersection of two real cycles is imaginary. The intersection of a real and an imaginary cycle is real.*

It is to be noted that the results of this section apply to an orientable manifold. The absence of orientation reversing paths in no wise invalidates the constructions. We have in this way a single theory including both types of manifolds.

The classical approach to intersection is to define directly the intersection of a p-chain f of K with a q-chain g of K^* to be a chain of the subdivision of K. We may do this here as follows. If σ is a p-simplex, σ' an $(n - q)$-face of σ and ϵ an oriented neighborhood of σ, define $\sigma\circ\mathfrak{D}_\epsilon\sigma'$ relative to ϵ in the usual way. Then define the intersection chain $f\circ g$ to have on $\sigma\circ\mathfrak{D}_\epsilon\sigma'$ the value $-f(\sigma)g(\mathfrak{D}_\epsilon\sigma')Z^n(\epsilon)$.

15. The Poincaré duality

If we assume that $\{G_x\}$, $\{H_x\}$ are character systems of one another with respect to the simple system $\{L_x\}$ of mod 1 groups, we may combine the results of §12 with those of §13 and §14 to obtain: *The* q^{th} *homology group of the manifold* K

coef. $\{G_x\}$ (coef. $\{G'_x\}$) *and the* $(n - q)^{\text{th}}$ *homology group* coef. $\{H'_x\}$ (coef. $\{H_x\}$) *are character groups of one another; the multiplication is determined by the scalar product* $(f^* \cap g)$ *where* f *is a* q*-chain of* K *and* g *is an* $(n - q)$*-chain of* K^*.

The results simplify in the orientable case if we note that $\{G_x\}$ and $\{G'_x\}$ are isomorphic, as also are $\{H_x\}$ and $\{G'_x\}$. In the non-orientable case, the results simplify if we note that $\{H'_x\}$ ($\{H_x\}$) is the character system of $\{G_x\}$ ($\{G'_x\}$) with respect to the system $\{L'_x\}$ of twisted mod 1 groups associated with the simple system $\{L_x\}$.

Before leaving the subject of duality, let us observe that the notion of local coefficients has nothing to add to the duality theorem of Alexander. A vital step in the argument of Alexander is the following: *A cycle in the closed set* R *in the n-sphere* S^n *is a cycle in* S^n *and is therefore the boundary of a chain in* S^n. Such a statement is valid only if the system of local coefficients used in R is part of a system in S^n; as S^n is simply-connected $(n > 1)$, the system of local coefficients must be simple.

16. Chain mapping and subdivision

A chain transformation is a homomorphism of the groups of chains of one complex on those of another which commutes with the boundary or coboundary operator or interchanges them (as in the case of the dual). This definition has meaning of course when local coefficients are used. All that we need to determine here is that chain transformation "S" with local coefficients exist in the usual circumstances.

Let a cell mapping $K' \to K$ be given which preserves the relation of incidence. Let $\{G_x\}$ be a system of local coefficients in K, and let $\{G'_y\}$ be the induced system in K' (see §7). If $\sigma' \to \sigma$, there is an attached isomorphism $G'_{\sigma'} \to G_\sigma$. A chain of K' which is zero except on a single σ' we call an *elementary chain*. Let f' be an elementary chain and $f'(\sigma') \neq 0$. If the image σ of σ' has a lower dimension, we define the image f of f' to be zero. If σ, σ' have the same dimension, the image f of f' is zero except on σ, and $f(\sigma)$ is the image of $f'(\sigma')$ under the isomorphism $G'_{\sigma'} \to G_\sigma$. An arbitrary chain of K' is uniquely a sum of elementary chains. Its image is defined as the sum of the images of its elementary parts. The resulting chain mapping we denote by τ. The inverse cochain mapping τ' attaches to an elementary chain of K the sum of the elementary chains of K' mapped into it by τ; we then extend τ' preserving linearity. That τ (τ') commutes with ∂ (δ) is proved by first establishing it in the usual way for elementary chains (of course (7.1) is used), and then applying the linearity of ∂ (δ).

It is necessary to use in K' the induced system $\{G'_y\}$ in order that τ, τ' shall exist in all dimensions. This is seen as follows. Let V' be a vertex of K' and V its image. Assuming τ, τ' defined for the elementary chains of V, V^1, we arrive at an isomorphism $G'_{V'} \to G_V$. Therefore $\{G'_y\}$ is the system induced by $\{G_x\}$ over the 0-dimensional part of K'. Suppose this is known for the q-dimensional part of K'. Any closed $(q + 1)$-cell is simply-connected, there is therefore just one system of local groups defined over it which agrees with a given system on its boundary, and that one is simple. We conclude that the given system and the induced system agree on each closed $(q + 1)$-cell, and finally over the whole of K'.

If K' is a subdivision of the simplicial complex K, we then have two systems in K': the given system $\{G_x\}$ for K, and the system $\{G'_x\}$ induced by the map $K' \to K$ defined by mapping each vertex of K' into a vertex of the simplex of K containing it. The two systems are isomorphic. The isomorphism is set up by using the line segments which join each point to its image point. The proof of the invariance under subdivision of the groups of K and their multiplications may now be completed in the standard way (see for example [17]).

17. Continuous cycles

Let $\{G_x\}$ be a system of local groups in a space R. A *continuous chain* in R is a collection composed of a complex K, a continuous map ϕ of K in R, and a chain Z in K with local coefficients in the system $\{G'_y\}$ induced by ϕ and $\{G_x\}$. If Z is a cycle, the collection (K, ϕ, Z) is called a *continuous cycle*. The boundary of (K, ϕ, Z) is $(K, \phi, \partial Z)$. Two continuous chains (K_i, ϕ_i, Z_i) $(i = 1, 2)$ are *added* by forming the abstract sum $K_1 + K_2$, defining $\phi = \phi_i$ on K_i, and adding Z_1 to Z_2. Two continuous cycles are *homologous* if there exists a chain (K, ϕ, Z) such that $K \supset K_1$ and K_2, $\phi = \phi_i$ on K_i, and $\partial Z = Z_1 - Z_2$. The cycles of a fixed dimension divide up into homology classes. Two classes are *added* by adding representative elements. In this way we define the homology groups of R based on continuous cycles with local coefficients $\{G_x\}$. That they are topological invariants of R and the system $\{G_x\}$ is an immediate consequence of the definition.

If R is the space of a complex, these groups of R are isomorphic to those of K with the same local coefficients. The identity map ϕ_1 of K attaches to a chain Z of K the continuous chain (K, ϕ_1, Z) of R. This chain mapping commutes with ∂, and therefore induces homomorphisms of the groups of K into those of R. That these are isomorphisms follows from the lemma: If (K', ϕ, Z') is a chain with boundary of the form (K, ϕ_1, Z), then there is a chain Z_1 of K such that $\partial Z_1 = Z$, and the difference $(K', \phi, Z') + (K, \phi_1, - Z_1)$ bounds a continuous chain in R. The lemma is proved in the usual way by using the simplicial approximation theorem to construct a map of the product complex $K' \times I$ $(I = (0, 1))$ into R. The needed chain is found in $K' \times I$ with local coefficients in the induced system.

18. Čech cycles

The only difficulty in the way of extending local coefficients to Čech cycles is that of constructing a system of local groups in the nerve K of a finite open covering when such a system is given in R. It is clear that the former must be chosen so as to induce the given system in R under the natural map $R \to K$. If R is sufficiently complicated, it is possible to construct in R a local system which is not induced by a local system in any nerve.[5] Therefore we are forced to restrict ourselves to a system $\{G_x\}$ induced in R by a system $\{G_x^0\}$ in a fixed nerve K^0. We then admit only those coverings which are refinements of K^0, and we use in them the local groups induced by their natural projections into K^0. With this modification, the definitions of the Čech homology and cohomology groups and their multiplications proceed as before.

If R is the space of a complex K, a local system in R is one in $K = K^0$. Using invariance under subdivision, one proves in the customary way (see [15], §9) that the groups of K and the Čech groups of R are isomorphic, and the isomorphisms preserve the multiplications. We are thus led to a proof that the homology theory (coef. $\{G_x\}$) of a complex K is a topological invariant of the space K with the local groups $\{G_x\}$.

19. Überdeckung

In a complex K choose a reference point o (preferably a vertex), and for each cell σ let α_σ be a path in K from o to a point $x(\sigma)$ in σ. If $\sigma' < \sigma$, let $\alpha_{\sigma\sigma'}$ be a path in the closure of σ from $x(\sigma)$ to $x(\sigma')$. The closed path $\alpha_\sigma\alpha_{\sigma\sigma'}\alpha_{\sigma'}^{-1}$ is abbreviated by $\gamma_{\sigma\sigma'}$. As elements of the fundamental group F of K (origin o), the γ's have the property

$$\gamma_{\sigma\sigma'}\gamma_{\sigma'\sigma''} = \gamma_{\sigma\sigma''} \qquad \text{for } \sigma'' < \sigma' < \sigma. \tag{19.1}$$

By means of the α's, we map isomorphically the chains of K with local coef. $\{G_x\}$ into ordinary chains with coefficients in G_o as follows. The transform $\bar{f}$ of the chain f with local coefficients is defined by $\bar{f}(\sigma) = \alpha_\sigma^{-1}(f(\sigma))$. By means of this isomorphism, we define operators $\bar{\partial}$, $\bar{\delta}$ for chains $\bar{f}$ by: $\bar{\partial}\bar{f} = \overline{\partial f}$, $\bar{\delta}\bar{f} = \overline{\delta f}$. From (9.3) we obtain

$$\begin{aligned} \bar{\partial}\bar{f}(\sigma') &= \textstyle\sum_\sigma[\sigma':\sigma]\gamma_{\sigma\sigma'}(\bar{f}(\sigma)), & \sigma' &< \sigma, \\ \bar{\delta}\bar{f}(\sigma'') &= \textstyle\sum_\sigma[\sigma:\sigma'']\gamma_{\sigma''\sigma}^{-1}(\bar{f}(\sigma)), & \sigma &< \sigma''. \end{aligned} \tag{19.2}$$

Thus the system of ordinary chains (coef. G_o) with the special operators $\bar{\partial}$, $\bar{\delta}$ are isomorphic to the system of chains with local coef. $\{G_x\}$ with the ordinary operators ∂, δ. They determine therefore isomorphic homology groups. It is a corollary that the homology groups determined by $\bar{\partial}$, $\bar{\delta}$ are independent of the choices of the α's.

The system $\bar{f}$, $\bar{\partial}$, $\bar{\delta}$ just described is called an *Überdeckung* of K by Reidemeister [10; 11]. An advantage of this approach is that it lends itself more readily to a computation of the homology groups. One may attempt to simplify the incidence matrices $[\sigma':\sigma]\gamma_{\sigma\sigma'}$, with elements in the group ring of F, by the usual methods of transforming bases and consolidation (see W. Franz [2]).

[5] This is the case if R is not locally simply-connected, and if $\{G_x\}$ is the system $\{G_x^L\}$ of local group rings of §8.

20. Zero and 1-dimensional groups

If the fundamental group F of K and the operations of F in G_o are given, then one may compute the 0 and 1-dimensional homology groups without further knowledge of K. Choose a basis $\alpha_1, \cdots, \alpha_h$ of F and a basis $r_1, \cdots, r_s$ for the relations in F (each r is a product of α's representing the unit). Construct a 2-complex K' consisting of one vertex o', one edge for each α_i (likewise denoted by α_i) with both end-points at o', and, for each r_i, a 2-cell E_i whose boundary is the product r_i of the α's. Clearly F is also the fundamental group of K'. The operations of F in G_o determine local coefficients in K' leading to 0 and 1-dimensional homology and cohomology groups which we shall prove isomorphic to those of K. By the duality of §12, it suffices to prove this for the homology groups.

Define a map ϕ of K' into K so that $\phi(o') = o$, $\phi(\alpha_i)$ represents α_i, and ϕ is continuous. It is readily seen that a map ψ of a complex K'' in K is homotopic to a map ψ' which, on the 1-dimensional part K_1'' of K'', can be expressed as a product $\phi\psi''$ of ϕ and a map ψ'' of K_1'' in K' which maps each vertex into o' and each edge into a product of α's. Thus every continuous 0 and 1-cycle is homologous to one of the form (K', ϕ, Z) (see §17). It is a further consequence that the 0-cycle Z bounds in K' if (K', ϕ, Z) bounds in K. This shows that the 0^{th} homology groups of K and K' are isomorphic. To prove the same for the 1-dimensional groups, we must show that a homology relation in K of the form $\partial(E, \psi', f(E) = g) = (K', \phi, Z)$, where E is a 2-cell and $\psi'(\partial E)$ is a product r of the α's, is a consequence of the relations in K'. Let E be regarded as a hemisphere of a 2-sphere S^2 and let E' be the other hemisphere. The map ψ'' of the equator in K' extends to a continuous map ψ'' of E' in K'; for the product r is expressible in terms of the r_i. The map ψ' of E and $\phi\psi''$ of E' define a map ψ' of S^2 in K, and thereby a 2-cycle $(S^2, \psi', f(E) = f(-E) = g)$. Thus $\partial(E', \psi', f(E') = g) = (K', \phi, Z)$, and ψ' factors into $\phi\psi''$.

Using the above results we may describe the 0-dimensional groups quite easily. Let G_o' be the subgroup of elements of G_o which are fixed under every automorphism $\alpha \in F$. *The* 0^{th} *cohomology group of* K *is isomorphic to the group* G_o'. Let G_o'' be the subgroup of elements of G_o expressible in the form $\sum_i(\alpha_i(g_i) - g_i)$ where $\alpha_i \in F$, $g_i \in G_o$. *The* 0^{th} *homology group of* K *is isomorphic to the difference group* $G_o - G_o''$.

It is worth noting that a continuous image of an n-sphere $(n > 1)$ in K determines a group of spherical n-cycles for any local coefficients; for the sphere is simply connected. However these cycles may or may not bound according to the structure of the system of local groups. Consider, as an example, the projective plane P^2 and the double covering of it by the 2-sphere S^2. With twisted integer coefficients (§14), the 2-cycles on P^2 form an infinite cyclic group and are nonbounding. The even multiples of the generator are images of the 2-cycles on S^2 with integer coefficients. Yet with simple integer coefficients in P^2, the mage of every 2-cycle on S^2 is bounding (algebraically zero).

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