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Periodicity of Branched Cyclic Covers of Manifolds with Open Book Decomposition

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Introduction

In this paper we study branched cyclic covers of manifolds with open book decomposition. Such a decomposition splits a manifold M in a binding K and a complement which fibres over S^1 with as fibre the page F such that the fibration is trivial in a neighbourhood of K. For example, if M is a sphere, we get a fibered knot K. Open book decompositions also arise in singularity theory: if X is a smoothing of the germ of an isolated complex singularity, i.e. a one-parameter deformation for which the general fibre is smooth, then the link M of the singular point in X has an open book decomposition with binding the link of the smoothed germ, while the fibration is the same as the Milnor fibration. The cyclic k-fold cover of M, branched along K, is induced from the k-fold cover of S^1 over S^1. We study these k-fold covers in terms of the monodromy $h: F \to F$ of the fibration and the topology of K.

There is a close connection between the topology of M and F if M has odd dimension $2n+1$ and F has the homotopy type of a n-dimensional CW-complex; the homology of F is then determined by M and vice versa except in the middle dimension, where we have the variation exact sequence:

$$0 \to H_{n+1}(M) \to H_n(F, \partial F) \xrightarrow{\operatorname{Var}(h)} H_n(F) \to H_n(M) \to 0 . \tag{1}$$

This sequence is the basis of our periodicity results. The map $\operatorname{Var}(h)$ is defined by assocating to a representative of a relative homology class z the class of $h(z) - z$. We first compare the homology of the branched cyclic covers M^k of k for different values of k; we suppose that the eigenvalues of the monodromy are d^{th} roots of unity for some d, which is always the case for smoothings of singularities. We find that M^k and M^{k+d} have the same rational homology. For periodicity in integral homology it is a necessary and sufficient condition that $\operatorname{Var}(h^d) = 0$.

In [2] Durfee and Kauffman prove homology periodicity for simple fibered knots [open book decompositions of S^{2n+1} with $(n-2)$-connected binding and $(n-1)$-connected page] under the assumption that the monodromy has finite order d and that the binding K is a rational homology sphere. For simple fibered knots this condition is equivalent to $\mathrm{Var}(h^d)=0$. They prove also that M^k and M^{k+d} are homeomorphic for odd $n \neq 1, 3, 7$ and they have results on the diffeomorphism type.

In our situation we can also use the classification of highly connected manifolds. To be able to apply such theorems we assume that M is $(n-1)$-connected and bounds a parallelizable manifold. For odd n we can express the invariants determining the homeomorphism type of M^k by means of the variation exact sequence; the diffeomorphism type has a period depending also on the signature of M^d and the group of homotopy spheres in an appropriate dimension. For n even we are able to prove homeomorphism periodicity with period $2d$ and diffeomorphism periodicity with period $4d$.

We conclude this paper with some examples of smoothings of singularities for which the relevant invariants are reasonably easy to calculate: smoothings of the A_2-singularity for n odd and Brieskorn complete intersections [4].

Unless otherwise stated all homology groups in this paper have integer coefficients.

1. Open Book Decompositions

In this section we discuss the exact sequence, describing the homology of open book decompositions, which will be the main tool in our study of periodicity of branched cyclic covers. A good reference for open book decompositions is a paper of Quinn [9].

Definition. An open book decomposition on a smooth m-dimensional manifold M is an isomorphism $M \cong E \cup_r (K \times D)$, where E is a m-manifold with boundary fibering over the circle and r is an isomorphism $r: \partial E \to K \times S^1$, $S^1 = \partial D$, such that the fibration corresponds to the projection on the second factor. The fibre F is called the page and $\partial F \cong K$ the binding.

All odd dimensional manifolds and many even dimensional manifolds have open book decompositions; each fibered knot gives one of the m-sphere. Other examples are given by links of smoothings of isolated singularities.

Definition. Let (X_0, p) be the germ of a complex-analytic space of dimension n with an isolated singularity at p. A smoothing of (X_0, p) is the germ of a flat morphism $f: (X, p) \to (\mathbb{C}, 0)$, together with an isomorphism $f^{-1}(0) \cong X_0$, such that $f^{-1}(t)$ is nonsingular for $t \neq 0$.

If $f: \mathbb{C}^{n+1} \to \mathbb{C}$ has an isolated critical point at the origin, then f is a smoothing of the hypersurface $\{f=0\}$. Similar to this case there is a Milnor fibration for general smoothings.

There are two ways to describe the Milnor fibration [3]. Choose a representative of the germ X, also called X, embedded in $\mathbb{C}^N$ with the singular point p at the origin. Let S_ε be the boundary of a small open ball B_ε in $\mathbb{C}^N$ with center 0 and radius $\varepsilon \ll 1$. The link of p in X is $M := X \cap S_\varepsilon$ and $K := X_0 \cap S_\varepsilon$ is the link of p in X_0. The complement of K in M fibres over the circle: the map $f/|f|: M \backslash K \to S^1$ (S^1 is the unit circle in the complex plane) defines a fibre bundle.

To define the second bundle, let $S^1(\alpha)$ be a small circle in the complex plane with radius $\alpha \ll \varepsilon$. The bundle

$$\phi := (1/\alpha) \cdot f : f^{-1}(S^1(\alpha)) \cap B_\varepsilon \to S^1$$

is fibre preserving diffeomorphic to the first bundle. In the second setting it is more natural to consider a fibering of manifolds with boundary:

$$\bar{\phi} : f^{-1}(S^1(\alpha)) \cap \bar{B}_\varepsilon \to S^1 .$$

We will call the fibre $F = \bar{\phi}^{-1}(1)$ of $\bar{\phi}$ the (closed) Milnor fibre. The boundary ∂F of F is diffeomorphic to K. The fibre F has the homotopy type of a finite CW-complex of dimension n [3, Satz 1.7; 7, (5.6)]. The second description shows that M has an open book decomposition with page F and binding K.

In an open book decomposition of a manifold M the manifold E is the suspension of a diffeomorphism $h : F \to F$, i.e. $E \cong F \times [0, 1]/(x, 0) \sim (h^{-1}(x), 1)$. We call h the geometric monodromy; the induced map $h_* : H_q(F) \to H_q(F)$ is the (algebraic) *monodromy*. Because the restriction of h to ∂F is the identity there exists a variation homomorphism $\operatorname{Var}(h) : H_q(F, \partial F) \to H_q(F)$, defined by the formula $\operatorname{Var}(h)([z]) = [h(z) - z]$, where $[z]$ denotes the homology class of a q-cycle z [6, 3.1]. This map fits into an exact sequence:

$$\ldots \to H_{q+1}(M) \to H_q(F, \partial F) \xrightarrow{\operatorname{Var}(h)} H_q(F) \to H_q(M) \to \ldots$$

which arises from the exact sequence for the pair (M, F) with the isomorphisms

$$\begin{aligned} H_q(M, F) &\cong H_q(M, F \cup (\partial F \times D)) \cong H_q(F \times I/\sim, F \cup (\partial F \times S^1)) \\ &\cong H_q(F \times I, (F \times \partial I) \cup (\partial F \times I)) \cong H_{q-1}(F, \partial F) . \end{aligned}$$

This sequence is especially useful if M is a $(2n+1)$-manifold and F has the homotopy type of a CW-complex of dimension n; the page F is then called *almost canonical* [9]. All odd dimensional manifolds have book decompositions of this type and smoothings of singularities lead to them. In this situation we have:

Proposition 1. *Let M be a $(2n+1)$-dimensional manifold with open book decomposition with almost canonical page F. Then the following sequence is exact:*

$$0 \to H_{n+1}(M) \to H_n(F, \partial F) \xrightarrow{\operatorname{Var}(h)} H_n(F) \to H_n(M) \to 0 . \tag{1}$$

Furthermore, $H_q(M) \cong H_q(F)$, $q < n$, *and* $H_q(F, \partial F) \cong H_{q+1}(M)$, $q > n$.

Remark. If $M = S^{2n+1}$, then $\operatorname{Var}(h)$ is an isomorphism in the middle dimension [3, 3.3]. In fact, in a suitable basis the matrix for $\operatorname{Var}(h)$ is up to sign the inverse of the matrix for the Seifert form [2, Sect. 4]. In [2] open book decompositions of S^{2n+1} with almost canonical page F, F simply connected if $n > 1$, are called *simple fibered knots.*

2. Branched Cyclic Covers and Homology Periodicity

In this section we generalize results of Durfee and Kauffman [2] on homology periodicity of simple fibered knots and we discuss the conditions for periodicity.

The k-fold cyclic cover M^k of M, branched along K, is the manifold with open book decomposition $(F \times I/(x,0) \sim (h^{-k}(x),1)) \cup_r (K \times D)$, so the monodromy is h^k.

If $X \to D$ is a smoothing of X_0, we get a new smoothing X^k by the following base change:

$$\begin{array}{ccc} X^k = X \times_D \tilde{D} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{D} & \xrightarrow{t \mapsto t^k} & D \end{array}$$

The link M^k of the singular point p in X^k is the smooth k-fold cyclic cover of the link M of p in X, branched along the link K of p in X_0. This can be proved along the following lines: embed X^k in $\mathbb{C}^N \times \mathbb{C}$ [which has coordinates (x,t)] such that the map $X^k \to D$ coincides with the restriction of the projection $(x,t) \mapsto t$. Define the link M^k by means of the function $r_k(x,t) = \|x\|^2 + |t|^{2k}$. The link of the origin in X^k can be defined as $\varrho^{-1}(\varepsilon) \cap X^k$ where $\varrho : \mathbb{C}^N \times \mathbb{C} \to [0,\infty)$ is any real-analytic function with $\varrho^{-1}(0) = (0,0)$ and the diffeomorphism type of this link does not depend on the particular choice [7, Proposition 2.5]. The map $(x,t) \mapsto (x,t^k)$ exhibits M^k as branched cover of M.

We are interested in the relations between the homology of M^k for different k. To obtain periodic behaviour we assume that the eigenvalues of h_* are roots of unity; then there exists a natural number d such that h_*^d is unipotent, i.e. $(h_*^d - \mathrm{id})^N = 0$ for some N.

Theorem 2. *Let the* $(2n+1)$*-manifold* M *have an open book decomposition with almost canonical page and suppose that* h_*^d *is unipotent. Then* $H_*(M^k, \mathbb{Q}) \cong H_*(M^{k+d}, \mathbb{Q})$ *and for* $k<d$, $H_*(M^k,\mathbb{Q}) \cong H_*(M^{d-k},\mathbb{Q})$.

Proof. We want to use the exact sequence (1), so we have to compare $\mathrm{Var}(h^k)$ and $\mathrm{Var}(h^{k+d})$. In general, if $g, h : (F, \partial F) \to (F, \partial F)$ are continuous maps with $g|\partial F = h|\partial F = \mathrm{id}$, then [6, 3.1.4]:

$$\mathrm{Var}(h \circ g) = \mathrm{Var}(h) + \mathrm{Var}(g) + \mathrm{Var}(h) \circ j_* \circ \mathrm{Var}(g), \tag{2}$$

where j_* is given by the inclusion $j : (F, \emptyset) \to (F, \partial F)$. From the definition of Var we also have:

$$\mathrm{Var}(h) \circ j_* = h_* - \mathrm{id}. \tag{3}$$

Therefore

$$\mathrm{Var}(h \circ g) = \mathrm{Var}(h) + h_* \circ \mathrm{Var}(g). \tag{4}$$

By Proposition 1 and duality we have to check the statements of the theorem only for H_{n+1}. Therefore we look at the kernels of the variation maps involved. From (4) we obtain

$$\mathrm{Var}(h^{k+d}) = \mathrm{Var}(h^k) + h_*^k \circ \mathrm{Var}(h^d), \tag{5}$$

$$\mathrm{Var}(h^d) = \mathrm{Var}(h^k) + h_*^k \circ \mathrm{Var}(h^{d-k}). \tag{6}$$

We claim that $\mathrm{Ker}\,\mathrm{Var}(h^k) \subset \mathrm{Ker}\,\mathrm{Var}(h^d)$ for all k; then

$$\mathrm{Ker}\,\mathrm{Var}(h^k) = \mathrm{Ker}\,\mathrm{Var}(h^{k+d})$$

and for $k<d$, $\operatorname{Ker}\operatorname{Var}(h^k)=h_*^k(\operatorname{Ker}\operatorname{Var}(h^{d-k}))$. To prove the claim we observe that

$$\begin{aligned}\operatorname{Ker}\operatorname{Var}(h^k) &\subset \operatorname{Ker}(\mathrm{id}+h_*^k+\ldots+(h_*^k)^{d-1})\circ\operatorname{Var}(h^k)=\operatorname{Ker}\operatorname{Var}(h^{kd})\\ &=\operatorname{Ker}(\mathrm{id}+h_*^d+\ldots+(h_*^d)^{k-1})\circ\operatorname{Var}(h^d)=\operatorname{Ker}\operatorname{Var}(h^d),\end{aligned}$$

where the first two equalities follow from (4) and the last holds because h_*^d is unipotent and $H_n(F,\mathbb{Z})$ is free. □

Proposition 3. *Let M be as in Theorem 2 with h_*^d unipotent. Then $H_*(M^k,\mathbb{Z})\cong H_*(M^{k+d},\mathbb{Z})$ for all $k>0$ if and only if $\operatorname{Var}(h^d)=0$. Moreover, if $\operatorname{Var}(h^d)=0$, then $h_*^d=\mathrm{id}$ and $H_n(M^d)\cong H_n(F)$ and for $0<k<d$, $H_*(M^{d-k})\cong H_*(M^k)$.*

Proof. If $\operatorname{Var}(h^d)=0$, then $h_*^d=\mathrm{id}$ by (3) and the isomorphisms follow immediately from (1), (5), and (6). Conversely, periodicity implies in particular that $\operatorname{Coker}\operatorname{Var}(h^{md})\cong\operatorname{Coker}\operatorname{Var}(h^d)$ for all m. We write g instead of h^d. By induction on m we find from $\operatorname{Var}(g^m)=\operatorname{Var}(g)+\operatorname{Var}(g^{m-1})+\operatorname{Var}(g)\circ j_*\circ\operatorname{Var}(g^{m-1})$ that $\operatorname{Im}\operatorname{Var}(g^m)\subset\operatorname{Im}\operatorname{Var}(g)$; the isomorphism of the cokernels gives $\operatorname{Im}\operatorname{Var}(g^m)=\operatorname{Im}\operatorname{Var}(g)$. Because g_* is unipotent we can choose an odd prime p with $(g_*-\mathrm{id})^{p-1}=0$. The binomial expansion of this gives a formula for $g_*^{p-1}+\mathrm{id}$, which can be substituted in the expression $\operatorname{Var}(g^p)=(\mathrm{id}+g_*+\ldots+g_*^{p-1})\circ\operatorname{Var}(g)$; so

$$\operatorname{Var}(g^p)=\left(pg_*+\ldots+\left(1-(-1)^i\binom{p-1}{i}\right)g_*^i+\ldots+pg_*^{p-2}\right)\circ\operatorname{Var}(g).$$

All the coefficients are divisible by p because the first one is and the difference of two consecutive ones is a binomial coefficient $\binom{p}{i}$. So $\operatorname{Im}\operatorname{Var}(g)=\operatorname{Im}\operatorname{Var}(g^p)\subset\operatorname{Im}p\operatorname{Var}(g)$ and therefore $\operatorname{Var}(g)=0$. □

Durfee and Kauffman prove periodicity for the integral homology of simple fibered knots under the assumption that K is a rational homology sphere and $h_*^d=\mathrm{id}$ [2, Corollary 3.2]. This implies our condition $\operatorname{Var}(h^d)=0$, because j_* is then a rational isomorphism, so $\operatorname{Var}(h^d)=(h_*^d-\mathrm{id})\circ j_*^{-1}=0$ over the rationals and, as $H_n(F)$ is free, also over $\mathbb{Z}$. In fact the two conditions are equivalent: by the remark following Proposition 1, $\operatorname{Var}(h)$ is an isomorphism, so by (3) we have that $\operatorname{Ker}j_*=\operatorname{Ker}(h_*-\mathrm{id})$; let $\operatorname{Var}(h)(y)=z\in\operatorname{Ker}j_*$, then $0=\operatorname{Var}(h^d)(y)=(\mathrm{id}+\ldots+h_*^{d-1})\circ\operatorname{Var}(h)(y)=dz$; therefore j_* is a rational isomorphism in the middle dimension. For simple fibered knots this implies that K is a rational homology sphere.

In the general case the conditions "$h_*^d=\mathrm{id}$" and "j_* is a rational isomorphism in the middle dimension" still imply that $\operatorname{Var}(h^d)=0$, but the converse is no longer true: take e.g. an open book for which the geometric monodromy is the identity and j_* arbitrary. Such an example cannot occur as the link of a singularity:

Lemma 4. *Let M be the link of a smoothing of an isolated singularity and let $\operatorname{Var}(h^d)=0$. Then j_* is a rational isomorphism.*

Proof. In this case the cohomology of the Milnor fibre F carries a mixed Hodge structure [12]. The weight filtration W is determined by j^* and a map

$$V: H^n(F;\mathbb{C})\to H^n(F,\partial F;\mathbb{C}).$$

For $k>0$ the map $V\circ(j^*\circ V)^{k-1}$ induces an isomorphism from $\mathrm{Gr}^W_{n+k}H^n(F)$ to $\mathrm{Gr}^W_{n-k}H^n(F,\partial F)$ and for $k\geq 0$ the map $(j^*\circ V)^k\circ j^*$ induces an isomorphism from $\mathrm{Gr}^W_{n+k}H^n(F,\partial F)$ to $\mathrm{Gr}^W_{n-k}H^n(F)$ [12, (2.6)]. The map V is defined as

$$\mathrm{Var}(g)+\sum(-1)^k\,\mathrm{Var}(g)\circ(g^*-\mathrm{id})^k/k+1\,,$$

where g is the monodromy of a semistable smoothing [12, (2.1)]; then g^* is unipotent. We may take g as some power of h^d. If $\mathrm{Var}(h^d)=0$, then also $\mathrm{Var}(g)=0$, so $V=0$, $\mathrm{Gr}^W_n H^n=H^n$ and j^* is an isomorphism over $\mathbb{C}$. Therefore j_* is a rational isomorphism. □

3. Diffeomorphism Periodicity for Odd n

In the previous section we proved homology periodicity for branched cyclic covers of open books with almost canonical page. These results can be extended to periodicity in k of the homeomorphism type and the diffeomorphism type of M^k, using the classification of highly connected manifolds [15, 1]. We assume from now on that the $(2n+1)$-manifold M and the page F are $(n-1)$-connected and that M bounds a parallelizable manifold. We call such an open book decomposition a *simple open book decomposition*. Examples are the simple fibered knots of [2] and links of smoothings of isolated complete intersection singularities [7, (5.8) and (2.11)]. Branched cyclic covers of simple open books have simple open book decompositions (Lemma 5 below), so we can apply the classification of highly connected manifolds. Here we have to distinguish the cases n is odd or even. In this section we prove that for odd n homology periodicity implies homeomorphism periodicity, generalizing the result for simple fibered knots in [2, Theorem 4.5].

Lemma 5. *Branched cyclic covers of simple open books have simple open book decompositions.*

Proof. If $n>1$, Van Kampen's theorem gives that M^k is simply connected; from the homology statements of Proposition 1 it follows then that M^k is $(n-1)$-connected. Let M be the boundary of a parallelizable manifold N. We construct a manifold N^k with boundary M^k as follows [5, p. 149]. We can push a copy of the page F into N keeping the boundary fixed: we obtain $F'\subset N$ with $F'\cap M=\partial F'=\partial F$. Then N^k is the k-fold cyclic cover of N, branched along F'. The manifold N^k is parallelizable: choose a trivialisation of the tangent bundle of N, which is a product on a tubular neighbourhood $F'\times D$ of F' (this is possible because F' itself is parallelizable); by pull-back we obtain a trivialization of the tangent bundle of the complement N^k_0 of $F'\times D$ in N^k. After a suitable rotation of the trivialization on $F'\times S^1$, depending only on S^1, the trivialisation can be extended over $F'\times D$; the rotation extends over N^k_0 because of the existence of a map $a: N^k_0\to S^1$ such that $a|F'\times S^1$ is the projection on the second factor [5, p. 150] (this map arises in the construction of N^k_0). □

For odd $n\neq 1, 3$ *or* 7 stably-parallelizable $(n-1)$-connected $(2n+1)$-manifolds are classified by the n^{th} homology group and a linking pairing and quadratic form on the torsion part of this group [1, Theorem 6.4]. We will express these invariants by means of the exact sequence (1).

Let M have a book decomposition with page F. Let $x, y \in H_n(M)$ be torsion elements with $rx = sy = 0$. Then there are $\xi, \eta \in H_n(F)$ with $i_*\xi = x$ and $i_*\eta = y$ and there is a $\xi' \in H_n(F, \partial F)$ with $\mathrm{Var}(h)(\xi') = r\xi$. Let $\langle\cdot,\cdot\rangle$ denote the intersection pairing between $H_n(F, \partial F)$ and $H_n(F)$.

Lemma 6. *The linking pairing can be computed by the formula:*

$$lk(x, y) = (1/r)\langle\xi', \eta\rangle \bmod 1$$

and the quadratic form by

$$q(x) = (1/r)\langle\xi', \xi\rangle + S(\xi) \bmod 2,$$

where $S(\xi)$ depends only on $H_n(F)$ and not on h.

Proof. i) We recall the geometric definition of the linking pairing [10, Sect. 77]: represent x and y by chains A and B. There exists an A' with $\partial A' = rA$. Then $lk(x, y) = (1/r)\mathfrak{S}(A', B) \bmod 1$ where $\mathfrak{S}(A', B)$ is the geometric intersection number of A' and B. In our case we may choose the chains in F. Under the isomorphism $H_n(F, \partial F) \cong H_{n+1}(M, F)$ ξ' corresponds to a relative cycle $\xi' \times c$ with boundary $r\xi$, and $\langle\xi', \eta\rangle = \mathfrak{S}(\xi' \times c, \eta)$.

ii) There is also a geometric description of the quadratic form [15, VI, p. 274]: represent x by an embedded sphere S. Let B be a tubular neighbourhood of S: this is a disc bundle over S. Let S_1 be a section of the associated sphere bundle and S_2 a fibre. Let λ be the geometric linking number of the cycles S and rS_1. The normal bundle of S_1 in the sphere bundle ∂B determines a characteristic class $\alpha \in \pi_{n-1}(SO_n)$. Sections of the sphere bundle are in $(1-1)$-correspondence with the integers. Changing the section S_1 to $S_1' \times S_1 + mS_2$ has the effect of changing λ and α to:

$$\lambda' = \lambda + mr$$

$$\alpha' = \alpha + \partial(m),$$

where $\partial : \pi_n(S^n) \to \pi_{n-1}(SO_n)$. If $n \neq 3, 7$ is odd, then there is a homomorphism $\phi : \pi_{n-1}(SO_n) \to \mathbb{Z}/2\mathbb{Z}$ and we require that $\phi(\alpha') = 0$. This determines the number $q(x) = \lambda'/r$ modulo 2.

In our situation we can represent x by an embedded sphere S in F and there is a natural choice for the section S_1, by transporting S to a nearby fibre. The geometric linking of rS_1 and S is given by $\langle\xi', \xi\rangle$. The condition $\phi(\alpha') = 0$ can be satisfied in a neighbourhood of S, so the correction term $S(\xi)$ depends only on $\xi \in H_n(F)$. □

Remark. For simple fibered knots $S(\xi)$ is the reduction mod 2 of the linking in $M = S^{2n+1}$. Because $q \equiv 0$ for $k = 1$, the quadratic form q_k on M^k is given by the following formula, which is essentially the same as the one on p. 168 of [2]:

$$q_k(x) = (1/r)\langle\xi', \xi\rangle - \langle\mathrm{Var}(h)^{-1}(\xi), \xi\rangle \bmod 2.$$

Theorem 7. *Let M^k be the k-fold branched cyclic cover of a $(2n+1)$-manifold M with simple open book decomposition, $n \neq 1, 3$ or 7 odd. If* $\mathrm{Var}(h^d) = 0$, *then M^k and M^{k+d} are (orientation preserving) homeomorphic, while M^k and M^{d-k} $(k < d)$ are orientation reversing homeomorphic.*

Proof. The invariants H_n, lk, and q classify the manifolds under consideration up to orientation preserving homeomorphism [1, Theorem 6.4]. If $\mathrm{Var}(h^d) = 0$, then M^k

and M^{k+d} agree in these invariants by Proposition 3 and Lemma 6. Orientation reversal changes the sign of the linking pairing and the quadratic form, so we have to show that the linking lk_k and the form q_k of M^k are the negative of lk_{d-k} and q_{d-k} of M^{d-k}. From the definition of the variation homomorphism it follows that $j\circ \mathrm{Var}(h)=h_*-\mathrm{id}$, where now h_* denotes the monodromy on $H_n(F,\partial F)$. With this formula instead of (3) and the analogue of (6) we find that

$$\mathrm{Var}(h^{d-k})=-\mathrm{Var}(h^k)\circ h_*^{d-k}$$

if $\mathrm{Var}(h^d)=0$. Now let $\mathrm{Var}(h^{k-d})(\xi')=r\xi$. Then $lk_{k-d}(x,y)=(1/r)\langle\xi',\eta\rangle$. So:

$$\begin{aligned} lk_k(x,y) &= -(1/r)\langle h_*^{d-k}\xi',\eta\rangle \\ &= -(1/r)\langle\xi',\eta\rangle-(1/r)\langle h_*^{d-k}\xi'-\xi',\eta\rangle \equiv -(1/r)\langle\xi',\eta\rangle \end{aligned}$$

for $(1/r)\langle h_*^{d-k}\xi'-\xi',\eta\rangle=(1/r)\langle j_*\circ\mathrm{Var}(h^{d-k})(\xi'),\eta\rangle=\langle j_*\xi,\eta\rangle\in\mathbb{Z}$. Furthermore, $\langle j_*\xi,\xi\rangle=0$ (n is odd) so $q_k(x)=-q_{d-k}(x)\bmod 2$. □

Remark. In the case $n=3,7$ Theorem 7 probably still holds. If H_n contains no 2-torsion, then M is determined by the homology and the linking pairing [17]. In particular, for all $n\geqq 3$ odd, M^{kd} is homeomorphic to the connected sum $(S^n\times S^{n+1})\#\ \dots\ \#(S^n\times S^{n+1})$, where the number of factors is equal to the rank of $H_n(F)$.

There is also a periodicity in the diffeomorphism type of the manifolds M^k with a period depending on the order of the finite group bP_{2n+2} of homotopy $(2n+1)$-spheres that bound parallelizable manifolds. Let Σ denote the generator of this group, $m\Sigma$ the connected sum of m copies of Σ and σ_k the signature of the manifold N^k, constructed in the proof of Lemma 5.

Proposition 8. *In the situation of Theorem 7, M^{k+d} is diffeomorphic to $(\sigma_d/8)\Sigma\# M^k$.*

Proof. By [1, Theorem 6.4] M^{k+d} is diffeomorphic to $1/8(\sigma_{k+d}-\sigma_k)\Sigma\# M^k$. To calculate $\sigma_{k+d}-\sigma_k$ we use a cut and paste description of N^{k+d} [5, p. 150]. In the process of pushing F into N, as in the proof of Lemma 5, we obtain a codimension one submanifold W with boundary $F\cup F'$, along which N can be cut open: this results in a N_0 with boundary $W_+\cup W_-\cup M_0$, where M_0 denotes M cut open along F. Let $W_0=W_+\cup W_-$, then $\partial W_0=W_0\cap M_0=\partial M_0=F_+\cup F_-$ and $F_+\cap F_-=\partial F$. Then N^k is obtained by pasting k copies of N_0 together, identifying W_- of one copy with W_+ of the next one. In the same way N^k and N^d can be cut open, so N^{k+d} is made by pasting N_0^k and N_0^d together along W. After "straightening the angle" N_0^k is diffeomorphic to N^k, so we can apply Wall's "non-additivity of the signature" formula [16]: $\sigma_{k+d}=\sigma_k+\sigma_d-\sigma(V;A,B,C)$, where the last term is the signature of a form on a certain subquotient of $V=H_n(\partial W_0)$. The spaces M_0^k, W_0, and M_0^d have F_+ as deformation retract and they have the same boundary $F_+\cup F_-=\partial W_0$. The inclusion of the boundary induces three maps $a,b,c\colon V\to H_n(F_+)$ with kernels A,B, and C. Using the exact sequence for $(\partial W_0,F_+)$ to identify V with $H_n(F_+)\oplus H_n(F_-,\partial F_-)$ we may write $a(x,y)=x+\mathrm{Var}(h^k)(y)$, $b(x,y)=x$ and $c(x,y)=x-h_*^{-d}\mathrm{Var}(h^d)(y)=x$. The subquotient of V is $B\cap(C+A)/((B\cap C)+(B\cap A))=0$, so $\sigma_{k+d}-\sigma_k=\sigma_d$. □

Remarks. 1. Let M^0 be the manifold with same page F as M^k, but for which the geometric monodromy is the identity. We have $M^0=\partial(F\times D)$, so $\sigma_0=0$ and the periodicity of the proposition extends to $k=0$; M^0 is diffeomorphic to $(S^n\times S^{n+1})\#\ldots\#(S^n\times S^{n+1})$. 2. It follows from a general signature periodicity theorem of Neumann [8] that $\sigma_{k+d}-\sigma_k$ is a constant whenever the eigenvalues of the monodromy are d^{th} roots of unity.

4. The Case n is Even

Homeomorphism periodicity for branched cyclic covers is more complicated if n is even than the case of the previous section. We start with an example, showing that the period is not the period of the monodromy.

Example [2, Sect. 6]. Consider smoothings of the ordinary double point. The monodromy has order 2. Let M^k be the link of $z_0^2+\ldots+z_n^2+t^k=0$. Then M^{k+8} is diffeomorphic to (notation: $\cong$) M^k and: $M^1\cong S^{2n+1}$, $M^2\cong T$, the tangent sphere bundle of S^{n+1}, $M^3\cong\Sigma$, the $(2n+1)$-dimensional Kervaire sphere, $M^4\cong(S^n\times S^{n+1})\#\Sigma$, $M^5\cong\Sigma$, $M^6\cong T\#\Sigma\cong T$, $M^7\cong S^{2n+1}$, and $M^8\cong S^n\times S^{n+1}$. If $n\neq 2,6$ then T is not equal to $S^n\times S^{n+1}$.

This example does however show homeomorphism periodicity with period $2d$ and diffeomorphism periodicity with period $4d$.

In the classification of $(n-1)$-connected $(2n+1)$-manifold with n even a new invariant enters, a class $\hat{\phi}$ in $H_n(M)\otimes\mathbb{Z}/2\mathbb{Z}$ [15, VI, p. 284]. It is this $\hat{\phi}$ that distinguishes between T and $S^n\times S^{n+1}$.

From the example we see that we cannot express $\hat{\phi}$ in terms of the variation map. Nevertheless, without explicitly determining $\hat{\phi}$, we are able to prove:

Theorem 9. *If for branched cyclic covers M^k of a $(2n+1)$-manifold M with simple open book decomposition* $\operatorname{Var}(h^d)=0$, *then M^k and M^{k+2d} are homeomorphic and M^k and M^{k+4d} are diffeomorphic. Moreover, if $n=2$ or 6, then M^k and M^{k+d} are diffeomorphic.*

Proof. The last assertion follows immediately from the fact that in these dimensions highly connected manifolds (that bound parallelizable manifolds) are already determined by their homology [11].

For the general case we will examine Wall's proof of the classification of highly connected manifolds [15, Sect. 13, 10, 11, 5] more closely.

Let M be a closed $(n-1)$-connected $(2n+1)$-manifold with $n\geqq 4$. If $H_n(M)$ is generated by μ elements, then there exists a handle decomposition of M with one zero-handle, μ n-handles, μ $(n+1)$-handles and one $(2n+1)$-handle. The union N of the zero handle and the n-handles is diffeomorphic to the union N' of the $(n+1)$-handles and the $(2n+1)$-handle. Attaching N' to N by means of a diffeomorphism $g:\partial N'=\partial N\to\partial N$ gives M. This decomposition of M is essentially unique; g is determined up to multiplication on the left and on the right by diffeomorphisms which extend to ones of N [15, VI, Sect. 13]. Now consider two manifolds M_1 and M_2, constructed from N by diffeomorphism g_1 and g_2. Then M_1 and M_2 are diffeomorphic if g_1 can be obtained by multiplying g_2 by diffeomorphisms of ∂N that extend to N.

We can describe an open book M in the above way by a modification of the construction in Sect. 1. In the same notations as there, let N be $(F\times[0,1/2])$

$\cup_r(\partial F \times D^+)$, where D^+ is the upper half of the disc$D$. Similarly $N' = (F \times [\frac{1}{2}, 1]) \cup_r(\partial F \times D^-)$. The boundary of N consists of two copies of F, glued together along their boundary by $\partial F \times [0,1]$. The attaching map $g: \partial N' \to \partial N$ is the identity on one copy of F and on $\partial F \times [0,1]$, and the geometric monodromy h on the other copy of F. Remark that the fibre F is a deformation retract of N.

Because the diffeomorphism classification of closed manifolds is rather subtle, we will first consider almost closed manifolds, i.e. manifolds whose boundary is a homotopy sphere. We can make a closed manifold M almost closed by removing an open $(2n+1)$-disc, which we put back on the end. As before we can split an almost closed manifold M in two diffeomorphic handle-bodies N and N' with an essential unique attaching diffeomorphism given on ∂N with the interior of a $2n$-disc removed; we will denote $\partial N \backslash \mathrm{Int}(D^{2n})$ by L.

So the problem is to decide when g_1 and g_2 are the same up to multiplication by diffeomorphisms of L that extend to N. According to the proof of [15, Theorem 7, Sect. 13] there are two types of obstructions. Firstly, if g_1 and g_2 agree in the invariants given in Lemma 31, which can be determined from homology invariants of M_1 and M_2, then g_2 can be modified by multiplication to achieve that g_1 and g_2 are homotopic. Secondly there are certain obstructions which by the proof of [15, Theorem 6, Sect. 11] are precisely the obstructions for g_1 and g_2 to be pseudo-isotopic, as given in Lemma 23: a homomorphism $S\beta: H_n(L) \to S\pi_n(SO_n)$, which may be simplified to a class $\hat{\beta} \in H^{n+1}(M; \pi_n(SO))$ and a class $\hat{\phi} \in H^{n+1}(M; \mathbb{Z}_2)$. This is the $\hat{\phi}$ mention before.

The invariant $S\beta$ appears in the following way: a pseudo-isotopy between g_1 and g_2 is a diffeomorphism $G: L \times [0,1] \to L \times [0,1]$ where G restricted to $L \times \{0\}$ is g_1 and restricted to $L\{1\}$ is g_2. The extension of g_1 and g_2, given on $L \times \{1\}$ to a diffeomorphism of $L \times [0,1]$ is achieved in five steps [15, Sect. 5]. In step 4 a diffeomorphism is given on the product of the interval with a disc and the core of the n-handles. Let D^n_x be such a core and x the corresponding generator of $H_n(L)$. To extend the diffeomorphism over a tubular neighbourhood of $D^n_x \times [0,1]$ (the objective of step 4), we need a trivialisation of the normal bundle. It is already given on the boundary of $D^n_x \times [0,1]$ [15, II, p. 267], so we find an obstruction $\beta(x) \in \pi_n(SO_n)$. The indeterminacy in these obstructions can be eliminated by using the suspension $S: \pi_n(SO_n) \to \pi_n(SO_{n+1})$; we get a map $S\beta: H_n(L) \to S\pi_n(SO_n)$. If $n \neq 2, 6$ and even, then $S: S\pi_n(SO_n) \to \pi_n(SO_{n+2})$ is a split surjection with kernel of order 2 ([15, Sect. 12B], where this is erroneously claimed for even $n \neq 2$, 4 or 8).

We are now going to compare the attaching maps g^k and g^{k+2d} for the manifolds M^k and M^{k+2d}. In our case the first type of obstruction reduces to the bilinear form on the torsion. From $\mathrm{Var}(h^d) = 0$ it follows that g^k and g^{k+d} agree in these invariants.

We may then try to extend g^k, given on $L \times \{0\}$, and g^{k+2d}, given on $L \times \{2\}$, to a diffeomorphism of $L \times [0,2]$. We first give g^{k+d} on $L \times \{1\}$, but this map may be changed during the construction. We may consider $L \times [1,2]$ as $L \times [0,1]$, shifted by g^d. The obstruction $\beta_{2d}(x)$ in $L \times [0,2]$ is the sum of $\beta_d(x)$ in $L \times [0,1]$ and $\beta_d((g^d)_*(x))$ in $L \times [1,2]$. But $(g^d)_* = id$ on homology so $\beta_{2d}(x) = 2\beta_d(x)$ and $S\beta_{2d} = 2S\beta_d$. The image of $S^2\beta$ in $\pi_n(SO_{n+2}) = \pi_n(SO)$ is zero because our manifolds are stably parallelizable. So the obstruction lies in a group of order two, so it vanishes.

Thus $M^k \backslash$disc is diffeomorphic to $M^{k+2d}\backslash$disc and therefore M^k and M^{k+2d} are homeomorphic.

From the above discussion we see that M^{k+2d} is diffeomorphic to $M^k \# \Sigma$ for some homotopy sphere Σ; moreover, $M^{k+4d} \cong M^k \# 2\Sigma$. We have to identify the homotopy sphere Σ. The manifolds M^k and M^{k+2d} bound parallelizable manifolds N_1 and N_2. We form a new manifold N by identifying the boundaries minus a disc of N_1 and N_2, using the diffeomorphism between $M^k\backslash$disc and $M^{k+2d}\backslash$disc; N is a n-connected $(2n+2)$-manifold with boundary Σ. From the classification of such manifolds [14] we see that for n not divisible by 8 the only possibilities for Σ are the standard S^{2n+1} and the Kervaire sphere, an element of order two in the group of homotopy spheres. The same conclusion holds if $n \equiv 0 \pmod 8$, provided another invariant $\chi = 0 \pmod 2$ [14]. Finding representatives of the homology of N with a Mayer-Vietoris argument from N_1, N_2, and $M^k\backslash$disc, we see that this condition is fulfilled because N_1 and N_2 are parallelizable, while the fact that $S\beta_{2d} = 0$ gives the right answer across $M^k\backslash$disc. □

5. Some Examples

Smoothings of the Cusp Singularity A_2

We consider smoothings of the singularity $z_0^3 + z_1^2 + \dots + z_n^2 = 0$ with n odd. After a change of co-ordinates we can write such a smoothing as $g(z, y, t) + z_2^2 + \dots + z_n^2 = 0$, where $g(z, y, t)$ is a smoothing of the plane curve singularity $z^3 + y^2 = 0$; the function on the total space of the smoothing is the restriction of the projection on the t co-ordinate. The invariants of the smoothings with $n > 1$ are the same as those of the smoothings $g(z, y, t)$, so we can do all our calculations with surfaces.

The link of $z^3 + y^2 = 0$ is a 1-sphere, so j_* is an isomorphism. The total space of a smoothing of the plane cusp has either a simple singularity or an elliptic Kulikov singularity, as studied in [13]. With such a singularity we can associate a degenerating family of elliptic curves and the monodromy of the smoothing is the same as that of the family of curves. The minimal resolution of the singularity has as exceptional divisor the curves in the special fibre of the family and a chain of $c-1$ rational (-2)-curves, intersecting the other configuration in a component with multiplicity one; this component has self-intersection one less in the exceptional divisor than in the special fibre of the family of elliptic curves. We denote this exceptional divisor by Kodaira's symbol for the fibre, followed by (c); e.g. $I_0(1)$ is an elliptic curve with self-intersection -1. In this dimension there is no homeomorphism periodicity, but we may say that the resolution is periodic, in the following sense: the special fibre of the family of curves associated to X^{k+d} is the same as that for X^k, but the chain of (-2)-curves is one longer. The signature can be computed from the resolution with a formula of Durfee [12, 2.27]: $\sigma = -8p_g - K^2 - b_2(E)$. For the singularities under consideration $p_g = c$ and $K^2 + b_2(E) = b - 1$, where b is the number of components of the special fibre of the associated family of elliptic curves [13, Sect. 4].

We now describe some smoothings with their branched covers. Let f_k be $y^2 + z^3 + t^k$. The monodromy of f_1 has order 6. For $2 \leqq k \leqq 5$ we obtain the simple singularities A_2, D_4, E_6, and E_8 with signatures -2, -4, -6, and -8. For f_6 we get $I_0(1)$ with $\sigma=-8$. Furthermore, f_{6d+1} gives $II(d)$, $f_{6d+2}: IV(d)$, $f_{6d+3}: I_0^*(d)$, $f_{6d+4}: IV^*(d)$, $f_{6d+5}: II^*(d)$, and f_{6d} gives $I_0(d)$.

Let g_k be $y^2+z^3+zt^k$. The monodromy of $g_1=A_1$ has order 4; $g_2=D_4$, $g_3=E_7$, $g_{4d}=I_0(d)$, $g_{4d+1}=III(d)$, $g_{4d+2}=I_0^*(d)$, and $g_{4d+3}=III^*(d)$. So the signatures are -1, -4, -7, -8, $-1-8, \ldots$ for $k=1,2,\ldots$.

As final example we consider smoothings with monodromy of infinite order. Let $p\geqq 1$ and $h_k=y^2+z^3+z^2t^{3k}+t^{9k+pk}$. We have $h_{2m}=I_{2mp}(3m)$ and $h_{2m+1}=I^*_{(2m+1)p}(3m+1)$. We see here the periodic behaviour of the signature, for $\sigma_k=-(12+p)k$ if k is odd and $\sigma_k=-(12+p)k+1$ if k is even.

Brieskorn Complete Intersections

In general it is difficult to determine the monodromy of a smoothing, but for quasi-homogeneous smoothings it has always finite order. The computations are rather easy in the special case of Brieskorn complete intersections. We recall some results from [4]. Let $\underline{a}=(a_1,\ldots,a_{n+p})$ be a collection of natural numbers. The Brieskorn complete intersection $X_p(\underline{a})$ is the singularity defined by $f:\mathbb{C}^{n+p}\to\mathbb{C}^p$ with $f_i(z_1,\ldots,z_{n+p})=c_{i,1}z_1^{a_1}+\ldots+c_{i,n+p}z_{n+p}^{a_{n+p}}$, $i=1,\ldots,p$, with all minors of the $p\times(n+p)$-matrix (c_{ij}) non-zero. This is an isolated complete intersection singularity because all maximal minors are non-zero.

There is a simple criterion to decide whether the link $K(\underline{a})$ of $X(\underline{a})$ is a rational homology sphere: construct the weighted graph $G(\underline{a})$ with $n+p$ vertices $p_1,\ldots,p_{n+p}$ with weights $a_1,\ldots,a_{n+p}$. The points p_i and p_j are to be joined by an edge if and only if the greatest common divisor (a_i,a_j) of a_i and a_j is bigger than one. Let $N(\underline{a})$ be the number of connected components of $G(\underline{a})$ consisting of an odd number of vertices and for which $(a_i,a_j)=2$ for every pair of different vertices in it.

If $N(a)\geqq p$, then $K(\underline{a})$ is a rational homology sphere and if $N(a)>p$, then $K(\underline{a})$ is an integral homology sphere. Furthermore, if $p\leqq n+2$ the converse statements are true.

For odd $n\geqq 3$ the link $K(\underline{a})$ is diffeomorphic to the Kervaire sphere when $G(\underline{a})$ has precisely $p+1$ connected components and the number of components, consisting of one point with $a_i\equiv\pm 3 \bmod 8$, is odd; in all other cases a homology sphere $K(\underline{a})$ is diffeomorphic to the standard sphere. For n even the signature can be determined with a formula of Hirzebruch [4, Satz 1.4].

We consider the smoothing $f_p: X_{p-1}(\underline{a})\to\mathbb{C}$ of $X_p(\underline{a})$. Hamm has determined the characteristic polynomial of the monodromy: if $d=LCM\,(a_i)$, then $h_*^d=\mathrm{id}$. As before we can consider the graph of f_p and write this smoothing as $pr: X_p(\underline{a},1)\to\mathbb{C}$, where pr is the projection on the last co-ordinate. The branched covers are given by $X_p(\underline{a},k)$.

Let $\underline{a}=(5,3,2,\ldots,2)$ and consider such smoothings of $X_2(\underline{a})$. If n is even, $K(\underline{a})$ is a rational homology sphere and the link $M^k=K(\underline{a},k)$ is never diffeomorphic to the Kervaire sphere. From the discussion in Sect. 4 we see that there is at least diffeomorphism periodicity with period 60. If n is odd, $K(\underline{a})$ is a topological sphere and there is homeomorphism periodicity with period 30.

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