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# Some applications of Tristram–Levine signatures and relation to Vassiliev invariants

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## Abstract

Tristram and Levine introduced a continuous family of signature invariants for knots. We show that any possible value of such an invariant is realized by a knot with given Vassiliev invariants of bounded degree. We also show that one can make a knot prime preserving Alexander polynomial and Vassiliev invariants of bounded degree. Finally, the Tristram–Levine signatures are applied to obtain a condition on (signed) unknotting number.

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*Keywords:* Unknotting number; Vassiliev invariant; Signature; Alexander polynomial

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## 1. Introduction

Consider for a knot  $K$  and a complex number  $\xi$  of unit norm the Hermitian forms  $M_\xi = (1 - \xi)A + (1 - \bar{\xi})A^T$ , where  $A$  is a Seifert matrix of  $K$ . The signatures of these forms  $M_\xi$  are the generalized or so-called Tristram–Levine signatures  $\sigma_\xi$  [40,23], of which the Murasugi [27] signature  $\sigma = \sigma_{-1}$  is a special case. These invariants have a variety of relations and applications, within and outside of knot theory. First, via the Tristram–Murasugi inequality [40,27], the signatures are related to the 4-genus, and hence unknotting number. More recently, they have been of some interest because of

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their application to the classification of zero sets of algebraic functions on projective spaces [31]. Tristram–Levine signatures have also close relationship to the zeros of the Alexander polynomial, which have been studied for a while and have importance for several subjects, including monodromy of fibered links [34], divisibility [28] and orderability [32] of knot groups, and statistical mechanical models of the Alexander polynomial [26]. Also some relations of these signatures to (a quantum version of) the Jones polynomial have become apparent [11].

Vassiliev invariants [5] have been introduced more recently, and their relations to more classical invariants have been sought. In this paper, we shall treat the possible relations between Vassiliev invariants and generalized signatures  $\sigma_\xi$  with  $\xi \in \mathbb{C}$  and  $|\xi| = 1$ . We extend the result on the Murasugi signature of [37] (and basically following also from [29]) to them, constructing knots of any given possible value of  $\sigma_\xi$  and Vassiliev invariants of bounded degree.

**Theorem 1.** *For any  $n \in \mathbb{N}$ , any  $\xi \in S^1 \setminus \{1\}$ , and any admissible value  $v$  of  $\sigma_\xi$ , one has a prime knot with given values of Vassiliev invariants up to degree  $n$ , realizing  $\sigma_\xi(K) = v$ .*

For the proof the notion of braiding sequences [39] is used. While for generic  $\xi$  (in particular if  $\xi$  is transcendental),  $\sigma_\xi$  admits—as the usual (Murasugi) signature—only even values on knots, the remaining cases (we describe them exactly) require an additional argument which we provide using Gousarov’s result [14] on the existence of  $n$ -inverses. The knots constructed in Section 3 are composite. In order to find prime examples, some additional work is needed. It will be carried out in Section 4, using the construction of  $n$ -trivadjacent knots of Askitas and Kalfagianni [4]. It shows also the following result, which may be independent interest.

**Theorem 2.** *If  $K$  is a knot and  $n > 0$ , then there exists a prime knot  $K'$   $n$ -similar to  $K$ , such that  $\Delta_{K'} = \Delta_K$ .*

We will have to show for this that a certain family of the knots arising in the construction of Askitas–Kalfagianni are non-trivial (a suggestive fact, which, however, does not follow from their results). In doing so, we give a new proof, using the Jones polynomial, that the (pure) braid groups are not nilpotent.

The result of Theorem 1 on the  $\sigma_\xi$  should be put in contrast to the fact that via the Alexander/Conway polynomial Vassiliev invariants may well pose conditions to the Seifert pairing  $A$ . Also, Ng’s result [29] that any concordance class contains a (possibly composite) knot with given Vassiliev invariants of bounded degree does not imply ours (even for composite knots), since by work of Levine [24], the signatures may not be concordance invariants in the zeros of the Alexander polynomial (see Remark 4). Namely, there are slice knots with non-zero (singular) signatures.

Albeit Levine writes down only the Seifert forms of such knots, we will later be able to find many concrete examples: any slice knot with zeros of  $\Delta$  on the unit circle, which has unknotting number one, turns out to be such. This follows from work given in an appendix, where we obtain a condition on signed unknotting number by analyzing

the eigenvalues of  $M_\xi$ . (This work should be differentiated from the rest of the paper, since it uses mainly the same background on the signatures, but not the arguments in the main part.) In particular, we will also show for an amphicheiral unknotting number one knot, no zeros of  $\Delta_K$  lie on the complex unit circle (Corollary 2).

It is a more challenging question (also pointed out by the referee) whether one can realize not just every individual signature, but rather every signature function, using an  $n$ -trivial knot. Although the methods we apply clearly provide a way to approach such a question, they do not put into perspective a rigorous, but elegant and short solution.

### 2. Preliminaries

We shall briefly introduce the main notions appearing in the sequel (and give a few additional references to those in the introduction for further details).

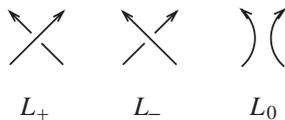
Recall that if  $A$  is a Seifert matrix of size  $2g \times 2g$  corresponding to a genus  $g$  Seifert surface of a knot  $K$ , then for any  $\xi \in \mathbb{C}$  with  $|\xi| = 1$  and  $\xi \neq 1$  we define

$$M_\xi(K) := (1 - \xi)A + (1 - \bar{\xi})A^T,$$

where bar denotes conjugation and, superscript  $T$  transposition. This is a Hermitian matrix, and all eigenvalues are real. By  $\sigma(M_\xi)$  and  $n(M_\xi)$  we denote the signature (sum of signs of eigenvalues) and nullity (number of zero eigenvalues) of  $M_\xi$ . They turn out to be independent in the surface and Seifert matrix, and are thus invariants of  $K$ , denoted by  $\sigma_\xi(K)$  and  $n_\xi(K)$  respectively.  $\sigma_\xi(K)$  is called a generalized or *Tristram–Levine signature*. It satisfies, as the usual signature  $\sigma = \sigma_{-1}$ , the rules

$$\begin{aligned} \sigma_\xi(L_+) - \sigma_\xi(L_-) &\in \{0, 1, 2\}, \\ \sigma_\xi(L_\pm) - \sigma_\xi(L_0) &\in \{-1, 0, 1\}, \\ \sigma_\xi(!L) &= -\sigma_\xi(L), \\ \sigma_\xi(L\#K) &= \sigma_\xi(L) + \sigma_\xi(K). \end{aligned} \tag{1}$$

(Whether to have  $\{0, 1, 2\}$  or  $\{0, -1, -2\}$  in (1) is a matter of convention.) Here  $L_+, L_0, L_-$  form a skein triple



and  $!L$  is the mirror image of  $L$ .  $K_1\#K_2$  denotes the connected sum of  $K_1$  and  $K_2$ , and  $\#^n K$  denotes the connected sum of  $n$  copies of  $K$ .

The main difference in the usual signature is that  $\sigma_\xi$  may be odd even on knots, and that nice combinatorial formulas, as for alternating links (see [27,18,13]), are lacking.

The (normalized) *Alexander polynomial* [3] can be defined from a Seifert matrix  $A$  by

$$\Delta_K(t) = t^{-g} \det(A - tA^T).$$

$\Delta$  satisfies the skein relation

$$\Delta\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right) - \Delta\left(\begin{array}{c} \nwarrow \\ \swarrow \end{array}\right) = (t^{1/2} - t^{-1/2})\Delta\left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}\right), \tag{2}$$

which defines it alternatively (up to a factor, fixed by demanding  $\Delta(\bigcirc) = 1$ ).

We will sometimes modify  $\Delta$  up to units in  $\mathbb{Z}[t, t^{-1}]$ , as in the original definition of Alexander.

Let  $\nabla_K$  denote the *Conway polynomial* [8], given by

$$\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t).$$

Consequently,  $\nabla$  also satisfies a skein relation, namely

$$\nabla\left(\begin{array}{c} \nearrow \\ \searrow \end{array}\right) - \nabla\left(\begin{array}{c} \nwarrow \\ \swarrow \end{array}\right) = z\nabla\left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array}\right). \tag{3}$$

A knot invariant is called *Vassiliev invariant* of degree  $n$ , if, when linearly extended to linear combinations of knots, it vanishes of the subspace of  $(n + 1)$ -singular knots, in which each singular knot is mapped to a linear combination of knots by the rule

$$\begin{array}{c} \nearrow \\ \searrow \end{array} = \begin{array}{c} \nwarrow \\ \swarrow \end{array} - \begin{array}{c} \nwarrow \\ \swarrow \end{array}.$$

See [5]. We consider Vassiliev invariants valued in  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  or  $\mathbb{C}$ .

‘R.h.s’ (resp. ‘l.h.s’) will abbreviate ‘right-hand side’ (resp. ‘left-hand side’). In the sequel the symbol ‘ $\subset$ ’ denotes a not necessarily proper inclusion. Finally, let  $\Re$  and  $\Im$  denote the real and imaginary part, respectively. We will also write  $i = \sqrt{-1}$  for the imaginary unit, in situations where no confusion (with the usage as index) arises.

### 3. Vassiliev invariants and generalized signatures

#### 3.1. Outline of results

Here, we consider the generalized signatures (see [23]) and show that they are all as independent from any finite number of Vassiliev invariants as the classical signature is.

First, we determine the value range of  $\sigma_\xi$  on knots. This result, albeit possibly known, was never stated explicitly in previous publications. We will give a proof of it, both because it involves some subtleties which are worth remarking, and because it demonstrates some facts used to prove the result on Vassiliev invariants stated below.

**Proposition 1.** *The value range  $V_\xi \subset \mathbb{Z}$  of  $\sigma_\xi$  ( $|\xi| = 1, \xi \neq 1$ ) on knots is given by*

$$V_\xi = \begin{cases} \mathbb{Z} & \text{if } \left(2\Re \frac{\xi-1}{|\xi-1|}\right)^{-2} \text{ is an algebraic integer} \\ 2\mathbb{Z} & \text{else} \end{cases} . \tag{4}$$

(An algebraic integer is the root of a polynomial in  $\mathbb{Z}[x]$  with leading coefficient 1.)

The main result we prove in this section is a weaker version of Theorem 1, without the primeness property.

**Theorem 3.** *Any  $s \in V_\xi$  is realizable as the value of  $\sigma_\xi(K)$  of some (possibly composite) knot  $K$  which is  $n$ -similar to any fixed knot  $K'$  for any fixed  $n$ .*

Here, we call a knot  $K$   $n$ -similar to  $K'$  in Gousarov’s [14] sense if the Vassiliev invariants of degree  $< n$  of  $K$  and  $K'$  coincide. We will sometimes write  $K \sim_n K'$ . For the proof of Theorem 3 we will need the construction of [37] recalled below.

#### 3.2. $n$ -trivial rational tangles

*Rational tangles* were introduced by Conway [8]. The *Conway notation* of a rational tangle is a sequence of integers, to which a canonical diagram of the tangle is associated. (The order of the numbers is a matter of convention, so that some authors use the reverse sequences.) In the tangle notation of Conway, shown in Fig. 1, this diagram corresponds to the expression

$$a_1 \cdots a_n = ((a_1 a_2) a_3 \dots) a_n$$

(that is, the ‘product’, which is not associative, is written as if it is left-associative); see [1, Section 2.3]. A *rational knot* is the closure of a rational tangle.

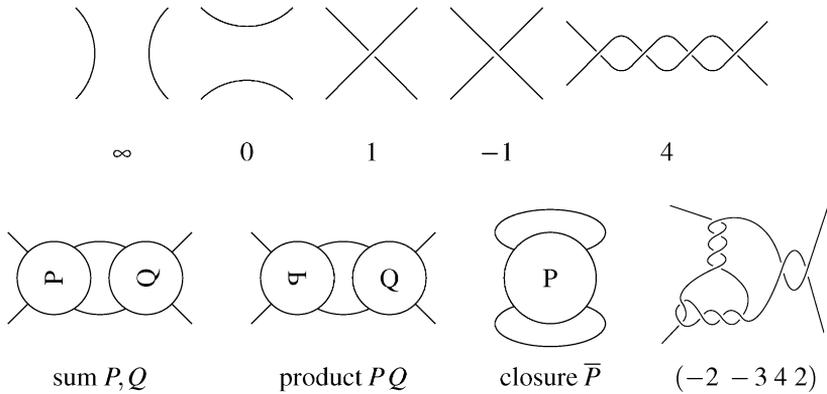


Fig. 1. Conway’s tangles and operations with them. (The designation ‘product’ is very unlucky, as this operation is neither commutative, nor associative, nor is it distributive with ‘sum’. Also, ‘sum’ is associative, but not commutative.)

Define the *iterated fraction* (IF) of a sequence of integers  $a = (a_1, \dots, a_n)$  recursively by

$$IF(a_1) := a_1, \dots, IF(a_1, \dots, a_{n-1}, a_n) := \frac{1}{IF(a_1, \dots, a_{n-1})} + a_n.$$

It will be helpful to extend the operations ‘+’ and ‘1/.’ to  $\mathbb{Q} \cup \{\infty\}$  by  $1/0 = \infty$ ,  $1/\infty = 0$ ,  $k + \infty = \infty$  for any  $k \in \mathbb{Q}$ . The reader may think of  $\infty$  as the fraction  $1/0$ , to which one applies the usual rules of fraction arithmetics and reducing. In particular reducing tells that  $-1/0 = 1/0$ , so that for us  $-\infty = \infty$ . This may appear at first glance strange, but has a natural interpretation in the rational tangle context. A rigorous account on this may be found in Krebs’s paper [22].

In this sense, *IF* is a map ( $\forall n \in \mathbb{N}$ )

$$IF : \mathbb{Z}^n \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

It is known [1], that diagrams of sequences of integers with equal *IF* belong to the same tangle (up to isotopy; where isotopy is defined by keeping the endpoints fixed). The correspondence is

$$a_1 \dots a_n \longleftrightarrow IF(a_1, \dots, a_n). \tag{5}$$

If  $IF(a_1, \dots, a_n) = p/q$  with  $(p, q) = 1$ , we call  $p$  the *numerator* of the tangle  $(a_1 \dots a_n)$  and  $q$  the *denominator*.

Using correspondence (5), one can convince himself, that a rational tangle  $T$  has a representation with all numbers of the same sign, or a different representation with all numbers even (and both signs), if one of numerator or denominator is even.

Define for a finite sequence of integers  $a = (a_1 \dots a_n)$  its reversion  $\bar{a} := (a_n \dots a_1)$  and its negation by  $-a := (-a_1 \dots -a_n)$ . For  $b = (b_1 \dots b_m)$  the term  $ab$  denotes the concatenation of both sequences  $(a_1 \dots a_n b_1 \dots b_m)$ .

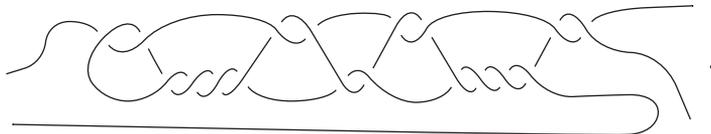
We call a tangle  $n$ -trivial, if all its Vassiliev invariants (defined analogously to the knot case) of degree  $< n$  are the same as for the 0-tangle. The following construction of such tangles was introduced in [37].

**Proposition 2** (Stoimenow [37]). *Fix some even  $a_1, \dots, a_n \in \mathbb{Z}$  and build inductively the integer sequences  $w_k = w_k(a_1, \dots, a_k)$  by*

$$w_1 := (a_1), \quad \dots \quad w_k := w_{k-1}(a_k) \overline{-w_{k-1}}. \tag{6}$$

*Then the rational tangles with Conway notation  $w_n$  are  $n$ -trivial, and, if all  $a_k \neq 0$ , non-trivial, i.e., not (isotopic to) the 0-tangle.*

**Example 1.** For  $a_1 = 2$ ,  $a_2 = -4$  and  $a_3 = 2$  we have  $w_1 = (2)$ ,  $w_2 = (2 \ -4 \ -2)$  and  $w_3 = (2 \ -4 \ -2 \ 2 \ 2 \ 4 \ -2)$ . The tangle  $w_3$  is shown below:



In other words, the replacement of a 0-tangle by a  $w_n$ -tangle in some diagram (possibly) changes the knot type, but preserves the values of Vassiliev (knot) invariants of degree  $< n$ .

### 3.3. Proof of Theorem 3

We start by a proof of a part of Proposition 1. Even although there are more direct arguments (coming from perturbation theory of linear forms [19]), we prefer not to be minimalistic, as we need to set up notations and tools needed in the following.

**Lemma 1.** *If for  $\zeta \in \mathbb{C}$  with  $|\zeta| = 1$ ,  $z_0 = \zeta^{1/2} - \zeta^{-1/2}$  is a simple zero of  $\nabla_K$  along the line between 0 and  $2i$ , where*

$$\zeta := - \left( \frac{1 - \bar{\zeta}}{|1 - \bar{\zeta}|} \right)^2 = \frac{\bar{\zeta} - 1}{1 - \zeta} = \zeta^{-1}, \tag{7}$$

*then  $\sigma_\zeta(K)$  is odd.*

**Proof.** Let

$$A_\xi = (1 - \xi)A + (1 - \bar{\xi})A^T$$

and  $\xi = \xi(t) = e^{2\pi it}$  (note that  $A = \bar{A}^T$  so that all eigenvalues of  $A_\xi$  are real). We have

$$\begin{aligned} \det(A_\xi) &= (1 - \xi)^{2g} \det\left(A - \frac{\bar{\xi} - 1}{1 - \xi} A^T\right) \\ &= (1 - \xi)^{2g} \left(\frac{\bar{\xi} - 1}{1 - \xi}\right)^g \Delta_K\left(\frac{\bar{\xi} - 1}{1 - \xi}\right) \\ &= (-1)^g |1 - \xi|^{2g} \Delta_K(\xi^{-1}) \\ &= (-1)^g |1 - \xi|^{2g} \nabla_K\left(2i \Im \sqrt{\xi^{-1}}\right) \\ &= (-1)^g |1 - \xi|^{2g} \nabla_K\left(2i \Re e \frac{\xi - 1}{|\xi - 1|}\right). \end{aligned}$$

As both maps  $t \mapsto |1 - \xi(t)|$  and

$$t \mapsto z(t) := 2i \Re e \frac{\xi(t) - 1}{|\xi(t) - 1|}$$

have non-zero derivatives for  $t \in (0, 1/2)$  ( $\xi = -1$  corresponds to the determinant which is never zero),  $t_0 = z^{-1}(z_0)$  is a simple zero of

$$t \mapsto \det A_{\xi(t)}.$$

This shows that

$$\left. \frac{\partial}{\partial t} \det A_{\xi(t)} \right|_{t=t_0} \neq 0.$$

Consider  $\chi(x, t) = \chi_{A_{\xi(t)}}(x)$  to be the characteristic polynomial of  $A_{\xi(t)}$  (whose absolute term is  $\det A_{\xi(t)}$ ). We have

$$\chi(0, t_0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \chi(0, t_0) \neq 0.$$

Then, by the implicit function theorem there is an  $\varepsilon > 0$  and a function  $\hat{t} : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$  with  $\hat{t}(0) = t_0$ , such that

$$\chi(x, t) = 0 \iff t = \hat{t}(x)$$

for  $x \in [-\varepsilon, \varepsilon]$  and  $t$  close to  $t_0$ . This means that, for  $t$  close to  $t_0$ , each eigenvalue  $\alpha_i = \alpha_i(\zeta(t))$  of  $A_{\zeta(t)}$  (with  $i = 1, \dots, 2g$ ) in  $[-\varepsilon, \varepsilon]$  is attained for only one value of  $t$ . Thus there is a *unique* eigenvalue of  $A_{\zeta(t)}$  in a neighborhood of 0 for  $t$  around  $t_0$ , and that eigenvalue indeed changes sign as  $\hat{t}$  exists for both positive and negative arguments. Finally, the eigenvalue may be a multiple, but it must have odd multiplicity because  $\nabla_K$  changes sign around  $z_0$  (and this multiplicity is locally constant because the dependence of the eigenvalues of  $A_{\zeta(t)}$  on  $t$  is at least continuous). Thus  $A_{\zeta(t_0)}$  has odd nullity, and hence also odd signature.  $\square$

**Remark 1.** One can show using [19] that in fact the multiplicity of the eigenvalue is indeed 1.

The fact that for a simple zero  $\zeta$  of  $\Delta$  on  $S^1$ ,  $\sigma_\zeta$  changes by  $\pm 2$  near  $\zeta = \overset{\circ}{\zeta}$  implies that the signature cannot vanish on both sides of  $\overset{\circ}{\zeta}$ . Thus we have

**Corollary 1.** *If  $K$  is achiral, then  $\Delta_K$  has no simple zero on  $S^1$ .*

The argument clearly shows the corollary also for slice knots, but in this case it easily follows from the Milnor–Fox condition  $\Delta(t) = f(t)f(1/t)$  [9]. It shows that in fact all the zeros of  $\Delta$  on  $S^1$  are of even order. It is tempting to conjecture that this is also true for achiral knots. It follows from [15] for strongly (positively or negatively) amphicheiral knots. It is also true for all the amphicheiral knots of up to 16 crossings. I have no proof in general, though.

**Lemma 2.** *If  $(2\Re \frac{\zeta-1}{|\zeta-1|})^{-2}$  is an algebraic integer, then  $V_\zeta$  contains some odd integer.*

**Proof.** If  $|\zeta| = 1$ , then  $\Delta_K(\zeta) = \nabla_K(2i\Im(\sqrt{\zeta}))$  for  $\zeta = \zeta^{-1}$ . Now, the polynomial  $\tilde{\nabla}_K(x) := \nabla_K(ix) \in 1 + x^2\mathbb{Z}[x^2]$ , and, as well-known, any polynomial in this affine ideal is  $\nabla_K$ , and hence  $\tilde{\nabla}_K$ , for some knot  $K$ . Therefore,  $x$  is a zero of  $\tilde{\nabla}_K$  for some  $K \iff 1/x^2$  is an algebraic integer. (Note, that here  $x = \pm 2\Im\sqrt{\zeta^{-1}} = \pm 2\Re(\frac{\zeta-1}{|\zeta-1|})$  is non-zero.) Moreover, in this case we can choose  $\tilde{\nabla}$  so as  $x$  to be a simple zero – as extensions of  $\mathbb{Q}$  are separable, simply take the minimal polynomial  $P_x$  of  $x$  if  $P_x = P_{-x}$ , or  $P_x P_{-x}$  otherwise. Then apply Lemma 1.  $\square$

**Remark 2.** Since for  $|\zeta| = 1$  we have  $\overset{\circ}{\zeta} = 1/\zeta$ , which is holomorphic and of non-zero derivative for  $\zeta \neq 1$ , and

$$\overset{\circ}{\zeta} \mapsto z(\overset{\circ}{\zeta}) = \overset{\circ}{\zeta}^{1/2} - \overset{\circ}{\zeta}^{-1/2}$$

is holomorphic and of non-zero derivative for  $\zeta \neq -1$ , we have for  $\zeta \neq \pm 1$  by Cauchy–Riemann that

$z(\zeta)$  is a simple zero of  $\nabla_K \iff \zeta$  is a simple zero of  $\Delta_K$  along  $S^1 \iff$   
 along line between 0 and  $2i$

$z(\zeta)$  is a simple zero of  $\nabla_K \iff \zeta$  is a simple zero of  $\Delta_K$ .

**Proof of Proposition 1.** We prove in (4) only the inclusion ‘ $\subset$ ’. The reverse inclusion will follow from Theorem 3, when taking the r.h.s. of (4) as a definition of  $V_\xi$ .

The case  $\zeta = 1$  is trivial, so we need to examine for which other  $\zeta$  the matrix  $(1 - \zeta)A + (1 - \bar{\zeta})A^T$  can be made singular (of odd nullity) for a Seifert matrix  $A$  of a knot  $K$ . This happens only if  $\Delta_K(\zeta^{-1}) = 0$  for some  $K$  (the zero being of odd order). But from the proof of Lemma 2, this is equivalent to  $(2\Re e^{\frac{\zeta-1}{|\zeta-1|}})^{-2}$  being an algebraic integer.  $\square$

**Remark 3.** The map  $\zeta \mapsto (2\Re e^{\frac{\zeta-1}{|\zeta-1|}})^{-2}$  for  $\zeta \in S^1 \setminus \{1\}$  is 2-1, the preimages of an element being conjugate and (up to this conjugacy) the dependence of the one quantity on the other is algebraic (in fact even quadratically radical, and the construction can be performed using ruler and compasses) so that the transcendency of the one is equivalent to the transcendency of the other. Therefore, in particular, as remarked in the introduction, for transcendent  $\zeta$ ,  $\sigma_\xi$  admits only even values on knots. (See [36] for more details on the algebraicity arguments.)

**Remark 4.** It is known that if  $\Delta_{K_1}(\zeta) \neq 0 \neq \Delta_{K_2}(\zeta)$ , and  $K_1$  is concordant to  $K_2$ , then  $\sigma_\xi(K_1) = \sigma_\xi(K_2)$ . However, by the examples of [24], there are slice knots with  $\sigma_\xi \neq 0$  for  $\Delta(\zeta) = 0$ . Ng showed [29] that any concordance class of knots contains one  $n$ -similar to a given knot  $K$  (modulo Arf invariant). This result implies for our values of  $\zeta$ , for which  $\Delta(\zeta)$  is never zero (on knots), e.g. for  $\zeta = -1$  or  $\zeta$  transcendental, but not for general  $\zeta$ . Also, since Ng’s knots are composite, her result does not imply any of the statements about prime knots we will make below.

Now we prove Theorem 3.

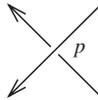
**Proof of Theorem 3.** We split the proof into three lemmas. For simplicity, we consider only the cases where  $(2\Re e^{\frac{\zeta-1}{|\zeta-1|}})^{-2}$  is an algebraic integer. The other cases are simpler and the argument for them is obtained by omitting irrelevant parts of the argument we describe for the  $V_\xi = \mathbb{Z}$  case. (As said, these cases can also be obtained from [29] using the results of [23].) Theorem 3 then follows by taking  $K$  to be the connected sum of  $K'$  with the knots constructed in the lemmas.

**Lemma 3.** *There is for any  $n$  an  $n$ -trivial knot  $K_n$  of odd  $\sigma_\xi$ .*

**Proof.** As we already saw in the proof of Lemma 2, there is a knot  $K$  with  $\sigma_\xi(K)$  odd. Let  $K'$  be an  $n$ -inverse of  $K$ , that is, a knot such that  $K' \# K$  is  $n$ -trivial ( $n$ -similar to the unknot). The existence of such  $n$ -inverses was proved by Gousarov [14]. Note, that a knot  $n$ -similar to an  $n$ -inverse is also an  $n$ -inverse. Thus we would be through (considering  $K_n = K \# K'$ ), if we show that  $K'$  can be chosen so that  $\Delta_{K'}(\xi^{-1}) \neq 0$ . This follows from the lemma below.  $\square$

**Lemma 4.** *For any knot  $K'$ , any  $n > 0$  and any  $\xi \in S^1 \setminus \{1\}$  there is a knot  $K$   $n$ -similar to  $K'$  such that  $\Delta_K(\xi^{-1}) \neq 0$ .*

**Proof.** By [30], we can assume up to  $n$ -similarity without loss of generality, that  $K'$  has unknotting number  $u(K') = 1$ . Assume that  $\Delta_{K'}(\xi^{-1}) = 0$ . Consider an unknotting number one diagram  $D$  of  $K'$  and a crossing  $p$  in  $D$  whose switch unknots  $K'$ .



(The sign choice of  $p$  here is irrelevant; the argument goes through also with the mirrored diagrams.)

Let  $D_0$  be the diagram obtained from  $D$  by smoothing out  $p$ , and  $K_0$  the 2 component link represented by  $D_0$ . As

$$\Delta_{K'}(\xi^{-1}) = 0 \neq 1 = \Delta_{K_0}(\xi^{-1}),$$

the skein relation (2) shows that

$$\Delta_{K_0}(\xi^{-1}) \neq 0.$$

Then consider the diagram

$$D_w = \text{[Diagram of a crossing } p \text{ with a rational tangle } w \text{ attached to the right strand]} \tag{8}$$

belonging to a knot  $K'_w$ , where  $w = w_n = w_n(a_1, \dots, a_n)$  is the  $n$ -trivial rational tangle considered before, depending on the even integers  $a_1, \dots, a_n$ . The new knot

$K'_w$  is  $n$ -equivalent to  $K'$ . Thus, we need to show that for some choice of the  $a_i$  the resulting  $K'_w$  has  $\Delta_{K'_w}(\xi^{-1}) \neq 0$ .

If we draw the tangle diagram of a rational tangle with all coefficients in the Conway notation even, it is easy to see that the half-twists counted by these coefficients are all reverse. Then the skein relation (2) for  $\Delta$  shows that, in analogy to the braiding polynomials considered in [39], any value of the Alexander polynomial of a rational knot depends polynomially on the coefficients in the Conway notation in the form with all coefficients even, and the dependence is linear in each single (even) coefficient.

Thus

$$(a_1, \dots, a_n) \mapsto P(a_1, \dots, a_n) = \Delta_{K'_w(a_1, \dots, a_n)}(\xi^{-1}) \in \mathbb{R}[a_1, \dots, a_n]$$

depends polynomially on the  $a_i$ 's, and the degree of  $a_i$  in this polynomial is less than or equal to the number of occurrences of  $a_i$  in the Conway notation of  $w_n$ , that is,  $2^{n-i}$ . The top degree coefficient of  $P(a_1, \dots, a_n)$ , that is, the coefficient of the monomial  $\prod_{i=1}^n a_i^{2^{n-i}}$ , is (up to a sign) equal to the product of a power of the (non-zero) number

$$(t^{1/2} - t^{-1/2}) \Big|_{t=\xi^{-1}}$$

with  $\Delta_{\tilde{D}_w}(\xi^{-1})$ . Here the link diagram  $\tilde{D}_w = D_{w(\infty, \dots, \infty)}$  arises from  $D_w$  by replacing  $w_n$  by  $w_\infty$ , and  $w_\infty$  is obtained in the same way as  $w_n$ , but formally setting all  $a_i = \infty$ , meaning composition with the  $\infty$ -tangle in the tangle calculus of Fig. 1.

Then  $w_\infty$  is just the  $\infty$ -tangle, and thus  $\tilde{D}_w$  depicts  $K_0$ , for which, as observed, we have  $\Delta_{K_0}(\xi^{-1}) \neq 0$ . Thus the top degree coefficient of  $P(a_1, \dots, a_n)$  is non-zero, so that for some choice of the  $a_i$ 's,  $P(a_1, \dots, a_n)$  will be non-zero, as desired.  $\square$

**Lemma 5.** *There exists an  $n$ -trivial (rational) knot  $\hat{K}_n$  of  $\sigma_\xi \in \{\pm 2\}$ .*

**Proof.** Consider for  $s \in 2\mathbb{Z}$  the knots  $K_{s, w_n} = \overline{(w_n, s)}$ . They unknot only switching crossings in a group of reverse twists (those counted by  $a_n$ ) and hence have  $|\sigma_\xi| \leq 2$  (such a move alters only a single diagonal entry in the Seifert matrix). We need to show that for some  $s$  and  $w_n$ ,  $\sigma_\xi(K_{s, w_n}) \neq 0$ .

The definitions of  $\sigma_\xi$  and  $\Delta$  in terms of Seifert matrices show (see the proof of Lemma 1), analogously to the classical case  $\xi = -1$  that, provided  $\Delta_K(\xi^{-1}) \neq 0$ ,  $\sigma_\xi(K)$  is even, and  $4 \mid \sigma_\xi(K)$  exactly if the (real) number  $\Delta_K(\xi^{-1})$  is positive, when  $\Delta$  is normalized so that  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ . Thus we would be through if for some  $s$  and  $w_n$  we could make  $\Delta_{K_{s, w_n}}(\xi^{-1}) < 0$ . Again this value depends polynomially on  $s$  and all  $a_i$  in  $w_n$ , and the top degree coefficient of this polynomial

$P$  is obtained, up to some non-zero multiple, by replacing  $s$  and all  $a_i$  by  $\infty$ . But the resulting diagram is an unknot diagram, so that the top degree coefficient of  $P$  is non-zero. As the top degree monomial of  $P$ , in which the  $a_i$  and  $s$  occur with the same multiplicities as in the notation  $(w_n, s)$ , is linear in  $s$  and  $a_n$ , so is  $P$  itself. But any non-trivial polynomial in  $k$  variables admitting only non-negative values on the whole  $(2\mathbb{Z})^k$  has even degree in *all* variables. This shows the lemma.  $\square$

To finish the proof of Theorem 3, now consider the families of knots

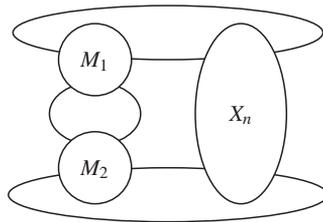
$$\begin{aligned} \mathcal{F}_1 &= \{K' \# K_n \#\#^l \hat{K}_n\}_{l \in \mathbb{N}}, \quad \mathcal{F}_2 = \{K' \#\#^l \hat{K}_n\}_{l \in \mathbb{N}}, \\ \mathcal{F}_3 &= \{K' \# K_n \#\#^l \hat{K}_n\}_{l \in \mathbb{N}}, \quad \text{and} \quad \mathcal{F}_4 = \{K' \#\#^l \hat{K}_n\}_{l \in \mathbb{N}}. \end{aligned} \tag{9}$$

Here  $K_n$  is the knot of Lemma 3, and  $\hat{K}_n$  the one of Lemma 5. Since any  $\sigma_\xi \in \mathbb{Z}$  is realized by some of the knots, one of the families contains the knot  $K$  we sought.  $\square$

#### 4. Constructing prime knots

The knots in the previous construction are composite. That we can modify them into prime knots we prove now. This is related to the proof of Theorem 2.

**Proof of Theorem 2.** We proceed as in [20]. Let  $K$  be a composite knot with prime factors  $K_1$  and  $K_2$  (more prime factors are dealt with inductively). Then we represent  $K_1$  and  $K_2$  as closures of prime tangles  $M_1$  and  $M_2$ . We consider a knot  $K' = (M_1, M_2)X_n$  for a suitable tangle  $X_n$ .



We need the following three properties of  $X_n$ :

1.  $X_n$  is prime,
2.  $X_n$  is  $n$ -trivial, and
3. replacing  $X_n$  for the 0-tangle preserves  $\Delta$ .

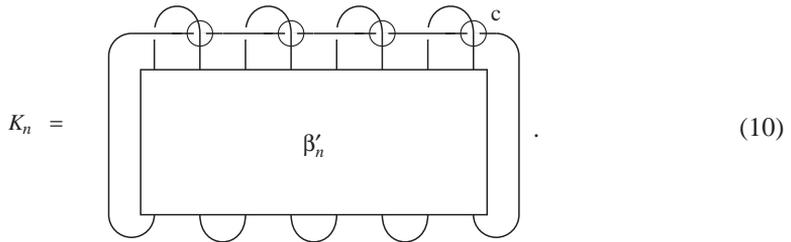
We recall from [20] that a tangle  $X \subset B^3$  is called prime, if it has no connected summand (a ball intersecting it in a knotted arc), and has no separating disk, i.e. a properly embedded disk  $D \subset B^3$ , with both components of  $B^3 \setminus D$  containing parts of  $X$ . The rational tangles are exactly those with no connected summand, but with a separating disk.

When  $X_n$  satisfies the above three conditions, then  $K'$  is prime from condition 1 by [20], and has the same Vassiliev invariants and Alexander polynomial by conditions 2 and 3.

To find  $X_n$ , we turn to a (special case of a) construction of  $n$ -trivadjacent knots due to Askitas and Kalfagianni [4].

Let  $B_n$  be the  $n$ -strand braid group and  $\sigma_i$  the Artin generators. Define a sequence of (pure) braids  $\beta_n \in B_n$  by  $\beta_2 = \sigma_1^{\pm 2}$  and inductively  $\beta_n = [\beta_{n-1}, \sigma_{n-1}^{\pm 2}]$ , where  $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$  is the commutator and  $\beta_{n-1} \in B_n$  is meant with respect to the canonical inclusion  $B_{n-1} \hookrightarrow B_n$ . (All the signs ‘ $\pm$ ’ can be chosen independently.) It is easy to see that  $\beta_n$  is “Brunnian”, that is, the removal of any strand(s) from  $\beta_n$  gives a trivial braid on the remaining strands.

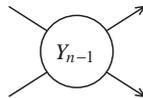
For  $\beta_n \in B_n$ , let  $\beta'_n \in B_{2n}$  be the doubled braid. (Each letter  $\sigma_i^{\pm 1}$  in  $\beta_n$  is turned into  $\sigma_{2i}^{\pm 1}\sigma_{2i-1}^{\pm 1}\sigma_{2i+1}^{\pm 1}\sigma_{2i}^{\pm 1}$ .) Consider the knot  $K_n$  built up from  $\beta'_n$  in the following obvious way (depicted for  $n = 4$ ):



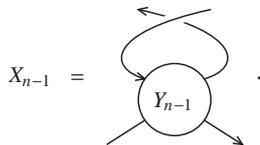
(Note, that here  $n$  of the strands of  $\beta'_n$  are reversely oriented.)

Such knots are constructed in [4].  $K_n$  is  $n$ -trivadjacent, i.e. for the set of  $n$  encircled crossings in (10) the switch of any non-empty set of these crossings gives an unknot diagram. Thus by [4, Theorem 5.2],  $\Delta_{K_n} = 1$  for  $n \geq 3$ .

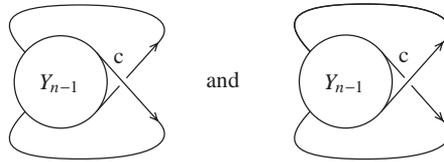
Let



be the complementary tangle to the rightmost encircled crossing  $c$  in (10). Let

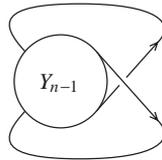


Since

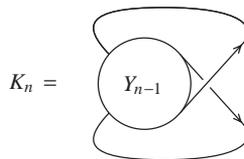


have unit Alexander polynomial, by a simple skein argument (see [6]) one sees that replacing  $Y_{n-1}$  for the 0-tangle preserves the Alexander polynomial, and hence the same is true for  $X_{n-1}$ . Also, one easily sees that  $Y_{n-1}$ , and hence  $X_{n-1}$ , are  $(n - 1)$ -trivial. Finally, we must show that  $X_{n-1}$ , or equivalently  $Y_{n-1}$ , is prime.

For this we must show that  $Y_{n-1}$  has no connected summand and is not rational. The first property is clear since



is the unknot. The second property would follow from the fact that



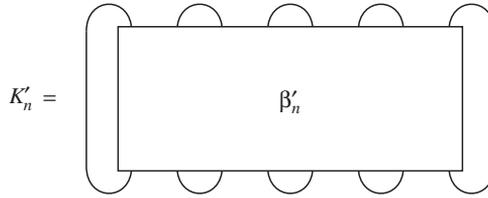
has unit Alexander polynomial, provided one can show that  $K_n$  is non-trivial, since no non-trivial rational knot has  $\Delta = 1$ . Thus the proof is concluded with the below lemma.  $\square$

**Lemma 6.**  $K_n$  is non-trivial.

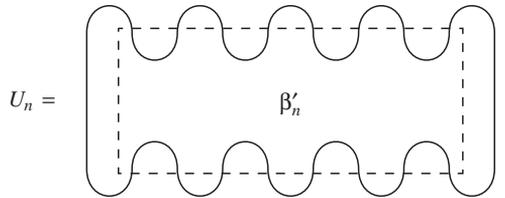
**Proof.** We prove by contradiction. Assume  $K_n$  is trivial.

We know from [35] that if in a skein triple  $L_+, L_0, L_-$  the links  $L_{\pm}$  are  $l$ -component unlinks, then  $L_0$  is the  $(l + 1)$ -component unlink. By iterating this argument, we

conclude that



is the  $(n + 1)$ -component unlink. Now,  $K'_n$  is the boundary of a disk  $D$  with  $n$  bands attached to it (corresponding to the doubled strands of  $\beta'_n$ ). The disk is obtained by removing the bands, i.e. replacing  $\asymp \cdots \asymp$  for  $\beta'_n$ :



Let  $U_n$  be the unknot bounding this disk. By [10] we know that if a link  $L$  is obtained from two (split) components  $L_1$  and  $L_2$  by band connecting,  $L$  has a minimal genus Seifert surface containing the band. Since the  $n$  bands attached to  $D$  can be viewed as connecting the  $(n + 1)$  components of  $K'_n$  to build an unknot, the assumption  $K_n$  is the unlink means that one must find a disk bounding  $U_n$  containing each band. As this disk  $D$  is unique up to isotopy, for each single band,  $D$  must be isotopable so as to contain it without intersecting the rest of the link  $K'_n$ . In particular, if one shrinks the bands into strands (i.e., ignores the framing), and connects their endpoints by arcs in  $D$ , one finds that the usual braid closure  $\hat{\beta}_n$  of  $\beta_n$  must be the  $n$ -component unlink.

To show that  $\hat{\beta}_n$  is non-trivial (and hence to have the contradiction we wish), consider the Jones polynomial value  $|V(e^{\pi i/3})|$  (or alternatively the number of torsion numbers of the double branched cover homology group divisible by 3) [25,12]. Since this is invariant under the insertion of  $\sigma_i^{\pm 3}$ , the assumption that

$$\hat{\beta}_n = \wedge ([\dots [\sigma_1^{\pm 2}, \sigma_2^{\pm 2}], \dots, \sigma_{n-1}^{\pm 2}])$$

is trivial means for

$$\tilde{\beta}_n = [\dots [\sigma_1^{\mp 1}, \sigma_2^{\mp 1}], \dots, \sigma_{n-1}^{\mp 1}]$$

that we have

$$|V_{\tilde{\beta}_n}(e^{\pi i/3})| = \sqrt{3}^{n-1}.$$

In particular, by [17, Proposition 15.3],  $b(\hat{\beta}_n) = n$ , with  $b$  denoting the braid index.



$\overline{S}_l \sim_n K$ . In general  $\Delta_{\overline{S}_l} \neq \Delta_K$ , but  $\Delta_{\hat{K}_n}(\xi) \neq 0$  implies that  $\xi$  is zero of all  $\Delta_{\overline{S}_l}$  of the same order. Hence still  $\sigma_\xi(K) - \sigma_\xi(\overline{S}_l)$  is even for all  $l$ . Also by the fact that  $T_n$  turns by undoing  $a_n$  reverse half-twists into the 0-tangle, we have  $|\sigma_\xi(\overline{S}_{l+1}) - \sigma_\xi(\overline{S}_l)| \leq 2$  (see proof of Lemma 5). Thus it suffices only to show that by adjusting  $l \in \mathbb{Z}$  one can make  $\sigma_\xi(\overline{S}_l)$  arbitrary large or small. But this is clear since by a crossing change one turns  $X_n$  into the 0-tangle, and hence  $\overline{S}_l$  into  $\#^l \hat{K}_n \# P$  (resp.  $\#^{-l} \hat{K}_n \# P$  for  $l < 0$ ), and  $\sigma_\xi(\hat{K}_n) \neq 0$ .

What remains is to justify our assumption that  $\xi$  is zero of  $\Delta_K$  of order 0 or 1. As  $\Delta_{\hat{K}_n}(\xi) \neq 0$ , this is equivalent under replacing  $\Delta_K$  by  $\Delta_P$ . That is, we must show that we can modify  $P$  up to  $n$ -similarity so as  $\xi$  to be a zero of order  $n\xi(K) \pmod 2$  of  $\Delta_P$ . This is proved in the below lemma. The knot found there may be composite, but applying the construction of Theorem 2 gives again a prime knot.  $\square$

**Lemma 7.** *Let  $K$  be a knot,  $\xi \in S_1 \setminus \{1\}$  and  $n > 0$ . Then there is a knot  $K' \sim_n K$  with  $\Delta_{K'}(\xi) \neq 0$ . Also there is a knot  $K' \sim_n K$  with  $\xi$  being a zero of  $\Delta_{K'}$  of order one, provided  $\Delta_K(\xi) = 0$ .*

**Proof.** That we can achieve  $\Delta_{K'}(\xi) \neq 0$  follows from the proof of Lemma 4.

Assume now  $\Delta_K(\xi) = 0$ . We would like to find  $K'$  such that  $\xi$  is a simple zero of  $\Delta_{K'}$  and  $K' \sim_n K$ .

Let  $K_1$  be a knot such that  $\xi$  is a simple zero of  $\Delta_{K_1}$ . That such  $K_1$  exists follows from the separable extension argument in the proof of Lemma 2, and the remark after this proof. Let  $K'_1$  be an  $n$ -inverse of  $K_1$ , chosen by the proof of Lemma 4 to have  $\Delta_{K'_1}(\xi) \neq 0$ . Finally, let  $\tilde{K}$  be a knot  $n$ -similar to  $K$  such that  $\Delta_{\tilde{K}}(\xi) \neq 0$ , also found by Lemma 4. Then take  $K' = \tilde{K} \# K_1 \# K'_1$ .  $\square$

### Appendix Unknotting numbers

We would like to note here an application of Tristram–Levine signatures independent from our previous constructions. It seems useful, and builds on some of the background we introduced, but is possibly too short to deserve a stand-alone exposition.

Let  $u(K)$  be the *unknotting number* of a knot  $K$ , the minimal number of crossing changes needed to make the unknot out of  $K$ . We call  $K = K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_n = \bigcirc$  ( $\bigcirc$  denoting the unknot) an *unknotting sequence* of  $K$ , if  $K_i$  and  $K_{i+1}$  differ by a crossing change. Thus the unknotting number of  $K$  is the minimal length  $n$  of an unknotting sequence of  $K$ .

We say that  $K$  has *positive unknotting number*  $n$  (denoted by  $u_+(K) = n$ ) if it unknots by switching  $n$ , but not less than  $n$ , positive crossings to negative. Similarly  $u_-(K) = n$  denotes the *negative unknotting number*. These concepts were introduced in [7]. Trivially  $u_+(K) = n \iff u_-(!K) = n$ ,  $!K$  being the mirror image of  $K$ , and  $u(K) = 1 \iff u_+(K) = 1$  or  $u_-(K) = 1$ . If in the unknotting sequence  $K = K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_n = \bigcirc$  we have that  $K_{i+1}$  differs from  $K_i$  by a change of

a positive crossing, then we call the unknotting sequence *positive*. In case  $K$  has no positive unknotting sequence, we set  $u_+(K) = \infty$ .

**Theorem 4.** *If there is a  $\zeta \in \mathbb{C}$  with  $|\zeta| = 1$  and  $\sigma_\zeta(K) = 0$  and  $\Delta_K(\overset{\circ}{\zeta}) = 0$  (with  $\overset{\circ}{\zeta}$  given as in (7)), then  $u_\pm(K) = \infty$ . In particular,  $u(K) \geq 2$ .*

**Proof.** Assume that  $u_+(K) = n < \infty$ . If  $K_+$  and  $K_-$  differ by a crossing which is positive resp. negative, we have  $A_\zeta(K_+) - A_\zeta(K_-) = \text{diag}(2 - 2\Re\zeta, 0, \dots, 0)$ . Then we know that if we order the eigenvalues of  $A_\zeta(K_\pm)$  non-increasingly  $\alpha_{1,\pm} \geq \alpha_{2,\pm} \geq \dots \geq \alpha_{2g,\pm}$ , we have  $\alpha_{i,+} \geq \alpha_{i,-}$ . This follows from a theorem attributed to Courant-Fischer and Weyl [42], and is now known as a special case of the complete description of inequalities for the eigenvalues of sums of Hermitian matrices due to Helmke–Rosenthal [16] and Klyachko [21]. Set e.g.  $j = 1$  in formula (3) of [2].

Then

$$\sigma_\zeta(K_+) = \sum_i \text{sgn}(\alpha_{i,+}) \geq \sum_i \text{sgn}(\alpha_{i,-}) = \sigma_\zeta(K_-).$$

Thus  $\sigma_\zeta(K_+) = \sigma_\zeta(K_-)$  only if  $\text{sgn}(\alpha_{i,+}) = \text{sgn}(\alpha_{i,-})$  for all  $i = 1, \dots, 2g$ . If  $\Delta_{K_+}(\overset{\circ}{\zeta}) = 0$ , then some of the  $\alpha_{i,+}$ , and hence  $\alpha_{i,-}$  vanishes, so that also  $\Delta_{K_-}(\overset{\circ}{\zeta}) = 0$ . If now  $K = K_0 \rightarrow K_1 \rightarrow \dots \rightarrow K_n = \bigcirc$  is a positive unknotting sequence, we have  $\sigma_\zeta(K_{i+1}) \leq \sigma_\zeta(K_i)$  with strict inequality at least once, as the zero  $\overset{\circ}{\zeta}$  of  $\Delta_K$  must disappear under some of the crossing changes. But then  $\sigma_\zeta(\bigcirc) < \sigma_\zeta(K) = 0$ , a contradiction.

The argument for  $u_-(K) = \infty$  is analogous.  $\square$

Since all Tristram–Levine signatures vanish on achiral knots, we have

**Corollary 2.** *If  $K$  is achiral and  $\Delta_K(S^1) \ni 0$ , then  $u(K) > 1$ .*

Unfortunately, this is not necessarily true for slice knots, as by the examples of [24], we may have  $\zeta$  with  $\sigma_\zeta \neq 0$  and  $\Delta(\overset{\circ}{\zeta}) = 0$ . In fact, Corollary 2 gives an easy way of finding such examples, without examining the Seifert form. (Note, that in [24], Levine just writes down Seifert forms, and claims they come from knots without giving the knots explicitly, though.)

**Example 2.** Consider, for example,  $8_{20}$ , which is slice and of unknotting number one, and  $\zeta = e^{\pi i/3}$ . As  $8_{20}$  has the Alexander polynomial of the square (and granny) knot,  $\Delta(\overset{\circ}{\zeta}) = 0$ , and one can in fact calculate that  $\sigma_\zeta(8_{20}) = -1$  and  $n_\zeta(8_{20}) = 1$ .

The signed unknotting number information of the theorem can be combined with other conditions.

**Proposition 3.** *Assume  $K$  satisfies the premise of Theorem 4. If additionally (a)  $V_K(e^{\pi i/3}) = -3$  or (b) there is some  $\zeta'$  with  $\sigma_{\zeta'}(K) = \pm 4$ , then  $u(K) > 2$ .*

**Proof.** If  $u(K) = 2$ , the additional conditions imply  $u_{\pm}(K) = 2$ , contradicting Theorem 4. (For condition (a) this follows from the inequality  $\sigma_{\xi'}(K_+) \leq \sigma_{\xi'}(K_-) + 2$ , analogous to Murasugi’s inequality for  $\xi' = -1$ , and for condition (b) from the argument of Traczyk [41].)  $\square$

To obtain some interesting examples, we consider twist knots. It is easy to see that for those of even crossing number, the Alexander polynomial has no zeros on the unit circle, while for those of odd crossing number, there in one pair of conjugate zeros, moving towards 1 when the crossing number increases.

**Example 3.**  $u(3_1\#3_1\#7_2) = 3$ . (Here the factor knots are so mirrored so as the knot to have signature 2.)

In particular, we have

**Corollary 3.** *If  $K$  is achiral and  $0 \in \Delta_K(S^1)$  and  $V_K(e^{\pi i/3}) = -3$ , then  $u(K) \geq 3$ .*

**Example 4.**  $u(3_1\#3_1\#4_1) = 3$ .

This solves one of the undecided numbers in the tables of [38]. No previous method seems to give this result.

**Remark 5.** As we proved above,  $\Delta(\xi) = 0$  and  $\sigma_{\xi} = 0$  imply that  $\xi$  is (at least) a double zero of  $\Delta$ . Already the existence of such a double zero, unfortunately, limits the space of applicable examples, and makes it most likely to obtain new information for composite knots, as we saw above.

The argument in the proof of Theorem 4 works in fact assuming a more general condition on the nullity  $n_{\xi}(K)$ .

**Theorem 5.** *Assume  $K$  is a knot for which there is a  $\xi \in \mathbb{C}$  with  $|\xi| = 1$  and  $\xi \neq 1$  with  $n_{\xi}(K) > \sigma_{\xi}(K)$ . Then  $u_+(K) = \infty$ .*

**Proof.** Assume the contrary. By the argument in the proof of Theorem 4, we obtain that the  $n_{\xi}(K)$  zero eigenvalues must become negative in  $K_n = \bigcirc$ , in order  $\Delta_{K_n}(\xi) \neq 0$ , but then  $0 = \sigma_{\xi}(K_n) \leq \sigma_{\xi}(K) - n_{\xi}(K) < 0$ , a contradiction.  $\square$

The condition of this theorem is unfortunately seldom satisfied. For example, if  $K = 3_1\#3_1\#3_1$  and  $\xi = e^{\pi i/3}$ , then  $n_{\xi} = 3 > 1 = \sigma_{\xi}$ . However, for the really interesting example  $K = 8_{10}$  (with the same Alexander polynomial), we have only  $n_{\xi} = 1$  (a single eigenvalue changes sign cubically), so that the desired conclusion  $u_+(K) > 1$  fails.

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