ALGEBRAIC COMPUTATIONS OF THE INTEGRAL CONCORDANCE AND DOUBLE NULL CONCORDANCE GROUP OF KNOTS

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I. INTRODUCTION

The first half of this note is a summary of the algebraic results concerned with the classification of isometric structures of the integers arising in knot theory under the concordance or metabolic equivalence relation. A detailed expostion can be found in the author's Memoir [St]. Briefly, the Seifert linking pairing, L, on the free submodule, M, of the middle dimensional homology of a Seifert manifold for an odd dimensional knot defines an endomorphism, t, of M by the equation

i) $L(x,y) = b(t(x),y)$

where b is the bilinear unimodular intersection pairing on M. From the symmetries satisfied by L we obtain the following relation

ii) $b(t(x),y) = b(x,(Id - t)(y))$

Such objects (M,b,t) are called isometric structures over the integers. It is called metabolic if there is a submodule N which is invariant under t and equal to its own annihilator under b, $N = N^{\perp} = \{m \text{ in } M \mid b(m,n) = 0 \text{ for each } n \text{ in } N\}$.

The algebraic technique of localization allows us to relate the integral case to the rational case which was computed by Levine [L]. Unlike the rational field case there are obstructions to the decomposition of an integral isometric structure according to the Z[X]-module structure induced by t which are measured by the coupling exact sequence. This reduces the explicit computation to modules over orders in some algeraic number field, where the final computations are made.

When (M,b,t) is metabolic on N, there is an exact sequence of Z[X]-modules

iii) $0 \longrightarrow N^{\perp} \longrightarrow M \longrightarrow \mathrm{Hom}_Z(N,Z) \longrightarrow 0.$

When this sequence splits, the isometric structure is called hyperbolic. This is

a necessary condition for the geometric condition of double null concordance of a knot studied by Dewitt Sumners [S] . Stabilization with this relation defines a new group of knots under the operation of connected sum which is much larger than the knot concordance group. In fact, a simple knot is trivial in this group only if it is (stably) isotopic to the connected sum of a knot with its inverse. Furthermore, the even dimensional group is non-trivial contrasting with the even-dimensional knot concordance group, which is zero [K]. The above techniques and ideas also apply to isometries of integral inner product spaces which arise geometrically in the bordism of diffeomorphsim question solved by Kreck [Kr]. The application may also be found in[St].
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II. THE METABOLIC CASE

Let R be a Dedekind domain, in particular the integers, Z, the rational field, Q, or a finite field with q elements, F_q. Let $\varepsilon = \pm 1$.

Definition 2.1 An ε-symmetric isometric structure over R is a triple (M,b,t) where M is a finitely generated R-module, b is an ε-symmetric bilinear form on M with values in R and t is an R-linear endomorphism of M satisfying:

i) (M,b) is an inner product space, that is the adjoint homomorphism, Ad $b: M \longrightarrow \mathrm{Hom}_R(M,R)$ given by $\mathrm{Ad}\, b(m) = b(m,-)$, is an isomorphism.

ii) $b(t(x),y) = b(x,(Id - t)(y))$

Let K denote the field of fractions of R. we will relate isometric structures over R and K by means of the following:

Definition 2.2 An ε-symmetric torsion isometric structure over R is a triple (T,b,t) where T is a finitely generated torsion R-module, b is an ε-symmetric bilinear form on T with values in the R-module K/R and t is an R-linear endomorphism of T with:

i) Ad $b: T \longrightarrow \mathrm{Hom}_R(T,K/R)$ is an isomorphism

ii) $b(t(x),y) = b(x,(Id - t)(y))$

An isomorphism of isometric structures must preserve the inner product and commute with the endomorphism. The isomorphism classes form a semigroup under the operation of orthogonal direct sum. We now define an equivalence relation so that the equivalence classes form a group.

Definition 2.3 An isometric structure is metabolic if there is an R-submodule N

i) N is t invariant, that is $t(N)\subseteq N$, and

ii) $N = N^{\perp} = \{m \text{ in } M: b(m,N) = \{ b(x,n): n \text{ in} N \} = \{0\}\}$, the annihilator of N under the inner product b. We call N a metabolic submodule or simply, metabolizer.

Examples 2.4

i) The diagonal D in $(M,b,t) + (M,-b,t)$ is a metabolic submodule

ii) Given an R-module N and an R-linear endomorphism s, where R is torsion free or completely torsion, the hyperbolic isometric structure $H(N,s) = (N + N^*, b, t)$ where $N^* = \mathrm{Hom}_R(N,R)$ in the torsion free case and $\mathrm{Hom}_R(N,K/R)$ is the torsion case with $b((x,f),(y,g)) = f(y) + \varepsilon g(x)$ and $t(x,f) = (s(x), f\circ(Id-s))$ has metabolic summands N and N^*.

iii) The torsion isometric structure $H(Z/(m^2), Id)$ has a metabolizer mT which is not a direct summand.

Definition 2.5 Two isometric structures, M and N, are Witt-equivalent (or concordant) if there are metabolic isometric structures, H and K, such that M + H is isometric with N + K. The set of equivalence classes form a group, denoted $C^{\varepsilon}(R)$ $(C^{\varepsilon}(K/R)$ in the torsion case), under orthogonal direct sum. The inverse of (M,b,t) is $(M,-b,t)$ as in example 2.4 i).

$C^{\varepsilon}(Z)$, which was first defined by Kervaire in [K], is well-known to be isomorphic to the geometric knot concordance group in dimensions above one and to have infinitely many elements of each possible order, two, four and infinite, [K,L] . We wish to further elucidate its structure. The first question we will solve is which rational isometric structures contain unimodular integral isometric structures?

Let (V,B,T) be an isometric structure on the field K, the fraction field of the Dedekind domain R. An R-lattice in V is a finitely generated R-submodule of V. An obvious necessary condition for an R-lattice L to be invariant under T, is that T satisfy a monic polynomial with coefficients in R. (If R = Z, then L is a free Z-module and this is the theorem of Cayley and Hamilton, in a general Dedekind domain, this applies to each localization.) Let $C_0^{\varepsilon}(K)$ be the Witt group of isometric structures over K satisfying $f(T)=0$ for some monic polynomial with coefficients in R. If $\{x_i\}$ is a basis for V, then the R-module generated by $T^j x_i$, $j < \deg f$, is an R-lattice invariant under T. Since K is the fraction field of R, we may scale the lattice L by the product of the denominators of the inner product B on the finite set $T^j x_i$ to obtain a new lattice dL on which the inner product B is R-valued. Define the lattice $L^\# = \{v \text{ in } V: B(v,l) \text{ is in } R, \text{ for every } l \text{ in } L\}$. This lattice is also invariant under T, so that T induces an endomorphism of the quotient $L^\#/L$, say t. Finally we define a K/R valued inner product on $L^\#/L$ by $b(\overline{u},\overline{v}) = B(u,v) \bmod R$. This is a torsion isometric structure and its equivalence class in $C^\varepsilon(K/R)$ is independent of the choice of the lattice L. We have defined a homomorphism $\partial: C^\varepsilon(K) \to C^\varepsilon(K/R)$, called the boundary.

<u>LOCALIZATION THEOREM 2.6</u> the following sequence is exact:

$$0 \longrightarrow C^\varepsilon(R) \xrightarrow{\ i\ } C^\varepsilon_0(K) \xrightarrow{\ \partial\ } C^\varepsilon(K/R)$$

In the case R = Z, the boundary is onto, making the sequence short exact. The map i is given by localization at zero (or tensor product with the field of fractions.)

<u>Example</u>: Let R = Z, and consider the rational isometric structure, given with respect to a basis, e_1 and e_2, by the matrices ($V = Q + Q$, $B = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Then $L^\# = 1/5\, e_1 + 1/5\, e_2$ and $L^\#/L = Z/(5) + Z/(5)$ with the form $\begin{pmatrix} 0 & 1/5 \\ -1/5 & 0 \end{pmatrix}$. The form has a one-dimensional self-annihilating subspace generated by $e_1 - e_2$, but no one dimensional line in $L^\#/L$ over the field F_5 can be invariant since the minimal polynomial is X^2-X+1, which is irreducible mod 5.

We now introduce a very powerful notion which simplifies the computation of the last two terms in the sequence.

<u>Definition 2.7</u> An isometric structure is <u>anisotropic</u>, if for any t invariant R-submodule N, $N \cap N^\perp = 0$

<u>Proposition 2.8</u> Every Witt equivalence class has an anisotropic representative.

<u>Proof</u>: Let L be an invariant pure R-submodule with $L \subset L^\perp$. Then $L^\perp/L$ inherits a quotient isometric structure and $N = \{(x,x+L): x \text{ is in } L\}$ is a metabolizer.

We now note that the endomorphism t must have kernel 0 in any anistropic representative, for $b(x,(1-t)y) = b(x,y) = b(tx,y) = b(0,y) = 0$ if x and y are in the kernel. Hence the kernel is invariant and self-annihilating. If R is a field or if (M,b,t) is a torsion structure on a finite module M, then t is invertible and $s = Id - t^{-1}$ is an isometry of b, by an easy verification.

Let Λ denote the polynomial ring $R[X]$. It has an involution induced by $X^* = 1-X$. An isometric structure over R can be viewed as a Λ-module in the traditional way, by allowing the indeterminate X to act as the endomorphism t. We will denote restrictions on the module structure by subscripts, for example, if I is an ideal in R, $C_{R/I}(K/R)$ will denote the Witt group of torsion isometric structures on R/I-modules, that is torsion R modules annihilated by I. As a second application of the existence of anisotropic representatives, we have the following:

Theorem 2.9 The following inclusions are isomorphisms:

$$\bigoplus_{I,\text{ maximal ideal in } R} C^{\varepsilon}_{R/I}(K/R) \subset C^{\varepsilon}(K/R)$$

$$\bigoplus_{I=I^*,\text{ invariant maximal ideal}} C^{\varepsilon}_{\Lambda/I}(F) \subset C^{\varepsilon}(F)$$

where R is a Dedekind ring of finite rank over Z and F is a field.

Proof: If N is a t invariant subspace in M, an anisotropic structure, then $N \cap N^{\perp} = 0$. If the module M is a vector space over a field or if M is a torsion module with finite cardinality (this is assured by the condition on R in the torsion case) then the exact sequence: $0 \longrightarrow N^{\perp} \longrightarrow M \longrightarrow \mathrm{Hom}_R(N,R) \longrightarrow 0$, where the second map is the adjoint homomorphism restricted to N, implies that $M = N + N^{\perp}$ by a dimension or cardinality argument. Hence the annihilator of an irreducible anisotropic must be a maximal ideal in the appropriate ring, R or Λ. Furthermore in the field case, the annihilator must be invariant under the involution for otherwise, if a is in the annihilator I, then $0 = b(ax,y) = b(x,a^*y)$ and if $a^*y \neq 0$ for some y then the form is singular. Injectivity is verified in [St].

In the field case the invariant maximal ideals in F[X]are generated by self-dual monic polynomials $p = p^*$, where $p^*(X) = (-1)^{\deg p} p(1-X)$. In the torsion case there is a further reduction. If Iis a maximal ideal, R/I is a field and if the torsion R-module is annihilated by I it must take its values in the submodule $I^{-1}/R = \{k : kI \subset R\}/R$ in K/R which is isomorphic to R/I by a choice of uniformizer. (If R = Z we make this choice canonical by choosing the positive prime element.) We then have $C^{\varepsilon}_{R/I}(K/R) = C^{\varepsilon}(R/I)$, and we may apply the field case of (2.9) to obtain a further decomposition according to the R/I[X]structure.

Corollary 2.10: The following inclusion is an isomorphism:

$$\underset{I=I^*,\ R[X]/I\ \text{finite}}{+} C^{\varepsilon}_{R[X]/I}(R/(I\cap R)) \longrightarrow C^{\varepsilon}(K/R)$$

in the integral case, R = Z, the involution on $Z[X]/I$ is trivial only if $X = X^*$ $= 1-X \bmod I$, that is $2X = 1 \bmod I$, so 2 is in I and $Z[X]/I = F_p$ for some odd prime p.

$$C^{\varepsilon}_{R[X]/I}(R/I)= \begin{cases} Z/(2) + Z/(2) & p = 1 \bmod 4 \\ Z/(4) & p = 3 \bmod 4 \end{cases}\ \varepsilon = +1 \quad \text{Trivial involution} \qquad \begin{cases} 0 & \varepsilon = -1 \text{ (trivial involution)} \\ Z/(2) & \text{if the involution is non-trivial} \end{cases}$$

The first three computations are the well-known computations of the Witt groups of finite fields, the last is a computation of the witt group of Hermitian forms over a finite field which will follow from the theory of the succeeding paragraphs.

We must now make use of the $R[X]$-module structure to make further computations. The setting we desire is specified as follows: Let S be an R-algebra, finitely generated as an R-module,Δ an S-module and L an R-module. Suppose also there is an R-linear map $s:\Delta \to L$ such that $s_*:S \times \Delta \to L$ given by $s_*(a,k) = s(ak)$ is non-singular (that is, both adjoints are R-module isomorphisms). Then we have the following trace lemma of Milnor [M]and Knebusch and Scharlau [KS]:

<u>TRACE LEMMA 2.11:</u> s_* induces an equivalence of the category of non-singular L-valued bilinear forms, (,), on R-modules with an S-module structure and the category of Δ-valued forms $\langle\ ,\ \rangle$ on S-modules. The equivalence is given by $s_*\langle\ ,\ \rangle = (\ ,\ )$. Furthermore, if there is an involution * on S, trivial on R such that $s(a^*) = s(a)$, then if (,) is ε-symmetric and satisfies $(ax,y) = (x,a^*y)$ then (,) corresponds to a ε-Hermitian form $\langle\ ,\ \rangle$.

<u>Examples 2.12 i)</u> If R is a field and S is a finite degree separable field extension, then $\Delta = S$, $L = R$ and $s = \text{trace}_{S/R}$ satisfies the conditions of the trace lemma (whence its name). This induces an isomorphism $C^{\varepsilon}_{R[X]/I}(R) = H^{\varepsilon}$ ($S = R[X]/I$), the Witt equivalence classes of ε-Hermitian forms over S. This completes the computation of $C^{\varepsilon}(Q)$ using Landherr's Theorem 2.14 (following).

ii) Let P be a prime ideal in R[X]and let S = R[X]/P. This is an R-order in a finite degree extension E of the fraction field K, provided that $P\cap R = 0$. Making this assumption, let $\Delta = \Delta^{-1}(S/R) = \{e \text{ in } E: \text{trace}_{E/K}(eS)\subset R\}$, the inverse different of S. By its very definition, the trace induces a nonsingular pairing $S \times \Delta \to R$. This example is very important in the further computation of $C^\varepsilon(Z)$.

iii) Let $P = (X^2-X+1)$ and $S = Z[X]/P$ with an integral basis 1 and $a = (1+\sqrt{-3})/2$. Then Δ is $(1/p'(a))S = (1/\sqrt{-3})S$ by a classical theorem of Euler. Consider the Δ-valued Hermitian form on the rank one free module S, given by $[x,y] = xy^*/\sqrt{-3}$. This is a non-singular form and corresponds to the symmetric isometric structure (S, b = trace $[\ ,\ ]$, t = multiplication by a) = $\left(Z + Z, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$ in matrix notation.

The beneficial decompostion of M as $N + N^\perp$ for any invariant submodule N of an anisotropic structure in the field and torsion case fails in the integral case. In $R[X]$the prime ideals result from an interaction between the primes in R and those in $K[X]$. This is featured in a second exact sequence which measures the failure of the above-mentioned decomposition.

<u>COUPLING EXACT SEQUENCE 2.13</u>: There is an exact sequence

$$0 \longrightarrow \bigoplus_{\substack{P=P^*,\ \text{prime in } R[X] \\ R\cap P = 0}} C^\varepsilon_{R[X]/P} \longrightarrow C^\varepsilon(R) \xrightarrow{\ C\ } \sum_P C^\varepsilon_{R[X]/P}(K/R)$$

The coupling map is defined as follows: Given (M,b,t) let $M_P = \{m : P^i m = 0$ for some $i\}$. The form b restricted to M is non-degenerate since if x is in M_P, there is a y in M so that $b(x,y) \neq 0$. But $0 = b(ax,y) = b(x,a^*y)$ hence y must be in M_P. Therefore M_P is a non-degenerate sub-lattice of M and we may define $c_p(M) = (M_P^\#/M_P, \overline{b}, \overline{t})$ as in the definition of boundary. The coupling map C is the direct sum of the c_P. In general the cokernel of C is known but it is not well-understood.

Now, by the trace lemma and the localization sequence, we have a commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & C^\varepsilon_{R[X]/P} = S^{(R)} & \longrightarrow & C^\varepsilon_{R[X]/P}(K) & \longrightarrow & C^\varepsilon_{R[X]/P}(K/R) \\
 & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & H^\varepsilon(\Delta^{-1}(S/R) = \Delta) & \longrightarrow & H^\varepsilon(E = K[X]/P) & \longrightarrow & H^\varepsilon(E/\Delta)
\end{array}$$

where the vertical arrows are isomorphisms. We will make a further reduction and compute the boundary homomorphism only for the maximal order in E, the ring of algebraic integers D, and obtain the computation for S from the appropriate commutative diagram of forgetful maps.

We now recall Landherr's theorem on the computation of the Witt group of Hermitian forms over an algebraic number field E with involution $*$. Denote by $Q(E,*)$ the semi-direct product $Z/(2) \times F^\cdot/NE^\cdot$ where $NE^\cdot = \{ee^*\}$is the image of the norm map $N_{E/F}: E^\cdot \to F^\cdot$

and the multiplication is given by $(e_1,d_1)(e_2,d_2) = (e_1 + e_2,(-1)^{e_1+e_2} d_1d_2)$. To each conjugate pair of equivariant embeddings $(E,*) \to (C, -)$ where C is the complex numbers with complex conjugation -, there is an associated signature homomorphism: σ_i: $H(E) \to H(C) = Z$.

Landherr's Theorem 2.14: There is an exact sequence:

$$0 \longrightarrow (4Z)^s \longrightarrow H(E,*) \longrightarrow Q(E,*) \xrightarrow{w} 0$$

where w is (rank mod 2, disc = $(-1)^{n(n-1)/2}$det) and n is the dimension of the form. The kernel of w is detected by the s signature homomorphisms which are $\equiv 0$ mod 4.

The group $F^{\cdot}/NE^{\cdot}$ is computed by the Hilbert symbols $(d,z^2)_{P_0} = \pm 1$ at all the prime ideals P_0 in the fixed field F, where $E = F(\sqrt{z^2})$, by the Hasse Cyclic Norm Theorem and can be realized arbitratily subject to Hilbert Reciprocity that only an even finite number can be - 1. For the next result we will suppose that no dyadic prime ramifies in the extension E/F. This is true in all the number fields arising in the knot concordance group. After performing the technical task of relating the boundary homomorphism to the Hilbert symbols we obtain:

Theorem 2.16: Let D be the ring of integers in an algebraic number field with involution. Then $H^{+1}(\Delta^{-1}(D/Z))$ is computed as follows:

i) there is a rank one form if and only if no prime ramifies in the extension E/F.

ii) Let JH be the subgroup of elements of even rank and H, the Hilbert reciprocity homomorphism. Then the following sequence is exact:

$$0 \longrightarrow JH^{+1}(\Delta^{-1}(D/Z)) \xrightarrow{\sigma} (2Z)^s \xrightarrow{H} (Z^{\cdot} = \pm 1) \longrightarrow 1$$

In particular there are no elements of order four and this is true in general for $H(\Delta^{-1}(S/Z)) \approx C_S^{+1}(Z)$.

Corollary 2.17: An element of order four in $C^{+1}(Z)$ must have a non-trivial coupling invariant and an Alexander polynomial with distinct factors.

We now give a complete description of two important subgroups of the knot concordance group: those whose minimal polynomials are a product of quadratic polynomials (this is related to low genus knots) and those whose Alexander polynomials are a product of cyclotomic polynomials (this is case for Milnor-Brieskorn knots).

Let λ_n be a primitive n^{th} root of unity and, denote by $p_n(X)$ the minimal polynomial of $(1 - \lambda_n)^{-1}$. This is a monic polynomial with integer coefficients if and only if n is composite. Let C be the semigroup of polynomials generated by the p_n under multiplication. Applying the coupling exact sequence 2.13) to this case, we have:

Cyclotomic Coupling Theorem 2.18: $C^{\varepsilon}_{\Lambda/(p_n)}(Z)$ is torsion free and the following is exact: $0 \to + C^{\varepsilon}_{\Lambda/p_n}(Z) \longrightarrow C^{\varepsilon}_{C}(Z) \longrightarrow + C^{\varepsilon}_{\Lambda/p_n}(Q/Z) \to C^{\varepsilon}(Q/Z) \to 0$

Any monic self-dual quadratic polynomial must be of the form: $X^2 - X + b$. Let d be the largest square free divisor of the discriminant and let a be the number of prime divisors of 1-4b not dividing d and b be the number of prime divisors of d. Denoting by T the semigroup generated by the quadratic self-dual polynomials, we have:

Quadratic Coupling Theorem 2.19: The following sequence is exact:

$$0 \longrightarrow \bigoplus_{p=p^*,\ \deg = 2} C^{\varepsilon}_{\Lambda/p}(Z) \longrightarrow C^{\varepsilon}_{T}(Z) \longrightarrow \bigoplus C^{\varepsilon}_{\Lambda/p}(Q/Z) \longrightarrow C^{\varepsilon}_{T}(Q/Z) \oplus \bigoplus \operatorname{Coker} c_p \longrightarrow 0$$

ε	Sign(1-4b)	$C^{\varepsilon}_{\Lambda/p}(Z)$	Coker c_p
+1	-	2Z	$(Z/(2))^{a+b-1}$
+1	+	0	$(Z/(2))^{a+b-1}$
-1	-	$Z + (Z/(2))^{a+b-1}$	0
-1	+	order = 2^{a+b} two torsion except single element of order 4 iff prime = 3(4) divides d	0

$$C^{\varepsilon}_{\Lambda/p}(F_q) = \begin{cases} Z/(2) & (1-4b|q) = -1 \\ 0 & (1-4b|q) = 1 \\ (Z/(2))^2 & p \equiv 1(4),\ 1-4b \equiv 0(q),\ \varepsilon = +1 \\ Z/(4) & p \equiv 3(4),\ 1-4b \equiv 0(q),\ \varepsilon = +1 \\ 0 & 1-4b \equiv 0(q),\ \varepsilon = -1 \end{cases}$$

where (|) is the Legendre symbol.

This has been a brief outline of the structure of $C^{\varepsilon}(Z)$ elucidated in complete detail in [St], together with many explicit examples of a more arithmetic nature and a geometric analog of the localization sequence.

III. THE HYPERBOLIC CASE

We now introduce another relation on the semigroup of isometric structures under the operation of orthogonal direct sum. Although this relation was first conceived algebraically, we will give a geometric interpretation in Theorem 3.13.

Definition 3.1: The hyperbolic ε-symmetric isometric structure on (N,s) where N is either a torsion or a torsion free R-module and s is an R-module endomorphism is the structure (M,b,t) where:

i) $M = N + \mathrm{Hom}_R(N,R)$ $(\mathrm{Hom}_R(N,K/R)$ in the torsion case$)$ $= N + N^*$

ii) $b((x,f),(y,g)) = f(y) + \varepsilon g(x)$

iii) $t(x,f) = (s(x), f \circ (\mathrm{Id} - s))$

As we have seen, the R[X]- module structure plays a strong role in the analysis of isometric structures. The following proposition relates the above definition to this structure.

Proposition 3.2: An ε-symmetric isometric structure over R is hyperbolic if and only if there is a metabolic submodule N such that the short exact sequence

$$0 \longrightarrow N^{\perp} \longrightarrow M \xrightarrow{\mathrm{Ad}|_N = p} \mathrm{Hom}_R(N, R \text{ or } K/R) \longrightarrow 0$$

splits as a sequence of R[X]- modules.

Proof: The isomorphism will be constructed in two stages. First, define the split isometric structure H(M,q) on $N + N^*$ depending on $q: N \to N^*$ satisfying $q = q^*$ $: N^* \to N^{**} = N$ by $b((x,f),(y,g)) = f(y) + \varepsilon g(x) + q(x)(y)$. If $a: N^* \to M$ splits the sequence there is an isometry $H(N,q) \to M$ where $q(x)(y) = b(a(x),a(y))$ given by $(x,f) \to x + a(f)$ because $b(x+a(f),y+a(g)) = b(x,y) + b(a(f),y) + b(x,a(g))$ $+ b(a(f),a(g)) = 0 + f(y) + \varepsilon g(x) + q(f)(g) = b((x,f),(y,g))$ since N is self-annihilating and a splits the adjoint map.

Next, we use the evenness property built into the fundamental relation of isometric structures: $b(x,y) = b(tx,y) + b(x,ty) = b(tx,y) + \varepsilon b(ty,x)$ to demonstrate the isomorphism: $H(N,q) = H(N,0) = H(N)$, given by $(x,f) \to (x, b(t(a(x)),a(-)) + g)$. For $b((x,b(t(a(x)),a(-)) + f), (y,b(t(a(y)),a(-)) + g)) = b(t(a(x),a(y)) + f(y) + \varepsilon g(x) +$ $b(t(a(y)),a(x)) = f(y) + \varepsilon g(x) + q(x)(y)$.

We now define an equivalence relation on the semigroup and demonstrate that this give a new group of isometric structures.

Definition 3.3: Two isometric structures M and N are (stably) hyperbolic equivalent if there are hyperbolic isometric structures H and K so that M + H is isometric to N + K.

As in the metabolic case, this is an equivalence on the isometry classes of isometric structures. Note that the trivial calss is the set of stably hyperbolic structures. Now the diagonal in M + -M is a metabolizer with an invariant complement (M, for instance). Hence by the proposition, the form on M + -M is hyperbolic, so inverses exists and the equivalences classes form a group under the orthogonal direct sum, which will be denoted $CH^{\varepsilon}(R)$. While the localization machinery of the metabolic case will give some results on this new group, many of the results fail to generalize.

When R is a field, we can make a complete analysis by using ancient and often reproven results concerning the classification of isometries of inner product spaces. To reduce to the case of isometries, we need the following:

Proposition 3.4: If F is a field, then every equivalence class in $CH^{\varepsilon}(F)$ has a representative (V,B,T) with T injective (and therefore an isomorphism.) This is also true for $CH^{\varepsilon}(K/R)$ when R is finitely generated over Z.

Proof: Let $H = \{v\colon T^N(v) = 0$ for some integer $N\}$ and let $K = \{v\colon (Id-T)^N(v) = 0\}$ Now, Id - T is invertible on H and T on K. Furthermore, $B(T^N x,y) = B(x,(Id-T)^N y)$. Hence, both H and K are self-annihilating. Furthermore, the above relation shows that H and K are dually paired, so that the form on H + K is hyperbolic. The orthogonal complement of H + K is the desired isometric structure.

By the standard verification, the monodromy, $s = Id - t^{-1}$ exists and is an isometry: $B(sx,sy) = B(x,y)$. We now consider a new involution on the polynomial ring: $P^*(X) = a_0 X^{\deg P} P(x^{-1})$, where a_0 is the zeroth coefficient. An isometry of an inner product space satisfies a self-dual polynomial, $p = p^*$. Fixing some self-dual polynomial p, define $V_k = \{v\colon p^k(T)(v) = 0\}$. and consider the form $b_k(v,w) = b(p^{k-1}v,w)$

Claim: $V_k^{\perp} = V_{k-1} + pV_{k+1}$

Proof: If v + pw is in the right hand side then $b(p^{k-1}u,v+pw) = b(u,p^{k-1}v + p^k w)$ = 0 so that the inclusion of the RHS in the left is obvious. Using the cyclic decomposition of F[X]- modules and computing dimensions, equality must hold. ($\dim V_k + \dim V_k^{\perp} = \dim V$.).

From the claim, it follows that the structure $(W_k = V_k/V_{k-1} + pV_{k+1}, B_k, T)$ is an inner product space under the induced form B_k and isomorphism T. Furthermore, if the original structure was hyperbolic, so is W_k. Observing that W_k is annihilated by p, we have defined a homomorphism: $\Phi: CH^{\varepsilon}(F) \longrightarrow +_{p=p^*} (\sum_{n=1} CH_p^{\varepsilon}(F))$

By the trace lemma, $CH_p^{\varepsilon}(F) = H^{\varepsilon}(F[X]/P(X))$. The following theorem computes $CH^{\varepsilon}(F)$.

Theorem 3.5 [Milnor[M]]Φ is an isomorphism.

One may also consider the n^{th} component of the right hand side as the hyperbolic equivalence classes of structures over F which are projective as $F[X]/p^n(X)$- modules.

This computation shows that the hyperbolic relation is infinitely finer than the metabolic relation. Furthermore, it is possible to show that stably hyperbolic implies hyperbolic in the field case. Hence, if a structure is in the kernel of Φ, it is isometric to a hyperbolic structure. This is an unknown and interesting question in the integral case.

In the integral case, the localization sequence degenerates to:

Theorem 3.6: $CH^{\varepsilon}(R) \longrightarrow CH^{\varepsilon}(K) \longrightarrow C^{\varepsilon}(K/R)$ is exact.

Proof: Proceed as in the localization theorem 2.6. The boundary is well-defined since metabolic is weaker than hyperbolic and its vanishing is the necessary condition (and sufficient also) for a rational structure to contain a unimodular lattice.

The previous injection of the localization homomorphism from the Dedekind Ring R to its field of fractions K is false. However, when the order $S = R[X]/P$ is maximal that is, S is the Dedekind ring of integers in its field of fractions, we have:

Theorem 3.7: $CH^{\varepsilon}_{R[X]/P}(R) = C^{\varepsilon}_{R[X]/P}(R) = H^{\varepsilon}(\Delta^{-1}(D/R))$

Proof: If M is metabolic then any metabolizer is torsion free over R and hence over D, the maximal order. Therefore the metabolizer is projective and the sequence of 3.2 splits.

Therefore no counterexample to the injection of (3.6) exists when the order S is maximal. The following example shows that the localization homomorphism fails to be injective in a slightly more restricted situation. We restrict ourselves to structures on projective S-modules and consider the case when S is local. Note then, the sequence of (3.2) splits if and only if the metabolizer is projective as an S-module.

Lemma 3.8: Let S be a local ring and E its field of fraction with a non-trivial involution $*$. Consider the homomorphism $\mathrm{Sym}(S^\cdot)/\mathrm{Norm}(S^\cdot) = \{s = s^*,\ s \text{ a unit}\}/\ \{uu^*\}$ $\to F^\cdot/NE^\cdot$ induced by the inclusion i. If $i(a) = 0$ then the form $<1> + <-a>$ is metabolic but not hyperbolic, even after the addition of a projective hyperbolic S-module.

Proof: The hermitian space is metabolic because it has rank two and its discriminant $= -(1)(-a) = a = ee^*$, a norm in $F^\cdot$ by assumption. An isotropic vector is given by (e,a). However the space annot be stably hyperbolic since the discriminant of a hyperbolic space is a norm in $S^\cdot$. Note that the discrinimant is defined because projective modules over a local ring are free.

The above space defines a self-dual S-lattice in a metabolic space V over E. Hence the boundary of this lattice is zero and we can find a non-singular lattice which localizes to the above lattice.

Example 3.9: Let $P(X) = X^2 - X - 11$ and consider the localization of the order S at the ideal $(3, X + 1)$. Then $i(-5) = 0$ because $-5 = (\sqrt{5})(-\sqrt{5})$, but $\sqrt{5}$, while integral, is not in S which is not a maximal order. By a straight-forward computation -5 is non-trivial in $\mathrm{Sym}(S^\cdot)/\mathrm{Norm}(S^\cdot)$. Therefore the form $<1> + <-5>$is metabolic but not hyperbolic at this localiztion and, by the above remarks, there is a unimodular lattice over the integers realizing this lattice.

In the hyperbolic case, the coupling exact sequence disintegrates. However, it still yields necessary conditions for a structure to be hyperbolic which are independent of the condition that the rational form be hyperbolic. Let $p(X)$ be a self-dual polynomial. Let $L_p = \{1 : p^N(t)(1) = 0\}$ and define $c_p(L) = (L_p^{\#}/L_p, b, t)$ be the usual alttice construction. If L had a hyperbolic splitting $L \simeq H + H^*$ and π_p be the projection of the vector space V containing L onto its P-primary component, then $L_p^{\#} = \pi_p(L)$, hence π_p induces a hyperbolic splitting of $c_p(L)$. This gives a well-defined homomorphism $c_p : CH^\varepsilon(R) \to CH^\varepsilon(K/R)$. As $CH^\varepsilon(F_q)$ is contained in $CH^\varepsilon(Q/Z)$ this is a much sharper invariant. Using these invariants it is possible to construct elements in the kernel of $CH^\varepsilon(Z) \to CH^\varepsilon(Q)$. We mention several other results that can be proven using these techniques. Let p and q be self-dual polynomials such that the associated orders are maximal.

Theorem 3.10: $0 \to CH_p^\varepsilon(Z) + CH_q^\varepsilon(Z) \to CH_{pq}^\varepsilon(Z) \xrightarrow{c_p + c_q} CH_p^\varepsilon(Q/Z) + CH_q^\varepsilon(Q/Z)$ is exact.

This is the one non-trivial case (in addition to the Dedekind case (3.7)) in which the integral hyperbolic case is computable.

A further complication occurs when the minimal polynomial of t is a power of an irreducible polynomial. In this case we have obtained the following partial results, without any Dedekind assumptions on the orders. Let $CH^{\varepsilon}_{p^n,0}(Z)$ be the subgroup of the group of isometric structures satisfying the condition that the condition that the rational structure is projective (and hence free) over $Q[X]/(P^n(X))$.

Theorem 3.11: There are homomorphisms $CH^{-\varepsilon}_p(Z) \to CH^{\varepsilon}_{p^{2n},0}(Z)$ and $CH^{\varepsilon}_p(Z) \to CH^{\varepsilon}_{p^{2n+1},0}(Z)$. The second homomorphism is split by the map $s(L) = (L_{p^n}/L_{p^n}, b, t)$ where L_{p^n} is the subgroup annihilated by p^n.

Hence the second map is an injection (the first is if the order is Dedekind). The surjectivity of either map is unknown for n greater than zero.

In answering a question posed by R.H. Fox at the Georgia Topology Conference in 1961, DeWitt Sumners introduced the notion of double null concordant knots [S]. Although the hyperbolic relation in the context of isometric structures first occurred in the algebraic context, the connection with the geometric notion of double null concordance was soon realized.

Definition 3.12: A knot (S^{n+2}, Σ^n) is doubly null concordant if it is the cross section of the trivial knot (S^{n+3}, S^{n+1}). (Equivalently, there is a smooth function $f: S^{n+3} \to R$ such that $f^{-1}(t)$ is the knot (S, Σ) for some regular value t.)

Theorem 3.13 (Sumners [S], Kearton [Ke]) A simple knot $(S^{2n+1}, \Sigma^{2n-1})$ is doubly null concordant if and only if the Seifert isometric structure $S(S,\Sigma)$ is hyperbolic, provided $n \neq 1$.

In the geometric setting, we can form a new group of knots from the semigroup of knots under connected sum by introducing the equivalence relation:

Definition 3.14: Two knots (S^{n+2}, Σ_0) and (S^{n+2}, Σ_1) are equivalent if there are doubly null concordant knots H and K such that the connected sums $\Sigma_0 \# H$ and $\Sigma_1 \# K$ are isotopic.

This is an equivalence relation and Corollary 2.9 of Sumners [S] verifies that -K is the inverse to K in the new group, denoted $\mathcal{CH}_n(Z)$. Let $\mathcal{CH}^{(q-1)}_n(Z)$ denote the group of knots with (q-1)-connected Seifert manifolds ((q-1)-simple) under the above

The above theorem and the realizeability of isometric structures gives:

<u>Theorem 3.15</u>: $\mathcal{CH}_{2n-1}^{(n-1)}(Z) \to CH^{\epsilon}(Z)$ is an isomorphism $(\epsilon = (-1)^n)$.

In the even dimensional case we have the surprising result that $\mathcal{CH}_{2n}(Z)$ is non-trivial, contrasting the triviality of the even dimensional knot concordance group due to M. Kervaire [K]. This result was prompted by a remark made to me by J. Levine at Les Plans. Dewitt Sumners has also communicated to me that he had an example (the double spun trefoil) of an even dimensional knot that could not be double null concordant.

According to Levine [L1], if (S^{n+2}, Σ^n) is an n-knot with complement $X = S^{n+2}\setminus\Sigma$ and universal abelian cover $\tilde{X}$, the torsion subgroup T_q of the knot module $A_q = H_q(\tilde{X};Z)$ has a nonsingular pairing $T_q \times T_{n-q} \to Q/Z$ satisfying $b(tx,y) = b(x,t^{-1}y)$ where t is the automorphism of T_q induced by the oriented generator of the group of covering translations. Furthermore, since X has the homology of a circle by Alexander duality, $Id - t$ is invertible (called type K by Levine). Then the isomorphism $T = (Id - t)^{-1}$ satisfies: $b(Tx,y) = b((1-t)^{-1}x,y) = b(x,(I-t^{-1})^{-1}y) = b(x,1-(1-t)^{-1}(y)) = b(x,(1-T)y)$ If $n = 2q$ then (T_q,b,T) is an $\epsilon = (-1)^{q+1}$-symmetric torsion isometric structure, and taking its equivalence class in $CH^{\epsilon}(Q/Z)$ we have a map S.

<u>Theorem 3.16</u>: There is a well-defined morphism $S: \mathcal{CH}_{2q}(Z) \to CH^{\epsilon}(Q/Z)$ which is an epimorphism for q greater than one.

<u>Proof;</u> By Theorem 13.1 of [L1] the map S is an epimorphism provided q is greater than one. To prove well-definedness it suffices to show that S vanishes on double null concordant knots. If (S^{2q+2}, Σ) is a cross-section of the trivial knot, the complement X of Σ splits the complement of the trivial knot, $S^1 x D^{2q+2}$ into two components, V and W. By the Mayer-Vietoris sequence: $H_{n+1}(S^1 x D^{2q+2}) \to H_n(X) \to H_n(V) + H_n(W) \to H_n(S^1 x D)$ we compute that V and W are also homology circles. Furthermore the Mayer-Vietoris sequence of the infinite cyclic covers:

$$0 = H_{n+1}(RxD^{2q+2}) \to H_n(\tilde{X}) \to H_n(\tilde{V}) + H_n(\tilde{W}) \to H_n(R \times D^{2q+2}) = 0$$

degenerates to an isomorphism at the middle morphism. Hence the $(i_*,-j_*)$ induced by the respective inclusions of X into V and W is an isomorphism. Let $H = \text{Kernel } i_* \cap T_q$ and $K = \text{kernel } j_* \cap T_q$. By the exactness of the homology exact sequence of the respective pairs, H and K are the torsion subgroup of the images of the respective boundaries. Note that H and K are invariant under t (and hence T) because the infinite cyclic cover of X is induced from that of V or W. Now H and K are disjoint and $T_q = H + K$ since $(i_*, -j_*)$ is an isomorphism. To complete the demonstration that our invariant is hyperbolic on double null concordant knots, it suffices to show that H and K are self-annihilating under the inner product b.

Let $\Theta = Z[X, X^{-1}]/(X^k - 1)$ where $t^k = id$ on T_q and let $I(\Theta)$ be the injective hull of the ring Θ. In Section Six of [L1] (particularly(6.4f) the pairing b is related to a pairing $\{\ ,\ \}$: $T_q \to \mathrm{Hom}(T_q, I(\Theta)/\Theta)$ with the following definition.

Consider the following sequence (numbered (6.4) in [L1]).

$$H_e^{q+2}(\tilde{X}) \xrightarrow{\delta_4} H_e^{q+1}(\tilde{X}, \Lambda_m) \xrightarrow{\delta_1} H_e^q(\tilde{X}, \Theta/m\Theta) \xrightarrow{e'} \mathrm{Hom}\,(A_q, \Theta/m\Theta)$$

If α is in T_{n-q} and β is in T_q then $\{\alpha,\beta\} = ire'(\alpha')(\beta)$ where α' in $H_e^q(\tilde{X}, \Theta/m\Theta)$ satisfies $-\delta_4\delta_1(\alpha') = \bar{\alpha}$, the dual of α. (Here, δ_1 and δ_4 are coboundaries induced by the appropriate coefficient sequences.) The map r is given by restriction to T_q and i is induced by the coeffieient map $\Theta/m\Theta \xrightarrow{1/m} Q(\Theta)/\Theta \subset I(\Theta)/\Theta$.

By the commutativity (up to sign) of the following diagram of duality maps

$$\begin{tikzcd}
H_e^{q+2}(\tilde{W}) \arrow[r, "i^*"] & H_e^{q+2}(\tilde{X}) \\
H_{q+1}(\tilde{W}, \tilde{X}) \arrow[u] \arrow[r, "\partial"] & H_q(\tilde{X}) \arrow[u]
\end{tikzcd}$$

if α,β are in the image of ∂ then their duals are in the image of i^*.

Now consider the following commutative diagram induced by inclusion and which exists since W is also a homology circle (so that δ_1 and δ_4 are defined).

$$\begin{tikzcd}
 & & H_e^{q+1}(\tilde{W}, \tilde{X}; \Theta/m\Theta) \\
H_e^{q+2}(\tilde{X}) \arrow[r, "\delta"] & H_e^{q+1}(\tilde{X}, \Lambda_m) \arrow[r, "\delta"] & H_e^q(\tilde{X}, \Theta/m\Theta) \arrow[u, "\delta"'] \\
H_e^{q+2}(\tilde{W}) \arrow[u, "i^*"'] \arrow[r, "\delta"] & H_e^{q+1}(\tilde{W}, \Lambda_m) \arrow[u, "i^*"'] \arrow[r, "\delta"] & H_e^q(\tilde{W}, \Theta/m\Theta) \arrow[u, "i^*"']
\end{tikzcd}$$

Therefore if α and β are in the image of the $\partial = \ker i_*$, then $\bar{\alpha}$, the dual of α is in the image of i^*, $\alpha = i^*(\gamma)$. Let γ' satisfy $-\delta_4\delta_1(\gamma') = \gamma$ so that $i^*(\gamma') = \alpha'$. Also let $\beta = \partial\eta$. Then we have $\{\alpha,\beta\} = ire'(\alpha')(\beta) = ire'(i^*\gamma')(\beta) = ire'(i^*\gamma')(\partial\eta)$
$= ire'(\delta i^*\gamma')(\eta)$ by the duality of δ and ∂ under the Kronecker pairing,
$= 0$ by the exactness of the cohomology sequence of $(\tilde{W}, \tilde{X})$. This completes the proof of the well-definedness of S.

This theorem should be compared with the stronger classification of Kearton of $(q-1)$-simple $2q$-knots when T_q is of odd order in [Ke2]. There is also a geometric long exact sequence for these new groups that can be developed according to that in Section Six of [St]. Finally, we raise the question of the relation between stably doubly null concordant and doubly null concordance or (equivalently, in the simple case), the relation between stably hyperbolic and hyperbolic isometric structures.

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