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A NOTE ON THE bP-COMPONENT OF $(4n-1)$-DIMENSIONAL HOMOTOPY SPHERES

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ABSTRACT. The bP-component of a $(4n-1)$-dimensional homotopy sphere $\Sigma \in \theta_{4n-1} \cong bP_{4n} \oplus (\operatorname{Coker} J)_{4n-1}$ bounding a spin manifold M is shown to be computable in terms of the signature and the decomposable Pontrjagin numbers of M.

Let θ_{m-1} be the group of h-cobordism classes of $(m-1)$-dimensional homotopy spheres and let $bP_m \subset \theta_{m-1}$ be the subgroup of those homotopy spheres bounding parallelizable m-manifolds. Using results of Kervaire and Milnor [5], G. Brumfiel showed that θ_{4n-1} has a direct sum decomposition

$$\theta_{4n-1} \cong bP_{4n} \oplus \pi^s_{4n-1}/\operatorname{im}(J),$$

where $J\colon \pi_{4n-1}(SO) \to \pi^s_{4n-1}$ is the stable J-homomorphism [1]. The group bP_{4n} is cyclic and its order $|bP_{4n}|$ can be expressed in terms of the nth Bernoulli number (see below). To define the projection map

$$s\colon \theta_{4n-1} \to bP_{4n} \cong \mathbf{Z}/|bP_{4n}|\mathbf{Z}$$

Brumfiel shows that every homotopy sphere $\Sigma \in \theta_{4n-1}$ bounds a spin manifold M with vanishing decomposable Pontrjagin numbers and that the signature of such an M is divisible by eight. Then he defines s by $s(\Sigma) := \frac{1}{8}\operatorname{sign}(M) \in \mathbf{Z}/|bP_{4n}|\mathbf{Z}$ [1].

The above definition is not suitable to compute $s(\Sigma)$ for a homotopy sphere Σ given explicitly by some geometric construction. The reason is that it is usually not possible to find an *explicit* spin manifold bounding Σ whose decomposable Pontrjagin numbers vanish. For example, if Σ is constructed by plumbing it bounds a manifold M by construction, but in general the decomposable Pontrjagin numbers of M do not vanish.

In this note we show how to compute $s(\Sigma)$ from the signature and the decomposable Pontrjagin numbers of a spin manifold M bounding Σ. To describe explicitly which linear combination of decomposable Pontrjagin numbers is involved, let $L(M)$ (resp. $\hat{A}(M)$) be the L-class (resp. the $\hat{A}$-class) of M, which are power series in the Pontrjagin classes of M [4]. For any power series $K(M)$ in the Pontrjagin classes, let $K_n(M)$ be its $4n$-dimensional component. Let $\operatorname{ph}(M)$ be the Pontrjagin character of M, i.e. the Chern character of the complexified tangent bundle of M. Here we think of the tangent bundle as an element of $\widetilde{KO}(M)$, in particular,

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$\mathrm{ph}_0(M) = 0$. Then define

$$S_n(M) = \frac{1}{8}L_n(M) + \frac{|bP_{4n}|}{a_n}\left(c_n\hat{A}_n(M) + (-1)^n d_n(\hat{A}(M)\,\mathrm{ph}(M))_n\right),$$

where $a_n = 1$ for n even, $a_n = 2$ for n odd, and c_n, d_n are integers such that

$$c_n \,\mathrm{num}(B_n/4n) + d_n \,\mathrm{denom}(B_n/4n) = 1.$$

Here B_n is the nth Bernoulli number and $\mathrm{num}(B_n/4n)$ (resp. $\mathrm{denom}(B_n/4n)$) denote the numerator (resp. denominator) of the irreducible fraction expressing the rational number $B_n/4n$. We will show below that $S_n(M)$ is a polynomial in $p_1, \dots, p_{n-1}$, i.e. $S_n(M)$ does not involve p_n. Note that for $1 \leqslant i < n$ we can interpret p_i as an element of $H^{4i}(M, \partial M)$, due to the isomorphism $H^{4i}(M) \cong H^{4i}(M, \partial M)$. Hence $S_n(M) \in H^{4n}(M, \partial M)$ and we can form the Kronecker product $\langle S_n(M), [M, \partial M]\rangle$ with the relative fundamental class of M.

THEOREM. *Let Σ be a $(4n-1)$-dimensional homotopy sphere bounding a spin manifold M. Then*

$$s(\Sigma) = \tfrac{1}{8}\,\mathrm{sign}(M) - \langle S_n(M), [M, \partial M]\rangle \mod |bP_{4n}|\mathbf{Z}.$$

This theorem generalizes some results of R. Lampe [**6**], who computed the bP-component of $(4n-1)$-dimensional homotopy spheres bounding $(2n-1)$-connected manifolds, and of G. Brumfiel, who obtained a formula for $s(\Sigma) - s(\Sigma')$, where Σ, Σ' are homotopy spheres bounding homotopy equivalent manifolds [**2**, Proposition 5.1, Corollary 5.8].

The expression $\frac{1}{8}\,\mathrm{sign}(M) - \langle S_n(M), [M, \partial M]\rangle$ can be viewed as a refinement of the μ-invariant of Eells-Kuiper [3]. They use the integrality of $\langle \hat{A}(W), [W]\rangle / a_n$ for closed spin manifolds W^{4n} to prove that their μ-invariant is well defined. We will use the integrality of $\langle \hat{A}(W), [W]\rangle / a_n$ and $\langle \hat{A}(W)\,\mathrm{ph}(W), [W]\rangle / a_n$ to show that

$$\tfrac{1}{8}\,\mathrm{sign}(M) - \langle S_n(M), [M, \partial M]\rangle \in \mathbf{Z}/|bP_{4n}|\mathbf{Z}$$

is independent of the choice of M.

My original motivation for this work comes from the study of highly connected smooth manifolds. There are classification results for highly connected 'almost closed' manifolds, i.e, manifolds whose boundaries are homotopy spheres [**9**, **10**]. To obtain results on closed manifolds, one has to determine whether the boundary of a given highly connected, almost closed manifold M is diffeomorphic to the standard sphere. In [**8**] it is shown that the cokernel J-component of ∂M often vanishes. Thus it remains to compute the bP-component which is easily done using the above theorem if M is $4n$-dimensional and using [**8**, 13] if M is $(4n+2)$-dimensional.

PROOF OF THE THEOREM. First we show that $S_n(M)$ does not involve p_n.

$$\hat{A}_n(M) = -\frac{B_n}{2(2n)!}p_n + \text{decomposables},$$

$$\mathrm{ph}_n(M) = \frac{(-1)^{n+1}}{(2n-1)!}p_n + \text{decomposables},$$

$$\begin{aligned}(\hat{A}(M)\,\mathrm{ph}(M))_n &= \hat{A}(M)_0\mathrm{ph}(M)_n + \hat{A}(M)_n\mathrm{ph}(M)_0 + \text{decomposables}\\ &= \frac{(-1)^{n+1}}{(2n-1)!}p_n + \text{decomposables}.\end{aligned}$$

Thus

$$\begin{aligned} &c_n\hat{A}_n(M) + (-1)^n d_n(\hat{A}(M)\,\mathrm{ph}(M))_n \\ &\quad\equiv -\frac{1}{(2n-1)!}\left(c_n\frac{B_n}{4n} + d_n\right)p_n \\ &\quad\equiv -\frac{1}{(2n-1)!\,\mathrm{denom}(B_n/4n)}\left(c_n\,\mathrm{num}\left(\frac{B_n}{4n}\right) + d_n\,\mathrm{denom}\left(\frac{B_n}{4n}\right)\right)p_n \\ &\quad\equiv -\frac{1}{(2n-1)!\,\mathrm{denom}(B_n/4n)}p_n \bmod \text{decomposables}. \end{aligned}$$

According to [**5**]

$$|bP_{4n}| = 2^{2n-2}(2^{2n-1}-1)\,\mathrm{num}(4B_n/n).$$

Using the facts that

$$4\,|\,\mathrm{denom}(B_n/n) \quad \text{for } n \text{ even},$$

and

$$2\,|\,\mathrm{denom}(B_n/n), \quad 4 \nmid \mathrm{denom}(B_n/n) \quad \text{for } n \text{ odd}$$

[**7**, p. 284] we conclude that

$$\mathrm{num}(4B_n/n) = a_n\,\mathrm{num}(B_n/n) = a_n\,\mathrm{num}(B_n/4n).$$

It follows that

$$\begin{aligned} &\frac{|bP_{4n}|}{a_n}\left(c_n\hat{A}_n(M) + (-1)^n d_n(\hat{A}(M)\,\mathrm{ph}(M))_n\right) \\ &\quad\equiv -\frac{2^{2n-2}(2^{2n-1}-1)}{(2n-1)!}\frac{B_n}{4n}p_n + \text{decomposables}. \end{aligned}$$

On the other hand

$$\frac{1}{8}L_n(M) = \frac{1}{8}\frac{2^{2n}(2^{2n-1}-1)}{(2n)!}B_n + \text{decomposables}$$

[**4**, p. 12], which shows that $S_n(M)$ is a polynomial of $p_1, \dots, p_{n-1}$.

The next step is to prove the equality

$$S(\Sigma) = \tfrac{1}{8}\,\mathrm{sign}(M) - \langle S_n(M), [M, \partial M]\rangle \quad \mathrm{mod}\,|bP_{4n}|\mathbf{Z}.$$

If the decomposable Pontrjagin numbers of M and hence $\langle S_n(M), [M, \partial M]\rangle$ vanish, the above equation holds by the definition of s. Thus we have to show that the right-hand side is independent of the spin manifold M. Let N be another spin manifold bounding Σ and let W be the closed spin manifold obtained by gluing M and $-N$ along Σ. Then

$$\mathrm{sign}(W) = \mathrm{sign}(M) - \mathrm{sign}(N)$$

and

$$\langle S_n(W), [W]\rangle = \langle S_n(M), [M, \partial M]\rangle - \langle S_n(N), [N, \partial N]\rangle.$$

It follows that

$$\frac{1}{8}\operatorname{sign}(M) - \langle S_n(M), [M, \partial M]\rangle - \left(\frac{1}{8}\operatorname{sign}(N) - \langle S_n(N), [N, \partial N]\rangle\right)$$

$$= \frac{1}{8}\operatorname{sign}(W) - \langle S_n(W), [W]\rangle$$

$$= \frac{1}{8}(\operatorname{sign}(W) - \langle L_n(W), [W]\rangle)$$

$$+ |bP_{4n}|\left(c_n \frac{1}{a_n}\langle \hat{A}_n(W), [W]\rangle + (-1)^n d_n \frac{1}{a_n}\langle (\hat{A}(W)\operatorname{ph}(W))_n, [W]\rangle\right) = 0$$

mod$|bP_{4n}|\mathbf{Z}$ since $\operatorname{sign}(W) = \langle L_n(W), [W]\rangle$ by Hirzebruch's signature theorem and since $\langle \hat{A}_n(W), [W]\rangle / a_n$ resp. $\langle (\hat{A}(W)\operatorname{ph}(W))_n, [W]\rangle / a_n$ are integers by the Hirzebruch-Riemann-Roch Theorem [**4**, Theorems 26.3.1 and 26.3.2]. Q.E.D.

References

1. G. Brumfiel, *On the homotopy groups of BPL and PL/O*, Ann. of Math. (2) **88** (1968), 291–311.

2. ______, *Homotopy equivalences of almost smooth manifolds*, Comment Math. Helv. **46** (1971), 381–407.

3. J. Eells and N. Kuiper, *An invariant for certain smooth manifolds*, Ann. Mat. Pura Appl. **60** (1962), 93–110.

4. F. Hirzebruch, *Topological methods in algebraic geometry*, 2nd corrected printing of the 3rd ed., Springer, 1978.

5. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

6. R. Lampe, *Diffeomorphismen auf Sphären und die Milnor-Paarung*, Diplomarbeit, Mainz, 1981.

7. J. Milnor and J. Stashoff, *Characteristic classes*, Ann. of Math. Studies, no. 76, Princeton Univ. Press, Princeton, N. J., 1974.

8. S. Stolz, *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*, Lecture Notes in Math., vol. 1116, Springer-Verlag, Berlin and New York, 1985.

9. C. T. C. Wall, *Classification of* $(n-1)$*-connected* $2n$*-manifolds*, Ann. of Math. (2) **75** (1962), 163–189.

10. ______, *Classification of* $(n-1)$*-connected* $(2n+1)$*-manifolds*, Topology **6** (1967), 273–296.

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