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Exotic structures on 4-manifolds detected by spectral invariants

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Contents

1. Introduction

The spectra of the signature operator and the Dirac operator on a spin manifold M with riemannian metric g contain subtle information about the diffeomorphism type of M. For example the 28 differentiable structures on the 7-sphere [D 1] and certain homeomorphic, but not diffeomorphic homogeneous spaces [KS] can be distinguished by their spectra, provided the metric is chosen such that the corresponding Pontrjagin forms vanish.

Following a suggestion of P. Gilkey [G 2] we study the spectrum of the twisted Dirac operator on the exotic projective space Q^4 constructed by Cappell-Shaneson [CS]. It is homeomorphic to the real projective space RP^4 by Freedman's topological s-cobordism theorem, but not diffeomorphic to it. We prove in this paper that the spectra of the twisted Dirac operator on RP^4 resp. Q^4 are never the same, independent of the metrics chosen.

To state more precise results we denote by $\eta(M, g, \phi) \in R$ the η-invariant of the twisted Dirac operator on a smooth closed 4-manifold with Riemannian metric g and pin^+-structure ϕ. Here the pin^+-structure is needed to define the twisted Dirac operator (like an orientation is needed to define the signature operator) and $\eta(M, g, \phi)$ is a measure for the asymmetry of the spectrum of this operator with respect to the origin.

Theorem A. *$\eta(RP^4, g, \phi) \neq \eta(Q^4, g', \phi')$ for all riemannian metrics g, g' and pin^+-structures ϕ, ϕ' on RP^4 resp. Q^4.*

Theorem B. *Let M, M' be smooth closed non-orientable 4-manifolds with fundamental group $Z/2$. Let g (resp. g') be a riemannian metric and let ϕ (resp. ϕ')*

be a pin$^+$*-structure on* M *(resp.* M'*). Then* M *is stably diffeomorphic to* M' *if and only if their Euler characteristics agree and* $\eta(M, g, \phi) = \pm\eta(M', g', \phi') \bmod 2Z$.

We recall that two manifolds are stably diffeomorphic if the connected sums with copies of $S^2 \times S^2$ are diffeomorphic.

The key to these results is the observation that $\eta(M, g, \phi) \bmod 2Z$ depends only on the pin$^+$-bordism class of (M, ϕ) (Prop. 4.3). This improves a result of P. Gilkey that $\eta(M, g, \phi) \bmod Z$ is a pin^c-bordism invariant [G2]. The improvement by the factor two is crucial for the following reason.

The bordism group of 4-manifolds with pin$^+$-structure is the cyclic group of order 16 generated by RP^4 (this follows modulo the identification of the bordism groups considered from [K2]) and $\eta(RP^4, g, \phi) = 1/8 \bmod Z$ by Gilkey's computations [G2]. Hence the η-invariant mod $2Z$ detects the pin$^+$-bordism classes. Moreover it follows from the theory of M. Kreck [K1], [K2] that the stable diffeomorphism type of closed non-orientable manifolds of dimension 4 with $\pi_1(M) \cong Z/2$ is determined by their pin$^+$-bordism class and their Euler characteristic thus proving theorem B. Theorem A is a corollary of this using the fact that RP^4 and Q^4 are not stably diffeomorphic [CS].

While this line of reasoning has the advantage of being short, I prefer to give a direct prove that RP^4 and Q^4 are not diffeomorphic *using* the η-invariant. The structure of the paper is as follows.

Sections 2 and 3 contain preliminaries about pin$^+$-structures and the twisted Dirac operator on $(8k+4)$-dimensional manifolds. In Sect. 4 we prove that $\eta(M, g, \phi) \bmod 2Z$ is a bordism invariant.

In Sect. 5 we show that $\eta(M, g, \phi)$ can be computed topologically by counting isolated fixed points on a manifold with $Z/2$-action bounding the orientation cover of M (Prop. 5.3). Alternatively, if M is orientable, $\eta(M, g, \phi) = 1/16$ $\operatorname{sign}(M) \bmod 2Z$ (Prop. 5.1).

Section 6 contains a discussion of pin$^+$-structures from a topologist's point of view. This is needed mainly to identify the bordism group of pin$^+$-manifolds with the bordism group relevant for the stable diffeomorphism classification which is carried out in Sect. 8.

In Sect. 7 we compute the η-invariant of Q^4 and other exotic structures by applying the formulas developed in §5.

Section 8 contains the proof of theorem B.

2. Pin$^+$-structures and pin$^-$-structures

The orthogonal group $O(n)$ has two double coverings $\sigma^+: \mathrm{Pin}^+(n) \to O(n)$ and $\sigma^-: \mathrm{Pin}^-(n) \to O(n)$. Restricted to $SO(n) \subset O(n)$ both give the universal covering $\mathrm{Spin}(n) \to SO(n)$ for $n \geq 3$. Using Clifford algebras [ABS] these coverings can be described as follows.

Let $C^+(R^n)$ (resp. $C^-(R^n)$) be the Clifford algebra generated by the elements $v \in R^n$ subject to the relation $v \cdot v = |v|^2 \cdot 1$) (resp. $v \cdot v = -|v|^2 \cdot 1$). $\mathrm{Pin}^+(n)$ (resp. $\mathrm{Pin}^-(n)$) is the subgroup of the units of $C^+(R^n)$ (resp. $C^-(R^n)$) generated by the elements $v \in S^{n-1}$. Note that if $w \in R^n$ then $v \cdot w \cdot v \in C^{\pm}(R^n)$ is in the subspace $R^n \subset C^{\pm}(R^n)$. This observation is used to define the homomorphisms $\sigma^{\pm}$: $\mathrm{Pin}^{\pm}(n) \to O(n)$. For $v \in S^{n-1} \subset \mathrm{Pin}^{\pm}(n)$ and $w \in R^n$ let $\sigma^+(v) \cdot w = -v \cdot w \cdot v \in R^n \subset C^+(R^n)$ (resp. $\sigma^-(v) \cdot w = v \cdot w \cdot v \in R^n \subset C^-(R^n)$). More geometrically both, σ^+ and σ^-, are the reflection at the hyperplane perpendicular to v.

If α is a n-dimensional vector bundle over some paracompact space X a **pin$^+$-structure** (resp. **pin$^-$-structure**) on α is a principal $\mathrm{Pin}^+(n)$-bundle (resp. $\mathrm{Pin}^-(n)$-bundle) P over X (with $\mathrm{Pin}^{\pm}(n)$ acting from the right on P) together with a vector bundle isomorphism

$$f \colon \alpha \cong P \times_{\mathrm{Pin}^{\pm}(n)} R^n,$$

where $\mathrm{Pin}^{\pm}(n)$ acts on R^n via the representation $\sigma^{\pm}$. More precisely a pin-structure on α is an equivalence class of such pairs (P, f) where two pairs are equivalent if there is an isomorphism of principal bundles compatible with the vector bundle isomorphisms.

A pin$^+$-structure (resp. pin$^-$-structure) on a manifold is a pin$^+$-structure (resp. pin$^-$-structure) on its tangent bundle.

3. The Dirac operator

The Dirac operator is a first order elliptic operator defined on riemannian manifolds with spin-structure [AS, §5]. It can be extended to manifolds with pin^c-structure [G2]. Below we define the Dirac operator and the twisted Dirac operator for manifolds with pin$^{\pm}$-structure. Our presentation follows closely the description of the Dirac operator in [AS].

Let Δ be a fixed module over $C^+(R^n)$ (resp. $C^-(R^n)$), and let $\mathrm{Pin}^{\pm}(n)$ act on R^n by $\sigma^{\pm}$. Note that the Clifford multiplication

$$c \colon R^n \otimes_R \Delta \to \Delta$$

is $\mathrm{Spin}(n)$-equivariant, but not $\mathrm{Pin}^{\pm}(n)$-equivariant. We can fix this by replacing Δ in the range of c by $\chi \otimes_R \Delta$, where χ is the 1-dimensional non-trivial representation of $\mathrm{Pin}^{\pm}(n)$.

Let M be a n-dimensional riemannian manifold with a pin$^+$-structure (resp. pin$^-$-structure) and let P be the corresponding principal $\mathrm{Pin}^{\pm}(n)$-bundle over M. If ζ is a representation of $\mathrm{Pin}^{\pm}(n)$ we write $E(\zeta)$ for the associated vector bundle $P \times_{\mathrm{Pin}^{\pm}(n)} \zeta$.

The **Dirac operator** is the first order elliptic operator

$$D \colon C^{\infty}(E(\Delta)) \to C^{\infty}(E(\chi \otimes \Delta))$$

defined as follows. The Levi-Civita connection on M induces a connection on P and hence a covariant derivative

$$\nabla \colon C^{\infty}(E(\Delta)) \to C^{\infty}(\tau^* \otimes E(\Delta)).$$

On the other hand $\tau^* \cong \tau \cong E(\sigma^\pm)$ using the metric resp. the pin$^\pm$-structure, and we have the homomorphism

$$C^\infty(E(\sigma^\pm) \otimes E(\Delta)) \to C^\infty(E(\chi \otimes \Delta))$$

induced by Clifford multiplication. The Dirac operator is the composite of these two homomorphisms. Thus in terms of an orthonormal basis $\{e_i\}$ we have

$$Ds = \sum_{i=1}^{n} e_i \nabla_{e_i} s.$$

If Δ extends to a module over $C^+(R^{n+1})$ (resp. $C^-(R^{n+1})$), denote by σ: $E(\Delta) \to E(\chi \otimes \Delta)$ the vector bundle isomorphism induced by the $\mathrm{Pin}^\pm(n)$-equivariant map $\Delta \to \chi \otimes \Delta$ given by the Clifford multiplication by $e_{n+1} \in R^{n+1}$. In this situation we define the **twisted Dirac operator**

$$\tilde{D}: C^\infty(E(\Delta)) \to C^\infty(E(\Delta))$$

to be the composition of D and σ^{-1}. $\tilde{D}$ is a first order selfadjoint elliptic operator.

Suppose the n-dimensional manifold M bounds a manifold W with pin$^\pm$-structure restricting to the given pin$^\pm$-structure on M. Furthermore suppose that the riemannian metric on W is the product metric in a collar neighbourhood of M. Then a $C^\pm(R^{n+1})$-module Δ leads to a Dirac operator D_W on W and a twisted Dirac operator $\tilde{D}_M$ on M. In the collar neighbourhood of M they are related by

$$D_W = \sigma\left(\frac{\partial}{\partial u} + \tilde{D}_M\right) \tag{3.1}$$

where u is the inward normal coordinate.

So far our definition of the (twisted) Dirac operator depends on the Clifford module Δ and we need to choose Δ to be able to speak about *the* (twisted) Dirac operator. Since we are mainly interested in the twisted Dirac operator on $(8k+4)$-dimensional pin$^+$-manifolds and the Dirac operator on $(8k+5)$-dimensional pin$^+$-manifolds it suffices for us to choose a $C^+(R^{8k+5})$-module Δ. According to Atiyah-Bott-Shapiro [ABS] the algebra $C^+(R^{8k+5})$ is isomorphic to $H(m) \oplus H(m)$, $m = 2^{4k+1}$, where $H(m)$ denotes the algebra of $(m \times m)$-matrices with quaternionic entries. Since $H(m)$ is a simple algebra there is exactly one irreducible $H(m)$-module, namely H^m with its natural (left) $H(m)$-action. Hence there are exactly two irreducible $C^+(R^{8k+5})$-modules which can be distinguished by the action of the central element $s_{8k+5} = e_1 \cdot \ldots \cdot e_{8k+5} \in C^+(R^{8k+5})$ whose square is the identity element.

We fix Δ to be the irreducible $C^+(R^{8k+5})$-module such that

$$s_{8k+5} \text{ acts trivially on } \Delta. \tag{3.2}$$

Note that Δ extends to a module over $C^+(R^{8k+5}) \otimes_R H$ if we let $h \in H$ act on $\Delta = H^m$ by right multiplication by its quaternionic conjugate $\bar{h}$.

It follows that $C^\infty(E(\Delta))$ and $C^\infty(E(\chi\otimes\Delta))$ are quaternionic vector spaces and that both, the Dirac operator on $(8k+5)$-dimensional and the twisted Dirac operator on $(8k+4)$-dimensional pin$^+$-manifolds, are H-linear.

4. The η-invariant

Let E be a hermitian vector bundle on a closed riemannian manifold M and let $D: C^\infty(E)\to C^\infty(E)$ be a selfadjoint elliptic operator. Then the sum

$$\eta(s)=\sum_{\lambda\neq 0} \operatorname{sign}\lambda \dim E(\lambda)\,|\lambda|^{-s}$$

where $E(\lambda)$ is the λ-eigenspace of D and λ runs through the non-zero eigenvalues converges for complex numbers s with $\operatorname{Re}(s)\gg 0$. Moreover $\eta(s)$ can be extended to a meromorphic function on the complex plane without a pole at $s=0$ [APS] [G1]. The number $\eta(0)$ can be interpreted as a measure for the asymmetry of the spectrum of D with respect to the origin.

The η-invariant plays a central role in the index theorem for manifolds with boundary. Let M be the boundary of a manifold W and let F, F' be vector bundles on W such that F restricts to E on the boundary. Suppose

$$D_W: C^\infty(F)\to C^\infty(F')$$

is an elliptic operator such that D_W restricted to a collar neighbourhood of M can be written in the form

$$D_W=\sigma\left(\frac{\partial}{\partial u}+D\right) \tag{4.1}$$

where u is the inward normal coordinate and $\sigma: F\to F'$ is some vector bundle isomorphism on the collar. Then the index of D_W with respect to certain global boundary conditions can be expressed as follows [APS]:

$$\operatorname{index}(D_W)=\int_W \alpha_0(x)\,dx-\eta(D) \tag{4.2}$$

Here following [G2] we use the notation $\eta(D)$ for

$$\frac{\dim\ker D+\eta_D(0)}{2}.$$

The integrand $\alpha_0(x)$ is constructed using eigenfunctions of $D_W^* D_W$ and $D_W D_W^*$. It vanishes identically if W is odd-dimensional [G2, lemma 1.5]. If M is a $(8k+4)$-dimensional manifold with riemannian metric g and pin$^+$-structure ϕ we write $\eta(M, g, \phi)$ for the η-invariant of the twisted Dirac operator on M.

The following proposition shows that $\eta(M,g,\phi)$ modulo $2Z$ is a pin$^+$-bordism invariant thus improving for pin$^+$-manifolds Gilkey's result that $\eta(M, g, \phi)$ modulo Z is a pin^c-bordism invariant [G2, lemma 1.7].

Proposition 4.3. *Let M be a $(8k+4)$-dimensional manifold with riemannian metric g and* pin$^+$*-structure ϕ. Suppose that M bounds a manifold W admitting a* pin$^+$*-structure restricting to ϕ on the boundary. Then $\eta(M, g, \phi)$ is an even integer.*

Proof. We choose a metric on W which is the product metric in a collar neighbourhood of M. Then according to (3.1) in this collar the Dirac operator D_W and the twisted Dirac operator $\tilde{D}_M$ are related by

$$D_W = \sigma\left(\frac{\partial}{\partial u} + \tilde{D}_M\right)$$

The index theorem then implies

$$\operatorname{index}(D_W) = -\eta(M, g, \phi).$$

As pointed out in §3 the Dirac operator D_W on $(8k+5)$-dimensional manifolds with pin$^+$-structure is H-linear and hence its index is even. Q.E.D.

5. Topological descriptions of the η-invariant

In this section we show how the η-invariant of a $(8k+4)$-dimensional manifold M can be computed topologically, namely by evaluating the $\hat{A}$-genus of M if M is orientable (Prop. 5.1) and in the general case by counting fixed points (with multiplicity) on a $Z/2$-manifold W bounding the orientation cover $\tilde{M}$ of M (Prop. 5.3).

Proposition 5.1. *Let M be a $(8k+4)$-dimensional closed manifold with riemannian metric g and* spin*-structure ϕ. Then $\eta(M, g, \phi) = 1/2\langle \hat{A}(M), [M]\rangle$* mod $2Z$.

Here $\hat{A}(M)$ is the $\hat{A}$-genus of M which is a polynomial in the Pontrjagin classes of M. If M has dimension 4 we compare

$$\langle \hat{A}(M), [M]\rangle = -1/24\langle p_1, [M]\rangle$$

with the signature of M

$$\operatorname{sign}(M) = \langle L(M), [M]\rangle = 1/3\langle p_1, [M]\rangle$$

and conclude:

Corollary 5.2. *If M is a closed 4-dimensional manifold with riemannian metric g and* spin*-structure ϕ then $\eta(M, g, \phi) = 1/16$* $\operatorname{sign}(M)$ mod $2Z$.

Remark. The right hand side of the above congruence depends on the choice of the orientation of M whereas the left hand side does not. Hence the corollary implies Rohlin's theorem that the signature of 4-dimensional spin-manifolds is divisible by 16.

Proof of proposition 5.1. First we show that the spectrum of the twisted Dirac operator $\tilde{D}$ is symmetric which implies $\eta(M, g, \phi) = 1/2 \dim \ker \tilde{D}$. Let

$$S: C^\infty(E(\Delta)) \to C^\infty(E(\Delta))$$

be the operator induced by the Spin(n)-equivariant map $\Delta \to \Delta$, $u \to s_n \cdot u$, where $s_n = e_1 \cdot \ldots \cdot e_n \in C_n$ and $e_1, \ldots, e_n$ is the standard basis of R^n, $n = 8k+4$. Note that this map is not Pin$^\pm(n)$-equivariant and thus the operator S can only be defined for spin-manifolds.

The identities $s_n \cdot w \cdot u = -w \cdot s_n \cdot u$ and $s_n \cdot s_n = 1$ for $w \in R^n$, $u \in \Delta$ imply $SD = -DS$ and $S^2 = \mathrm{id}$. Moreover since $s_n \cdot e_{n+1} = e_{n+1} \cdot s_n$ the isomorphism σ induced by multiplication by e_{n+1} commutes with S and hence $S\tilde{D} = -\tilde{D}S$ for the twisted Dirac operator $\tilde{D} = \sigma^{-1} \circ D$. Thus S interchanges the eigenspaces of $\tilde{D}$ corresponding to the eigenvalues $\pm\lambda$ proving that the spectrum of $\tilde{D}$ is symmetric.

Next we relate $1/2 \dim \ker \tilde{D}$ to the spinor index of M which is defined as follows [AS, §5]. Let $\Delta = \Delta^+ \oplus \Delta^-$ be the decomposition of Δ into the $\{\pm 1\}$-eigenspaces of the above map. This is a decomposition of Δ as a representation of Spin(n). The Clifford multiplication $R^n \otimes_R \Delta \to \Delta$ interchanges Δ^+ and Δ^- and thus the Dirac operator decomposes in the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}: C^\infty(E(\Delta^+)) \oplus C^\infty(E(\Delta^-)) \to C^\infty(E(\Delta^+)) \oplus C^\infty(E(\Delta^-))$$

Let $h^+ = \dim \ker D^+ = \dim \operatorname{coker} D^-$ and $h^- = \dim \ker D^- = \dim \operatorname{coker} D^+$. Index $D^+ = h^+ - h^-$ is called the spinor index of M and can be computed using the index theorem [ABS, Thm. 5.3]:

$$\operatorname{Index} D^+ = \langle \hat{A}(M), [M] \rangle.$$

The numbers $h^\pm$ are even since $D^\pm$ is H-linear and we conclude

$$\dim \ker \tilde{D} = \dim \ker D = h^+ + h^- \equiv h^+ - h^- = \langle \hat{A}(M), [M] \rangle \bmod 4Z \quad \text{Q.E.D.}$$

There is an other situation where the η-invariant can be computed topologically.

Suppose $\tilde{M}$, the orientation cover of M, is the boundary of a $(n+1)$-dimensional manifold W with involution $T: W \to W$ extending the non-trivial covering transformation of the double covering $\tilde{M} \to M$. Assume that T has only isolated fixed points. Furthermore suppose that W has a $Z/2$-equivariant pin$^+$-structure extending the $Z/2$-equivariant pin$^+$-structure on $\tilde{M}$ determined by ϕ. Here an equivariant pin$^+$-structure on W is a pin$^+$-structure (P, ϕ) on the tangent bundle τ of W together with a Pin$^+(n+1)$-equivariant involution $\bar{T}: P \to P$ making the diagram

$$\begin{tikzcd} \tau \arrow[r, "f", "\cong"'] \arrow[d, "dT"'] & P \times_{\mathrm{Pin}^+(n+1)} R^{n+1} \arrow[d, "\bar{T} \times \mathrm{id}"] \\ \tau \arrow[r, "f", "\cong"'] & P \times_{\mathrm{Pin}^+(n+1)} R^{n+1} \end{tikzcd}$$

commutative.

If $x \in W$ is a fixed point $dT\colon \tau_x \to \tau_x$ is multiplication by -1 and hence $\bar{T}\colon P_x \to P_x$ is left-multiplication by $s_{n+1} = e_1 \cdot \ldots \cdot e_{n+1}$ or $-s_{n+1}$ (here we identify P_x with $\mathrm{Pin}^+(n+1)$). We attach to x an index $\iota_x \in \{\pm 1\}$ according to the sign of $\bar{T}$.

Proposition 5.3. *If M is a $(8k+4)$-dimensional riemannian manifold with* pin^+*-structure ϕ and W is as above then*

$$\eta(M, \phi, g) = 2^{-(4k+3)} \sum \iota_x \bmod 2Z$$

where the sum extends over all fixed points of W.

This result can be used to compute the η-invariant of RP^{8k+4} whose orientation cover S^{8k+4} bounds the disk D^{8k+5} with its antipodal involution. We can extend this involution to $P = D^{8k+5} \times \mathrm{Pin}^+(8k+5)$ by left-multiplication with s_{n+1} on the second factor and there is an obvious vector bundle isomorphism between the associated vector bundle $E(\sigma^+)$ and the tangent bundle of D^{8k+5}. This $Z/2$-equivariant Pin^+-structure induces a Pin^+-structure ϕ on RP^{8k+4}.

Corollary 5.4. *$\eta(RP^{8k+4}, g, \phi) \equiv 2^{-(4k+3)} \bmod 2Z$, where ϕ is the* pin^+*-structure described above.*

The proof of proposition (5.3) is essentially identical with Gilkey's computation of the η-invariant of RP^{2l} [G2, Thm. 3.3] except that working modulo $2Z$ we have to be slightly more careful.

The Dirac operator D_W and the twisted Dirac operator $\tilde{D}_{\tilde{M}}$ are $Z/2$-equivariant with respect to the $Z/2$-actions on the vector spaces $C^\infty(E(\Delta))$ resp. $C^\infty(E(\chi \otimes \Delta))$ induced by $\bar{T}$. To compute $\eta(\tilde{D}_M)$ we can identify the eigenspaces of $\tilde{D}_M$ with the $Z/2$-invariant subspaces of the corresponding eigenspaces of $\tilde{D}_{\tilde{M}}$. For $g \in Z/2$ denote by $\eta(\tilde{D}_{\tilde{M}}, g)$ the real number defined analogously to $\eta(\tilde{D}_{\tilde{M}})$ but replacing the dimension of vector spaces by the trace of the element g acting on it. In particular $\eta(\tilde{D}_{\tilde{M}}, 1) = \eta(\tilde{D}_{\tilde{M}})$ and $\eta(\tilde{D}_M) = 1/2(\eta(\tilde{D}_{\tilde{M}}, 1) + \eta(\tilde{D}_{\tilde{M}}, t))$, where 1 (resp. t) is the identity element (resp. the non-trivial element) of $Z/2$ (written multiplicatively). To compute $\eta(\tilde{D}_{\tilde{M}}, g)$ we can use the equivariant version of the Atiyah-Patodi-Singer index theorem as proved by Donnelly [D2]:

$$\mathrm{index}(D_W, g) = \int_F \beta_0(y)\, dy - \eta(\tilde{D}_{\tilde{M}}, g)$$

Here $\mathrm{index}(D_W, g) = \mathrm{Trace}(g \text{ on } \ker D_W) - \mathrm{Trace}(g \text{ on } \operatorname{coker} D_W)$, F is the fixed point set of g and $\beta_0(y)$ is a local invariant.

For $g = 1$ this reduces to the non-equivariant index theorem and we get

$$\mathrm{index}(D_W, 1) = -\eta(\tilde{D}_{\tilde{M}}, 1).$$

For $g = t$ the fixed point set F consists of isolated points. The contribution of β_0 at an isolated fixed point x was analyzed by Kotake [Ko] who showed

$$\int_x \beta_0(y)\, dy = \det(1 - dT_x)^{-1} \{\mathrm{Tr}(t \text{ on } E(\Delta)_x) - \mathrm{Tr}(t \text{ on } E(\chi \otimes \Delta)_x)\}$$

As discussed above dT_x is multiplication by -1 and t acts by multiplication by $\iota_x e_1 \cdot \ldots \cdot e_{n+1}$ on P_x and hence by multiplication by ι_x (resp. $-\iota_x$) on $E(\Delta)_x$ (resp. $E(\chi \otimes \Delta)_x$) due to our conventions (3.2) and the fact that $e_1 \cdot \ldots \cdot e_{n+1}$ acts by -1 on χ since $n+1=8k+5$ is odd. The complex dimension of Δ is 2^{4k+2} and thus

$$\int_x \beta_0(y)\, dy = 2^{-(n+1)}(2^{4k+2}\iota_x - (-2^{4k+2}\iota_x)) = 2^{-(4k+2)}\iota_x.$$

Putting everything together we have

$$\begin{aligned}
\eta(\tilde{D}_M) &= 1/2(\eta(\tilde{D}_{\tilde{M}}, 1) + \eta(\tilde{D}_{\tilde{M}}, t)) \\
&= 1/2(-\operatorname{index}(D_W, 1) + 2^{-(4k+2)}\textstyle\sum \iota_x - \operatorname{index}(D_W, t)) \\
&= 2^{-(2k+3)}\textstyle\sum \iota_x - [\dim(\ker D_W)^{Z/2} - \dim(\operatorname{coker} D_W)^{Z/2}] \\
&\equiv 2^{-(4k+3)}\textstyle\sum \iota_x \bmod 2Z
\end{aligned}$$

since the $Z/2$-invariant subspaces $(\ker D_W)^{Z/2}$ (resp. $(\operatorname{coker} D_W)^{Z/2}$) are H-vector spaces and hence even dimensional. Q.E.D.

6. Classification of pin$^+$-structures and pin$^-$-structures

In this section we give a topological description of pin$^\pm$-structures which we mainly need for the proof of theorem B in Sect. 8.

The exact sequence of groups

$$1 \longrightarrow Z/2 \longrightarrow \operatorname{Pin}^\pm(n) \xrightarrow{\sigma} O(n) \longrightarrow 1$$

induces a fibration

$$BZ/2 \longrightarrow B\operatorname{Pin}^\pm(n) \xrightarrow{B\sigma} BO(n).$$

It follows from the properties of classifying spaces that pin$^\pm$-structures on a vector bundle α as defined in Sect. 2 correspond bijectively to homotopy classes of lifts $\bar{c}$ in the diagram

$$\begin{tikzcd} & B\operatorname{Pin}^\pm(n) \arrow[d] \\ X \arrow[ur, "\bar{c}"] \arrow[r, "c(\alpha)"] & BO(n) \end{tikzcd}$$

Here $c(\alpha)\colon X \to BO(n)$ is the classifying map of α.

The fibrations $B\operatorname{Pin}^\pm(n) \to BO(n)$ are compatible for various n and hence they are pull backs of a fibration $B\sigma\colon B\operatorname{Pin}^\pm \to BO$ via the natural map $BO(n) \to BO$.

To identify the fibration $B\operatorname{Pin}^\pm \to BO$ we use the following lemma which is easily proved using obstruction theory (compare [K 1, §1]).

Lemma 6.1. *Let $B \to BO$ and $B' \to BO$ be two fibrations such that the homotopy groups of both fibers vanish above dimension k. Suppose that there is a homotopy equivalence between the $(k+1)$-skeleta of B (resp. B') compatible with the projection maps. Then they are fibre homotopy equivalent.*

Let H be the Hopf line bundle over $RP^\infty = BZ/2$ and let γ be the universal spin bundle over $B\mathrm{Spin}$. For a natural number n let $c(\gamma \times nH)$: $B\mathrm{Spin} \times RP^\infty \to BO$ be the classifying map of $\gamma \times nH$ and denote by $B(n) \to BO$ the associated fibration.

Proposition 6.2. i) *$B\mathrm{Pin}^+ \to BO$ is fibre homotopy equivalent to $B(1) \to BO$.*

ii) *$B\mathrm{Pin}^- \to BO$ is fibre homotopy equivalent to $B(3) \to BO$.*

Proof. The exact sequence of groups

$$1 \longrightarrow \mathrm{Spin}(n) \xrightarrow{i} \mathrm{Pin}^+(n) \xrightarrow{p} Z/2 \to 1$$

splits. A splitting homomorphism s^+: $Z/2 \to \mathrm{Pin}^+(n)$ is defined by sending the generator to $e_1 \in S^{n-1} \subset \mathrm{Pin}^+(n) \subset C^+(R^n)$, where $\{e_1, \ldots, e_n\}$ denotes the standard basis of R^n. Analogously the exact sequence

$$1 \longrightarrow \mathrm{Spin}(n) \xrightarrow{i} \mathrm{Pin}^-(n) \xrightarrow{p} Z/2 \to 1$$

splits for $n \geqq 3$. A splitting map s^-: $Z/2 \to \mathrm{Pin}^-(n)$ is defined by sending the generator to $e_1 \cdot e_2 \cdot e_3 \in \mathrm{Pin}^-(n) \subset C^+(R^n)$. Moreover $B\mathrm{Spin}(n)$ is 3-connected for $n \geqq 3$ and hence we can conclude from the fibration

$$B\mathrm{Spin}(n) \xrightarrow{Bi} B\mathrm{Pin}^\pm(n) \xrightarrow{Bp} BZ/2$$

that $Bs^\pm: BZ/2 \to B\mathrm{Pin}^\pm(n)$ is a 3-equivalence. The bundle over $BZ/2$ corresponding to the composition

$$BZ/2 \xrightarrow{Bs^\pm} B\mathrm{Pin}^\pm(n) \xrightarrow{B\sigma} BO(n)$$

is the vector bundle associated to the representation σ^+ (resp. σ^-) restricted to $Z/2 \subset \mathrm{Pin}^\pm(n)$. Since this $Z/2$-action on R^n is non-trivial on $R^1 \subset R^n$ (resp. $R^3 \subset R^n$) the vector bundle in question is $H \oplus (n-1)\varepsilon$ (resp. $3H \oplus (n-3)\varepsilon$), where ε is the trivial line bundle.

This shows that the 2-skeleton of $B\mathrm{Pin}^\pm$ can be identified with RP^2 such that the map to BO corresponds to the classifying map of H (resp. $3H$). The proposition then follows from 6.1. Q.E.D.

If w is an element of $H^2(BO; Z/2)$ let $BO\langle w\rangle \to BO$ be the pull back of the path fibration $PK(Z/2, 2) \to K(Z/2, 2)$ via the map $BO \to K(Z/2, 2)$ determined by w.

Proposition 6.3. i) $BO\langle w_2\rangle \to BO$ is fibre homotopy equivalent to $B(1) \to BO$.

ii) $BO\langle w_2 + w_1^2\rangle \to BO$ is fibre homotopy equivalent to $B(3) \to BO$.

Proof. It follows from the fibre homotopy exact sequence of the fibration $BO\langle w\rangle \to BO$ that the composition

$$BO\langle w\rangle \longrightarrow BO \xrightarrow{c(w_1)} RP^\infty$$

is a 3-equivalence. Here $w = w_2$ (resp. $w = w_2 + w_1^2$) and $c(w_1)$ is the map corresponding to the cohomology class w_1. Moreover the pull back of H (resp. $3H$) via this composition has the same first and second Stiefel Whitney classes as the pull back of the universal bundle. Thus we can identify the 2-skeleton of $BO\langle w\rangle$ with RP^2 and the map to BO with the classifying map of H (resp. $3H$) and hence the claim follows using 6.1. Q.E.D.

As a consequence of 6.2 and 6.3 we see that the fibration $B\mathrm{Pin}^\pm(n) \to BO(n)$ is induced from the path fibration over $K(Z/2, 2)$ and we conclude

Corollary 6.4. *Let α be an n-dimensional vector bundle over some space X.*

i) *α has a* $\mathrm{Pin}^+(n)$*-structure (resp.* $\mathrm{Pin}^-(n)$*-structure)* iff $w_2(\alpha) = 0$ *(resp.* $w_2(\alpha) + w_1^2(\alpha) = 0$*).*

ii) *If α has a* $\mathrm{Pin}^\pm(n)$*-structure then $H^1(X; Z/2)$ acts transitively and effectively on the set of* $\mathrm{Pin}^\pm(n)$*-structures on α.*

If α is a vector bundle over some space X with a lift $\bar{c}$: $X \to BO\langle w\rangle$ of its classifying map $c(\alpha)$: $X \to BO$ we call $\bar{c}$ a $\langle w\rangle$**-structure** on α. If M is a manifold we define a tangential $\langle w\rangle$-structure (resp. normal $\langle w\rangle$-structure) on M to be a $\langle w\rangle$-structure on its tangent bundle τ (resp. its normal bundle ν). One can translate back and forth between tangential and normal structures in the following way.

Let g: $BO \to BO$ be the H-space inverse, i.e. the classifying map of the inverse bundle. Then the pull-back fibration $g^* BO\langle w\rangle \to BO$ can be identified with $BO\langle g^* w\rangle$ and the commutative diagram

$$\begin{tikzcd} & BO\langle g^* w\rangle \arrow[r, "\bar{g}"] \arrow[d] & BO\langle w\rangle \arrow[d] \\ M \arrow[ur, "\bar{c}"] \arrow[r, "c(\nu)"] & BO \arrow[r, "g"] & BO \end{tikzcd}$$

shows that $\bar{c} \to \bar{g}\bar{c}$ gives a bijection between normal $\langle g^* w\rangle$-structures and tangential $\langle w\rangle$-structures, since $gc(\nu)$ is the classifying map of the tangent bundle. An easy computation shows $g^*(w_2) = w_2 + w_1^2$ and we conclude using (6.4):

Corollary 6.5. Pin^+*-structures (resp.* pin^-*-structures) on a manifold are in* 1 $-$ 1*-correspondence with normal $\langle w_2 + w_1^2\rangle$-structures (resp. $\langle w_2\rangle$-structures).*

7. Exotic structures on 4-manifolds

In this section we describe two methods to produce exotic differentiable structures on 4-manifolds due to Cappell-Shaneson [CS] respectively M. Kreck [K2]. Moreover we show that in some cases this exotic structure can be detected by the η-invariant. We first describe the exotic structure due to M. Kreck.

Let K be the Kummer surface which is a simply connected closed 4-manifold with signature 16 and let S be the connected sum of eleven copies of $S^2 \times S^2$. If M is a non-orientable closed smooth 4-manifold, it is an easy consequence of Freedman's results that the connected sum $M \# K$ is homeomorphic to $M \# S$. On the other hand there are many examples where $M \# K$ is not diffeomorphic to $M \# S$ [K2, Thm. 1].

In some cases $M \# K$ and $M \# S$ can be distinguished using the η-invariant as follows. Using the result (6.4) we see that K and S have unique pin^+-structures ϕ_K resp. ϕ_S, since they are simply connected and their intersection form is even which implies the vanishing of the second Stiefel-Whitney class. Their η-invariants can be computed via their signatures (Cor. 5.2). If M has a pin^+-structure ϕ then

$$\eta(M \# K, \phi \# \phi_K) = \eta(M, \phi) + \eta(K, \phi_K) = \eta(M, \phi) + 1 \bmod 2Z$$

and

$$\eta(M \# S, \phi \# \phi_S) = \eta(M, \phi) + \eta(S, \phi_S) = \eta(M, \phi) \bmod 2Z,$$

since the connected sum of two manifolds is bordant to their disjoint union and the η-invariant is clearly additive with respect to disjoint unions.

Note that this does *not* imply that $M \# K$ and $M \# S$ aren't diffeomorphic since the η-invariant involves the choice of a pin^+-structure. By Corollary 6.4 the possible choices of pin^+-structures on a manifold N are parametrized by $H^1(N; Z/2)$. Hence if we assume $H^1(N; Z/2) \cong Z/2$ there are only two pin^+-structures. Moreover if ϕ is a pin^+-structure on N and N is non-orientable the only other pin^+-structure is $-\phi$ with $(N, -\phi)$ representing the inverse of (N, ϕ) in the bordism group of manifolds with pin^+-structure. In particular $\eta(N, \phi) = \pm \eta(N, \phi') \bmod 2Z$ for any pin^+-structure ϕ'. We conclude:

Theorem 7.1. *Let M be a non-orientable closed 4-manifold with* pin^+*-structure ϕ and assume $H^1(M; Z/2) \cong Z/2$ and $\eta(M, \phi) \neq \pm 1/2 \bmod 2Z$. Then $\eta(M \# K, \phi') \neq \eta(M \# S, \phi'')$ for all* pin^+*-structures ϕ', ϕ'' on $M \# K$ (resp. $M \# S$).*

Corollary 7.2. *$\eta(RP^4 \# K, \phi') \neq \eta(RP^4 \# S, \phi'')$ for all* pin^+*-structures ϕ', ϕ'' on $RP^4 \# K$ (resp. $RP^4 \# S$). In particular $RP^4 \# S$ has an exotic differentiable structure detected by the η-invariant.*

Now we turn to the exotic structures of Cappell-Shaneson. A matrix $A \in \mathrm{GL}(3; Z)$ induces a diffeomorphism

$$\psi_A \colon T^3 \to T^3$$

by viewing the 3-torus T^3 as the quotient of R^3 by the integral lattice Z^3. Let M_A be the mapping torus of ψ_A, i.e. M_A is obtained from $T^3 \times [0, 1]$ by identifying $(x, 0)$ with $(\psi_A x, 1)$.

If M is a closed non-orientable 4-manifold, α is an embedded, orientation reversing circle and $A \in \mathrm{GL}(3; Z)$ is a matrix with $\det A = -1$ denote by $M \#_\alpha M_A$ the manifold constructed by deleting a tubular neighbourhood of α resp. of the circle $* \times \{t\} \subset (T^3 \times [0, 1])/_\sim = M_A$ and glueing those two manifolds along their common boundary.

Then there is a simple homotopy equivalence

$$f\colon M \#_\alpha M_A \to M$$

provided $\det(\mathrm{id} - A^2) = \pm 1$ [CS, Thm. 3.1]. Moreover the normal invariant of f is the unique non-trivial element in the kernel of $[M, \mathrm{G/O}] \to [M, \mathrm{G/TOP}]$.

Using the surgery exact sequence [W, Thm. 10.3 and Thm. 10.5]

$$\to \mathrm{L}_5(Z\pi_1 M, w_1(M)) \to \mathcal{S}^{\mathrm{TOP}}(M) \to [M, \mathrm{G/TOP}] \to$$

we conclude that $M \#_\alpha M_A$ is topologically s-cobordant and hence by Freedman homeomorphic to M if the surgery group vanishes. This is the case for example if $\pi_1(M) \cong Z/2$ [W, Thm. 13A.1.]. The following result is the key step to distinguish M and $M \#_\alpha M_A$ by means of the η-invariant.

Proposition 7.3. *M_A has a* pin$^+$*-structure ϕ_A such that $\eta(M_A, \phi_A) = 1$.*

Proof. The basic idea to compute $\eta(M_A, \phi_A)$ is to use a combination of (5.2) and (5.3). Let W be the standard bordism between two copies of the mapping torus of the diffeomorphism ψ_{-A} and the mapping torus of the composition $\psi_{-A} \circ \psi_{-A} = \psi_{A^2}$, which is constructed as follows. Let D' be the 2-dimensional disk with two smaller disks deleted and cut open along the lines L and L' as indicated in the picture.

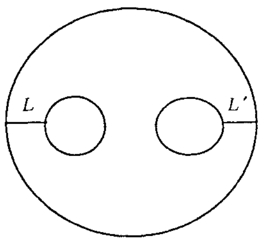

Fig. 1

Then W is the product $T^3 \times D'$ glued along $T^3 \times L$ (resp. $T^3 \times L'$) using the diffeomorphism $\psi_{-A} \times \mathrm{id}$. Let $T\colon W \to W$ be the involution induced by the involution

$$\psi_{-\mathrm{id}} \times -\mathrm{id}\colon T^3 \times D' \to T^3 \times D'.$$

It has exactly eight fixpoints of the form $([x], 0)$ where $x \in R^3$ is half integral, i.e. $2x \in Z^3$. In particular the involution is free on ∂W and the corresponding quotient is the disjoint union of M_A and M_{-A}. We assume for the moment that W has a $Z/2$-equivariant pin$^+$-structure ϕ and that the eight fixed points have index $+1$. Then ϕ induces pin$^+$-structures ϕ_A (resp. ϕ_{-A}) on M_A (resp. M_{-A}) and using (5.3) we conclude:

$$\eta(M_A, \phi_A) + \eta(M_{-A}, \phi_{-A}) = 2^{-3} \sum \iota_x = 1 \bmod 2Z.$$

Note that M_{-A} is an orientable manifold and hence we can use (5.2) to conclude $\eta(M_{-A}, \phi_{-A}) = 1/16 \operatorname{sign}(M_{-A}) = 0 \bmod 2Z$ since the signature of a mapping torus vanishes by Novikov additivity.

It remains to be shown that W has an equivariant pin$^+$-structure. We observe that $T^3 \times D'$ has a pin$^+$-structure induced by the natural trivializations of the tangent bundles of T^3 and D', i.e. $P = T^3 \times D' \times \mathrm{Pin}^+(5)$ and the vector bundle isomorphism f is given by those trivializations. Moreover we can turn this into a $Z/2$-equivariant pin$^+$-structure by defining

$$\bar{T} = -\mathrm{id} \times -\mathrm{id} \times s_5 \colon T^3 \times D' \times \mathrm{Pin}^+(5) \to T^3 \times D' \times \mathrm{Pin}^+(5),$$

where $s_5 = e_1 \cdot \ldots \cdot e_5 \in \mathrm{Pin}^+(5)$. To extend this $Z/2$-equivariant pin$^+$-structure to W we have to study the derivative of the glueing diffeomorphism. With respect to the trivialization f it is given by multiplication by $A \in \mathrm{GL}(3; Z) \subset \mathrm{GL}(5; R)$, but using a path between A and some element $A' \in O(5)$ we can change f without loosing the equivariance such that the derivative of the glueing map is given by multiplication by A' with respect to this new trivialization. Choosing an element $A'' \in \mathrm{Pin}^+(5)$ projecting down to A' we can extend the glueing to the principal bundle P by left multiplication by A'' on the factor $\mathrm{Pin}^+(5)$. The vector bundle isomorphism f then extends by construction and $\bar{T}$ extends to this bundle since $s_5 \in \mathrm{Pin}^+(5)$ commutes with all other elements. Moreover by construction of $\bar{T}$ all fixed points have index $+1$. Q.E.D.

If M is a closed non-orientable 4-manifold, α is an embedded, orientation reversing circle and ϕ is a pin$^+$-structure on M restricting to the trivial pin$^+$-structure on α (i.e. the pin$^+$-structure extending to the disk) then ϕ and ϕ_A fit together to give a pin$^+$-structure $\phi \#_\alpha \phi_A$ on $M \#_\alpha M_A$. Moreover $(M \#_\alpha M_A, \phi \#_\alpha \phi_A)$ is pin$^+$-bordant to the disjoint union of (M, ϕ) and (M_A, ϕ_A) (a bordism is provided by $M \times I \cup_T T \times I \cup_T M_A \times I$, where I is the unit interval $[0, 1]$ and T is the tubular neighbourhood of the circle in M (resp. M_A)). Using the same arguments as for theorem 7.1 we conclude:

Theorem 7.4. *Let M be a non-orientable closed 4-manifold with pin$^+$-structure ϕ and assume $H^1(M; Z/2) \cong Z/2$ and $\eta(M, \phi) \neq \pm 1/2 \bmod 2Z$. Then $\eta(M \#_\alpha M_A, \phi') \neq \eta(M, \phi)$ for all pin$^+$-structures ϕ' on $M \#_\alpha M_A$.*

Corollary 7.5. *Let Q^4 be $RP^4 \#_\alpha M_A$. Then $\eta(Q^4, \phi') \neq \eta(RP^4, \phi'')$ for all pin$^+$-structures ϕ', ϕ'' on Q^4 resp. RP^4. In particular RP^4 has an exotic differentiable structure detected by the η-invariant.*

8. Proof of theorem B

Let M be a closed non-orientable 4-manifold with fundamental group $Z/2$ and pin$^+$-structure ϕ. Denote by $\bar{c}$ the normal $\langle w \rangle$-structure, $w = w_2 + w_1^2$, corresponding to ϕ using (6.5).

$$\begin{tikzcd} & BO\langle w \rangle \arrow[d] \\ M \arrow[ur, "\bar{c}"] \arrow[r, "c(v)"] & BO \end{tikzcd}$$

Note that $\pi_1(BO\langle w \rangle) \cong Z/2$ and $\pi_2(BO\langle w \rangle) = 0$ which follows from the fibre homotopy exact sequence of the fibration

$$K(Z/2, 1) \to BO\langle w \rangle \to BO.$$

Hence $\bar{c}$ induces an isomorphism on π_1 (since M is non-orientable) and is a surjection on π_2. Thus the pair $(M, \bar{c})$ is a normal 1-smoothing in the sense of M. Kreck [K 1], [K 2] and it follows from his results that two such pairs are stably diffeomorphic if and only if their Euler characteristics agree and they represent the same element in the bordism group $\Omega_4^{\langle w \rangle}$ of manifolds with normal $\langle w \rangle$-structure.

We claim that the map induced by the η-invariant $\eta: \Omega_4^{\langle w \rangle} \to R/2Z$ is injective. To prove it we use the Pontrjagin Thom construction to identify $\Omega_4^{\langle w \rangle}$ with $\pi_4 MO[w]$, the Thom spectrum associated to $BO\langle w \rangle \to BO$. It follows from (6.3) that $MO[w]$ is homotopy equivalent to $M\mathrm{Spin} \wedge M(3H)$ where $M(3H)$ is the Thom spectrum of $3H$ and $MS\mathrm{pin}$ is the Thom spectrum associated to $BS\mathrm{pin} \to BO$.

Following [K 2] the inspection of the relevant terms in the Atiyah Hirzebruch spectral sequence shows that $\Omega_4^{\langle w \rangle} \cong \pi_4 MO[w] \cong \pi_4(M\mathrm{Spin} \wedge M(3H))$ is a group of order at most 16. On the other hand $\eta(RP^4, g, \phi) = 1/8 \bmod 2Z$ by corollary 5.4 and hence RP^4 represents an element of order 16 in $\Omega_4^{\langle w \rangle}$. It follows that $\Omega_4^{\langle w \rangle}$ is isomorphic to $Z/16$ and that the homomorphism η: $\Omega_4^{\langle w \rangle} \to R/2Z$ is injective.

Finally if M^4 is a non-orientable manifold with fundamental group $Z/2$ and twisted pin-structure ϕ according to (6.4) the only other twisted pin-structure is $-\phi$. Since $[M, -\phi]$ is the inverse of $[M, \phi] \in \Omega_4^{\langle w \rangle}$ we find $\eta(M, -\phi) = -\eta(M, \phi)$. This proves theorem B.

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