Bordism and involutions

By R. E. Stong

1. Introduction

Let X be a topological space with $A \subset X$ a subspace, and let $\tau: (X, A) \to (X, A)$ be an involution; i.e., a continuous map $\tau: X \to X$ with square the identity and such that $\tau A \subset A$. Combining the notions of bordism (Atiyah [1]) and of differentiable periodic maps (Conner and Floyd [4]), one may define bordism groups of the involution (X, A, τ).

Specifically, a (free) equivariant bordism class of (X, A, τ) is an equivalence class of triples (M, μ, f) with M a compact differentiable manifold with boundary, $\mu: M \to M$ a differentiable (fixed-point free) involution on M, and $f: (M, \partial M) \to (X, A)$ a continuous equivariant map $[\tau f = f\mu]$ sending ∂M into A. Two triples (M, μ, f) and (M', μ', f') are equivalent, or bordant, if there is a 4-tuple (W, V, ν, g) such that W and V are compact differentiable manifolds with boundary, $\partial V = \partial M \cup \partial M'$ and $\partial W = M \cup M' \cup V/\partial M \cup \partial M' \equiv \partial V$, $\nu: (W, V) \to (W, V)$ is a differentiable (fixed-point free) involution restricting to μ on M and μ' on M', and $g: (W, V) \to (X, A)$ is a continuous equivariant map $[\tau g = g\nu]$ restricting to f on M and f' on M'.

The disjoint union of triples induces an operation on the set of (free) equivariant bordism classes of (X, A, τ) making this set into an abelian group. This is a graded group, where the grading is given by the dimension of the manifold M, and one lets $\mathfrak{N}_*(X, A, \tau)$ be the group of equivariant bordism classes of (X, A, τ), and one lets $\hat{\mathfrak{N}}_*(X, A, \tau)$ be the group of free equivariant bordism classes of (X, A, τ). If A is empty, one writes $\mathfrak{N}_*(X, \tau)$ and $\hat{\mathfrak{N}}_*(X, \tau)$ for these groups.

Letting X be a point, with A empty and τ the identity map, this reduces to the situation studied by Conner and Floyd [4], with $\mathfrak{N}_*(\mathrm{pt}, 1) = I_*(Z_2)$ and $\hat{\mathfrak{N}}_*(\mathrm{pt}, 1) = \mathfrak{N}_*(Z_2)$ in their notation.

The main purpose of this paper is to compute the groups $\mathfrak{N}_*(X, \tau)$ and $\hat{\mathfrak{N}}_*(X, \tau)$ and to explore their interrelationships. As always, however, the study of an involution (X, τ) is really the study of a pair (X, F_τ, τ), with F_τ the fixed-point set of τ, and one is forced to consider pairs in order to study the absolute case. To study more general pairs tends to force a study of 4-tuples, and this additional complication will naturally be avoided.

The major portion of this paper is a geometric analysis of bordism of involutions, following the Conner and Floyd approach. The main results are

PROPOSITION 1. *Equivariant bordism and free equivariant bordism are equivariant generalized homology theories on the category of pairs with involution.*

PROPOSITION 2. *There is an exact sequence*

$$\begin{tikzcd} \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[rr, "k^*"] & & \mathfrak{N}_*(X, A, \tau) \arrow[dl, "F"] \\ & \bigoplus_{k=0}^{*} \mathfrak{N}_k\big(F_\tau \times BO(* - k), (A \cap F_\tau) \times BO(* - k)\big) \arrow[ul, "S"] & \end{tikzcd}$$

where k_* forgets freeness, F is obtained from analysis of the fixed-point set, and S is obtained from a sphere bundle construction.

PROPOSITION 3. $\hat{\mathfrak{N}}_*(X, A, \tau) \cong \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a)$ *where a is the antipodal involution on the infinite sphere.*

PROPOSITION 4. *For any involution (X, τ), the equivariant bordism exact sequence of the involution (X, F_τ, τ) is naturally split. Thus*

$$\mathfrak{N}_*(X, \tau) \cong \mathfrak{N}_*(F_\tau, 1) \oplus \hat{\mathfrak{N}}_*(X, F_\tau, \tau)$$

or $\mathfrak{N}_n(X, \tau)$ is isomorphic to

$$\Big\{\bigoplus_{\substack{k=0 \\ k \neq n-1}}^{n} \mathfrak{N}_k\big(F_\tau \times BO(n - k)\big)\Big\} \oplus \mathfrak{N}_n\big(X \times S^\infty/\tau \times a, F_\tau \times RP(\infty)\big) .$$

This last makes extremely precise the heretofore vague feeling that the general involution is a combination of a trivial involution and a free involution. In particular, understanding these two special cases really gives complete information about the general case.

Exploring the bordism analogue of the Smith homomorphism one has

PROPOSITION 5. *There is an exact sequence*

$$\begin{tikzcd} \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[r, "\mathfrak{L}_*"] & \mathfrak{N}_*(X, A) \arrow[r, "1 + \tau_*"] & \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[ll, bend left, "\Delta"] \end{tikzcd}$$

where $\mathfrak{L}_*$ forgets equivariance, $1 + \tau_*$ converts any map to an equivariant map (sending $f\colon M \to X$ to $g\colon M \times Z_2 \to X$ with $g(m, 1) = f(m)$, $g(m, -1) = \tau f(m)$), and Δ is a Smith homomorphism of degree -1.

The general development ends with an equivariant formulation of spectra, aimed primarily at obtaining free equivariant bordism in analogy with the classical spectra theoretic approach to bordism. This approach differs from that of Bredon [3, Chapter III], which is inadequate for free bordism. The main result justifying this approach is a Poincaré duality theorem.

PROPOSITION 6. *If X is a closed n-dimensional differentiable manifold with differentiable involution τ, then*

$$\widehat{\mathfrak{N}}^q(X, \tau) \cong \widehat{\mathfrak{N}}_{n-q}(X, \tau) ,$$

the "cobordism set" being formed with a suitable equivariant Thom spectrum.

As an application of the equivariant bordism concept, following a suggestion of Professor P. E. Conner, the final section of the paper considers the bordism classification of 4-tuples $(M, \sigma, \xi, \sigma^*)$ where M is a closed differentiable manifold, $\sigma\colon M \to M$ is a differentiable involution, ξ is a real n-plane bundle over M, and σ^* is an involution of ξ by a real bundle map covering σ. Since n-plane bundles of this type are classified by equivariant maps into $BO(n)$ with appropriate involution τ, the problem is to analyze the equivariant bordism group $\mathfrak{N}_*(BO(n), \tau)$. The main results obtained are

PROPOSITION 7. *The correspondence*

$$(M, \sigma, \xi, \sigma^*) \longrightarrow \big(M/\sigma \xrightarrow{f} BO(1) \times BO(n)\big) ,$$

where f classifies the line bundle associated to the double cover of M/σ by M and the n-plane bundle ξ/σ^* over M/σ, defines an isomorphism

$$q\colon \widehat{\mathfrak{N}}_*\big(BO(n), \tau\big) \xrightarrow{\cong} \mathfrak{N}_*\big(BO(1) \times BO(n)\big) .$$

PROPOSITION 8. *The fixed point set of the involution $\tau\colon BO(n) \to BO(n)$ is the disjoint union of the spaces $BO(j) \times BO(n-j)$ for $0 \leqq j \leqq n$, and the exact sequence of Proposition 2 becomes a split exact sequence*

$$0 \longrightarrow \mathfrak{N}_*\big(BO(n), \tau\big) \xrightarrow{F} \bigoplus_{k=0}^{*} \bigoplus_{j=0}^{n} \mathfrak{N}_k\big(BO(*-k) \times BO(j) \times BO(n-j)\big) \xrightarrow{S} \mathfrak{N}_*\big(BO(1) \times BO(n)\big) \longrightarrow 0 .$$

In particular, the fixed point information completely determines the bordism class of the 4-tuple $(M, \sigma, \xi, \sigma^*)$. Most of the Conner and Floyd analysis of involutions on manifolds may then be carried through for such 4-tuples.

A closely related application is the bordism classification of Atiyah real bundles over manifolds given by $\mathfrak{N}_*\big(BU(n), c\big)$ where c is the involution induced by complex conjugation. These groups are also studied together with their relation to $\mathfrak{N}_*\big(BO(2n), \tau\big)$.

The author is indebted to the National Science Foundation and Princeton University for financial support during this work, and to Professors P. E. Conner and Larry Smith for many useful discussions concerning these ideas.

2. Bordism as an homology theory

Let $\mathcal{P}$ denote the category of pairs with involution, (X, A, τ), and equivariant maps of pairs. Let $\mathcal{H}_n(X, A, \tau)$ denote one of $\mathfrak{N}_n(X, A, \tau)$ and $\hat{\mathfrak{N}}_n(X, A, \tau)$. If $f: (X, A, \tau) \to (Y, B, \sigma)$, let $\mathcal{H}_n(f): \mathcal{H}_n(X, A, \tau) \to \mathcal{H}_n(Y, B, \sigma)$ be the homomorphism sending $[(M, \mu, g)]$ to $[(M, \mu, f \cdot g)]$. Let $\partial_n: \mathcal{H}_n(X, A, \tau) \to \mathcal{H}_{n-1}(A, \tau)$ be the homomorphism sending $[(M, \mu, g)]$ to $[(\partial M, \mu, g)]$. One then has

PROPOSITION 1. *$\{\mathcal{H}_n, \partial_n\}$ is a sequence of covariant functors $\mathcal{H}_n: \mathcal{P} \mapsto$ Abel, together with a sequence of natural transformations $\partial_n: \mathcal{H}_n(X, A, \tau) \to \mathcal{H}_{n-1}(A, \tau)$ satisfying*

(1) *If f_0, f_1 are equivariantly homotopic maps, then $\mathcal{H}_n(f_0) = \mathcal{H}_n(f_1)$.*

(2) *If U is an invariant open set with closure contained in the interior of the closed set A, then the inclusion $i: (X - U, A - U) \to (X, A)$ induces an isomorphism*

$$\mathcal{H}_n(i): \mathcal{H}_n(X - U, A - U, \tau) \longrightarrow \mathcal{H}_n(X, A, \tau) .$$

(3) *The sequence*

$$\cdots \longrightarrow \mathcal{H}_n(A, \tau) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(X, \tau) \xrightarrow{\mathcal{H}_n(j)} \mathcal{H}_n(X, A, \tau) \xrightarrow{\partial_n} \mathcal{H}_{n-1}(A, \tau) \longrightarrow \cdots$$

with $(A, \varnothing, \tau) \xrightarrow{i} (X, \varnothing, \tau) \xrightarrow{j} (X, A, \tau)$ the inclusions, is exact.

PROOF. This is, of course, very easy to prove and also fairly standard. To see (1), one notes that an equivariant homotopy $G: (M \times I, \partial M \times I) \to (X, A)$ gives a cobordism from $(M, \mu, G(\,, 0))$ to $(M, \mu, G(\,, 1))$. To see (2), one notes that, if $g: (M, \partial M) \to (X, A)$, there is an equivariant submanifold with boundary $M' \subset M$ with $M - M'$ also a manifold with boundary and such that $M' \subset g^{-1}(X - \bar{U})$, $M - M' \subset g^{-1}(\text{interior } A)$ [One need only split M at a regular value of $\frac{1}{2}[h + h \circ \mu]$ where $h: M \to [0, 1]$ is differentiable with $h^{-1}(0) \supset g^{-1}(\bar{U})$ and $h^{-1}(1) \supset X - g^{-1}(A^0)$] and then $g: (M', \partial M') \to (X - U, A - U)$ is bordant in (X, A) to (M, μ, g) by $(M \times I, \partial M \times I \cup (M - M') \times 1, \mu \times 1, g \circ \pi_1)$, so $\mathcal{H}_n(i)$ is epic. The same construction applied to a bordism shows that $\mathcal{H}_n(i)$ is monic. To prove (3) requires six steps.

(1) $\mathcal{H}_n(j)\mathcal{H}_n(i) = 0$, for $\mathcal{H}_n(j)\mathcal{H}_n(i)\alpha$ is represented by a map $g: (M, \partial M) \to (X, A)$ with $g(M) \subset A$, and then $(M \times I, \partial M \times I \cup M \times 1, \mu \times 1, g \circ \pi_1)$ is a bordism to zero.

(2) $\ker \mathcal{H}_n(j) \subset \operatorname{im} \mathcal{H}_n(i)$ for if $g: M \to X$ bounds (V, W, ν, h) then $\partial W = \partial M = \varnothing$, so $h: W \to A$ is a bordism element equivalent to (M, μ, g).

(3) $\partial_n \mathcal{H}_n(j) = 0$ for $\partial_n \mathcal{H}_n(j)\,[M, \mu, g]$ is represented by $(\partial M, \mu, g)$ and ∂M is empty.

(4) $\ker \partial_n \subset \operatorname{im} \mathcal{H}_n(j)$ for if $\partial_n[M, \mu, g]$ is zero, then $(\partial M, \mu, g)$ is the boundary of some map $h: (V, \nu) \to (A, \tau)$, and letting $W = M \cup V/\partial M \equiv \partial V$,

$$\big(W \times I,\ V \times 0,\ (\mu \cup \nu) \times 1,\ (g \cup h) \circ \pi_1\big)$$

is a bordism of (M, μ, g) with $(W, \mu \cup \nu, g \cup h)$ which is a representative coming from $\mathcal{H}_n(X, \tau)$.

(5) $\mathcal{H}_{n-1}(i)\partial_n = 0$ for $(\partial M, \mu, g)$ bounds $(M, \varnothing, \mu, g)$.

(6) $\ker \mathcal{H}_{n-1}(i) \subset \operatorname{im} \partial_n$, for if $[M, \mu, g] \in \mathcal{H}_{n-1}(A, \tau)$ is zero in $\mathcal{H}_{n-1}(X, \tau)$ there is an $h\colon V \to X$, with $\partial(V, \varnothing, \nu, h) = (M, \mu, g)$, so $[V, \nu, h] \in \mathcal{H}_n(X, A, \tau)$ with $\partial_n[V, \nu, h] = [M, \mu, g]$.**

This makes both $\mathfrak{N}_*$ and $\widehat{\mathfrak{N}}_*$ equivariant generalized homology theories on the category of pairs with involution, as defined by Bredon [3, Ch. I, § 2], for example. Slightly more details for the proof of the proposition may be found in Conner and Floyd [4, Ch. I, § 5].

3. Nature of the fixed-point set

If $\tau\colon X \to X$ is an involution, the fixed-point set of τ, F_τ, is the set of points $x \in X$ with $\tau(x) = x$.

Being given a class $\alpha \in \mathfrak{N}_n(X, A, \tau)$ represented by (M^n, μ, f), one has the equivariant map $f\colon (M, \partial M) \to (X, A)$. If $m \in F_\mu$, $\tau f(m) = f(\mu m) = f(m)$, so $f(m) \in F_\tau$, defining a map $f\,|_{F_\mu}\colon (F_\mu, F_\mu \cap \partial M) \to (F_\tau, F_\tau \cap A)$.

As in Conner and Floyd [4, § 22], F_μ is a manifold with boundary, ∂F_μ being $F_\mu \cap \partial M$, and one lets F_μ^k be the union of the k-dimensional components of F_μ. The normal bundle of F_μ^k, ν_k, is an $(n-k)$ plane bundle classified by a map $\nu_k\colon F_\mu^k \to BO(n-k)$. One then has a map

$$\varphi = \textstyle\bigcup_{k=0}^{n} (f\,|_{F_\mu^k} \times \nu_k) : \bigcup_{k=0}^{n} F_\mu^k \longrightarrow \bigcup_{k=0}^{n} F_\tau \times BO(n-k)$$

and sending $\bigcup_{k=0}^{n} \partial(F_\mu^k)$ into $\bigcup_{k=0}^{n} (F_\tau \cap A) \times BO(n-k)$.

Assigning to (M^n, μ, f) the class of this map defines an element of

$$\textstyle\bigoplus_{k=0}^{n} \mathfrak{N}_k\big(F_\tau \times BO(n-k),\ (F_\tau \cap A) \times BO(n-k)\big)\ ,$$

which may be seen to depend only on α by making the same construction on a bordism. This defines the homomorphism

$$F\colon \mathfrak{N}_*(X, A, \tau) \longrightarrow \textstyle\bigoplus_{k=0}^{*} \mathfrak{N}_k\big(F_\tau \times BO(*-k),\ (F_\tau \cap A) \times BO(*-k)\big)\ .$$

The critical point in extending the Conner and Floyd results to bordism of involutions is the observation that, up to bordism, the map φ completely determines the nature of the map $f\colon M \to X$ in a neighborhood of the fixed-point set F_μ.

Specifically, there is a map $g\colon (M, \partial M) \to (X, A)$ equivariantly homotopic (through such maps) to f, with $g\,|_{F_\mu} = f\,|_{F_\mu}$ and such that on a tubular neighborhood V_k of F_μ^k

$$g\,|_{V_k} = f\,|_{F^k} \circ \pi : V_k \cong D(\nu_k) \xrightarrow{\ \pi\ } F_\mu^k \xrightarrow{\ f|_{F_\mu^k}\ } F_\tau\ ,$$

π being the projection of the disc bundle of ν_k, $D(\nu_k)$, on F^k_μ, V_k being identified with $D(\nu_k)$ in the standard fashion so that $\mu\,|_{V_k}$ coincides with multiplication by -1 in the fibers of $D(\nu_k)$.

To see that this is possible, one identifies some tubular neighborhood V'_k of F^k with $D'(\nu_k)$ equivariantly and applies a fiberwise deformation in $D'(\nu_k)$ fixed on a neighborhood of the sphere bundle $S'(\nu_k)$ and shrinking a smaller neighborhood $D(\nu_k)$ to its zero section, F^k_μ.

Being given $\beta \in \mathfrak{N}_k(F_\tau \times BO(n-k), (F_\tau \cap A) \times BO(n-k))$ represented by a map $g: H^k \to F_\tau \times BO(n-k)$, let ρ be the $(n-k)$ plane bundle over H^k induced by $\pi_2 \circ g$ from the canonical bundle over $BO(n-k)$. One then has $f: D(\rho) \to F_\tau$ defined by $\pi_1 \circ g \circ \pi$, with π the projection of $D(\rho)$. Using the involution of $D(\rho)$ given by -1 in the fibers, f gives an equivariant map of $D(\rho)$ into X. The restriction of the involution to $S(\rho)$, the sphere bundle of ρ, is fixed-point free, and sends the boundary of $S(\rho)$ (the inverse image of ∂H^k under π, or $S(\rho\,|_{\partial H^k})$) into A, so $f\,|_{S(\rho)}: S(\rho) \to X$ defines a class $S(\beta) \in \hat{\mathfrak{N}}_{n-1}(X, A, \tau)$.

Letting $k_*: \hat{\mathfrak{N}}_*(X, A, \tau) \to \mathfrak{N}_*(X, A, \tau)$ be the homomorphism induced by forgetting the free condition, one then has

PROPOSITION 2. *The sequence*

$$\begin{tikzcd}[column sep=small] \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[rr, "k_*"] & & \mathfrak{N}_*(X, A, \tau) \arrow[dl, "F"] \\ & \bigoplus_{k=0}^{*} \mathfrak{N}_k(F_\tau \times BO(*-k), (F_\tau \cap A) \times BO(*-k)) \arrow[ul, "S"] & \end{tikzcd}$$

is exact.

PROOF. The proof is an obvious repetition of the proof given by Conner and Floyd [4, § 28], making use of the observations about fixed-point structure.

(1) $Fk_* = 0$. If $\gamma = [M, \mu, f] \in \hat{\mathfrak{N}}_*(X, A, \tau)$, F_μ is empty so $Fk_*(\gamma) = 0$.

(2) $SF = 0$. If $\alpha = [M, \mu, f] \in \mathfrak{N}_*(X, A, \tau)$, with fixed set F_μ and tubular neighborhoods $V_k = D(\nu_k)$ on which $f\,|_{V_k} = f\,|_{F^k_\mu} \circ \pi$, then $SF(\alpha)$ is represented by $\bigcup_{k=0}^n f\,|_{S(\nu_k)} : \bigcup_{k=0}^n S(\nu_k) \to X$, and

$$(M - \bigcup V_k^\circ, \partial M - \bigcup (V_k \cap \partial M)^\circ, \mu, f)$$

is a bordism of this to zero (where $^\circ$ denotes interior).

(3) $k_*S = 0$. If

$$\beta = [H, g] \in \mathfrak{N}_k(F_\tau \times BO(n-k), (F_\tau \cap A) \times BO(n-k)),$$

$k_*S(\beta)$ is represented by $(S(\rho), -1, f\,|_{S(\rho)})$ and $(D(\rho), D(\rho\,|_{\partial H}), -1, f)$ is a bordism of this to zero in $\mathfrak{N}_{n-1}(X, A, \tau)$.

(4) $\ker F \subset \operatorname{im} k_*$. If $\alpha = [M, \mu, f] \in \mathfrak{N}_n(X, A, \tau)$ with f being $f\,|_{F^k_\mu} \circ \pi$

on $V_k = D(\nu_k)$ and $F(\alpha) = 0$, there are manifolds W_k with boundary $F^k_\mu \cup W'_k$ and maps $g_k: (W_k, W'_k) \to (F_\tau, F_\tau \cap A)$ extending $f|_{F^k_\mu}$ and $(n-k)$ plane bundles ρ_k over W_k restricting to ν_k on F^k_μ. Let W be formed from $M \times I \cup \bigcup_{k=0}^n D(\rho_k)$ by identifying $V_k \times 1$ with $D(\rho_k)|_{F^k_\mu} \cong D(\nu_k)$. W has an involution ω given by $\mu \times 1$ on $M \times I$ and -1 in the fibers of $D(\rho_k)$, and there is an equivariant map $h: W \to X$ given by $f \circ \pi_1$ on $M \times I$ and by $g_k \circ \pi$ on $D(\rho_k)$. Then $\big(W, \partial M \times I \cup \bigcup_{k=0}^n D(\rho_k \mid W'_k), \omega, h\big)$ is a bordism of (M, μ, f) and the free involution induced on $\{M - \bigcup_{k=0}^n V^\circ_k\} \cup \{\bigcup_{k=0}^n S(\rho_k)\}/\partial V_k \equiv S(\rho_k|_{F^k_\mu})$, and so $\alpha \in \operatorname{im} k_*$.

(5) $\ker S \subset \operatorname{im} F$. If $\beta = [\bigcup_{k=0}^n (H^k, g_k)]$ with $S(\beta) = 0$, there is a free manifold W^n with $\partial W = \{\bigcup S(\rho_k)\} \cup W'/\partial W' \equiv \bigcup S(\rho_k|_{\partial H^k})$ and an equivariant map $h: (W, W') \to (X, A)$ extending $\bigcup f_k|_{S(\rho_k)}$. Let M^n be formed from $W \cup \bigcup D(\rho_k)$ by identifying the two copies of $\bigcup S(\rho_k)$, with $f: M \to X$ given by h on W and $\bigcup f_k$ on $\bigcup D(\rho_k)$, and with involution μ given by that of W and -1 in the fibers of $D(\rho_k)$. The fixed-point set of M is precisely the union of the zero sections in the $D(\rho_k)$; i.e., $\bigcup H^k$, with normal bundles ρ_k classified by $\pi_2 \circ g_k$ and maps to F_τ given by $f_k|_{H^k} = \pi_1 \circ g_k$. Thus $\beta = F[M, \mu, f]$.

(6) $\ker k_* \subset \operatorname{im} S$. If $\gamma = [M, \mu, f] \in \widehat{\mathfrak{N}}_n(X, A, \tau)$ and $k_*(\gamma) = 0$, there is a manifold W with $\partial W = M \cup W'$ with involution ω extending μ and an equivariant map $h: (W, W') \to (X, A)$ extending f. The fixed-point set of ω is a submanifold with boundary, with boundary contained in the interior of W', and not meeting M. One may then deform h, supposing it kept the same on M so that $h|_{V_k} = h|_{F^k_\omega} \circ \pi$ on tubular neighborhoods V_k of F^k_ω, identifying V_k with $D(\nu_k)$. Then $(W - \bigcup V^\circ_k, W' - \bigcup (W' \cap V_j)^\circ, \omega, h)$ is a free bordism of (M, μ, f) with $\big(\bigcup_k S(\nu_k), -1, \bigcup_k (h|_{F^k_\omega} \circ \pi)\big)$ and hence γ is in the image of S.

Notes. (1) If $A \supset F_\tau$, then from the sequence k_* is an isomorphism. One may see this by noting that for (M, μ, f) one may suppose $f|_{V_k} = f|_{F^k_\mu} \circ \pi$ and then

$$\big(M \times I, \partial M \times I \cup \bigcup_k (V_k \times 1), \mu \times 1, f \circ \pi_1\big)$$

is a bordism of (M, μ, f) and the free action $(M - \bigcup V^\circ_k, \mu, f)$.

If $\tau: X \to X$ is free, then any equivariant map into X is free, so k_* is an isomorphism in this case also.

(2) If $\tau = 1$, $F_\tau = X$, and for $[M, \mu, f] \in \widehat{\mathfrak{N}}_n(X, A, 1)$, $f: M \to X$ factors through M/μ giving a map $\bar{f}: M/\mu \to X$. Letting ξ be the line bundle defined by the principal Z_2 bundle $M \to M/\mu$ one has $\bar{f} \times \xi : M/\mu \to F_\tau \times BO(1)$ with $(S(\xi), -1, \bar{f} \circ \pi) = (M, \mu, f)$. Thus S is epic and the sequence becomes

$$0 \longrightarrow \mathfrak{N}_n(X, A, 1) \xrightarrow{F} \bigoplus_{k=0}^{n} \mathfrak{N}_k(X \times BO(n-k), A \times BO(n-k)) \xrightarrow{S} \hat{\mathfrak{N}}_{n-1}(X, A, 1) \longrightarrow 0 .$$

Assigning to $[M, \mu, f]$ the map $\bar{f} \times \xi : M/\mu \to X \times BO(1)$ defines a splitting homomorphism

$$\theta: \hat{\mathfrak{N}}_{n-1}(X, A, 1) \xrightarrow{\cong} \mathfrak{N}_{n-1}(X \times BO(1), A \times BO(1)) .$$

It should be noted that this splitting is the natural splitting for this sequence, and gives rise to an exact sequence ($A = \emptyset$, $X = \mathrm{pt}$)

$$0 \longrightarrow \mathfrak{N}_{n-1}(Z_2) \xrightarrow{\theta} \bigoplus_{k=0}^{n} \mathfrak{N}_k(BO(n-k)) \xrightarrow{\rho} I_n(Z_2) \longrightarrow 0$$

where ρ is the homomorphism constructed by Conner and Floyd (§ 28). In this respect, θ is much nicer than the splitting $K: \mathfrak{N}_{n-1}(Z_2) \to \bigoplus_{k=0}^{n} \mathfrak{N}_k(BO(n-k))$ which they defined, giving an isomorphism

$$\rho: \bigoplus_{\substack{k=0 \\ k \neq n-1}}^{n} \mathfrak{N}_k(BO(n-k)) \xrightarrow{\cong} I_n(Z_2) .$$

4. Calculation of free bordism

The main result of this section is

PROPOSITION 3. *$\hat{\mathfrak{N}}_*(X, A, \tau) \cong \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a)$ where a is the antipodal involution on the infinite sphere.*

PROOF. Basically this is the restatement for bordism of Theorem (19.1) of Conner and Floyd.

Let $\alpha \in \hat{\mathfrak{N}}_n(X, A, \tau)$ be represented by (M, μ, f). Then the principal Z_2 bundle $M \to M/\mu$ is induced by a map $\bar{\varphi}: M/\mu \to RP(\infty)$ with equivariant covering map $\varphi: M \to S^\infty$, S^∞ being given the antipodal involution. One then has an equivariant map $f \times \varphi: (M, \partial M) \to (X \times S^\infty, A \times S^\infty)$ and

$$\overline{f \times \varphi}: (M/\mu, \partial(M/\mu)) \longrightarrow (X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) .$$

The assignment

$$(M, \mu, f) \longrightarrow [M/\mu, \overline{f \times \varphi}] \in \mathfrak{N}_n(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a)$$

defines a homomorphism

$$\sigma: \hat{\mathfrak{N}}_n(X, A, \tau) \to \mathfrak{N}_n(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) .$$

Being given $\bar{g}: (N, \partial N) \to (X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a)$ there is an induced double cover

$$\begin{array}{ccc} \tilde{N} = \bar{g}^*(\pi) & \xrightarrow{g'} & X \times S^\infty \\ \downarrow & & \downarrow \\ N & \xrightarrow{g} & X \times S^\infty/\tau \times a, \end{array}$$

and letting $g = \pi_1 \circ g' \colon \tilde{N} \to X$ and $\tilde{\nu} \colon \tilde{N} \to \tilde{N}$ being the involution which interchanges sheets of the double cover, $(\tilde{N}, \tilde{\nu}, g)$ is a free bordism element of (X, A, τ). The assignment $(N, \bar{g}) \to (\tilde{N}, \tilde{\nu}, g) \in \hat{\mathfrak{N}}_n(X, A, \tau)$ induces a homomorphism inverse to σ.

Notes. (1) If X is a point, $A = \varnothing$, this gives $\mathfrak{N}_*(Z_2) \cong \mathfrak{N}_*(RP(\infty))$ as in Conner and Floyd. For $\tau = 1$, this is

$$\hat{\mathfrak{N}}_*(X, A, 1) \cong \mathfrak{N}_*\big(X \times RP(\infty),\ A \times RP(\infty)\big)$$

and σ coincides with the isomorphism

$$\theta \colon \hat{\mathfrak{N}}_*(X, A, 1) \cong \mathfrak{N}_*\big(X \times BO(1),\ A \times BO(1)\big)\,,$$

as mentioned in the notes of the previous section.

(2) If $\tau \colon X \to X$ is fixed point free, one may apply the above arguments to prove that $\hat{\mathfrak{N}}_n(X, A, \tau) \cong \mathfrak{N}_n(X/\tau, A/\tau)$, sending (M, μ, f) to $(M/\mu, \bar{f})$ where $\bar{f} \colon M/\mu \to X/\tau$ is induced by f. The map $\pi_1 \colon X \times S^\infty \to X$ induces $\bar{\pi}_1 \colon X \times S^\infty/\tau \times a \to X/\tau$ which is the fiber bundle with contractible fiber S^∞ associated to the Z_2 bundle $X \to X/\tau$. If X is a reasonably decent space, there is an equivariant map $h \colon X \to S^\infty$ classifying this Z_2 bundle, and $\overline{1 \times h} \colon X/\tau \to X \times S^\infty/\tau \times a$ is a homotopy inverse of the homotopy equivalence $\bar{\pi}_1$.

COROLLARY. *The natural homomorphism* $\mathfrak{N}_*(pt, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A) \to \mathfrak{N}_*(X, A, 1)$ *sending* $[M, \mu] \otimes [N, f] \to [M \times N, \mu \times 1, f \circ \pi_2]$ *is an isomorphism.*

PROOF. Denote this homomorphism by r and let

$$s \colon \hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A) \longrightarrow \hat{\mathfrak{N}}_*(X, A, 1)$$

be similarly defined, with

$$t \colon \textstyle\bigoplus_{k=0}^{*} \mathfrak{N}_k\big(BO(* - k)\big) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A) \longrightarrow \textstyle\bigoplus_{k=0}^{*} \mathfrak{N}_k\big(X \times BO(* - k),\ A \times BO(* - k)\big)$$

by sending $[M, g] \otimes [N, f]$ $\big(g \colon M \to BO(p),\ f \colon (N, \partial N) \to (X, A)\big)$ to the class of $f \times g \colon N \times M \to X \times BO(p)$. One then has the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{N}_*(\mathrm{pt}, 1) \otimes \mathfrak{N}_*(X, A) \arrow[r, "F \otimes 1"] \arrow[d, "r"] & \bigoplus_{k=0}^{*} \mathfrak{N}_k\big(BO(* - k)\big) \otimes \mathfrak{N}_*(X, A) \arrow[d, "t"] \\
0 \arrow[r] & \mathfrak{N}_*(X, A, 1) \arrow[r, "F"] & \bigoplus_{k=0}^{*} \mathfrak{N}_k\big(X \times BO(* - k),\ A \times BO(* - k)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "S \otimes 1"] & \hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \otimes \mathfrak{N}_*(X, A) \arrow[r] \arrow[d, "s"] & 0 \\
{} \arrow[r, "S"] & \hat{\mathfrak{N}}_*(X, A, 1) \arrow[r] & 0\,.
\end{tikzcd}$$

Since s and t are isomorphisms by the Künneth theorem [4, (29.1)] where

$$\begin{aligned}\hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A) &\cong \mathfrak{N}_*(RP(\infty)) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A) \\ &\cong \mathfrak{N}_*(X \times RP(\infty), A \times RP(\infty)) \cong \hat{\mathfrak{N}}_*(X, A, 1)\end{aligned}$$

with the proposition, one applies the five lemma to see that r is an isomorphism.

5. Calculation of unrestricted bordism

The main result of this section is

PROPOSITION 4. *The (unrestricted) equivariant bordism exact sequence of the involution* (X, F_τ, τ) *is split exact.*

PROOF. One has the homomorphisms

$$\mathfrak{N}_n(X, \tau) \xrightarrow{\mathfrak{N}_n(j)} \mathfrak{N}_n(X, F_\tau, \tau) \underset{\cong}{\xleftarrow{k_*}} \hat{\mathfrak{N}}_n(X, F_\tau, \tau) ,$$

and it suffices to define a homomorphism $q\colon \hat{\mathfrak{N}}_n(X, F_\tau, \tau) \to \mathfrak{N}_n(X, \tau)$ with $\mathfrak{N}_n(j) \circ q(\alpha) = k_*(\alpha)$ for all α. Being given $\alpha \in \hat{\mathfrak{N}}_n(X, F_\tau, \tau)$ represented by (M, μ, f), one has a closed manifold $\bar{M}$ obtained from M by identifying each $m \in \partial M$ with $\mu m \in \partial M$. (This is the manifold obtained from M by attaching the disc bundle $D(\xi)$ of the line bundle ξ associated to the double cover $\partial M \to \partial M/\mu$ along their common boundary.) Since f is equivariant and $f(\partial M) \subset F_\tau$, $f(m) = f(\mu m)$ for $m \in \partial M$, and f factors through $\bar{f}\colon \bar{M} \to X$, this being equivariant if $\bar{M}$ is given the involution $\bar{\mu}$ induced by μ. Letting $q(\alpha)$ be the class of $(\bar{M}, \bar{\mu}, \bar{f})$ defines the homomorphism $q\colon \hat{\mathfrak{N}}_n(X, F_\tau, \tau) \to \mathfrak{N}_n(X, \tau)$.

Now $k_*(\alpha)$ and $\mathfrak{N}_n(j) \circ q(\alpha)$ are represented by (M, μ, f) and $(\bar{M}, \bar{\mu}, \bar{f})$ respectively, in $\mathfrak{N}_n(X, F_\tau, \tau)$. Let $H\colon \bar{M} \times I \to X$ be a homotopy of the map $\bar{f} = H(\ , 0)$ to a map $g = H(\ , 1)$ with $g\,|_V = \bar{f}\,|_{F_{\bar{\mu}}} \circ \pi$ where $V \cong D(\nu)$ is a tubular neighborhood of $F_{\bar{\mu}}$, constructed by the standard radial deformation. Then $F_{\bar{\mu}} = \partial M/\mu$ with $\nu \cong \xi$, and one may find a map $h\colon M \to \bar{M} \times 1$ identifying M with $\bar{M} - V^\circ$ and such that $gh = f$. Then $(\bar{M} \times I, V \times 1, \bar{\mu} \times 1, H)$ is a bordism of $(\bar{M}, \bar{\mu}, \bar{f})$ and (M, μ, f), so $k_*(\alpha) = \mathfrak{N}_n(j) \circ q(\alpha)$.

COROLLARY. $\mathfrak{N}_*(X, \tau) \cong \mathfrak{N}_*(F_\tau, 1) \oplus \hat{\mathfrak{N}}_*(X, F_\tau, \tau)$ *or*

$$\mathfrak{N}_n(X, \tau) \cong \Big\{\textstyle\bigoplus_{\substack{k=0 \\ k \neq n-1}}^{n} \mathfrak{N}_k(F_\tau \times BO(n-k))\Big\} \oplus \mathfrak{N}_n(X \times S^\infty/\tau \times a, F_\tau \times RP(\infty)) .$$

Notes. (1) Applying the Thom isomorphism in the first summand, one may rewrite the above in a highly palatable form as

$$\begin{aligned}\mathfrak{N}_n(X, \tau) &\cong \textstyle\bigoplus_{\substack{s=0 \\ s\neq 1}}^{n} \tilde{\mathfrak{N}}_n((F_\tau/\varnothing) \wedge TBO(s)) \oplus \tilde{\mathfrak{N}}_n((X \times S^\infty/\tau \times a)/F_\tau \times RP(\infty)) , \\ &\cong \tilde{\mathfrak{N}}_n(\{(X \times S^\infty/\tau \times a)/F_\tau \times RP(\infty)\} \vee \textstyle\bigvee_{\substack{s=0 \\ s\neq 1}}^{\infty} \{(F_\tau/\varnothing) \wedge TBO(s)\})\end{aligned}$$

giving the reduced bordism of a modification of $\bigvee_{s=0}^{\infty}\{(F_\tau/\varnothing)\vee TBO(s)\}$ obtained by changing the $s=1$ term.

Being given $[M,\mu,f]\in\mathfrak{N}_n(X,\tau)$ with $f|_{V_k}=f_{F_\mu^k}\circ\pi$ on $V_k=D(\nu_k)$ one may map

$$M\longrightarrow M/\bigcup_k\partial V_k\cong\frac{M-V^\circ}{\partial(M-V^\circ)}\vee\bigvee_{k=0}^{r}V_k/\partial V_k$$

$$\xrightarrow{\ b\ }\frac{((M-V^\circ)/\partial V)}{\mu}\vee\bigvee_{\substack{k=0\\k\neq n-1}}^{n}V_k/\partial V_k=\widetilde{M}$$

where $V=\bigcup_k V_k$, with b obtained by dividing out the action of μ on the first wedge term and collapsing $V_{n-1}/\partial V_{n-1}$ to the base point. $\widetilde{M}$ is a wedge of manifolds modulo their boundaries, with $V_k/\partial V_k$ being also the Thom space of the normal bundle of F_μ^k. There is naturally induced a map of $\widetilde{M}$ into

$$\{(X\times S^\infty/\tau\times a)/F_\tau\times RP(\infty)\}\vee\bigvee_{s\neq 1}\{(F_\tau/\phi)\wedge TBO(s)\}$$

which gives the asserted isomorphism in a natural and geometric fashion.

The $TBO(s)$ formulation is essentially the Wall approach to the cobordism of a manifold M with imbedded submanifolds F_μ^k (see [7]).

(2) A highly sophisticated approach to

$$k_*^{-1}\mathfrak{N}_n(j):\mathfrak{N}_n(X,\tau)\longrightarrow\widehat{\mathfrak{N}}_n(X,F_\tau,\tau)$$

may be described as follows. Beginning with (M,μ,f) one may form a manifold with boundary $\widehat{M}$ by giving M a riemannian metric invariant under μ and letting $\widehat{M}$ be the set of points $m\in M-F_\mu$ and the set of pairs (m,v) with $m\in F_\mu$ and v a unit vector in the tangent space to M at m which is orthogonal to F_μ. $\widehat{M}$ has the structure of a manifold with boundary making it diffeomorphic to $M-\bigcup V_k^\circ$. $\widehat{M}$ has an involution $\hat{\mu}$ (μ on $M-F_\mu$, $(m,v)\longrightarrow(m,-v)$ on the boundary) and an equivariant map $p\colon\widehat{M}\longrightarrow M$ (identity on $M-F_\mu$, $(m,v)\longrightarrow m$ on the boundary). Then $(\widehat{M},\hat{\mu},f\circ p)$ represents $k_*^{-1}\mathfrak{N}_n(j)[M,\mu,f]$ and is obtained by "resolving the fixed-point set of the involution μ". $\overline{M}$ is obtained by collapsing $\partial\widehat{M}$ under $\hat{\mu}$ and may be thought of as "blowing up" F_μ to the projective space bundle of its normal bundle. Thus $qk_*^{-1}\mathfrak{N}_n(j)$ is obtained by applying the monoidal transformation along the fixed point set (see Hirzebruch [6, pp. 175–176]).

(3) It is vaguely interesting to note that $q(\widehat{\mathfrak{N}}_*(X,F_\tau,\tau))$ is the direct summand of $\mathfrak{N}_*(X,\tau)$ formed by using elements (M,μ,f) for which the fixed-point set has codimension one (allowing the empty manifold of codimension 1, of course).

(4) The splitting given by Proposition 4 is not a splitting as $\mathfrak{N}_*(\mathrm{pt},1)$ modules, but only as $\mathfrak{N}_*$ modules. One may describe the $\mathfrak{N}_*(\mathrm{pt},1)$ module structure in the following way. Writing $a\in\mathfrak{N}_*(X,\tau)$ as (a',a'') in

$\mathfrak{N}_*(F_\tau, 1) \oplus \hat{\mathfrak{N}}_*(X, F_\tau, \tau)$, with $\alpha \in \mathfrak{N}_*(\mathrm{pt}, 1)$, $\alpha \cdot a = (\alpha a' + \varphi(\alpha \otimes a''), \alpha a'')$ where

$$\varphi \colon \mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(X, F_\tau, \tau) \longrightarrow \mathfrak{N}_*(F_\tau, 1)$$

describes the difference between the two ways of closing up, and sends $[N, \nu] \otimes [M, \mu, f]$ to the class of (V, ρ, g) with V obtained from $N \times \partial M \times [-1, 1]$ by identifying the points $(n, m, -1)$ with $(\nu n, \mu m, -1)$ and $(n, m, 1)$ with $(n, \mu m, 1)$, ρ being given by $\nu \times \mu \times 1$, and g by $f \circ \pi_2$. [*Note.* V is obtained by joining the disc bundles of $N \times \partial M \to N \times \partial M/\nu \times \mu$ and $N \times \partial M \to N \times (\partial M/\mu)$ along their common boundary.] φ may also be described by

$$\begin{array}{ccc}
\mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(X, F_\tau, \tau) & \xrightarrow{1 \otimes \partial} & \mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(F_\tau, 1) \\
 & & \cong \big\downarrow \\
 & & \mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F_\tau) \\
 & & \mathcal{P} \otimes 1 \big\downarrow \\
\mathfrak{N}_*(F_\tau, 1) & \xleftarrow{\cong} & \mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(F_\tau) ,
\end{array}$$

where $\mathcal{P} \colon \mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \to \mathfrak{N}_*(\mathrm{pt}, 1)$ is the homomorphism of degree $+1$ sending $[N, \nu] \otimes [S, \sigma]$ into $[W, \omega]$ where $W = N \times S \times [-1, 1]$ with $(n, s, -1) \sim (\nu n, \sigma s, -1)$, $(n, s, 1) \sim (n, \sigma s, 1)$ and ω is induced by $\nu \times \sigma \times 1$ as above. This is obtained by the fact that $[N \times S, \nu \times \sigma]$ bounds in two ways. Using the Conner and Floyd exact sequence, one may write this

$$\begin{array}{ccc}
\mathfrak{N}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \hat{\mathfrak{N}}_*(\mathrm{pt}, 1) & \xrightarrow{F \otimes \theta} & \bigoplus_{k=0}^{*} \mathfrak{N}_k(BO(* - k)) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(BO(1)) \\
\big\downarrow \mathcal{P} & & \big\downarrow \mathcal{P}' \\
\mathfrak{N}_*(\mathrm{pt}, 1) & \xleftarrow{\rho} & \bigoplus_{k=0}^{*} \mathfrak{N}_k(BO(* - k)) ,
\end{array}$$

where $\mathcal{P}'$ is obtained from the product in $\bigoplus_{s=0}^{\infty} \mathfrak{N}_*(BO(s))$ given by the Whitney sum maps $BO(s) \times BO(s') \to BO(s + s')$, θ and ρ being the splitting maps. [To see that the diagram commutes one need only check the fixed-point set of codimension more than 1 in W; $N \times S/\nu \times \sigma$ has codimension 1 in W, so the fixed set of interest is just $F_\nu \times S/\sigma \subset N \times S/1 \times \sigma$ with normal bundle given by the Whitney sum of the normal bundle of F_ν and the line bundle over S/σ.]

6. The Smith homomorphism

Let $\alpha \in \hat{\mathfrak{N}}_n(X, A, \tau)$ be represented by (M, μ, f), and let $\bar{N} \subset M/\mu$ be a submanifold (with boundary) defining the double cover $\pi \colon M \to M/\mu$ with $\bar{\pi} \colon N \to \bar{N}$ the induced double cover and $i \colon N \to M$ the inclusion. The involution μ on M induces an involution $\mu|_N$ on N and i is equivariant. Let $\Delta(\alpha) \in \hat{\mathfrak{N}}_{n-1}(X, A, \tau)$ be the class of $(N, \mu|_N, f \circ i)$.

The homomorphism $\Delta\colon \hat{\mathfrak{N}}_*(X, A, \tau) \to \hat{\mathfrak{N}}_*(X, A, \tau)$, of degree -1, is called the Smith homomorphism, extending the Smith homomorphism $\Delta\colon \mathfrak{N}_*(Z_2) \to \mathfrak{N}_*(Z_2)$ of Conner and Floyd § 26 to bordism.

One may also define homomorphisms $\mathfrak{L}_*\colon \hat{\mathfrak{N}}_*(X, A, \tau) \to \mathfrak{N}_*(X, A)$ sending $[M, \mu, f] \to [M, f]$ by forgetting equivariance, and $1 + \tau_*\colon \mathfrak{N}_*(X, A) \to \hat{\mathfrak{N}}_*(X, A, \tau)$ by sending $[M, f]$ to $[M \times Z_2, 1 \times (-1), g]$ where $g(m, 1) = f(m)$, $g(m, -1) = \tau f(m)$.

One then has

PROPOSITION 5. *The sequence*

$$\begin{tikzcd} \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[r, "\mathfrak{L}_*"] & \mathfrak{N}_*(X, A) \arrow[r, "1 + \tau_*"] & \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[ll, bend left, "\Delta"] \end{tikzcd}$$

is exact.

PROOF. (1) $(1 + \tau_*)\mathfrak{L}_* = 0$. If $[M, \mu, f] \in \hat{\mathfrak{N}}_n(X, A, \tau)$, let

$$\sigma\colon M \times I \longrightarrow M \times I\colon (m, s) \longrightarrow (\mu m, -s)\ ,$$

$$h\colon M \times I \longrightarrow X\colon (m, s) \longrightarrow f(m)\ ,$$

and $\rho\colon M \times Z_2 \to M \times I\colon (m, 1) \to (m, 1), (m, -1) \to (\mu m, -1)$. Then $(M \times I, \partial M \times I, \sigma, h)$ has boundary isomorphic via ρ with $(M \times Z_2, 1 \times (-1), g)$ which represents $(1 + \tau_*)\mathfrak{L}_*[M, \mu, f]$.

(2) $\Delta(1 + \tau_*) = 0$. If $[M, f] \in \mathfrak{N}_n(X, A)$, with $g\colon M \times Z_2 \to X$, then $\pi\colon M \times Z_2 \to M \times Z_2/1 \times (-1) \cong M$ is the trivial double cover and is defined by the empty submanifold N. Thus $\Delta(1 + \tau_*)[M, f]$ is represented by the empty manifold, so is zero.

(3) $\mathfrak{L}_*\Delta = 0$. If $[M, \mu, f] \in \hat{\mathfrak{N}}_n(X, A, \tau)$ and $\bar{N} \subset M/\mu$ defines the double cover $M \xrightarrow{\pi} M/\mu$, then over $M/\mu - \bar{N}$ the double cover is trivial, so M is formed as the union of two manifolds M_0, M_1, joined (by $\mu\,|\,_N$) along copies of N imbedded in the boundary of each, (M_i being diffeomorphic to M/μ − (interior of tubular neighborhood of $\bar{N}$)), with $\mu(M_0) = M_1$. Then $(M_0, \partial M_0$-interior $(N), f\,|\,_{M_0})$ has boundary $(N, f \circ i)$ so $\mathfrak{L}_*\Delta[M, \mu, f] = [N, f \circ i] = 0$.

(4) $\ker \mathfrak{L}_* \subset \operatorname{im} \Delta$. If $[M, \mu, f] \in \hat{\mathfrak{N}}_*(X, A, \tau)$ and $[M, f] = 0$ in $\mathfrak{N}_*(X, A)$, there is a manifold W with boundary $M \cup M'/\partial M \equiv \partial M'$ and a map $g\colon (W, M') \to (X, A)$ extending f. Let $T = W \times Z_2/(m, 1) \equiv (\mu m, -1)$ for $m \in M$, $\nu\colon T \to T$ the involution induced by $1 \times (-1)$ on $W \times Z_2$ and $h\colon T \to X$ by $h(w, 1) = g(w)$, $h(w, -1) = \tau g(w)$. Then (T, ν, h) is a free bordism element of (X, A, τ) and the double cover $T \to T/\nu$ is defined by M/μ with covering M. Thus $\Delta[T, \nu, h] = [M, \mu, f]$.

(5) $\ker \Delta \subset \operatorname{im}(1 + \tau_*)$. If $[M, \mu, f] \in \hat{\mathfrak{N}}_*(X, A, \tau)$ and $\Delta[M, \mu, f] = 0$,

$(N, \mu|_N, f\circ i)$ is the boundary of some element (V, W, σ, g) with

$$\partial V = W \cup N/\partial W \equiv \partial N,\ \sigma|_N = \mu|_N,\ g|_N = f\circ i\ ,$$

$g\colon (V, W)\to(X, A)$ being equivariant. The normal bundle of N in M is trivial and one may identify a tubular neighborhood Q of N in M with $N\times[-1, 1]$ with involution $\mu|_N\times(-1)$, and may by a homotopy of f suppose $f|_Q = f|_N\circ\pi_1$. Let T be formed from $M\times[0, 1]\cup V\times[-1, 1]$ by identifying $N\times[-1, 1]\times 1$ with $N\times[-1, 1]$, with involution $\rho=\mu\times 1\cup\sigma\times(-1)$ and map $h\colon T\to X$ given by $f\circ\pi_1\cup g\circ\pi_1$. Then $(T, \partial M\times[0, 1], \rho, h)$ has boundary the disjoint union of (M, μ, f) and (M', μ', f') where M' is formed from $(M-Q^\circ)\cup V\times\{-1, 1\}$ by identifying the two copies of $N\times\{-1, 1\}$. The double cover $T\to T/\rho$ is defined by the submanifold $N\times[0, 1]\cup V$, and in particular $M'\to M'/\mu'$ is then a trivial double cover, or (M', μ', f) represents a class in $\mathrm{im}\,(1+\tau_*)$.

(6) $\ker(1+\tau_*)\subset\mathrm{im}\,\mathfrak{L}_*$. If $[M, f]\in\mathfrak{N}_*(X, A)$ and $(1+\tau_*)[M, f]=0$, then $(M\times Z_2, 1\times(-1), g)$ bounds, or there is a manifold $V, \partial V = M\times Z_2\cup W/\partial M\times Z_2\equiv\partial W$, a free involution $\sigma\colon V\to V$ extending $1\times(-1)$, and an equivariant map $h\colon(V, W)\to(X, A)$. Let $N\subset V$ be a submanifold obtained by lifting $\bar N\subset V/\sigma$ which defines the double cover. $\pi\colon M\times Z_2\to M\times Z_2/1\times(-1)$ being the trivial cover, $\partial N\subset$ interior W and $N\subset V-M\times Z_2$. Now $(N, \sigma|_N, h|_N)$ is a free equivariant bordism element of (X, A, τ), with $V=V_0\cup V_1$ being the union of two manifolds with boundary (each diffeomorphic to $V/\sigma-$(interior of tubular neighborhood of $\bar N$)) joined along two copies of N, and with $V_0\supset M\times 1$, $V_1\supset M\times(-1)$. Then $(V_0, \partial V_0-(M\times 1)^\circ - N^\circ, h|_{V_0})$ is a bordism of (M, f) and $(N, h|_N)$, so

$$[M, f] = [N, h|_N] = \mathfrak{L}_*[N, \sigma|_N, h|_N]\ .$$

One may give a different proof of exactness by using Proposition 3.

(1) Letting $p\in S^\infty$, one has

$$(X, A)\underset{\pi_0}{\overset{1\times p}{\rightleftarrows}}(X\times S^\infty, A\times S^\infty)\xrightarrow{\pi_1}(X\times S^\infty/\tau\times a, A\times S^\infty/\tau\times a)$$

with $1\times p$ and π_0 being inverse homotopy equivalences. Since π_1 pulls the double cover π_1 back to a trivial double cover, the composite

$$\mathfrak{N}_*(X, A)\cong\mathfrak{N}_*(X\times S^\infty, A\times S^\infty)\xrightarrow{\pi_{1*}}$$
$$\mathfrak{N}_*(X\times S^\infty/\tau\times a, A\times S^\infty/\tau\times a)\cong\hat{\mathfrak{N}}_*(X, A, \tau)$$

coincides with $1+\tau_*$.

(2) Let ξ denotes the line bundle of the double cover π_1 and ξ' the restriction of ξ to $A\times S^\infty/\tau\times a$. One has

$$\begin{tikzcd}[column sep=small]
\hat{\mathfrak{N}}_*(X, A, \tau) \cong \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) \arrow[r, shift left, "\pi_{2*}"] & \mathfrak{N}_*(D(\xi), D(\xi')) \arrow[l, shift left, "0_*"] \arrow[r, "i_*"] & \mathfrak{N}_*(D(\xi), S(\xi) \cup D(\xi')) \arrow[d, "\cong"] \\
\hat{\mathfrak{N}}_*(X, A, \tau) \cong \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) \cong \mathfrak{N}_*(B(\xi), B(\xi')) & & \mathfrak{N}_*(T(\xi), T(\xi')) \arrow[ll, "\Phi^{-1}"']
\end{tikzcd}$$

where T is Thom space, B is base space, π_2 is the projection, 0 the zero section, i the inclusion, and Φ the Thom isomorphism. The composite is just Δ, for being given $[M, \mu, f] \in \hat{\mathfrak{N}}_*(X, A, \tau)$ or $\bar{f}\colon M/\mu \to X \times S^\infty/\tau \times a$, $\Phi^{-1} i_* 0_*$ is the Euler class construction, obtained by making $\bar{f}$ transverse regular on $X \times S^\infty/\tau \times a \subset D(\xi)$ (or $T(\xi)$), and this just constructs $\bar{N}$.

(3) One has the composite

$$\begin{tikzcd}
\hat{\mathfrak{N}}_*(X, A, \tau) \cong \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) \arrow[r, "\Phi"] & \mathfrak{N}_*(T(\xi), T(\xi')) \\
& \mathfrak{N}_*(D(\xi), S(\xi) \cup D(\xi')) \arrow[u, phantom, sloped, "\cong"] \arrow[d, "\partial"] \\
\mathfrak{N}_*(S(\xi), S(\xi')) & \mathfrak{N}_*(S(\xi) \cup D(\xi'), D(\xi')) \arrow[l, "\cong", "\text{(excision)}"'] \\
\mathfrak{N}_*(X, A) \cong \mathfrak{N}_*(X \times S^\infty, A \times S^\infty) \arrow[u, phantom, sloped, "\cong"] &
\end{tikzcd}$$

and for $[M, \mu, f] \in \hat{\mathfrak{N}}_*(X, A, \tau)$ let ρ, ρ' be the line bundles over M/μ and $\partial M/\mu$. Then $\Phi[M, \mu, f]$ is represented by the induced map $\big(D(\rho), \partial D(\rho)\big) \to \big(T(\xi), T(\xi')\big)$ with boundary the map

$$(M, \partial M) = \big(S(\rho), S(\rho')\big) \longrightarrow \big(S(\xi), S(\xi')\big),$$

which is just the bordism class of $[M, f]$. Thus this composite is $\mathfrak{L}_*$.

Using these one has

$$\begin{tikzcd}[column sep=small]
& \hat{\mathfrak{N}}_*(X, A, \tau) & \\
& \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) \arrow[u, phantom, sloped, "\cong"] & \\
& \mathfrak{N}_*(T(\xi), T(\xi')) \arrow[u, phantom, sloped, "\cong"] & \\
& \mathfrak{N}_*(D(\xi), S(\xi) \cup D(\xi')) \arrow[u, phantom, sloped, "\cong"] \arrow[dl, "\partial"'] & \\
\mathfrak{N}_*(S(\xi) \cup D(\xi), D(\xi')) \arrow[rr] & & \mathfrak{N}_*(D(\xi), D(\xi')) \arrow[ul, "i_*"'] \\
\mathfrak{N}_*(S(\xi), S(\xi')) \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\pi_{1*}"] & & \mathfrak{N}_*(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a) \arrow[u, phantom, sloped, "\cong"] \\
\mathfrak{N}_*(X \times S^\infty, A \times S^\infty) \arrow[u, phantom, sloped, "\cong"] & & \hat{\mathfrak{N}}_*(X, A, \tau) \arrow[u, phantom, sloped, "\cong"] \\
\mathfrak{N}_*(X, A) \arrow[u, phantom, sloped, "\cong"] & &
\end{tikzcd}$$

for the exact sequence of the triple. This gives the Smith sequence by chasing around the diagram.

In particular, the Smith sequence is just the relative Gysin sequence of the double cover by $(X \times S^\infty, A \times S^\infty)$ over $(X \times S^\infty/\tau \times a, A \times S^\infty/\tau \times a)$, or the exact sequence of the cofibration giving the relative Thom space

$$\big(S(\xi), S(\xi')\big) \longrightarrow \big(D(\xi), D(\xi')\big) \longrightarrow \big(T(\xi), T(\xi')\big) .$$

7. Equivariant spectra and free bordism

The object of this section is to sketch an approach to equivariant spectra suitable for dealing with free equivariant bordism. The approach taken differs from that of Bredon [3, Ch. III § 3], in that suspensions are taken with a non-trivial action.

Definition. Let X be a space with involution τ.

(1) The (unreduced) suspension of (X, τ) is the space ΣX formed from $X \times [-1, 1]$ by identifying $X \times \{-1\}$ to a point $-\infty$, and $X \times \{1\}$ to a point $+\infty$, and with involution $\Sigma\tau$ induced by $\tau \times (-1)$.

(2) If $* \in X$ is a base point fixed under τ, the (reduced) suspension of $(X, *, \tau)$ is the space ΣX formed from $X \times [-1, 1]$ by identifying $\{* \times [-1, 1]\} \cup \{X \times Z_2\}$ to a base point $*$, and with involution $\Sigma\tau$ induced by $\tau \times (-1)$.

Definition. An equivariant (reduced) spectrum $\mathbf{E} = \{(E_n, t_n), f_n\}$, $n \in Z$, consists of a collection spaces E_n (with base points $*$) with involutions $t_n\colon E_n \to E_n$ (fixing base points) and equivariant (base point preserving) maps $f_n\colon \Sigma E_n \to E_{n+1}$.

Examples. (1) $\mathbf{S} = \{(S^n, a), i_n\}$ with S^n the n-sphere, a the antipodal involution, and $i_n\colon \Sigma S^n \to S^{n+1}$ induced by $(x, t) \to (\sqrt{1-t^2}\cdot x, t)$ is the equivariant sphere spectrum with antipodal involution.

(2) The equivariant (reduced) Thom spectrum

$$\mathbf{TBO} = \{(TBO_n \times S^\infty/\infty \times S^\infty, 1 \times a), h_n\} .$$

is formed by constructing a map h_n as follows. Let

$$TBO_n \times S^\infty \times [-1, 1] \xrightarrow{a_n} TBO_n \times \big(S^\infty \times [-1, 1]/a \times (-1)\big) = TBO_n \times D(\xi)$$

with ξ the non-trivial line bundle over $RP(\infty)$, a_n being the collapse;

$$TBO_n \times D(\xi) \xrightarrow{b_n} TBO_n \times \big(D(\xi)/S(\xi)\big) = TBO_n \times T(\xi) = TBO_n \times TBO_1 ;$$

$$TBO_n \times TBO_1 \xrightarrow{c_n} TBO_n \wedge TBO_1 \xrightarrow{d_n} TBO_{n+1} ,$$

with c_n the collapse of $\infty \times TBO_1 \cup TBO_n \times \infty$, and d_n induced by the Whitney sum;

$$TBO_n \times S^\infty \times [-1, 1] \xrightarrow{e_n} S^\infty \times [-1, 1] \xrightarrow{i} S^\infty$$

with e_n the projection and $i(x, t) = (t, \sqrt{1 - t^2}\cdot x)$; and

$$\pi_n\colon TBO_{n+1} \times S^\infty \longrightarrow TBO_{n+1} \times S^\infty/\infty \times S^\infty = E_{n+1} .$$

One then lets

$$\tilde{h}_n = \pi_n \circ ((d_n \circ c_n \circ b_n \circ a_n) \times (i \circ e_n))\colon TBO_n \times S^\infty \times [-1, 1] \longrightarrow E_{n+1} ,$$

and this is compatible with the identifications, to define a map $h_n\colon \Sigma E_n \to E_{n+1}$. (*Note.* $E_n = TBO_n \times S^\infty/\infty \times S^\infty$ is homotopy equivalent to TBO_n with $1 \times p\colon TBO_n \to E_n\colon x \to (x, p)$ and $\pi\colon E_n \to TBO_n$ being inverse homotopy equivalences. Then

$$\begin{array}{c} S^1 \wedge TBO_n \xrightarrow[\cong]{1 \times p \times 1} \Sigma E_n \xrightarrow{h_n} E_{n+1} \xrightarrow[\cong]{\pi} TBO_{n+1} \\ \| \\ \dfrac{TBO_n \times [-1, 1]}{\infty \times [-1, 1] \cup TBO_n \times Z_2} \end{array}$$

is just the standard "suspension-map" defining the ordinary Thom spectrum. Thus, one has given the ordinary Thom spectrum an involution which is free except at the fixed base point.)

(3) The equivariant (reduced) Eilenberg-MacLane spectrum $\mathbf{K}(Z_2) = \{(K(Z_2, n) \times S^\infty/* \times S^\infty, 1 \times a), k_n\}$ is defined much as **TBO**, by

$$K(Z_2, n) \times S^\infty \times [-1, 1] \to K(Z_2, n) \wedge T(\xi) \xrightarrow{=} K(Z_2, n) \wedge K(Z_2, 1) \longrightarrow K(Z_2, n + 1)$$

and

$$K(Z_2, n) \times S^\infty \times [-1, 1] \longrightarrow S^\infty \times [-1, 1] \xrightarrow{i} S^\infty .$$

This gives an involution, free except at a fixed base point, on the standard Eilenberg-MacLane spectrum. (*Note.* Any involution of a $K(Z_2, n)$ is homotopic to the identity since every automorphism of the n^{th} homotopy group is trivial.)

(4) If (X, A, τ) is any involution pair, with $\mathbf{E}$ an equivariant reduced spectrum, then $X/A \wedge \mathbf{E} = \{((X/A) \wedge E_n, \tau \wedge t_n), 1 \wedge f_n\}$ is also an equivariant reduced spectrum.

In order to compute with spectra of type 2 or 3, one has

LEMMA. *Let X be a space with base point $*$, and suppose there is a homotopy $\Lambda\colon X \times I \to X$ (rel$*$) with $\Lambda(\ , 0) = 1$ and so that $\Lambda(\ , 1) = \lambda$ satisfies $\lambda^{-1}(*)$ is a neighborhood of $*$. Then the set of equivariant homotopy classes of maps of (Y, σ) into $(X \times S^\infty/* \times S^\infty, 1 \times a)$ is in one-to-one correspondence with the set of homotopy classes of maps of $(Y/\sigma, F_\sigma/\sigma)$ into $(X, *)$.*

PROOF. If $f\colon (Y, \sigma) \to (X \times S^\infty/* \times S^\infty, 1 \times a)$ is an equivariant map, then $\pi_1 \circ f\colon Y \to X$ sends F_σ to $*$ and maps (Y, σ) equivariantly into $(X, 1)$, so induces a map $\bar{f}\colon (Y/\sigma, F_\sigma/\sigma) \to (X, *)$, with $f \to \bar{f}$ defining a function

$$\rho\colon [(Y, \sigma), (X \times S^\infty/* \times S^\infty, 1 \times a)] \longrightarrow [(Y/\sigma, F_\sigma/\sigma), (X, *)] .$$

Let $g\colon Y - F_\sigma \to S^\infty$ be any equivariant map ($\bar{g}\colon Y - F_\sigma/\sigma \to RP(\infty)$ classifying the double cover), and to each $\varphi\colon (Y/\sigma, F_\sigma/\sigma) \to (X, *)$ assign the equivariant map $\tilde{\varphi}\colon (Y, \sigma) \to (X \times S^\infty/* \times S^\infty, 1 \times a)$ with

$$\tilde{\varphi}(y) = \begin{cases} (\lambda \circ \varphi \circ \pi(y), g(y)) , & y \notin F_\sigma , \\ * \times S^\infty , & y \in F_\sigma . \end{cases}$$

Since $(\lambda \circ \varphi \circ \pi)^{-1}(*)$ is a neighborhood of F_σ, this is continuous. Clearly $\rho(\tilde{\varphi})$ is represented by $\lambda \circ \varphi$ and so also φ. Thus ρ is epic, and let

$$\kappa\colon [(Y/\sigma, F_\sigma/\sigma), (X, *)] \longrightarrow [(Y, \sigma), (X \times S^\infty/* \times S^\infty, 1 \times a)]$$

be the function defined by $\kappa([\varphi]) = [\tilde{\varphi}]$.

Now let $\pi_2 \circ f\colon Y - f^{-1}(* \times S^\infty) \xrightarrow{f} (X - *) \times S^\infty \xrightarrow{\pi_2} S^\infty$ and let $H\colon (Y - f^{-1}(* \times S^\infty)) \times I \to S^\infty$ be an equivariant homotopy with $H(\ , 0) = g$, $H(\ , 1) = \pi_2 \circ f$ (possible since both maps "classify" the cover),

$$\begin{aligned} K\colon Y \times I \to X \times S^\infty/* \times S^\infty\colon (y, t) &\longrightarrow (\lambda \circ \bar{f} \circ \pi(y), H(y, t)) , & y \notin f^{-1}(* \times S^\infty) , \\ (y, t) &\longrightarrow * \times S^\infty , & y \in f^{-1}(* \times S^\infty) , \end{aligned}$$

and

$$L\colon Y \times I \xrightarrow{f \times 1} (X \times S^\infty/* \times S^\infty) \times I \xrightarrow{\overline{\Lambda \times 1}} X \times S^\infty/* \times S^\infty$$

where $\overline{\Lambda \times 1}$ is induced by

$$X \times S^\infty \times I \longrightarrow X \times S^\infty\colon (x, s, t) \longrightarrow (\Lambda(x, t), s) .$$

Being the composition of continuous maps L is continuous, and K is continuous since $(\lambda \circ \bar{f} \circ \pi)^{-1}(*)$ is a neighborhood of $f^{-1}(* \times S^\infty)$. Then

$$f = L(\ , 0) \sim L(\ , 1) = K(\ , 1) \sim K(\ , 0) = (\lambda \circ \bar{f})^\sim ,$$

so $\kappa\rho[f] = [(\lambda \circ \bar{f})^\sim] = [f]$ or ρ is 1–1. *Note.* This assumes that Y is a reasonably decent space.

As a special case, one has

COROLLARY. $[(Y, \sigma), K(Z_2, n) \times S^\infty/* \times S^\infty, 1 \times a)] \cong H^n(Y/\sigma, F_\sigma/\sigma; Z_2)$.

LEMMA. *If (X, τ) is any involution, $(\Sigma^r(X \cup *), \Sigma^r(\tau \cup 1))$ coincides with $(X \times D^r/X \times S^{r-1}, \tau \times (-1))$.*

PROOF.

$$\begin{aligned}\Sigma(X \times D^r/X \times S^{r-1})\\ &= \left(\frac{X \times D^r}{X \times S^{r-1}}\right) \times [-1, 1]\Big/\frac{X \times D^r}{X \times S^{r-1}} \times Z_2 \cup \frac{X \times S^{r-1}}{X \times S^{r-1}} \times [-1, 1]\,,\\ &= X \times D^r \times [-1, 1]/X \times S^{r-1} \times [-1, 1] \cup X \times D^r \times Z_2\,,\\ &= X \times D^{r+1}/X \times S^r\end{aligned}$$

with the involution being obvious.

PROPOSITION 6. *If X is a closed n-dimensional differentiable manifold with differentiable involution τ, then*

$$\lim_{r\to\infty}[(\Sigma^r(X \cup *), \Sigma^r(\tau \cup 1)), (TBO_{r+t} \times S^\infty/\infty \times S^\infty, 1 \times a)] \cong \hat{\mathfrak{N}}_{n-t}(X, \tau)\,,$$

where the limit is taken via the spectral maps h_s.

PROOF. An equivariant map $f\colon \Sigma^r(X \cup *) \to TBO_{r+t} \times S^\infty/\infty \times S^\infty$ gives a map $f'\colon X \times D^r/X \times S^{r-1} \to TBO_{r+t} \times S^\infty/\infty \times S^\infty$ and hence

$$\bar{f}'\colon (X \times D^r/\tau \times (-1), X \times S^{r-1}/\tau \times (-1) \cup F_\tau \times 0) \longrightarrow (TBO_{r+t}, \infty)\,.$$

Since the first pair is a relative manifold, one may make the map $\bar{f}'$ transverse regular on $BO(r+t)$ giving a closed submanifold of codimension $r+t$ in $X \times D^r - F_\tau \times 0 - X \times S^{r-1}/\tau \times (-1)$ and thus a closed submanifold $M^{n-t} \hookrightarrow X \times D^r$, invariant under $\tau \times (-1)$ and acted upon freely by $\tau \times (-1)$ since it does not meet $F_\tau \times 0$. The composite $M \hookrightarrow X \times D^r \xrightarrow{\pi} X$ then gives a free equivariant bordism element of (X, τ). Exactly the same construction converts an equivariant homotopy to a bordism.

To reverse the steps, it suffices to represent $[M, \mu, f]$ as a composite $M \hookrightarrow X \times D^r \xrightarrow{\pi} X$, or to find an equivariant map $j\colon M \to D^r$ so that $f \times j\colon M \to X \times D^r$ is an imbedding. Since M is freely acted upon by μ, one may find an equivariant imbedding $i\colon M \to S^s$ (with antipodal map) and composing with the inclusion of S^s as the sphere of radius $\frac{1}{2}$ in D^{s+1} defines such a j.

If one defines $\hat{\mathfrak{N}}^t(X, A, \tau)$ by

$$\lim_{r\to\infty}[(\Sigma^r(X/A), \Sigma^r(\tau)), (TBO_{r+t} \times S^\infty/\infty \times S^\infty, 1 \times a)]$$

for any pair with involution, this becomes a Poincaré duality theorem :

For X a closed n-manifold, $\hat{\mathfrak{N}}^t(X, \tau) \cong \hat{\mathfrak{N}}_{n-t}(X, \tau)$.

8. Bordism of bundles with involution

As an application of the equivariant bordism technique, one may consider the classification of bundles with involution over manifolds with involution. Specifically, one considers 4-tuples $(M, \sigma, \xi, \sigma^*)$ with M a closed differentiable manifold, $\sigma: M \to M$ a differentiable involution, ξ a real n-plane bundle over M, and $\sigma^*: E(\xi) \to E(\xi)$ an involution given by a real vector bundle map covering σ. Two such 4-tuples are bordant if there is a similar 4-tuple (V, ρ, η, ρ^*) with base V being a manifold with boundary and with $(\partial V, \rho|_{\partial V}, \eta|_{\partial V}, \rho^*)$ being the disjoint union of the two given 4-tuples.

Taking the operation induced by disjoint union, the equivalence classes of such 4-tuples form an abelian group given by $\mathfrak{N}_*(BO(n), \tau)$ where $(BO(n), \tau)$ is described as follows.

Let $BO(n)$ be the grassmannian of n-dimensional real subspaces of the infinite dimensional euclidean space $R^\infty \times R^\infty$. The linear transformation $t: R^\infty \times R^\infty \to R^\infty \times R^\infty: (x, y) \to (x, -y)$ induces an involution $\tau: BO(n) \to BO(n)$ by sending an n-dimensional subspace into its image under t.

Over $BO(n)$ one has the universal n-plane bundle γ^n consisting of pairs (α, a) with α an n-plane in $R^\infty \times R^\infty$ and a a vector in α, with t inducing an involution τ^* on the total space of γ^n by $\tau^*(\alpha, a) = (\tau\alpha, ta)$. By taking the induced bundle and involution one establishes a one-to-one correspondence between the equivariant homotopy classes of maps of (X, ρ) into $(BO(n), \tau)$, with X a compact Hausdorff space and ρ an involution of X, and the isomorphism classes of n-plane bundles over X with bundle involution covering ρ. (*Note.* The crucial point is that Z_2 has only two irreducible real representations given by $+1$ and -1 on R and so any Z_2 bundle over X is a subbundle of $X \times R^\infty \times R^\infty$ with involution $\rho \times 1 \times (-1)$.)

In order to analyze the group $\mathfrak{N}_*(BO(n), \tau)$ one considers first the free equivariant bordism $\hat{\mathfrak{N}}_*(BO(n), \tau)$. This is clearly equivalent to the study of manifold-bundle 4-tuples $(M, \sigma, \xi, \sigma^*)$ in which σ is fixed point free, and one has

PROPOSITION 7. *The homomorphism $q: \hat{\mathfrak{N}}_*(BO(n), \tau) \to \mathfrak{N}_*(BO(1) \times BO(n))$ induced by sending $(M, \sigma, \xi, \sigma^*)$ to the class of the map $f \times g: M/\sigma \to BO(1) \times BO(n)$, with f inducing the double cover $M \to M/\sigma$ and g classifying the n-plane bundle $E(\xi)/\sigma^* \to M/\sigma$, is an isomorphism.*

PROOF. Being given $h: N \to BO(1) \times BO(n)$, one has the double cover $\tilde{N} \xrightarrow{\pi} N$ induced from $\pi_1 \circ h$, with an involution ν given by interchanging the sheets of the cover and letting ξ be the n-plane bundle over N induced by $\pi_2 \circ h$, $\pi^*\xi$ has an involution ν^* induced by restriction of $\nu \times 1$ on $\tilde{N} \times E(\xi)$ to $E(\pi^*\xi)$. The homomorphism

$$q': \mathfrak{N}_*(BO(1) \times BO(n)) \longrightarrow \hat{\mathfrak{N}}_*(BO(n), \tau)$$

defined by $q'([N, h]) = [(\tilde{N}, \nu, \pi^*\xi, \nu^*)]$ is clearly inverse to q.

The fixed point set of the involution $\tau: BO(n) \to BO(n)$ clearly consists of

n-planes in $R^\infty \times R^\infty$ invariant under t and an invariant n-plane is clearly of the form $V \times W$ where V is a subspace of $R^\infty \times 0$ and W is a subspace of $0 \times R^\infty$, with $\dim V + \dim W = n$. Thus F_τ is clearly the disjoint union of the subspaces $BO(j) \times BO(n-j)$ for $0 \leqq j \leqq n$. Applying Proposition 2 one has an exact sequence

$$\begin{tikzcd} \widehat{\mathfrak{N}}_*(BO(n),\tau) \arrow[r, "k_*"] & \mathfrak{N}_*(BO(n),\tau) \arrow[r, "F"] & \bigoplus_{k=0}^{*}\bigoplus_{j=0}^{n}\mathfrak{N}_k\big(BO(*-k)\times BO(j)\times BO(n-j)\big) \arrow[ll, bend left, "S"] \end{tikzcd} .$$

Fixing attention on the summand $\mathfrak{N}_{m-1}\big(BO(1)\times BO(n)\times BO(0)\big)$ one considers a map $h\colon N^{m-1} \to BO(1) \times BO(n) \times BO(0)$ which maps under S to the class of $(S(\eta), a, i\circ(\pi_2\circ h)\circ\pi)$ where η is induced by $\pi_1\circ h$, a is the antipodal involution on the sphere bundle $S(\eta)$, and

$$S(\eta) \xrightarrow{\pi} N \xrightarrow{\pi_2\circ h} BO(n) \xrightarrow{i} BO(n)$$

with i being the inclusion of $BO(n) = BO(n)\times BO(0)$ in $BO(n)$ as part of the fixed point set of τ. Since $BO(n) \times BO(0)$ is covered by the universal bundle with trivial involution, this is precisely $(S(\eta), a, \pi^*\xi, a^*)$ where ξ is induced by $i\circ(\pi_2\circ h)$ and a^* by $a \times 1$ on $S(\eta) \times E(\xi)$. Then q' coincides with

$$S\colon \mathfrak{N}_{m-1}\big(BO(1) \times BO(n)\times BO(0)\big) \to \widehat{\mathfrak{N}}_{m-1}(BO(n), \tau)$$

and q defines a splitting of the above sequence, which proves

PROPOSITION 8. *The homomorphism q defines a splitting of the sequence of Proposition 2 for the involution $(BO(n), \tau)$ and hence*

$$\mathfrak{N}_m(BO(n), \tau) \overset{F}{\cong} \bigoplus_{\substack{k=0\\(k,j)\neq(m-1,n)}}^{m} \bigoplus_{j=0}^{n} \mathfrak{N}_k\big(BO(m-k) \times BO(j) \times BO(n-j)\big) .$$

This is clearly the manifold-bundle analog of the splitting in Theorem 28.1 of Conner and Floyd [4]. The groups involved are of course well known since $\mathfrak{N}_*\big(BO(k)\big)$ is known and the Künneth theorem holds for ordinary bordism.

A splitting homomorphism

$$\rho\colon \bigoplus_{k=0}^{*}\bigoplus_{j=0}^{*} \mathfrak{N}_k\big(BO(*-k) \times BO(j) \times BO(n-j)\big) \longrightarrow \mathfrak{N}_*(BO(n), \tau)$$

may be constructed by sending a map $h\colon N^k \to BO(m-k) \times BO(j) \times BO(n-j)$ to the class of $(RP(\xi_1 \oplus 1), \sigma, \pi^*\xi_2 \oplus \lambda \otimes \pi^*\xi_3, \sigma^*)$ where ξ_i is the bundle induced by $\pi_i\circ h(i = 1, 2, 3)$, $RP(\xi_1 \oplus 1)$ is the projective space bundle of lines in the fibers of the Whitney sum of ξ_1 with a trivial bundle, σ is the involution induced on the space of lines by $s = (-1) \times 1$ on $E(\xi_1) \times R = E(\xi_1 \oplus 1)$, λ is the canonical line bundle over $RP(\xi_1 + 1)$ with total space consisting of pairs (α, a) with α a line in a fiber and a a vector in that line, $\pi\colon RP(\xi_1 \oplus 1) \to N$ is the projection, and σ^* is the involution described as follows. On the summand $\pi^*\xi_2$, σ^* is given by $\sigma \times 1$, and on $\lambda \otimes \pi^*\xi_3$, σ^* is the tensor product of the involutions s^* on λ given by $s^*(\alpha, a) = (\sigma\alpha, sa)$ and $\sigma \times (-1)$ on $\pi^*\xi_3$.

To verify that this is a splitting one simply looks at the fixed point set, formed as the disjoint union of $RP(\xi_1)$ and $RP(1)$ contained in $RP(\xi_1 \oplus 1)$. For the $RP(1)$ component, one identifies $RP(1)$ with N by π, and the normal bundle is identified with ξ_1, while λ is the trivial bundle with $s^* = 1$, so the n-plane bundle is $\xi_2 \oplus \xi_3$ with involution $1 \oplus (-1)$, and this fixed component recovers (N, h). The component $RP(\xi_1)$ is of codimension 1, with normal bundle $\lambda' = \lambda|_{RP(\xi_1)}$ with s^* acting as -1 in the fibers of λ', so that σ^* is the trivial involution given by the identity on the bundle $\pi^*\xi_2 \oplus \lambda' \otimes \pi^*\xi_3$, and thus one obtains a component in the summand $\mathfrak{N}_{m-1}(BO(1) \times BO(n) \times BO(0))$, which is the image of the splitting q as defined above. In particular, $F\rho F(M, \sigma, \xi, \sigma^*)$ and $F(M, \sigma, \xi, \sigma^*)$ differ only in $(k = * -1, j = n)$ term, so $\rho F = 1$.

Note. This construction may also be described as taking the bundle $\xi = \pi^*\xi_2 \oplus \pi^*\xi_3$ over $D(\xi_1)$, with involution induced by $(-1) \times 1 \times (-1)$ on $D(\xi_1) \times E(\xi_2) \times E(\xi_3)$ and then collapsing $S(\xi_1)$ and $\xi|_{S(\xi_1)}$ by means of the involutions.

One may now duplicate much of the analysis of involutions from Conner and Floyd, Chapter IV, in the case of involutions on manifold-bundles. For example, $SF = 0$ is their result (24.1), giving

LEMMA 1. *If $(M, \sigma, \xi, \sigma^*)$ is a manifold-bundle 4-tuple and if $(S, \sigma|_S, \xi|_S, \sigma^*)$ is the induced 4-tuple on the normal sphere bundle of the fixed point set, then*

$$[S, \sigma|_S, \xi|_S, \sigma^*] = 0$$

in $\hat{\mathfrak{N}}_*(BO(n), \tau)$.

The major portion of their (24.2) is contained in.

LEMMA 2. *Let $(M^m, \sigma, \xi, \sigma^*)$ be a manifold-bundle 4-tuple, with η the normal bundle to the fixed point set of σ. If*

$$\alpha \in \bigoplus_{k=0}^{m} \bigoplus_{j=0}^{n} \mathfrak{N}_k(BO(m+1-k) \times BO(j) \times BO(n-j))$$

classifies the Whitney sum of η with a trivial line bundle and $\xi|_{F_\sigma}$, then

$$S(\alpha) = [Z_2, -1] \times [M, \xi]$$

in $\hat{\mathfrak{N}}_*(BO(n), \tau)$, where $[Z_2, -1]$ is the free involution on the two point set, and the product is given by the obvious pairing

$$\hat{\mathfrak{N}}_*(\mathrm{pt}, 1) \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(BO(n)) \longrightarrow \hat{\mathfrak{N}}_*(BO(n), \tau) .$$

PROOF. If (X, ρ) is an involution on a manifold, an involution pair $K_1(X, \rho)$ is formed as follows: $X \times S^1$ has involutions T', S' given by $T'(x, z) =$

$(\rho x, -z)$ and $S'(x, z) = (x, \bar{z})$. Since T' and S' commute, S' induces an involution S on $X \times S^1/T'$ and one lets $K_1(X, \rho) = (X \times S^1/T', S)$. This construction is natural in (X, ρ), so $K_1(E(\xi), \sigma^*)$ is a real n-plane bundle over $K_1(M, \sigma)$, giving a manifold-bundle 4-tuple $K_1(M, \sigma, \xi, \sigma^*)$. The fixed point set of $K_1(M, \sigma)$ is M (image of $M \times \{\pm 1\}$) and F_σ (image of $F_\sigma \times \{\pm i\}$) with normal bundle a trivial line bundle and $\eta \oplus 1$ respectively, with the inverse image of this fixed set in $K_1(E(\xi), \sigma^*)$ being given by $E(\xi)$ with trivial involution, and $E(\xi|_{F\sigma})$ with involution induced by σ^*

$$[(x, z) \longrightarrow (x, \bar{z}) = (x, -z) = (\sigma^* x, z) \text{ if } z = \pm i] .$$

Thus the involution on $K_1(M, \sigma, \xi, \sigma^*)$ has induced involution on the normal sphere bundle of the fixed point set given by $S(\alpha) \cup [Z_2, -1] \times [M, \xi]$. Applying Lemma 1 completes the proof.

Note. The construction is precisely that used in the proof of (24.2) and was also explored more fully in Conner and Floyd [5], where the K_1 notation was introduced.

Let $I_*: \mathfrak{N}_p(BO(m) \times BO(j) \times BO(n-j)) \rightarrow \mathfrak{N}_p(BO(m+1) \times BO(j) \times BO(n-j))$ be the homomorphism induced by the inclusion of $BO(m)$ in $BO(m+1)$ classifying the Whitney sum of the universal bundle and a trivial line bundle. One then has the analogues of (26.3) and (26.4).

LEMMA 3. *I_* is an isomorphism for $p \leqq m$.*

LEMMA 4. *The diagram*

$$\begin{array}{ccc} \mathfrak{N}_p(BO(m) \times BO(j) \times BO(n-j)) & \xrightarrow{S} & \hat{\mathfrak{N}}_{p+m-1}(BO(n), \tau) \\ \downarrow I_* & & \uparrow \Delta \\ \mathfrak{N}_p(BO(m+1) \times BO(j) \times BO(n-j)) & \xrightarrow{S} & \hat{\mathfrak{N}}_{p+m}(BO(n), \tau) \end{array}$$

commutes, Δ being the Smith homomorphism.

PROOF. Application of the Smith homomorphism to $(S(\xi \oplus 1), a)$ is just given by restriction to $(S(\xi), a)$.

This completes the translation of the results needed for the proof of the analogue of Conner and Floyd's (27.1), and using the obvious notational modifications in their proof gives

PROPOSITION 9. *Let k and n be non-negative integers. There exists an integer $\varphi(k, n)$ such that, if $(M, \sigma, \xi, \sigma^*)$ is a manifold-bundle 4-tuple with $[M, \xi]$ a non-zero element in $\mathfrak{N}_*(BO(n))$ and* $\dim M > \varphi(k, n)$, *then the dimension of some component of the fixed point set F_σ is greater than k.*

Note. M may bound when $[M, \xi] \neq 0$, and hence this does not follow directly from (27.1).

As a somewhat curious situation, consider a manifold-bundle 4-tuple $(M^m, \sigma, \xi^n, \sigma^*)$ and suppose that F_σ is a finite set of points. At each point $p \in F_\sigma$, σ^* induces a representation on the n-dimensional vector space $\xi|_p$, being of the form $R^j \times R^{n-j}$, $0 \leqq j \leqq n$, with $\sigma^* = 1 \times (-1)$, and the point p will then be said to have type $(j, n-j)$. It is, of course, standard that F_σ has an even number of points if $m > 0$, and in the manifold-bundle case one finds

PROPOSITION 10. *If $0 < n < m$ and $(M^m, \sigma, \xi^n, \sigma^*)$ is a manifold-bundle 4-tuple such that F_σ is a finite set of points, then, for each j with $0 \leqq j \leqq n$, the number of points of type $(j, n-j)$ is even.*

PROOF. If $p \in F_\sigma$ has type $(j, n-j)$, then p represents the generator α_j of $\mathfrak{N}_0(BO(m) \times BO(j) \times BO(n-j))$ after "applying F", and $S(\alpha_j)$ in $\mathfrak{N}_{m-1}(BO(1) \times BO(n)) = \hat{\mathfrak{N}}_{m-1}(BO(n), \tau)$ is the bordism class of the map $h_j: RP(m-1) \to BO(1) \times BO(n)$ with $\pi_1 \circ h_j$ the usual inclusion and $\pi_2 \circ h_j$ classifying $(n-j)\lambda \oplus j$ with λ the canonical line bundle over $RP(m-1)$. To prove the proposition, it suffices to show that the elements $S(\alpha_j)$ are linearly independent. For this one computes the Stiefel-Whitney numbers $h_j^*(w_1^{m-1-k} \otimes w_k)[RP(m-1)]$ which are given by the binomial coefficient $\binom{n-j}{k}$ for $k \leqq n-j$ and zero for $k > n-j$. Thus the matrix of these numbers for $0 \leqq k, j \leqq n < m$ is triangular with 1's along the diagonal, and hence the bordism classes $[RP(m-1), h_j]$ are linearly independent.

For $n \geqq m$, this phenomenon breaks down, and one has

Assertion. For $n = m > 0$, there is a manifold-bundle 4-tuple (M^m, μ, ξ^m, μ^*) with F_μ consisting precisely of $\binom{m}{j}$ fixed points of type $(j, m-j)$ for each $0 \leqq j \leqq m$.

PROOF. Let σ be the involution on $S^1 = RP(R \times R)$ induced by $1 \times (-1)$ on $R \times R$, with σ^* the induced involution on the canonical line bundle λ. Then $(S^1, \sigma, \lambda, \sigma^*)$ has fixed point set consisting of one point of each of the types $(1, 0)$ and $(0, 1)$. Let M^m be the product of m copies of S^1 with $\mu = \sigma \times \cdots \times \sigma$, and take $\xi^m = \bigoplus_{i=1}^m \pi_i^*(\lambda)$ with μ^* being the product involution of the σ^*. Then (M^m, μ, ξ^m, μ^*) has precisely $\binom{m}{j}$ fixed points of type $(j, m-j)$ for $0 \leqq j \leqq m$.

Notes. (1) As in the proof of Proposition 10, the elements $[RP(m-1), h_j]$ in $\mathfrak{N}_{m-1}(BO(1) \times BO(m))$ for $1 \leqq j \leqq m$ are lineary independent, and by the assertion

$$[RP(m-1), h_0] = \sum_{i=0}^{m-1} \binom{m}{j} [RP(m-1), h_{m-i}] \, .$$

In particular, any 4-tuple (M^m, μ, ξ^m, μ^*) with F_μ of dimension zero is then bordant to an integral multiple of the example given in the assertion.

(2) The manifold-bundle 4-tuple (M^m, μ, ξ^m, μ^*) of the assertion is non-

zero in $\mathfrak{N}_m(BO(m))$ since $w_m(\xi^m)[M^m] = 1$. This shows that $\varphi(0, n) = n$ in the notation of Proposition 9.

As another sample analysis, consider a 4-tuple $(RP(2), \tau, \xi^n, \tau^*)$ with τ the standard involution $\tau[z_0, z_1, z_2] = [z_0, z_1, -z_2]$. The fixed point set is then the union of $S^1 = RP(1)$ and an isolated point $p = RP(0)$ with trivial normal 2-plane bundle and type $(k, n-k)$. S^1 has normal bundle λ and (ξ^n, τ^*) restricts to the sum of a j-plane bundle ξ^j_+ with trivial involution and an $(n-j)$-plane bundle ξ^{n-j}_- with involution given by -1 in the fibers. One then has

Assertion. If $(M^2, \sigma, \xi^n, \sigma^*)$ has fixed point information $(S^1, \lambda, \xi^j_+, \xi^{n-j}_-)$ and $(p, 2, k, n-k)$, then

$$w_1(\xi) = w_1(\xi^j_+ \oplus \xi^{n-j}_-) = (k-j)\alpha \qquad \text{in } H^1(S^1; Z_2)$$

(α being the standard generator) and any fixed point data satisfying these relations comes as the fixed point data of a 4-tuple with base $(RP(2), \tau)$.

PROOF. $S[(p, 2, k, n-k)]$ is given by $h_k: RP(1) \to BO(1) \times BO(n)$ as noted above, while applying ρ one sees that $S[(S^1, \lambda, \xi^j_+, \xi^{n-j}_-)]$ is given by $h: RP(1) = S^1 \to BO(1) \times BO(n)$ with $\pi_1 \circ h$ the inclusion and $\pi_2 \circ h$ classifying $\xi^j_+ \oplus (\lambda \otimes \xi^{n-j}_-)$. In order that these be bordant, it is necessary and sufficient that $w_1(k + (n-k)\lambda) = w_1(\xi^j_+ \oplus (\lambda \otimes \xi^{n-j}_-))$ or $(n-k)\alpha = w_1(\xi^j_+) + ((n-j)\alpha + w_1(\xi^{n-j}_-))$, giving the relation. If the relation holds then h_k and h are homotopic so that in fact the pullback to $D(\lambda)$ of $\xi^j_+ \oplus \xi^{n-j}_-$ with involution and to D^2 of $k \oplus (n-k)$ with involution are isomorphic bundles with involution over the common boundary S^1 with antipodal map. These bundles may then be joined over $(S^1, -1)$ to give a bundle over $(RP(2), \tau)$ having the given fixed point data. (*Note.* Since $2\lambda \cong 2$ on $RP(1)$, the classes obtained from the fixed components by applying S depend only on the classes of k and j mod 2. Thus, having only a 2 primary relation is not unreasonable.)

Remark. This is a partial analogue of Conner and Floyd's (27.6). Specifically, if $(M^m, \sigma, \xi^n, \sigma^*)$ has as fixed point set the union of a point and a circle, then $m = 2$ and $(M^m, \sigma, \xi^n, \sigma^*)$ is bordant as 4-tuple to a manifold-bundle 4-tuple with base $(RP(2), \tau)$. This is immediate from the above since $m = 2$ and that the normal bundle to the fixed circle is λ are given in the proof of (27.6). Then the fixed data is the same as that of a 4-tuple with base $(RP(2), \tau)$.

As a final result in this vein, one notes that (24.4) also generalizes to give

PROPOSITION 11. *Let $(M^{2n}, \sigma, \xi^{2k}, \sigma^*)$ be a 4-tuple in which M^{2n} is an almost complex manifold with σ a conjugation, and ξ^{2k} is a complex vector bundle such that σ^* is a conjugation* (i.e., $\sigma^* i = -i\sigma^*$). *The fixed point set $F_{\sigma^*} \subset E(\xi)$ is then a real k-plane bundle η over the n-manifold $F_\sigma \subset M^{2n}$, and $(M^{2n}, \sigma, \xi^{2k}, \sigma^*)$ is bordant as 4-tuple to the "square of its real-fold",*

$$(F_\sigma \times F_\sigma, \tau, \pi_1^*(\eta) \oplus \pi_2^*(\eta), \tau^*)$$

with τ and τ^* being the interchange involutions.

PROOF. The fixed point data in both cases is given by the fixed set F_σ with normal bundle isomorphic to the tangent bundle of F_σ and with the restriction of the bundle given by the complexification of η and the involution being complex conjugation.

This leads at once to another closely related application. One may consider the bordism classification of 4-tuples (M, μ, ξ, μ^*) with M a closed differentiable manifold, $\mu\colon M \to M$ a differentiable involution, ξ a complex n-plane bundle over M, and μ^* *a* conjugation on ξ covering μ; i.e., a real vector bundle involution covering μ and such that $\mu^* i = -i\mu^*$. These are "real" vector bundles in the sense of Atiyah [2].

Bundles with conjugation are classified by equivariant homotopy classes of maps into $(BU(n), c)$ where $BU(n)$ is the grassmannian of n-dimensional complex subspaces of the infinite dimensional complex vector space C^∞, and c is the involution induced by complex conjugation in C^∞. Thus the bordism classification of bundles with conjugation is the equivariant bordism group $\mathfrak{N}_*(BU(n), c)$.

PROPOSITION 12. *The fixed point set of $(BU(n), c)$ is the space $BO(n)$. Further, the exact sequence of Proposition 2 becomes*

$$0 \longrightarrow \mathfrak{N}_*(BU(n), c) \xrightarrow{F} \bigoplus_{k=0}^{*} \mathfrak{N}_k\big(BO(*-k) \times BO(n)\big) \xrightarrow{S}$$
$$\mathfrak{N}_*(BU(n) \times S^\infty/c \times a) \longrightarrow 0\,.$$

PROOF. Clearly the n-planes in C^∞ fixed under conjugation are precisely those with a basis consisting of real vectors, which form $BO(n)$ as a subspace of $BU(n)$. To complete the result, it suffices to show that

$$S\colon \mathfrak{N}_{m-1}\big(BO(1) \times BO(n)\big) \longrightarrow \mathfrak{N}_{m-1}(BU(n) \times S^\infty/c \times a)$$

is epic. This will require some peripheral steps.

LEMMA 1. *The fibration*

$$BU(n) \xrightarrow{j} BU(n) \times S^\infty/c \times a \xrightarrow{p} BO(1) = S^\infty/a$$

is totally non-homologous to zero mod 2. *In particular $H^*(BU(n) \times S^\infty/c \times a; Z_2)$ is the Z_2 polynomial ring on the classes $w_1(\xi)$ and $w_{2i}(\zeta)$ where ξ is the line bundle of the double cover $BU(n) \times S^\infty \to BU(n) \times S^\infty/c \times a$, and ζ is the real $2n$-plane bundle $E(\bar\gamma^n) \times S^\infty/c^* \times a \to BU(n) \times S^\infty/c \times a$ with c^* the involution on the universal bundle $\bar\gamma^n$ over $BU(n)$ induced by conjugation on C^∞.*

PROOF. Let $q\colon BU(n) \times S^\infty/c \times a \to BO(2n)$ classify ζ. The composite

$q \cdot j \colon BU(n) \to BO(2n)$ then classifies $\bar{\gamma}^n$ and so $(q \cdot j)^*$ maps $Z_2[w_{2i} \mid i \leq n]$ isomorphically onto $H^*(BU(n); Z_2)$. Thus j^* is epic or the fibration is totally non-homologous to zero mod 2. Thus $H^*(BU(n) \times S^\infty/c \times a; Z_2)$ is the free module over $H^*(BO(1); Z_2)$ *via* p^* on the monomials in the $w_{2i}(\zeta)$, so the asserted polynomial algebra maps isomorphically onto $H^*(BU(n) \times S^\infty/c \times a; Z_2)$.

LEMMA 2. *$S \colon \mathfrak{N}_*(BO(1) \times BO(n)) \to \mathfrak{N}_*(BU(n) \times S^\infty/c \times a)$ is the homomorphism induced by the inclusion $\mathfrak{L}$ of $F_c \times S^\infty/1 \times a = BO(n) \times BO(1)$ in $BU(n) \times S^\infty/c \times a$. Further, the composite*

$$f = (p \times q) \cdot \mathfrak{L} \colon BO(1) \times BO(n) \longrightarrow BO(1) \times BO(2n)$$

has $f^*(\gamma^1) = \gamma^1$ and $f^*(\gamma^{2n}) = \gamma^n \oplus (\gamma^1 \otimes \gamma^n)$.

PROOF. If $g \colon M \to BO(1) \times BO(n)$ with $g^*(\gamma^1) = \eta$, $g^*(\gamma^n) = \rho$, then $S(g)$ is represented by $(S(\eta), a, \pi^*(\rho \otimes C), a^*)$ with a^* induced by $a \times$ conjugation on $S(\eta) \times E(\rho \otimes C)$, or by the equivariant map $h \colon (S(\eta), a) \to (BU(n) \times S^\infty, c \times a)$ classifying $\pi^*(\rho \otimes C)$, which is the composite $S(\eta) \to BO(n) \times S^\infty \xrightarrow{\mathfrak{L}'} BU(n) \times S^\infty$ induced by the inclusion. Passing to quotients gives

$$M = S(\eta)/a \xrightarrow{g} BO(n) \times BO(1) \xrightarrow{\mathfrak{L}} BU(n) \times S^\infty/c \times a$$

representing $S(g)$ as a bordism class. Since the map $\pi_1 f \colon BO(1) \times BO(n) \to BO(1)$ classifies the double cover, $f^*(\gamma^1) = \gamma^1$, while $\pi_2 f$ classifies the quotient of

$$S^\infty \times E(\gamma^n \otimes C) \subset S^\infty \times E(\gamma^n) \times E(\gamma^n)$$

with involution $a \times 1 \times (-1)$ over $BO(1) \times BO(n)$, which is the bundle $\gamma^n \oplus (\gamma^1 \otimes \gamma^n)$.

Now considering $f \colon BO(1) \times BO(n) \to BO(1) \times BO(2n)$, one has $f^*(w_1 \otimes 1) = w_1 \otimes 1$ and $f^*(1 \otimes w_{2i}) = 1 \otimes w_i^2 +$ terms divisible by $w_1 \otimes 1$, so image f^* has dimension at least as large as $Z_2[x_1, x_{2i} \mid i \leqq n]$. Since this factors through $BU(n) \times S^\infty/c \times a$, whose cohomology is of precisely this dimension, $\mathfrak{L}^*$ is monic. Thus

$$\mathfrak{L}_* \colon H_*(BO(1) \times BO(n); Z_2) \longrightarrow H_*(BU(n) \times S^\infty/c \times a; Z_2)$$

is epic, and the induced bordism homomorphism S is also epic.

Notes. (1) Since the homology homomorphism $\mathfrak{L}_*$ is epic, this sequence in fact splits as $\mathfrak{N}_*$ module, but the splitting is not geometrically defined.

(2) $BU(n) \times S^\infty/c \times a$ is the classifying space for real $2n$-plane bundles whose structure group is the group of real linear transformations $T \colon C^n \to C^n$ which preserve inner products and satisfy $Ti = \pm iT$. The existence and relation to conjugations of this classifying space was pointed out to me by Professor P.S. Landweber.

One may consider C^∞ as $R^\infty \times R^\infty$, by taking real and imaginary parts, and conjugation is then the involution $1 \times (-1)$. Considering a complex n-dimensional subspace as a real subspace of dimension $2n$ then gives an equivariant inclusion $s: (BU(n), c) \to (BO(2n), \tau)$. On the bordism level this just ignores the complex structure of the bundle and considers the conjugation as only an involution.

PROPOSITION 13. *The homomorphisms*

$$s_*: \mathfrak{N}_*(BU(n), c) \longrightarrow \mathfrak{N}_*(BO(2n), \tau)$$

and

$$s_*: \hat{\mathfrak{N}}_*(BU(n), c) \longrightarrow \hat{\mathfrak{N}}_*(BO(2n), \tau)$$

are monic.

PROOF. One has induced a map of the exact sequences given by Proposition 2 and since these sequences are split exact, it suffices to prove

$$s_*: \bigoplus_{k=0}^{*} \mathfrak{N}_k\big(BO(*-k) \times BO(n)\big) \longrightarrow \bigoplus_{k=0}^{*} \bigoplus_{j=0}^{2n} \mathfrak{N}_k\big(BO(*-k) \times BO(j) \times BO(2n-j)\big)$$

is monic. This is induced by the inclusion of the fixed point sets and is given by the diagonal inclusion of $BO(n)$ in $BO(n) \times BO(n) \subset F_\tau$. Clearly then

$$\mathfrak{N}_*\big(BO(p) \times BO(n)\big) \xrightarrow{1 \times \Delta^*} \mathfrak{N}_*\big(BO(p) \times BO(n) \times BO(n)\big)$$

is monic.

Notes. (1) $s_*: \hat{\mathfrak{N}}_*(BU(n), c) = \mathfrak{N}_*(BU(n) \times S^\infty/c \times a) \to \hat{\mathfrak{N}}_*(BO(2n), \tau) = \mathfrak{N}_*\big(BO(1) \times BO(2n)\big)$ is just the homomorphism induced by the map $p \times q$.

(2) It is well known that the inclusion $BU(n) \to BO(2n)$ induces monomorphisms on ordinary bordism, and this is the obvious equivariant analogue.

UNIVERSITY OF VIRGINIA

REFERENCES

[1] M. F. ATIYAH, *Bordism and cobordism*, Proc. Camb. Phil. Soc. **57** (1961) 200-208.
[2] ———, *K-theory and reality*, Quart. J. Math. **17** (1966), 367-386.
[3] G. E. BREDON, Equivariant Cohomology Theories, Springer-Verlag, Berlin, 1967.
[4] P. E. CONNER and E. E. FLOYD, Differentiable Periodic Maps, Springer-Verlag, Berlin, 1964.
[5] ———, *Fibring within a cobordism class*, Michigan Math. J. **12** (1965), 33-47.
[6] F. HIRZEBRUCH, Topological Methods in Algebraic Geometry, Springer-Verlag, Berlin, 1966.
[7] C. T. C. WALL, *Cobordism of pairs*, Comment. Math. Helv. **35** (1961), 136-145.

(Received August 30, 1968)