ON THE STRUCTURE OF MANIFOLDS.*

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In this paper, we prove a number of theorems which give some insight into the structure of differentiable manifolds.

The methods, results and some notation of [13], hereafter referred to as GPC, and [12] will be used. These two papers and [14] can be considered as a starting point for this one. The main theorems in these papers are special cases of the theorems here.

Among the most important theorems in this paper are 1.1 and 6.1.

Some conversations with A. Haefliger were helpful in the preparation of parts of this paper.

Everything will be considered from the differentable, equivalently C^∞, point of view; manifolds, imbeddings, and isotopes will be C^∞.

Section 1. We give a necessary and sufficient condition for two closed simply connected manifolds of dimension greater than four to be diffeomorphic. The condition is *h-cobordant*, first defined by Thom [16] for the combinatorial case, and developed by Milnor [9], and Kervaire and Milnor [7] for the differentiable case (sometimes previously h-cobordant has been called J-equivalent). It involves a combination of homotopy theory and cobordism theory. More precisely, two closed connected oriented manifolds M_1^n, M_2^n are h-cobordant if there exists an oriented compact manifold W with ∂W (the boundary of W) diffeomorphic to the disjoint union of M_1 and $-M_2$, and each component of ∂W is a deformation retract of W.

THEOREM 1.1. *If $n \geqq 5$, and two closed oriented simply connected manifolds M_1^n and M_2^n are h-cobordant, then M_1 and M_2 are diffeomorphic by an orientation preserving diffeomorphism.*

It has been asked by Milnor whether h-cobordant manifolds in general are diffeomorphic, problem 5, [9]. Subsequently, Milnor himself has given a counter-example of 7-dimensional manifolds with fundamental group Z_7, h-cobordant but not diffeomorphic [10]. Thus the condition of simple-connectedness is necessary in Theorem 1.1.

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Theorem 1.1 was proved in special cases in [13] and [14]. These special cases were applied to show that every sphere not of dimension four or six has a finite number of differentiable structures. The six dimensional case is taken care of by the following.

COROLLARY 1.2. *Every homotopy 6-sphere is diffeomorphic to* S^6.

This follows from 1.1 and the result of Kervaire and Milnor [7] that every homotopy 6-sphere is h-cobordant to S^6.

COROLLARY 1.3. *The semigroup of 2-connected closed 6-manifolds is generated by* $S^3 \times S^3$.

This follows from 1.2 and [15].

Haefliger [2] has extended the notion of h-cobordant to the relative case. Let V_1, V_2, M_1, M_2 be closed oriented, connected manifolds with $V_i \subset M_i$, $i = 1, 2$. According to Haefliger (M_1, V_1), (M_2, V_2) are *h-cobordant* if there is a pair (M, V) (i.e., $V \subset M$) with $\partial M = M_1 - M_2$, $\partial V = V_1 - V_2$ and $M_i \to M$, $V_i \to V$ homotopy equivalences. Then 1.1 can be extended to the relative case.

THEOREM 1.4. *Suppose* $(M_1{}^n, V_1{}^k)$ *and* $(M_2{}^n, V_2{}^k)$ *are h-cobordant,* $k \geqq 5$, $\pi_1(V_i) = \pi_1(M_i - V_i) = 1$. *Then there is an orientation preserving diffeomorphism of* M_1 *onto* M_2 *sending* V_1 *onto* V_2.

By taking V_i empty (the proof of 1.4 is valid for this case also), one can consider 1.1 as a special case of 1.4.

Actually we obtain much stronger theorems which will imply 1.4. The proof of 1.4 is completed in Section 3.

It would not be surprising if the hypothesis of simple connectedness in these theorems could be weakened using torsion invariants (see [10], for example).

Theorem 1.4 has application in the theory of knots except in codimension two.

Section 2. The main theorem we prove in this section is the following. Here we use the notation of GPC.

THEOREM 2.1. *Let* M^n *be a compact manifold with a simply connected boundary component* Q. *Let* $V = \chi(M, Q; f; m)$ *where* $f: \partial D_0{}^m \times D_0{}^{n-m} \to Q$ *is an embedding,* $m > 2$, $n - m > 3$. *Suppose* $W = \chi(V, Q_1; g_1, \cdots, g_r; m+1)$ *where* Q_1 *is the component of* ∂V *corresponding to* Q *and suppose that* $H_m(W, M)$ *is zero. Then* W *is of the form* $\chi(M, Q; g'_1, \cdots, g'_{n-1}; m+1)$.

Note that an example of Mazur [8] shows that the dimensional restriction is necessary here.

For the proof we use several lemmas.

LEMMA 2.2. *Let M^n be a compact manifold, Q a component of ∂M, $n - m > 1$, $V = \chi(M, Q; f; m)$, $W = \chi(V, Q_1; g_1, \cdots, g_r; m+1)$ where Q_1 is the component of ∂V corresponding to Q and $f\colon \partial D_0^m \times D_0^{n-m} \to Q$, $g_i\colon \partial D_i^{m+1} \times D_i^{n-m-1} \to Q$, are imbeddings. Let $F = q \times D_0^{n-m} \subset V$ with $q \notin \partial D_0^m$, $\partial F \subset \partial V$. Suppose ∂F does not intersect $g_i(\partial D_i^{m+1} \times 0)$, $i = 1, \cdots, r-1$* and *$g_r(\partial D_r^{m+1} \times 0)$ intersects ∂F transversally in a single point. Then W is of the form*

$$\chi(M, Q; g'_1, \cdots, g'_{r-1}; m+1).$$

Proof 2.2. In the proof of 2.2, we use without further mention, the fact that the diffeomorphism type of an n-manifold is not changed when an n-disk is adjoined by identifying an $(n-1)$ disk on the boundary of each under a diffeomorphism. See GPC, 3.4, J. Milnor, "Sommes de varietes différentiables et structures différentiables des sphères," *Bulletin de la Societe Mathematique de France*, vol. 87 (1959), pp. 439-444 and R. Palais, "Extending diffeomorphism," *Proceedings of the American Mathematical Society*, vol. 11 (1960), pp. 274-277.

We may assume, using the uniqueness of tubular neighborhoods that ∂F does not intersect $g_i(\partial D_i^{m+1} \times D_i^{n-m-1})$, $i = 1, \cdots, r-1$.

Since $g_r(\partial D^{m+1} \times 0)$ is transversal to ∂F in ∂V, there exists a disk neighborhood L of $o = g_r(\partial D^{m+1} \times 0) \cap F$, $L = A^m \times B^{n-m-1}$, where $A^m \times 0$ is a disk neighborhood of o in $g_r(\partial D^{m+1} \times 0)$, $0 \times B^{n-m-1}$ a disk neighborhood of o in ∂F, with (o, o) corresponding to o.

Now there exists a disk neighborhood D_a^m of the point $F \cap (D_0^m \times 0)$ in $D_0^m \times 0$ so small that if $N = D_a^m \times D_0^{n-m} \subset V$, then

(1) $N \cap \text{image}\, g_i = \emptyset$, $i = 1, \cdots, r-1$, and

(2) $N \cap \text{image}\, g_r \subset L$.

Since both $D_a^m \times o$ and $A^m \times o$ (i.e. $A^m \times 0$) are transversal to ∂F in ∂V, we may assume using a diffeomorphism of V, and restricting L, that $A^m \times o$, $D_a^m \times o$ coincide, and that L coincides with image $g_r \cap N$.

The following statements are made under the assumptions that corners are smoothed via "straightening the angle," Section 1 of GPC or better [9]. Let $K = N \cup D_r^{n-m-1} \subset W$.

We claim that $K \cap Cl(W - K)$ is diffeomorphic to an $(n-1)$-disk. First $K \cap Cl(W - K)$ is $\partial D_a^m \times (\partial D_r^{m+1} \times D_r^{n-m-1})$ — interior L or

$$\partial D_a^m \times D_0^{n-m} \cup D_b^m \times D_r^{n-m-1}$$

where D_b^m is ∂D_r^{m+1} minus the interior of an m-disk. Furthermore $K \cap Cl(W - K)$ may be described as $\partial D_a^m \times D_0^{n-m}$ with $D_b^m \times D_r^{n-m-1}$ attached by an embedding $h\colon \partial D_b^m \times D_r^{n-m-1} \to \partial D_a^m \times D_0^{n-m}$ with the property that $h(\partial D_b^m \times 0)$ coincides with $\partial D_a^m \times c$ for some point $c \in D_0^{n-m}$. In fact, h is the restriction of g_r. This is the situation in the proof of 3.3 of GPC, where it was shown that the resulting manifold was a disk. Thus $K \cap Cl(W - K)$ is indeed an $(n-1)$-disk.

Since K is an n-disk, $K \cap Cl(W - K)$ an $(n-1)$-disk, we have that W is diffeomorphic to $Cl(W - K)$. On the other hand it is clear from the previous considerations that $Cl(W - K)$ is of the form

$$\chi(M, Q; g'_1, \cdots, g'_{r-1}, m+1).$$

This proves 2.2.

The next lemma follows from the method of Whitney [18] of removing isolated intersection points. The paper of A. Shapiro [11] makes this apparent (apply 6.7, 6.10, 7.1 of [11]).

LEMMA 2.3. *Suppose N^{n-m} is a closed submanifold of the closed manifold X^n and $f\colon M^m \to X$ is an imbedding of a closed manifold. Suppose also that M, N are connected, X is simply connected, $n - m > 2$, $m > 2$ and $b = f(M) \circ N$ is the intersection number of $f(M)$ and N. Then there exists an imbedding $f'\colon M \to X$ isotopic to f such that $f'(M)$ intersects N in b points, each with transversal intersection.*

LEMMA 2.4. *Let F_0^{n-m-1} be a submanifold of Q where Q is a component of the boundary of a compact manifold V^n, $n - m > 2$. Let $W = \chi(V, Q; g; m+1)$ where $g\colon \partial D_0^{m+1} \times D_0^{n-m-1} \to Q$ is an imbedding with b the intersection number $g(\partial D_0^{m+1} \times 0) \circ F_0$. For an imbedding $h\colon S^m \to Q \cap \partial W$, there is an imbedding $h'\colon S^m \to Q \cap \partial W$, isotopic to h in ∂W with $h'(S^m) \circ F_0 = h(S^m) \circ F_0 \pm b$, sign prescribed.*

Proof. Let D be the closed upper hemisphere of S^m, $\chi_0 \in \partial D_0^{n-m-1}$ and H^+, H^- be the closed upper, lower hemisphere respectively of $g(\partial D_0^{m+1} \times \chi_0)$. Then h is isotopic in $\partial W \cap Q$ to an imbedding $\bar{h}'\colon S^m \to \partial W \cap Q$, with $\bar{h}'(S^m) \cap g(\partial D_0^{m+1} \times \chi_0)$ equal H^+ with the orientation determined by the $\pm b$ of 2.4. This follows essentially from R. Palais, Extending Diffeomorphism, Proc. AMS, vol. 11 (1960), pp. 274-277, Theorem B, Corollary 1.

Next let $\bar{h}$ be $\bar{h}'$ followed by the reflection map $H^+ \to H^-$, so that $\bar{h}$, $\bar{h}'$: $D \to \partial W$ are naturally topologically isotopic. However $\bar{h}'$ is an angle on ∂D. By the familiar process of "straightening the angle" we modify $\bar{h}'$: $S^m \to \partial W \cap Q$ to an embedding h': $S^m \to \partial W \cap Q$. Our construction makes it clear that h' and h are isotopic in ∂W, and that h' has the desired property of 2.4.

We now prove 2.1. Let F be as in 2.2 and b_i be the algebrais intersection number $g_i(\partial D^{m+1} \times 0) \circ \partial F$, $i = 1, \cdots, r$. We first note that the b_i are relatively prime. This in fact follows from the homology hypothesis of the theorem.

The proof proceeds by induction on $\sum_{i=1}^{r} | b_i |$ and is started by 2.3 and 2.2. Suppose 2.1 is true in case $\sum_{i=1}^{r} | b_i |$ is $p - 1 > 0$.

We can say from the homotopy structure of W that $H_m(W, M)$ is $H_m(V, M)$ wtih the added relations $[\partial D_i^{m+1}] = 0$, $i = 1, \cdots, r$, where $[\partial D_i^{m+1}] \subset H_m(V, M) = Z$ and $H_m(V, M)$ is generated by $(D_0^m, \partial D_0^m)$.

Since $H_m(W, M) = 0$, $[\partial D_i^{m+1}]$ are relatively prime. On the other hand, since $D_0^m \times 0 \circ F = 1$, we have that $[\partial D_i^{m+1}] = b_i$. So the b_i, $i = 1, \cdots, r$ are relatively prime.

Since the b_i are relatively prime, there exists, i_0, i_1, $i_0 \neq i_1$ with $| b_{i_0} | \geqq | b_{i_1} | > 0$. One now applies 2.4 to reduce $| b_{i_0} |$ by $| b_{i_1} |$ using the covering homotopy property as in Section 2 of GPC. The induction hypothesis applies and we have proved 2.1.

LEMMA 2.5. *Let $n \geqq 2m + 1$, $(n, m) \neq (4, 1)$, (3.1), (5.2), (7.3), M^n be a compact manifold with a simply connected boundary component Q and $V = \chi(M, Q; f; m)$ where f: $\partial D^m \times D^{n-m} \to Q$ is a contractible imbedding. Let Q_1 be the component of ∂V corresponding to Q and $W = \chi(V, Q_1; g; m + 1)$ where g: $\partial D_1^{m+1} \times D_1^{n-m} \to Q_1$. Then if the homomorphism $\pi_m(V, M) \to \pi_m(W, M)$ induced by inclusion is zero, W is diffeomorphic to M.*

We use the following for the proof of 2.5.

LEMMA 2.6. *Let Y be a simply connected polyhedron and Z an $(m - 1)$-connected polyhedron. Then $\pi_m(Y \vee Z) = \pi_m(Y) + \pi_m(Z)$.*

This is a standard fact in homotopy theory. For example it follows from [6], V. 3.1 and the relative Hurewicz theorem.

Using 2.6 it follows easily that $\pi_m(Q_1) = \pi_m(Q) + \pi_m(S^m)$.

Then from the homotopy hypothesis it follows that the homotopy class γ of g restricted to $\partial D_1^{m+1} \times 0$ is of the form $a + g_1$ where $a \in \pi_m(Q)$ and g_1 generates $\pi_m(S^m)$.

Since P is contractible, $V = M + H$, where H is an $(n - m)$-cell bundle over S^m, and also $Q_1 = Q + \partial H$. Then let $g'_1\colon \partial D^{m+1} \to Q$ be an imbedding representing a and $g'_2\colon \partial D^{m+1} \to \partial H$ an imbedding intersecting ∂F transversally in a single point where F is the same as n 2.2. Then by the sum construction we obtain $g'\colon \partial D^{m+1} \times 0 \to Q_1$ realizing γ with the property that $g'(\partial D_1{}^{m+1} \times 0)$ intersects ∂F transversally in a single point where F is the same as in 2.2. Application of 2.4 of GPC and 2.2 finishes the proof.

Section 3. Among other things, we apply the theory of Section 2 to obtain 1.4.

THEOREM 3.1. *Let W^n be a manifold (not necessarily compact), $n > 5$, with ∂W the disjoint union of simply-connected manifolds M_1 and M_2 where the inclusion $M_i \to W$ are homotopy equivalences. Suppose $j\colon V_0 \to M_1$ is the inclusion of a compact manifold V_0 into M which is a homotopy equivalence and there is an imbedding $\alpha\colon Cl(M_1 - V_0) \times [1,2] \to W$ such that* a) *the complement of the image of α has compact closure and* b)

$$\alpha(Cl(M_1 - V_j) \times n) \subset M_i, \qquad i = 1, 2,$$

α restricted to $Cl(M_1 - V_0) \times 1$ is j. Then α can be extended to a diffeomorphism $M_1 \times [1,2] \to W$.

Proof of 3.1. *Let* $I_0 = [-\tfrac{1}{2}, n + \tfrac{1}{2}]$ and replace $[1,2]$ in the statement of 3.1 by I_0, denoting the projection $Cl(M_1 - V_0) \subset I_0 \to I_0$ by f_0. We may suppose that points under α have been identified so that $Cl(M_1 - V_0) \times I_0 \subset W$. Then by the results of [12] one can find a non-degenerate C^∞ real function f on W such that a) f restricted to $Cl(M_1 - V_0) \times I_0$ is f_0, b) at a critical point the value of f is the index and c) $f(M_1) = -\tfrac{1}{2}$, $f(M) = n + \tfrac{1}{2}$.

Let $X_p = f^{-1}[-\tfrac{1}{2}, p + \tfrac{1}{2}]$. We will show inductively that by suitable modifications of f which also satisfy a), b) and c), we can assume X_p is a product $M_1 \times I$ (or equivalently the modified f has no critical points of index $\leqq p$).

First by 5.1 of GPC, note that we may assume that the function f has no critical points of index 0. Next by the method in Section 7 of GPC, using the fact that $\pi_1(M_1) = \pi_1(W) = 1$, we can similarly assume that there are no critical points of f of index 1.

We are not quite yet in the dimension range where 2.1 applies, but we apply 2.5 to eliminate a critical point of f of index 2 if it occurs, as follows.

We have that $X_2 = \chi(X_1, Q_1; f_1, \cdots, f_k; 2)$, $X_3 = \chi(X_2, Q_2; g_1, \cdots, g_r; 3)$ where $Q_1 = f^{-1}(1\tfrac{1}{2})$, $Q_2 = f^{-1}(2\tfrac{1}{2})$. It follows from the homotopy hypothesis that each f_i is contractible in Q_1 so that X_2 is of the form $X_1 + H$, $H \in \mathcal{H}(n, k, 2)$ (following notation of GPC).

The g_i's induce a homomorphism $G_r \rightarrow \pi_2(Q_2)$. Let ϕ be the composition

$$G_r \rightarrow \pi_2(Q_1) \rightarrow \pi_2(X_2) \rightarrow \pi_2(H)$$

where the last homomorphism it obtained by identifying X_1 to a point in X_2.

Assertion. ϕ is an epimorphism.

Suppose the assertion is false and $\alpha \in \pi_2(H)$ is not in the image of ϕ. Then since $\pi_2(X_2) = \pi_2(X_1) + \pi_2(H)$ (by 2.6), the image of α under $\pi_2(H) \rightarrow \pi_2(X_2) \rightarrow \pi_2(X_3)$ is not in the image of $\pi_2(X_1) \rightarrow \pi_2(X_2) \rightarrow \pi_2(X_3)$. But the last composition is an isomorphism since $X_1 = M_1 \times I$, thus contradicting the existence of such an α. Hence the assertion is true.

Let $\gamma_1, \cdots, \gamma_k$ be the generators of $\pi_2(H)$ corresponding to $f_1, \cdots, f_k$. Then by 4.1 of GPC there is an automorphism β of G_r such that $\phi\beta(g_i) = \gamma_i$, $i \leqq k$, $\phi\beta(g_i) = 0$, $i > k$. By 2.1 of GPC it can be assumed that the g_i are such that $\phi(g_i) = \gamma_i$, $i \leqq k$, $\phi(g_i) = 0$, $i < k$.

Now apply 2.5 with W, V, M corresponding to

$$\chi(X_2, Q_2; g_k), \chi(X_1, Q_1; f_1, \cdots, f_k), \chi(X_1, Q_1; f_1, \cdots, f_{k-1})$$

respectively. This eliminates the critical point of f corresponding to f_k and by induction all the critical points of index two are eliminated.

Applying some of the previous considerations to $n - f$ we eliminate the critical points of f of index n, $n - 1$.

Now more generally suppose f on X_{p-1} has no critical points where $p \leqq n - 3$. Then since $H_p(X_{p+1}, X_p) = 0$, 2.1 applies to eliminate the critical points of index p. Thus we obtain by induction a function f on W with critical points only of index $n - 2$, and which satisfies the conditions a)-c) above. By 7.5 of GPC, f has no critical points at all. This proves 3.1.

COROLLARY 3.2. *Suppose W^n is compact, $n > 5$, ∂W the disjoint union of closed manifolds M_1, M_2 with each $M_i \rightarrow W$ a homotopy equivalence. Suppose also $V \subset W$ with*

$$\partial V = V_1 \cup V_2,\ V_i \subset M_i,\ V = V \times I \text{ and } \pi_1(M_i - V_i) = 1.$$

Then $i\colon \rightarrow W$ can be extended to a diffeomorphism of $M_1 \times I$ onto W.

Proof of 3.2. First i may be extended to $T \times I$ where T is a tubular neighborhood of V_1 in M_1. Then apply 3.1 to $W - V$ to get 3.2.

We now can prove 1.4. First by 3.2 with V empty applied to V of 1.4 yields that V is diffeomorphic to $V_1 \times I$. Now 3.2 applies to yield 1.4.

Section 4. The following is quit a general theorem and in fact contains 1.1 as a special case with $k = n - 1$.

THEOREM 4.1. *Suppose $W^n \supset M^k$ where W is a compact connected manifold and M is a closed manifold. Furthermore suppose*

(a) $\pi_1(\partial W) = \pi_1(M) = 1$

(b) $n > 5$

(c) *The inclusion of M into W is a homotopy equivalence. Then W is diffeomorphic to a closed cell bundle over M, in particular to a tubular neighborhood of M in W.*

We need a lemma.

LEMMA 4.2. *Suppose B is a compact connected n-dimensional submanifold of a compact connected manifold V^n with $\partial B \cap \partial V = \emptyset$, $\pi_1(\partial B) = \pi_1(\partial V) = 1$ and $H_*(B) \to H_*(V)$ induced by inclusion is bijective. Then $Q = Cl(V - B)$ has boundary consisting of ∂V, ∂B with the inclusions of ∂V, ∂B into Q homotopy equivalences.*

For the proof of 4.2 we use the following version of the Poincaré Duality Theorem, which follows from the Lefschetz Duality Theorem.

THEOREM 4.3. *Suppose W^n is a compact manifold ∂W the disjoint union of manifolds M_1 and M_2 (possibly either or both empty). Then for all i, $H^i(W, M_1)$ is isomorphic to $H_{n-i}(W, M_2)$.*

To prove 4.2 note $H_i(Q, \partial B) = H_i(V, B) = 0$ and $H^i(Q, \partial B) = H^i(V, B) = 0$ for all i. By 4.3 then $H_i(Q, \partial V) = 0$ for all i also. By the Whitehead theorem we get 4.2.

The proof of 4.1 then goes as follows. We can first suppose that M is disjoint from the boundary of W. Now let T be a tubular neighborhood of M which is also disjoint from ∂W. Now apply 4.2 and 3.2 to $Cl(W - T)$ with V of 3.2 empty. This yields tht $Cl(W - T)$ is diffeomorphic to $\partial T \times I$ and hence W is diffeomorphic to T. We have proved 4.1.

THEOREM 4.4. *Suppose $2n \geqq 3m + 3$ and a compact manifold W^n has the homotopy type of a closed manifold M^m, $n > 5$, with $\pi_1(\partial W) = \pi_1(M) = 1$. Then W is diffeomorphic to a cell-bundle over M.*

Proof. Let $f: M \to W$ be a homotopy equivalence. By Haefliger [1], f is homotopic to an embedding $g: M \to W$. Now 4.1 applies to yield 4.4.

Section 5. We continue with some consequences of 4.1. The next theorem is a strong form of the Generalized Poincaré Conjecture for $n > 5$ and was first proved in [14] except for $n = 7$. This theorem follows from 4.1 by taking M to be a point.

THEOREM 5.1. *Suppose C^n is a compact contractible manifold with $\pi_1(\partial C) = 1$ and $n > 5$. Then C is diffeomorphic to the n-disk D^n.*

For $n = 5$, if one knows in addition that ∂C is diffeomorphic to S^4, then using the theorem of Milnor $\Theta^5 = 0$ and 1.1, one obtains that C is diffeomorphic to D^5.

The following is a weak unknotting theorem in the differentiable case. Haefliger [2] has given an imbedding (differentiable) of S^3 in S^6 which does not bound an imbedded D^4. On the other hand we have:

THEOREM 5.2. *Suppose $S^k \subset S^n$ with $n - k > 2$. Then the closure of the complement of a tubular neighborhood T of S^k in S^n is diffeomorphic to $S^{n-k-1} \times D^{k+1}$.*

The proof of 5.2 is as follows (the case $n \leqq 5$ is essentially contained in Wu Wen Tsun [19]). It is well-known and easy to prove that if $X = Cl(S^n - T)$, X has the homotopy type of S^{n-k-1}. In fact T is diffeomorphic to a cell bundle over S^k and the inclusion of the boundary of a fiber S_0^{n-k-1} into X induces the equivalence. Furthermore the normal bundle of S_0^{n-k-1} in S^n is trivial because S_0^{n-k-1} bounds a disk in S^n. Now 4.1 applies to yield Theorem 5.2.

One can also prove some recent theorems of M. Hirsch [5], replacing his combinatorial arguments by application of the above theorems.

THEOREM 5.3 (Hirsch). *If $f: M_1^n \to M_2^n$ is a homotopy equivalence of simply connected closed manifolds such that the tangent bundle of M_1 is equivalent to the bundle over M_1 induced from the tangent bundle of M_2 by f, then $M_1 \times D^k$ and $M_2 \times D^k$ are diffeomorphic for $k > n$.*

One obtains 5.3 by imbedding M_1 in $M_2 \times D^k$ approximating the homotopy equivalence and applying 4.1. The tangential property of f is used to conclude that a tubular neighborhood of M_1 in $M_2 \times D^k$ is a product neighborhood.

THEOREM 5.4 (Hirsch). *If the homotopy sphere M^n bounds a parallelizable manifold, then $M^n \times D^3$ is diffeomorphic to $S^n \times D^3$.*

One first proves that M^n can be imbedded in S^{n+3} with trivial normal bundle by following Hirch [4] or using "handlebody theory." Then apply the argument in 5.2 to obtain the complement of a tubular neighborhood of M^n is diffeomorphic to $S^2 \times D^{n+1}$. The closure of the complement of $S^2 \times D^{n+1}$ in S^{n+3} is $S^n \times D^3$, thus proving 5.4.

Section 6. The main goal of this section is the following theorem.

THEOREM 6.1. *Let M be a simply connected closed manifold of dimension greater than five. Then on M there is a non-degenerate C^∞ function with the minimal number of critical points consistent with the homology structure.*

One actually obtains such a function with the additional property that at a critical point its value is the index.

6.2. We make more explicit the conclusion of 6.1. Suppose for each $0 \leqq i \leqq n$, $\sigma_{i1}, \cdots, \sigma_{ip(i)}, \tau_{i1}, \cdots, \tau_{iq(i)}$ is a set of generators for a corresponding direct sum decomposition of $H_i(M)$, σ_{ij} free, τ_{ij} of finite order. Then one can obtain the function of 6.1 with type numbers satisfying

$$M_i = p(i) + q(i) + q(i-1).$$

By taking the $q(i)$ minimal, the M_i becomes minimal.

In the case there is no torsion in the homology of M, 6.1 becomes.

THEOREM 6.3. *Let M be a simply connected closed manifold of dimension greater than five with no torsion in the homology of M. Then there is a non-degenerate function on M with type numbers equal the betti numbers of M.*

We start the proof of 6.1 with the following lemma.

LEMMA 6.4. *Let M^n be a simply connected compact manifold, $n > 5$, $n \geqq 2m$. Then there is an n-dimensional simply connected compact manifold X_m such that:*

a) $H_j(X_m) = 0$, $j > m$

b) *There is a "nice" function on X_m, minimal with respect to its homology structure. In other words there is a C^∞ non-degenerate function on X_m, value at a critical point equal the index, equal to $m + \frac{1}{2}$ on ∂X_m, regular in a neighborhood of ∂X_m and the k-th type number M_k is minimal in the sense of 6.2.*

c) *There is an imbedding $i: X_m \rightarrow M^n$ such that*

$$i(\partial X_m) \cap = M = \emptyset, \qquad i_{\#}: H_j(X_m) \rightarrow H_j(X_m)$$

is bijective for $j < m$ and surjective for $j = m$.

Proof. The proof goes by induction on m, starting by taking X_1 to be an n-disk. Suppose, X_{k-1}, $i_0\colon X_{k-1}\to M$ have been constructed satisfying a)-c) with $2k\leqq n$. For convenience we identify points under i_0 so that $X_{k-1}\subset M$. We now construct X_k, $i\colon X_k\to M$ satisfying a)-c).

By the relative Hurewicz theorem the Hurewicz homomorphism

$$h\colon \pi_k(M,X_{k-1})\to H_k(M,X_{k-1})$$

is bijective.

For the structure of $H_k(M,X_{k-1})$ consider the exact sequence

$$0\to H_k(M)\to H_k(M,X_{k-1})\to H_{k-1}(X_{k-1})\xrightarrow{j} H_{k-1}(M)\to 0.$$

Let $\gamma_1,\cdots,\gamma_p$ be a set of generators of $H_k(M,X_{k-1})$ corresponding to a minimal set of generators of $H_k(M)$ together with a minimal set for $\operatorname{Ker} j$.

Represent the elements $h^{-1}(\gamma_1)\cdots h^{-1}(\gamma_p)$ by imbeddings

$$\bar g_i\colon (D^k,\partial D^k)\to (Cl(M-X_{k-1},\partial X_{k-1})$$

with $\bar g_i(D^k)$ transversal to ∂X_{k-1} along $\bar g_i(\partial D^k)$, for example following Wall [17], proof of Theorem 1.

In the extreme case $n=2k$, the images of $\bar g_i$ generically intersect each other in isolated points. These points can be removed by pushing them along arcs past the boundaries. Still following [17], the $\bar g_i$ can be extended to tubular neighborhoods,

$$g_i\colon (D^k,\partial D^k)\times D^{n-k}\to (Cl(M-X_{k-1}),\partial X_{k-1}).$$

Then we take X_k to be $\chi(X_{k-1};g'_1,\cdots,g'_p;k)$ where $g'_i\colon \partial D^k\times D^{n-k}\to \partial D_{k-1}$ is the restriction of g_i. It is not difficult to check that X_k has the desired properties a)-c). This proves 6.4.

To prove 6.1, let M^n be as in 6.1 with $n=2m$ or $2m+1$. Let $X_m\subset M$ as in 6.4, f the nice function on X_m and $K=Cl(M-X_m)$. Then $H_i(M,X)=0$, $i\leqq m$, so by duality $H^j(K)=0$, $j\geqq n-m$. By the Universal Coefficient Theorem this implies that $H_{n-m-1}(K)$ is torsion free. Let $Y_{n-m-1}\subset K$ be again given by 6.4 with g the nice function on Y_{n-m-1}. By 4.2 and 3.2 we can in fact assume that K and Y_{n-m-1} are the same, so $M=X_m\cup Y_{n-m-1}$. Let f_0 be the function which is f on X_m and $n-g$ on Y_{n-m-1}. By smoothing f_0 along ∂X_m we obtain a C^∞ function f'. It is not difficult using the Universal Coefficient Theorem and Poincaré Duality to show that f' may be taken as the desired function of 6.1.

The previous results of this section may be extended to manifolds with boundary.

By the previous methods one may prove the following generalization of 6.1. We leave the details to the reader.

THEOREM 6.5. *Suppose W^n is a simply connected manifold with simply-connected boundary, $n > 5$. Then there is a nice function f on W (non-degenerate, value $n + \frac{1}{2}$ on ∂W, regular in a neighborhood of ∂W, value at a critical point is the index) with type numbers minimal with respect to the homology structure of $(W, \partial W)$.*

Section 7. The goal of this section is to prove the following.

THEOREM 7.1. *Let $f\colon W_1{}^n \to W_2{}^n$ be a homotopy equivalence between two manifolds such that the tangent bundle T_1 of W_1 is equivalent to $f^{-1}T_2$. Suppose also $n > 5$, $n \geqq 2m + 1$, $H^i(W_1) = 0$, $i > m$, $\pi_1(W_1) = \pi_1(\partial W_1) = \pi_1(\partial W_2) = 1$. Then W_1 and W_2 are diffeomorphic by a diffeomorphism homotopic to f.*

Let g be a nice function on W_1 with no critical points of index greater than m, whose existence is implied by 6.5. Thei we let $X_k = g^{-1}[0, k + \frac{1}{2}]$, $k = 0, 1, \cdots, m$ with $X_m = W_1$. By 3.2 and 4.2 is is sufficient to imbed X_m in W_2 by a map homotopic to f.

Suppose inductively we have defined a map $f_{k-1}\colon X_k \to W_2$ homotopic to f with the property f_{k-1} is an imbedding, $k \geqq m$. Let X_k be written in the form $\chi(X_{k-1}; g_1, \cdots, g_p; k)$ where $g_i\colon \partial D^k \times D^{n-k} \to \partial X_{k-1}$. Using the Whitney imbedding theory we can find $f'_{k-1}\colon X_k \to W_2$ homotopic to f_{k-1}, which is an imbedding on X_{k-1} and on the images $g_i(D^k \times 0)$ in X_k as well. It remains to make f'_{k-1} an imbedding on a tubular neighborhood of each of the $g_i(D^k \times 0)$, or equivalently on each of the $g_i(D^k \times D^{n-k})$.

This can be done for a given i if and only if an element γ_i in $\pi_{k-1}(0(n-k))$ defined by f'_{k-1} in a neighborhood of $g_i(\partial D^k \times 0)$ is zero. But the original tangential assumptions on f insure $\gamma_i = 0$ in this dimension range. The arguments in proving these statements are so close to the arguments in Hirsch [3] Section 5, that we omit them. This finishes the proof of 7.1.

Section 8. We note here the following theorem.

THEOREM 8.1. *Let M^{2m+1} be a closed simply connected manifold, $m > 2$, with $H_m(M)$ torsion free. Then there is a compact manifold W^{2m+1}, uniquely determined by M and a diffeomorphism $h\colon \partial W \to \partial W$ such that M is union of two copies of W with points identified under h.*

Proof. Let $W_1^{2m+1} \subset M$ be the manifold given by 6.4. Let $W_2^{2m+1} \subset M - W$ be also given by 6.4. Then it is not difficult using homotopy theory to show that W_1, W_2 satisfy the hypotheses of 7.1. Also by previous arguments W_2 is diffeomorphic to $Cl(M - W_1)$. The uniqueness of $W_1 = W_2$ is also given by 7.1. Putting these facts together we get 8.1.

Remark. I don't believe the condition on $H_m(M)$ is really necessary here. Also in a different spirit, 8.1 is true for the cases $m = 1$, $m = 2$.

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