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Infinitely many ribbon knots with the same fundamental group

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1. *Introduction*

We work in the DIFF category. A knot $K = (S^{n+2}, S^n)$ is a *ribbon knot* if S^n bounds an immersed disc $D^{n+1} \to S^{n+2}$ with no triple points and such that the components of the singular set are n-discs whose boundary $(n-1)$-spheres either lie on S^n or are disjoint from S^n. Pushing D^{n+1} into D^{n+3} produces a *ribbon disc pair* $D = (D^{n+3}, D^{n+1})$, with the ribbon knot (S^{n+2}, S^n) on its boundary. The double of a ribbon $(n+1)$-disc pair is an $(n+1)$-ribbon knot. Every $(n+1)$-ribbon knot is obtained in this manner.

The *exterior* of a knot (disc pair) is the closure of the complement of a tubular neighbourhood of S^n in S^{n+2} (of D^{n+1} in D^{n+3}). By the usual abuse of language, we will call the homotopy type invariants of the exterior the homotopy type invariants of the knot (disc pair). We study the question of how well the fundamental group of the exterior of a ribbon knot (disc pair) determines the knot (disc pair).

L. R. Hitt and D. W. Sumners [**14**], [**15**] construct arbitrarily many examples of distinct disc pairs (D^{n+2}, D^n) with the same exterior for $n \geqslant 5$, and three examples for $n = 4$. S. P. Plotnick [**24**] gives infinitely many examples for $n \geqslant 3$. For $n = 3$, his proof requires Freedman's solution of the four-dimensional Poincaré conjecture, so he only gets results in TOP. We prove:

THEOREM 1·1. *There exist infinitely many distinct ribbon disc pairs* (D^{n+2}, D^n), $n \geqslant 3$, *with the same exterior.*

A nice feature of these disc knots is that π_1 is the trefoil knot group. The difference comes from the fact that their meridians are not equivalent under any automorphism of π_1.

In [**9**], C. McA. Gordon gives three examples of knots in S^4 with isomorphic π_1 but different π_2 (viewed as $\mathbb{Z}\pi_1$-modules). Plotnick [**23**] generalizes this to arbitrarily many knots. In [**24**], he produces infinitely many examples in the TOP category. Analysing the boundaries of the discs provided by Theorem 1·1 for $n = 3$, we prove:

THEOREM 1·2. *There exist infinitely many ribbon knots in* S^4 *with fundamental group the trefoil knot group, but with non-isomorphic* π_2 (*as* $\mathbb{Z}\pi_1$-*modules*).

The exteriors of these knots are fibred over S^1, with the same fibre

$$S^1 \times S^2 \# S^1 \times S^2 - \mathring{D}^3$$

as that of the spun trefoil, but with monodromy suitably modified. As π_2 of the fibre is not generated by the boundary 2-sphere, we are unable to use the techniques in [**9**], [**24**] to distinguish among $\mathbb{Z}\pi_1$-module structures on π_2. Accordingly, we give a presentation of π_2 coming from a surgery description of the knots, and reduce the problem to a question about 2×2 matrices.

This paper is organized as follows. In § 2 we discuss several definitions of ribbon discs and knots. § 3 gives a method for computing π_2 of a ribbon 2-knot. In § 4 we construct our examples and prove Theorem 1·1. § 5 contains the proof of Theorem 1·2. Finally, in § 6, we derive a few consequences and make some comments.

Notation. All R-modules are left-modules. An element $u \in R$ induces the R-module map $u: R \to R$ via right multiplication. Vectors in R^n are row vectors and matrices with entries in R act on the right.

2. *Ribbon discs and knots*

Ribbon n-knots were first defined by Fox [7], for $n = 1$, and Yajima [28], for $n = 2$. A ribbon knot (S^{n+2}, S^n), $n \geqslant 2$, is the double of a ribbon disc pair (D^{n+2}, D^n). It is determined by its equatorial cross section $S^{n-1} = \partial D^n$, which consists of disjoint spheres $S_0^{n-1}, \ldots, S_m^{n-1}$ (with meridians x_i), joined together by m bands running from S_0^{n-1} to S_i^{n-1} $(1 \leqslant i \leqslant m)$. Fig. 1 shows the 'motion picture' of a ribbon disc pair.

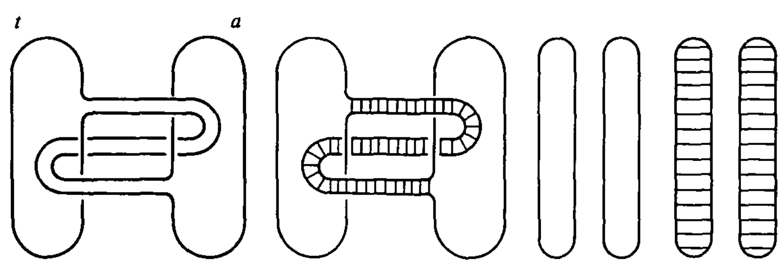


Fig. 1

The fundamental group of the ribbon knot can be computed from the motion picture (see [7], [29]):

$$\pi_1(S^{n+2} - S^n) = \pi_1(D^{n+2} - D^n)$$
$$= (x_0, x_1, \ldots, x_m \mid x_0 = w_i x_i w_i^{-1},\ 1 \leqslant i \leqslant m),$$

where w_i is the word in $x_0, \ldots, x_m$ that records the way the ith band links the S_j^{n-1}'s homotopically. We call such a presentation of π_1 a 'ribbon presentation'. For example, the ribbon 2-knot with cross section the stevedore knot (Fig. 1) has

$$\pi_1 = (t, a \mid t = (at^{-1})\, a (at^{-1})^{-1}) = (t, x \mid txt^{-1} = x^2)$$

(compare ([7], p. 136)). The spun trefoil, with equatorial section the square knot (Fig. 2), has $\pi_1 = (t, x \mid t = (xt)\, x (xt)^{-1}) = (t, x \mid txt = xtx)$.

Here is another construction of ribbon discs and knots, first described in [16]. Start with (D^{n+3}, D^{n+1}), the standard disc pair, with meridian t. Add 1-handles h_i^1 $(1 \leqslant i \leqslant m)$ to D^{n+3}, with core circles x_i, and 2-handles h_i^2 along curves r_i in $\partial(D^{n+3} \cup \{h_i^1\}) - \partial D^{n+1}$, with r_i isotopic in $\partial(D^{n+3} \cup \{h_i^1\})$ to a circle which intersects the cocore of h_i^1 in a single point and is disjoint from the cocore of h_j^1, $j \neq i$.

By the handle cancelling theorem, $D^{n+3} \cup \{h_i^1\} \cup \{h_i^2\} = D^{n+3}$ and we get a new disc pair (D^{n+3}, D^{n+1}), with $\pi_1(D^{n+3} - D^{n+1}) = (t, x_1, \ldots, x_m \mid r_1, \ldots, r_m)$. The procedure is illustrated in Fig. 3.

THEOREM 2·1 ([4], [13]). *A disc pair is ribbon if and only if it can be obtained from the standard disc pair by adding* 1- *and* 2-*handles in the above manner.*

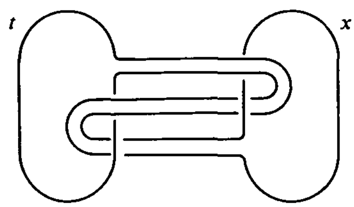


Fig. 2

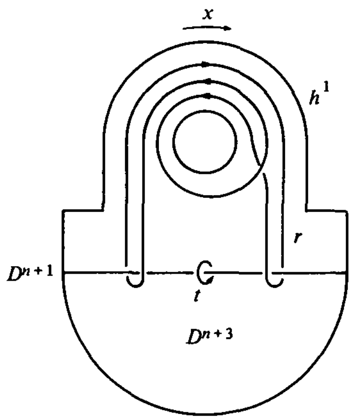


Fig. 3

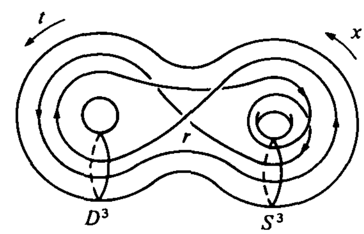


Fig. 4

In practice, one passes from the above presentation of π_1 to a ribbon presentation through Andrews–Curtis moves [1], and then draws the ribbon knot prescribed by this presentation. An example of the procedure is given in §4. The exterior of the ribbon knot $(S^{n+2}, S^n) = \partial(D^{n+3}, D^{n+1})$ is obtained from $S^1 \times D^{n+1} \# (\#_1^m S^1 \times S^{n+1})$ by performing surgery on the curves r_i. For example, the ribbon 2-knot with cross section the stevedore knot is the boundary of the disc pair in Fig. 3, and can be constructed by surgery on $r = txt^{-1}x^{-2}$ in $S^1 \times D^3 \# S^1 \times S^3$ (Fig. 4).

3. π_2 *of a ribbon* 2-*knot*

This section gives a method for calculating π_2 of a ribbon 2-knot as a $\mathbb{Z}\pi_1$-module. This method, briefly sketched in [25], yields explicit results for one-relator ribbon knots and spun knots. Let X be the exterior of a one-relator ribbon knot. It is obtained by surgery on a simple closed curve r in $S^1 \times D^3 \# S^1 \times S^3$, where $\pi_1(S^1 \times D^3) = \mathbb{Z}(t)$, $\pi_1(S^1 \times S^3) = \mathbb{Z}(x)$, and $r(t,x)$ has exponent sum ± 1 in x. We write $\pi = \pi_1 X = (t, x | r)$. As the exponent sum of x is ± 1, the relation r is not a proper power. Hence, by Lyndon's theorem [21], π is torsion-free. In case x has finite order, $\pi = \mathbb{Z}(t)$ and Milnor duality in the universal cover shows the knot to be homotopically trivial (see, for instance, [4]).

Assume x has infinite order in π. Let M be the cover of $S^1 \times D^3 \# S^1 \times S^3$ corresponding to the kernel of $\mathbb{Z} * \mathbb{Z} \to (\pi = \mathbb{Z} * \mathbb{Z}/\langle r \rangle)$. If we perform equivariant surgery on the lifts of r in M, we get $\tilde{X}$, the universal cover of X. M consists of copies of $\mathbb{R} \times D^3$, indexed by the cosets $\pi/\mathbb{Z}(t)$, and copies of $\mathbb{R} \times S^3$, indexed by the cosets $\pi/\mathbb{Z}(x)$, tubed together by 'connectors' $S^3 \times I$, indexed by π. Fig. 5 depicts the cover, together with three lifts of the surgery curve $r = txt^{-1}x^{-2}$. The lifts of the 'fibre' S^3, $gS^3 = S^3_g$, are indexed by π. The lifts of r are indexed by their basepoints $g \in \pi$.

Let
$$M = M_0 \underset{\coprod_\pi S^1 \times S^2}{\cup} (\coprod_\pi S^1 \times D^3),$$
$$\tilde{X} = M_0 \underset{\coprod_\pi S^1 \times S^2}{\cup} (\coprod_\pi D^2 \times S^2).$$

The Mayer–Vietoris sequences corresponding to these decompositions yield

$$0 \to \oplus H_3(S^1 \times S^2) \to H_3(M_0) \to H_3(M) \xrightarrow{\phi} \oplus H_2(S^1 \times S^2)$$
$$\to H_2(M_0) \to H_2(M) \to 0 \to H_1(M_0) \xrightarrow{\cong} H_1(M) \to 0$$

and

$$0 \to \oplus H_3(S^1 \times S^2) \to H_3(M_0) \to H_3(\tilde{X}) \to \oplus H_2(S^1 \times S^2)$$
$$\to \oplus H_2(D^2 \times S^2) \oplus H_2(M_0) \to H_2(\tilde{X}) \to \oplus H_1(S^1 \times S^2) \xrightarrow{\psi} H_1(M_0) \to 0.$$

Notice that $H_2(M) = 0$ and $H_3(M) = \mathbb{Z}\pi$, generated by the lifts of S^3. These sequences simplify to give

$$H_3(\tilde{X}) = \ker(\mathbb{Z}\pi \xrightarrow{\phi} \mathbb{Z}\pi), \tag{1}$$

$$0 \to \operatorname{coker}\phi \to H_2(\tilde{X}) \to \ker\psi \to 0. \tag{2}$$

Let $X_r = e^0 \cup e^1_t \cup e^1_x \cup e^2_r$ be the 2-complex associated to the presentation $\pi = (t, x \mid r)$. The reduced chain complex of its universal cover is (see ([5], pp. 45–46))

$$\mathbb{Z}\pi \xrightarrow{\partial_2 = \left(\frac{\partial r}{\partial t}\ \frac{\partial r}{\partial x}\right)} \mathbb{Z}\pi \oplus \mathbb{Z}\pi \xrightarrow{\partial_1 = \binom{t-1}{x-1}} \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{3}$$

where ∂_2 is the matrix of Fox derivatives. By Lyndon's theorem [**21**], X_r is aspherical, that is, $\partial_2^{X_r} = (\partial r/\partial t\ \partial r/\partial x)$ is a monomorphism.

To compute ϕ, note first that the 'fibre' S^3 is a dual cycle to x. Hence, the algebraic sum of the lifts of S^3 cut by the lift of r at 1 equals $(\partial r/\partial x).S^3$. Therefore $\phi(S^3_1)$, which is the algebraic sum of the lifts of r which intersect S^3_1, equals $\overline{\partial r/\partial x}$, where

$$\overline{\Sigma n_g g} = \Sigma n_g g^{-1}.$$

That is to say, $\phi = \overline{\partial r/\partial x}$: $\mathbb{Z}\pi \to \mathbb{Z}\pi$. For example, if $r = txt^{-1}x^{-2}$, $\phi(1) = t^{-1} - x^{-1} - 1$, which can be seen directly in Fig. 5.

LEMMA 3·1. *Let $g \in G$ be an element of infinite order in a group G. Then $\mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G$ is a monomorphism.*

Proof. Suppose $(\Sigma n_h h).(g-1) = 0$. Then $n_{hg^{-1}} - n_h = 0$, and so

$$n_h = n_{hg^{-1}} = n_{hg^{-2}} = \dots,$$

an infinite sequence of equalities. Hence, $n_h = 0$. ∎

The exact sequence (3) gives $\partial r/\partial t.(t-1) + \partial r/\partial x.(x-1) = 0$. From the lemma and the injectivity of $(\partial r/\partial t\ \partial r/\partial x)$ we deduce that $\partial r/\partial x$ is injective. Hence, ϕ is a monomorphism, and $H_3(\tilde{X}) = 0$ (that is to say, the knot is *quasi-aspherical* [**20**]).

Lyndon's theorem also shows that the relation module $H_1(M_0)$ is freely generated by the lifts of r, so that ψ: $\mathbb{Z}\pi \to \mathbb{Z}\pi$ is an isomorphism. Hence $\ker\psi = 0$, and (1) and (2) combine to give the exact sequence

$$0 \longrightarrow \mathbb{Z}\pi \xrightarrow{\overline{\partial r/\partial x}} \mathbb{Z}\pi \longrightarrow \pi_2 X \longrightarrow 0.$$

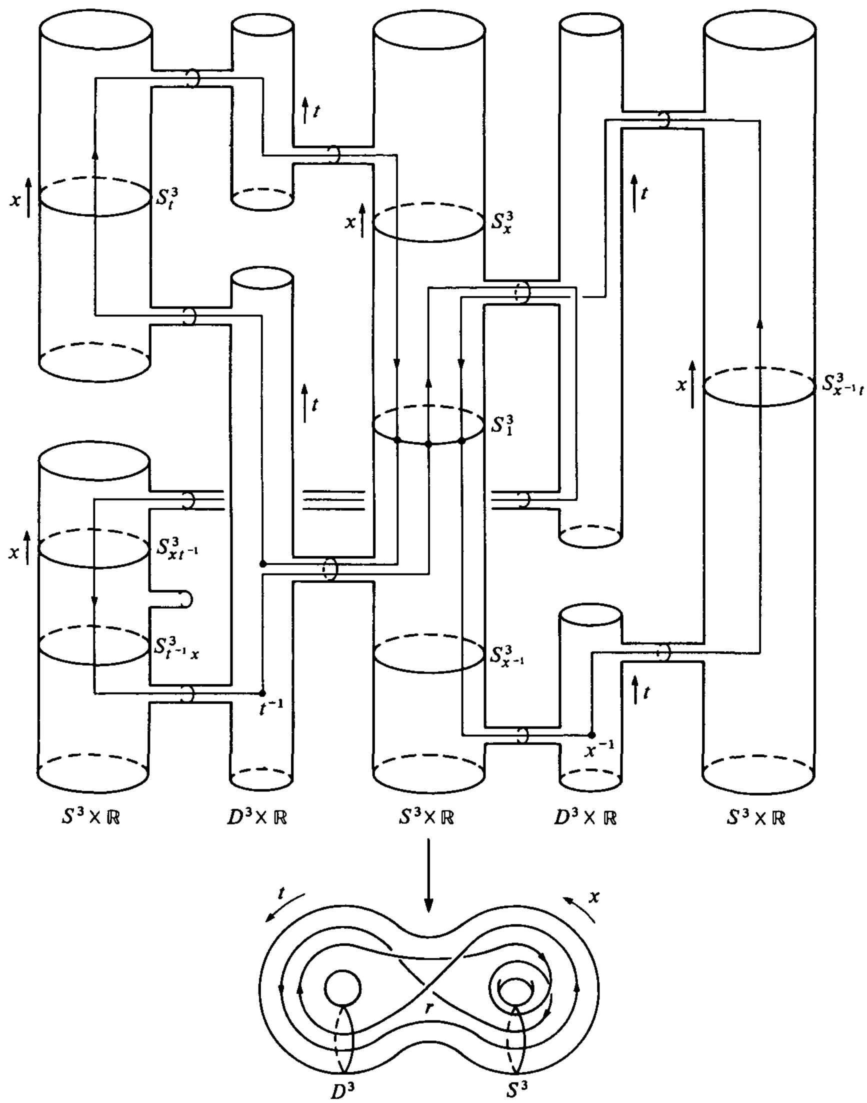


Fig. 5

PROPOSITION 3·2. *One relator ribbon 2-knots are quasi-aspherical, with* $\pi_2 = \mathbb{Z}\pi/\overline{(\partial r/\partial x)}$, *where* $\pi_1 = (t, x|r)$. ∎

For example, the knot in Fig. 4, with $\pi_1 = (t, x|txt^{-1} = x^2)$, has

$$\pi_2 = \mathbb{Z}\pi/(1+x^{-1}-t^{-1}),$$

whereas the spun trefoil (Fig. 2), with $\pi_1 = (t, x|txt = xtx)$, has

$$\pi_2 = \mathbb{Z}\pi/(1+t^{-1}x^{-1}-t^{-1}).$$

These calculations check the ones in ([**20**], appendix B).

4. *Meridians and ribbon discs*

In this section we produce the examples for Theorem 1·2. The $(n-2)$-spun trefoil, $n \geqslant 3$, is a fibred knot with fibre $(S^1 \times S^{n-1} \# S^1 \times S^{n-1}) - \mathring{D}^n$ (see [3]). If u and v generate π_1 of the fibre, the monodromy σ is given by $\sigma(u) = v$, $\sigma(v) = u^{-1}v$. This knot bounds a fibred ribbon disc pair $D_0 = (D^{n+2}, D^n)$, with fibre $V^{n+1} = S^1 \times D^n \natural S^1 \times D^n$ and monodromy σ. The exterior $V \times_\sigma S^1$ has meridian t and π_1 the trefoil knot group $(t, u, v | tut^{-1} = v, tvt^{-1} = u^{-1}v)$.

We now construct other disc pairs D_k, with the same exterior, but different meridians. Add a 2-handle h^2 to $V \times_\sigma S^1$ along a simple closed curve representing $t_k = u^k t$, with either framing. Since t_k is homologous to t in $V \times_\sigma S^1$, the Mayer–Vietoris sequence shows that the resulting manifold $\mathscr{D}^{n+2}$ is acyclic. Its fundamental group is Andrews–Curtis equivalent to the trivial group:

$$\begin{aligned}\pi_1(\mathscr{D}^{n+2}) &= (t, u, v | tut^{-1} = v,\ tvt^{-1} = u^{-1}v,\ u^k t = 1)\\ &= (u, v | u^{-k}uu^k = v, u^{-k}vu^k = u^{-1}v)\\ &= (v | v^{-k}vv^k = v^{-1}v)\\ &= (v | v = 1).\end{aligned}$$

By a standard argument [1], $\mathscr{D}^{n+2}$ is diffeomorphic to D^{n+2}.

Then $(D^{n+2}$, cocore of 2-handle) is a knotted disc pair $D_k = (D^{n+2}, D^n)$ with exterior $V \times_\sigma S^1$, and meridian t_k. The fundamental group is

$$\pi_1 = \pi_1(V \times_\sigma S^1) = (t, u, v | tut^{-1} = v,\ tvt^{-1} = u^{-1}v),$$

which is Andrews–Curtis equivalent to:

$$\begin{aligned}&(t, u, v, t_k | tut^{-1} = v,\ tvt^{-1} = u^{-1}v,\ t_k = u^k t)\\ &\qquad = (t_k, u, v | u^{-k}t_k u t_k^{-1} u^k = v, u^{-k}t_k v t_k^{-1} u^k = u^{-1}v)\\ &\qquad = (t_k, u | u^{-k}t_k u^{-k} t_k u t_k^{-1} u^k t_k^{-1} u^k = u^{-1}u^{-k} t_k u t_k^{-1} u^k)\\ &\qquad = (t_k, u | t_k^{-1} u t_k u^{-k} t_k u t_k^{-1} u^{k-1} = 1).\end{aligned}$$

Hence $V \times_\sigma S^1$ has a handle decomposition $h^0 \cup h^1_{t_k} \cup h^1_u \cup h^2_r$, with h^2_r attached along a simple closed curve representing $r = r(t_k, u) = t_k^{-1} u t_k u^{-k} t_k u t_k^{-1} u^{k-1}$ with the property $r(1, u) = u$. Now

$$D^{n+2} = V \times_\sigma S^1 \cup h^2 = (h^0 \cup h^1_{t_k} \cup h^2) \cup h^1_u \cup h^2_r = D_0^{n+2} \cup h^1_u \cup h^2_r,$$

with D^n = cocore h^2 = standard n-disc in $D_0^{n+2} = h^0 \cup h^1_{t_k} \cup h^2$ and r isotopic to u in $D_0^{n+2} \cup h^1_u$. By Theorem 2·1, D_k is a ribbon disc pair.

The pairs D_0 and D_1 are equivalent, since the conjugation map μ_v: $V \to V$ extends to a diffeomorphism of $V \times_\sigma S^1$ taking t to t_1. The boundary of $D_0 = (D^5, D^3)$ is the spun trefoil (Fig. 2). In order to picture the other disc pairs, we give here a ribbon presentation of π_1.

$$\begin{aligned}\pi_1 &= (t_k, u | t_k^{-1} u t_k u^{-k} t_k u^k u^{-k+1} t_k^{-1} u^{k-1} = 1)\\ &= (t_k, u, c, d | t_k^{-1} u t_k c d^{-1} = 1, c = u^{-k} t_k u^k, d = u^{-k+1} t_k u^{k-1})\\ &= (t_k, c, d | d = t_k (dc^{-1})^{-k+1} . t_k . (dc^{-1})^{k-1} t_k^{-1}, c = t_k c d^{-1} t_k^{-1} . d . t_k d c^{-1} t_k^{-1}).\end{aligned}$$

Fig. 6 depicts the boundary of $D_2 = (D^5, D^3)$ – a ribbon knot in S^4 with its equatorial cross-section drawn.

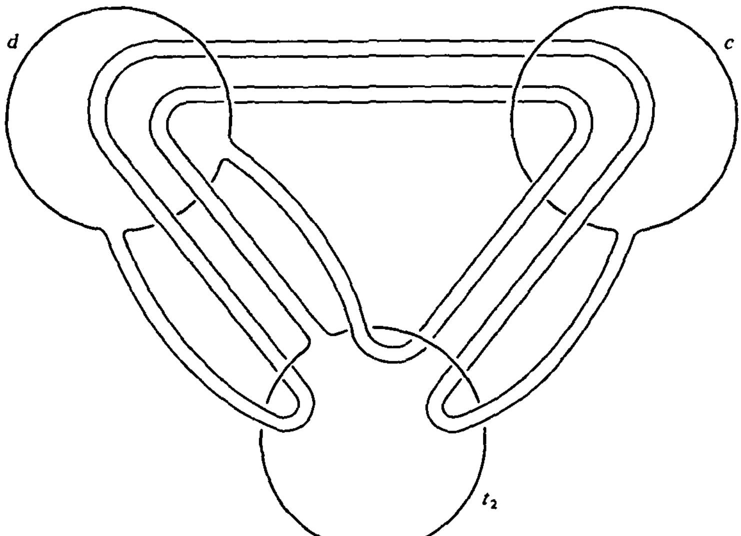


Fig. 6

The ribbon disc pairs $D_k = (D^{n+2}, D^n), k \geqslant 1$, have the same exterior $V \times_\sigma S^1$. To prove Theorem 1·1, we have to show that they are all distinct. A diffeomorphism of pairs $D_k \to D_l$ restricts to a diffeomorphism of $V \times_\sigma S^1$ preserving meridians, thus taking t_k to $t_l^{\pm 1}$. It induces an automorphism of $\pi_1 = \pi_1(V \times_\sigma S^1)$ taking t_k to $t_l^{\pm 1}$. Rewriting π_1 as:

$$\pi_1 = (t, u, v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v) \underset{\substack{u = t^{-1}x \\ v = xt^{-1}}}{=} (t, x \mid txt = xtx) \underset{\substack{t = b^{-1}a \\ x = ab^{-1}}}{=} (a, b \mid a^2 = b^3),$$

gives $t_k = u^k t = (a^{-1}bab^{-1})^k b^{-1}a$. It is well known that $\pi/Z(\pi) \cong PSL(2, \mathbb{Z})$, under the isomorphism $a \mapsto A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b \mapsto B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. The centre being characteristic, we are left with proving:

LEMMA 4·1. *Let $T_k = (A^{-1}BAB^{-1})^k B^{-1}A \in PSL(2, \mathbb{Z})$. There is no automorphism of $PSL(2, \mathbb{Z})$ taking T_k to $T_l^{\pm 1}$ for $k, l \geqslant 1$, $k \neq l$.*

Proof. We compute

$$A^{-1}BAB^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad (A^{-1}BAB^{-1})^k = \begin{pmatrix} a_{2k} & -a_{2k-1} \\ -a_{2k-1} & a_{2k-2} \end{pmatrix},$$

where $a_{-3} = -1$, $a_{-2} = 1$, $a_{-1} = 0$, $a_0 = 1$, $a_k = a_{k-1} + a_{k-2}$ are the Fibonacci numbers. Therefore

$$T_k = \begin{pmatrix} a_{2k} & a_{2k-2} \\ -a_{2k-1} & -a_{2k-3} \end{pmatrix} \quad \text{and} \quad \operatorname{tr}(T_k^{\pm 1}) = a_{2k} - a_{2k-3} = 2a_{2k-2}.$$

An automorphism of $PSL(2, \mathbb{Z})$ has the form $A \to HAH^{-1}$, $B \to HB^{\pm 1}H^{-1}$ (O. Schreier ([**26**], Hilfssatz 3)). As $A \to A$, $B \to B^{-1}$ is given by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we are done. ∎

5. *Ribbon knots with different* π_2

In the previous section we produced ribbon disc pairs $D_k = (D^5, D^3)$. The boundary of D_k is a ribbon knot $K_k = (S^4, S^2)$. We show in this section that the knots K_k provide the examples for Theorem 1·2. The exterior X_k is obtained from

$$(S^1\times S^2 \# S^1\times S^2)\times_\sigma S^1$$

by deleting a neighbourhood of the curve $t_k = u^k t$. Actually, X_k is fibred over S^1, with fibre $S^1\times S^2 \# S^1\times S^2 - \mathring{D}^3$ and monodromy $\sigma_k = \mu_{u^k}\sigma$. As explained in [24], we 'untwist' the deleted curve, thereby 'twisting' the monodromy. The fundamental group is

$$\begin{aligned}\pi_1 = \pi_1 X_k &= (u, v, t_k \mid t_k u t_k^{-1} = u^k v u^{-k},\ t_k v t_k^{-1} = u^{k-1} v u^{-k})\\ &= (t_k, u \mid r_k),\end{aligned}$$

where $r_k = u t_k u^{-k} t_k u t_k^{-1} u^{k-1} t_k^{-1}$. We saw that

$$\pi_1 X_k \cong \pi = (t, u, v \mid tut^{-1} = v,\ tvt^{-1} = u^{-1}v),$$

the trefoil knot group.

Proposition 3·2 gives the following presentation for $\pi_2 X_k$, $k \geqslant 1$:

$$0 \to \mathbb{Z}\pi \xrightarrow{w_k} \mathbb{Z}\pi \to \pi_2 X_k \to 0,$$

where

$$\begin{aligned} w_k = \overline{\frac{\partial r_k}{\partial u}} &= \overline{1 - u t_k(u^{-1} + \ldots + u^{-k}) + u t_k u^{-k} t_k + t_k u^{-k+1}(1 + \ldots + u^{k-2})}\\ &= 1 - (u + \ldots + u^k)\, t_k^{-1} u^{-1} + t_k^{-1} u^k t_k^{-1} u^{-1} + (u + \ldots + u^{k-1})\, t_k^{-1}\\ &= u[u^k - (1 + \ldots + u^{k-1})\, t^{-1} + u^{-1} t^{-2} + (1 + \ldots + u^{k-2})\, t^{-1} u]\, u^{-k-1}.\end{aligned}$$

(It is understood that $1 + \ldots + u^{k-2} = u + \ldots + u^{k-1} = 0$ when $k = 1$.)

Remark. Given a knot group π, the abelianization map $\gamma\colon \pi \to \mathbb{Z}$ induces a ring homomorphism $\gamma\colon \mathbb{Z}\pi \to \mathbb{Z}\mathbb{Z}$, which takes the Jacobian matrix of Fox derivatives to the Alexander matrix. In our situation, $\gamma(w_k) = 1 - t^{-1} + t^{-2}$, the Alexander polynomial of the trefoil knot. By Levine duality [18], the knots K_k have the same Alexander invariants. We thus have to look at non-abelian invariants in order to distinguish among our knots.

We have the following result, which proves Theorem 1·2:

LEMMA 5·1. *Let* $\alpha\colon \pi_1 X_k \to \pi_1 X_l$ *be an isomorphism,* $k, l \geqslant 1, k \neq l$. *There is no* α*-isomorphism* $\beta\colon \pi_2 X_k \to \pi_2 X_l$.

Proof. An automorphism of $\pi = (a, b \mid a^2 = b^3)$ has the form $a \to h a^\epsilon h^{-1}$, $b \to h b^\epsilon h^{-1}$, where $\epsilon = \pm 1$ and $h \in \pi$ [26]. Therefore, any automorphism inducing the identity on $\pi/\pi' = \mathbb{Z}$ is an inner automorphism.

LEMMA 5·2. *There is a diffeomorphism* $F\colon X_k \to X_k$ *inducing* -1 *on* $H_1(X_k; \mathbb{Z}) = \mathbb{Z}$.

Proof. We define $f \in \operatorname{Aut}(\mathbb{Z} * \mathbb{Z})$ via

$$f\colon \begin{cases} u \to v u^{-1}\\ v \to u v u^{-1}.\end{cases}$$

We check that $f = \sigma_k f \sigma_k$:

$$\begin{aligned}\sigma_k f\sigma_k(u) &= \sigma_k f(u^k v u^{-k}) = \sigma_k((vu^{-1})^k\, uvu^{-1}(uv^{-1})^k)\\ &= \sigma_k((vu^{-1})^{k-1}\, v(uv^{-1})^{k-1}) = u^{-k+1}u^{k-1}vu^{-k}u^{k-1} = vu^{-1}\\ \sigma_k f\sigma_k(v) &= \sigma_k f(u^{k-1}vu^{-k}) = \sigma_k((vu^{-1})^{k-1}\, uvu^{-1}(uv^{-1})^k)\\ &= \sigma_k((vu^{-1})^{k-2}\, v(uv^{-1})^{k-1}) = u^{-k+2}u^{k-1}vu^{-k}u^{k-1} = uvu^{-1}.\end{aligned}$$

f can be realized by a diffeomorphism of $S^1 \times D^3 \natural S^1 \times D^3$ by handle slides and inversions (see ([**17**], lemma 2)). Up to conjugation, σ_k consists of handle slides and inversions also; hence $f = \sigma_k f \sigma_k$ geometrically. Restrict f to the boundary $S^1 \times S^2 \# S^1 \times S^2$. We can assume that f fixes a ball D^3 and that the relation still holds. The required diffeomorphism is $F(x,t) = (f(x), 1-t)$. ∎

Replacing the given isomorphism by $\alpha \circ F_*$ if need be, we may assume that α induces $+1$ on $\mathbb{Z}$. Then $\alpha = \mu_h$, conjugation by an element $h \in \pi$.

We have the central extension

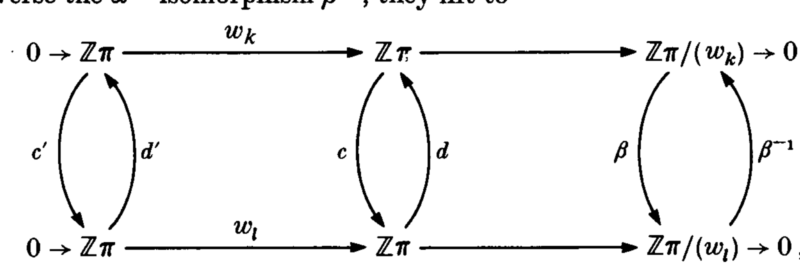
$$\begin{array}{ccccccc} 1 \to \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & SL(2,\mathbb{Z}) \to 1. \\ \| & & \| & & \| \\ (a^4) & & (a,b \mid a^2 = b^3) & & (a,b \mid a^2 = b^3,\ a^4 = 1) \end{array}$$

The automorphism $\alpha = \mu_h$ of π induces $\mu_h \in \mathrm{Aut}\,(SL\,(2,\mathbb{Z}))$, which extends to an automorphism of $\mathbb{Z}(SL(2,\mathbb{Z}))$. Define the ring homomorphism $\Phi\colon \mathbb{Z}(SL(2,\mathbb{Z})) \to \mathcal{M}(2,\mathbb{Z})$ by adding up the matrices in the formal sum. Then $\mu_h \in \mathrm{Aut}\,(\mathbb{Z}(SL(2,\mathbb{Z})))$ extends via Φ to $\mu_h \in \mathrm{Aut}\,(\mathcal{M}(2,\mathbb{Z}))$.

We now turn to studying isomorphisms of π_2. Given an α-isomorphism

$$\beta\colon \pi_2 X_k \to \pi_2 X_l,$$

with inverse the α^{-1}-isomorphism β^{-1}, they lift to



where $c, d, c', d' \in \mathbb{Z}\pi$. From the commutativity of the diagram, we find

$$\begin{cases} \alpha(w_k) \,.\, c = c' \,.\, w_l, \\ \alpha^{-1}(w_l) \,.\, d = d' \,.\, w_k, \\ \alpha(d) \,.\, c = y \,.\, w_l + 1, \\ \alpha^{-1}(c) \,.\, d = z \,.\, w_k + 1, \end{cases}$$

for some $y, z \in \mathbb{Z}\pi$. Projecting these equations to $\mathbb{Z}(SL(2,\mathbb{Z}))$, and then mapping them to $\mathcal{M}(2,\mathbb{Z})$ via Φ, we find

$$\begin{cases} hW_k h^{-1} \,.\, C = C' \,.\, W_l, \\ h^{-1} W_l h \,.\, D = D' \,.\, W_k, \\ hDh^{-1} \,.\, C = Y \,.\, W_l + I, \\ h^{-1} C h \,.\, D = Z \,.\, W_k + I. \end{cases}$$

These equations provide the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}^2 & \xrightarrow{\;W_k\;} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2/(W_k) \to 0 \\
{\scriptstyle h^{-1}C'}\downarrow\uparrow{\scriptstyle hD'} & & {\scriptstyle h^{-1}C}\downarrow\uparrow{\scriptstyle hD} & & {\scriptstyle B}\downarrow\uparrow{\scriptstyle B^{-1}} \\
\mathbb{Z}^2 & \xrightarrow{\;W_l\;} & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2/(W_l) \to 0,
\end{array}$$

showing that

$$\mathbb{Z}^2/(W_k) \cong \mathbb{Z}^2/(W_l). \tag{*}$$

Now recall that the projection $\pi \to SL(2,\mathbb{Z})$ takes $t = b^{-1}a$ to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u^k = (a^{-1}bab^{-1})^k$ to

$$U^k = \begin{pmatrix} a_{2k} & -a_{2k-1} \\ -a_{2k-1} & a_{2k-2} \end{pmatrix}.$$

Hence

$$\begin{aligned}
U^{-1}W_k U^{k+1} &= U^{k-1}(U-T^{-1}) - (I+U+\ldots+U^{k-2})T^{-1}(I-U) + U^{-1}T^{-2} \\
&= \begin{pmatrix} a_{2k-2} & -a_{2k-3} \\ -a_{2k-3} & a_{2k-4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} a_{2k-3} & 1-a_{2k-4} \\ 1-a_{2k-4} & 1+a_{2k-5} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2a_{2k-1} & -1-a_{2k-3} \\ 2-2a_{2k-2} & -1+a_{2k-4} \end{pmatrix},
\end{aligned}$$

which gives

$$\begin{aligned}
\det W_k &= 2[a_{2k-1}(a_{2k-4}-1) + 1 - a_{2k-2} + a_{2k-3} - a_{2k-2}a_{2k-3}] \\
&= 2[(a_{2k-1}-1)(a_{2k-4}-1) - a_{2k-2}a_{2k-3}] \\
&= 2[(a_{2k-2}+a_{2k-3}-1)(a_{2k-2}-a_{2k-3}-1) - (a_{2k-1}-a_{2k-3})a_{2k-3}] \\
&= 2(a_{2k-2}^2 - 2a_{2k-2} + 1 - a_{2k-1}a_{2k-3}) \\
&= 4(1-a_{2k-2}),
\end{aligned}$$

where we used

$$\begin{aligned}
a_k^2 - a_{k+1}a_{k-1} &= a_k^2 - (a_k + a_{k-1})a_{k-1} = a_k(a_k - a_{k-1}) - a_{k-1}^2 \\
&= -(a_{k-1}^2 - a_k a_{k-2}) = \begin{cases} +1, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

This contradicts (*), thus proving Lemma 5·1. ∎

6. *Comments*

We now derive some consequences of the computations done in the previous sections. The exterior of a ribbon disc is homotopy equivalent to the 2-complex associated to the presentation of its fundamental group. Hence, by Lyndon's theorem,

COROLLARY 6·1 ([**12**], p. 169). *One-relator ribbon disc exteriors are aspherical.* ∎

COROLLARY 6·2. *Let X_1 and X_2 be the exteriors of one relator ribbon 2-knots. If $\pi_1 X_1 \cong \pi_1 X_2$ and $\pi_2 X_1 \cong \pi_2 X_2$ (as $\mathbb{Z}\pi_1$-modules), then $X_1 \simeq X_2$.*

Proof. As π_1 has a 2-dimensional $K(\pi_1, 1)$, $H^3(\pi_1, \pi_2) = 0$ and the first k-invariant vanishes. Since X_i are quasi-aspherical, a theorem of Lomonaco [20] implies $X_1 \simeq X_2$. ∎

It is claimed in [4] that the conclusion of Corollary 6·1 is valid for arbitrary ribbon discs. The proof rests on a proposition erroneously attributed to Lomonaco, which amounts to showing $\ker \psi = 0$, for an arbitrary ribbon 2-knot. The asphericity of ribbon discs is implied by the Whitehead Conjecture (see ([12], Conjecture 6·5)).

Given a knot $K = (S^3, S^1)$, the *n-spin* of K, $n \geqslant 1$, is the $(n+1)$-knot

$$\sigma_n(K) = \partial(\bar{K} \times D^{n+1}),$$

where $\bar{K} = K - \text{standard}\,(\mathring{D}^3, \mathring{D}^1)$. For $n = 1$, we get the usual spin of K. Let X be the exterior of K, and $\pi = (t, x_1, \ldots, x_m | r_1, \ldots, r_m)$ be a Wirtinger presentation of $\pi_1 X$. The spin of K is a ribbon 2-knot with exterior X_1 obtained from $S^1 \times D^3 \# (\#_1^m S^1 \times S^3)$ by surgery on the curves r_i. With notation as before, we compute

$$\phi = \overline{(\partial r_i/\partial x_j)}\colon (\mathbb{Z}\pi)^m \to (\mathbb{Z}\pi)^m.$$

X is an aspherical 2-complex [22], with $\partial_2^{\tilde{X}} = (\partial r_i/\partial t \;\; \partial r_i/\partial x_j)$. As in the proof of Proposition 3·2, ϕ is a monomorphism. Thus $H_3(\tilde{X}_1) = 0$, as a result of Gordon ([9], theorem 4·1). It also follows that the map $(\partial r_i/\partial x_j)\colon H_1(M_0) \to (\mathbb{Z}\pi)^m$ ([5], pp. 43–46), is a $\mathbb{Z}\pi$-isomorphism. Therefore ψ is an isomorphism. We have proved:

Proposition 6·3. *Spun 2-knots are quasi-aspherical, with* $\pi_2 = (\mathbb{Z}\pi)^m/\overline{(\partial r_i/\partial x_j)}$, *where* $\pi_1 = (t, x_1, \ldots, x_m | r_1, \ldots, r_m)$. ∎

This complements Andrews' and Lomonaco's computation $\pi_2 = (\mathbb{Z}\pi)^m/(\partial r_i/\partial x_j)^t$ [2], [19].

The asphericity of classical knots [22] implies that n-spun knots with isomorphic π_1 have homotopy equivalent exteriors. It seems reasonable to conjecture that they are actually equivalent. This is supported by

Proposition 6·4. *Let* K_1 *and* K_2 *be knots in* S^3 *with* $\pi_1 X_1 \cong \pi_1 X_2$. *Assume* K_i *are not* (p, q)*-cables*, $|p| \leqslant 2$, *of a non-trivial knot. Then* $\sigma_n(K_1) = \sigma_n(K_2)$.

Proof. Results of Johannson, Feustel, Whitten, Burde and Zieschang (see [11], pp. 9–10), imply that either (i) K_i are prime knots, with $X_1 = X_2$ or (ii) K_i are composite knots, with the prime factors equal, up to orientations. In case (i), $\sigma_n(X_1) = \sigma_n(X_2)$, and by Gluck [8], for $n = 1$ and Cappell [6], for $n > 1$, $\sigma_n(K_1) = \sigma_n(K_2)$. In case (ii), the argument in Gordon [10] yields the equivalence of $\sigma_n(K_i)$. ∎

Combining this with Theorem 1·2, we get

Corollary 6·5. *There are infinitely many distinct knots in* S^4 *which are not spun but have the fundamental group of the spun trefoil.* ∎

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