Surgery of non-simply-connected manifolds

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The method of surgery, initiated by Milnor [12] in 1959, has proved a powerful tool in differential topology (see, for example, [10], [14]). The key idea can be expressed as follows. Let M^m be a smooth (or piecewise linear) manifold, $f: M \rightarrow X$ a map of M to any CW-complex X. Then we wish to change (M, f), by performing a series of spherical modifications [25] to M, to make f a homotopy equivalence. In the first instance, X was taken as a point.

We shall regard this problem with manifolds as analogous to the same problem for CW-complexes. This latter problem is trivial; our proof of this (Lemma 1.1) gives a pattern of attack on the case of manifolds. Now a spherical modification on an $(r-1)$-sphere of M^m has an effect similar to attaching simultaneously two cells e^r and e^{m-r-1}. For $2r \leqq m$, our CW-analogue is satisfactory; however, a detailed investigation of duality is necessary before we can pursue our programme in the middle dimensions. We feel, however, that although the algebraic problems raised are of considerable difficulty, the method of setting up the algebra gives a clearer insight into the problems raised than previous expositions of the topic.

Our exposition of the technique of surgery is given in this paper; applications are reserved for a planned second part. Our approach is general enough to cover essentially all published applications of surgery other than in codimension one (which seems to be somewhat easier). The paper is divided as follows:

§ 1 Surgery below the middle dimension.
§ 2 Duality and maps of degree 1.
§ 3 Surgery in the middle dimension: case $m = 2k$.
§ 4 The algebraic problem.
§ 5 Surgery in the middle dimension: case $m = 2k - 1$.
§ 6 The case $m = 2k - 1$ and π of order 2.
§ 7 Relative surgery.

The first section sets the scene for the main part of the paper, as well as obtaining a strong result on surgery below the middle dimension, which already has useful applications. The second defines Poincaré complexes, and introduces the K notation which is much used later. In § 3 the geometric problem of surgery is reduced (when $m = 2k$) to a purely algebraic problem. This is discussed in § 4, and solved when the fundamental group is cyclic of prime order. The treatment of the case $m = 2k - 1$ is much less satisfactory, and the author now feels that the linking numbers here used should be replaced by a different algebraic setting altogether. Publication of the present account is perhaps justified by the solution in § 6 of all the algebraic problems when the fundamental group is cyclic of order 2 (this permits an application to diffeomorphism classification of homotopy projective spaces, for example). The paper closes with a brief description of surgery on manifolds with boundary.

1. Surgery below the middle dimension

Our surgery will follow the following pattern. We have a map $\psi: M \to X$, and we perform surgery on M to make the map ψ highly-connected. Recall that the relative group $\pi_{r+1}(\psi)$ is a set of equivalence classes of pairs (f, g) of maps which form a commutative diagram

$$\begin{array}{ccc} S^r & \subset & D^{r+1} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{\psi} & X \end{array}$$

and that there is an exact sequence

$$\cdots \pi_r(M) \xrightarrow{\psi_*} \pi_r(X) \longrightarrow \pi_r(\psi) \longrightarrow \pi_{r-1}(M) \cdots .$$

First of all we describe a CW-analogue of surgery, and prove a lemma which contains all the necessary homotopy theory.

Suppose given an element α of $\pi_{r+1}(\psi)$. Let (f, g) be a representative pair of maps, as above. Form M' from M by attaching an $(r + 1)$-cell with attaching map f. Let $F: D^{r+1} \to M'$ be the characteristic map of the cell, so that $F \mid S^r = f$. Then we extend ψ to a map $\psi': M' \to X$ by setting $\psi' \mid F(D^{r+1}) = g \circ F^{-1}$: on $M \cap F(D^{r+1}) = F(S^r)$, this is $g \circ f^{-1} = \psi$, so we have a well-defined, continuous map. The natural inclusion of ψ in ψ' takes α to zero, since both triangles commute in the representative diagram

$$\begin{array}{ccc} S^r & \subset & D^{r+1} \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle F} & \downarrow{\scriptstyle g} \\ M' & \xrightarrow{\psi'} & X . \end{array}$$

We describe this process as *attaching an* $(r + 1)$*-cell to* M *by* α, or *to kill* α.

LEMMA 1.1. *Suppose* M *and* X *finite* CW*-complexes,* $\psi: M \to X$ *a map. Then we can attach to* M *a finite number of cells of dimension* $\leq k$ *to make* ψ k*-connected.*

REMARK. This result was obtained in [23], using a lemma of Whitehead. We now give a direct and simple proof. Note that it is enough for the k-skeleton of X to be finite.

PROOF. We may suppose that ψ is a cellular map (if necessary, alter ψ by a homotopy). Replace X by the mapping cylinder of ψ (which is homotopy equivalent to it), then ψ is homotopic to an inclusion. So we may suppose M a subcomplex of X.

Now all we have to do is to attach to M in turn (in order of increasing dimension) the cells of $X - M$ of dimension $\leqq k$, at each stage extending ψ by the inclusion map. We end with a subcomplex M' of X, which contains the k-skeleton, and the inclusion ψ' of M' in X. Then ψ' is k-connected, for any map $f: (D^i, S^{i-1}) \to (X, M')$ is homotopic to a cellular map which, if $i \leqq k$, has image in M'.

Observe that although for the proof it was convenient to use different models for X and ψ, the cell attachment was made on the given M.

In order to apply this result, we need a construction which, given $\alpha \in \pi_{i+1}(\varphi)$, attaches an $(i+1)$-handle to a manifold corresponding to the cell attachment above. We discuss first the absolute case. If $f: S^i \times D^{m-i} \to \partial V^{m+1}$ is an imbedding, and we form a manifold from $V^{m+1} \cup (D^{i+1} \times D^{m-i})$ by identifying $S^i \times D^{m-i}$ by f, the result is said to be obtained from V by attaching an $(i+1)$-handle. If the components of ∂V are partitioned, so that $\partial V = \partial_- V \cup \partial_+ V$ is a splitting into disjoint open sets, or more generally, if $\partial_- V$ and $\partial_+ V$ are m-manifolds which meet only along their common boundary, we call V a cobordism. We write τ_V for the tangent bundle of V.

LEMMA 1.2. *Suppose V^{m+1} a cobordism, obtained from $\partial_- V$ by attaching i-handles with $i \leqq m - r$. Let $\psi_V: V \to X$, $\omega: X \to B(O_{m+1})$ be such that $\omega \circ \psi_V$ is a classifying map for τ_V. Let $\alpha \in \pi_{r+1}(\psi_V)$. Then if $m \geqq 2r+1$, we can attach a handle along $\partial_+ V$ by α, giving V' and $\psi_{V'}: V' \to X$, so that $\omega \circ \psi_{V'}$ classifies $\tau_{V'}$.*

LEMMA 1.3. *With the notations of 1.2, but $V \cong \partial_- V \times I$, $m = 2r$, $r \geqq 2$, α determines a regular homotopy class of immersions $S^r \to \partial_+ V$, in the homotopy class $\partial_* \alpha$, and the construction is possible if and only if this class contains an imbedding.*

PROOF OF 1.2. Set $N = \partial_+ V$. Then V is obtained from $N \times I$ by attaching handles of dimension $\geqq (m+1) - (m-r) = r+1$, so $\pi_r(N)$ maps onto $\pi_r(V)$. Hence we can represent $\partial_* \alpha \in \pi_r(V)$ by a map $\bar{f}: S^r \to N$ which, since $\dim N = m \geqq 2r+1$, may be supposed a smooth imbedding. Let $(\bar{f}, \bar{g})$ represent α. Then $\omega \circ \bar{g}$ induces the trivial bundle over D^{r+1}, and its restriction to S^r gives a framing of $\bar{f}^*\tau_V$.

Let ε denote the field of inward unit normals to N in V. Then we identify $\bar{f}^*\varepsilon$ with the field η of unit outward normals to S^r in D^{r+1}, and so $\bar{f}^*(\tau_{\bar{f}(S^r)} + \varepsilon)$ with $\tau_{S^r} + \eta$, the bundle of vectors tangent to D^{r+1}, which has a standard framing. This defines a field of $(r+1)$-frames in $\bar{f}^*\tau_V$, which is framed, and hence an element of $\pi_r(V_{m+1,r+1})$. But since $m \geqq 2r+1$, this group vanishes. So we can extend our $(r+1)$-frame to a framing of $\bar{f}^*\tau_V$, homotopy equivalent to the given framing. The remaining $(m-r)$ vector fields frame the normal bundle of $\bar{f}(S^r)$ in N.

Extend $\bar{f}$ to a tubular neighbourhood $f\colon S^r \times D^{m-r} \to N$ using this framing, and take f as the attaching map of a handle $D^{r+1} \times D^{m-r}$. Extend ψ_V over $D^{r+1} \times 0$ by $\bar{g}$, and over $D^{r+1} \times D^{m-r}$ using a retraction on $D^{r+1} \times 0 \cup S^r \times D^{m-r}$. It remains only to check that $\omega \circ \psi_{V'}$ classifies $\tau_{V'}$, and this is known on V. But we chose the normal framing of $\bar{f}(S^r)$ in N precisely so that the induced framing of $S^r \times D^{m-r}$ was given on $S^r \times 0$ by $\omega \circ \bar{g}$. The result now follows.

PROOF OF 1.3. There are just two points at which the proof of 1.2 breaks down when $m = 2r$: first, where we say that $\bar{f}$ may be supposed a smooth imbedding; and second, the group $\pi_r(V_{m+1,r+1})$ no longer vanishes. We shall reverse our procedure.

Since $V \cong N \times I$, $\partial_*\alpha \in \pi_r V \cong \pi_r N$, so $\bar{f}^*\tau_N$ is well determined. As before, α induces a framing of $\bar{f}^*\tau_V$. We wish $\tau_{S^r} + \eta$, with the standard framing, to be imbedded by the first $(r+1)$ fields of the framing of $\bar{f}^*\tau_V = \bar{f}^*\tau_N + \varepsilon$. This determines an imbedding, unique up to homotopy, of τ_{S^r} in $\bar{f}^*\tau_N$ (since the single vector field ε, or η, in $\bar{f}^*\tau_V$ is unique up to homotopy, and the homotopy between two such is also unique up to homotopy if $r \geqq 2$). By a theorem of Hirsch [9], this determines a regular homotopy class of immersions $S^r \to N$, in the homotopy class $\partial_*\alpha$. It is now clear that, to proceed with the construction above, to attach a handle, we need an imbedding in this regular homotopy class.

REMARK. Similarly if $m = 2r+1$, $r \geqq 2$, we can perform surgery using any imbedding in a well-defined regular homotopy class.

THEOREM 1.4. *Suppose M^m compact, X a CW-complex with X^k finite, $\psi_M\colon M^m \to X$ and $\omega\colon X \to B(O_{m+1})$ such that $\omega \circ \psi_M$ classifies $\tau_M + \varepsilon$, and $m \geqq 2k$. Then we can attach to $M \times I$ a finite set of handles of dimensions $\leqq k$, forming W, and extend ψ_M to a k-connected map $\psi_W\colon W \to X$ such that $\omega \circ \psi_W$ classifies τ_W. If $N = \partial_+ W$, $\psi_N = \psi_W \mid N$, then ψ_N is also k-connected.*

PROOF. Lemma 1.1 shows how cells can be attached to W to make ψ_W

k-connected, and Lemma 1.2 states that each r-cell attachment may be represented by a handle attachment (with the additional condition about the tangent bundle), provided $m \geqq 2r + 1$. This proves all but the last clause. Now W is obtained from N by attaching handles of dimensions $\geqq (m + 1) - k \geqq k + 1$, so the inclusion of N in W is k-connected. Since ψ_W is also k-connected, so is the composite ψ_N.

REMARK. This result can also be extended to the non-compact case, provided that X^k is locally finite and ψ_M proper.

We now consider the relative case, when M^m lies in a containing manifold L^{m+c}. We need to be able to represent any $\alpha \in \pi_{r+1}(\psi_M)$ by a handle attached inside $L \times I$, and so by an element of $\pi_{r+1}(L, M)$, so we must relate X to L. We also have to keep a disc D^{r+1} disjoint from M in L.

LEMMA 1.5. *Suppose given maps* $\psi_M: M^m \to X$, $\upsilon: X \to L^{m+c}$, $\omega: X \to B(O_c)$ *and a smooth imbedding* $\iota: M^m \to L^{m+c}$, *such that* $\upsilon \circ \psi_M \simeq \iota$, *and* $\omega \circ \psi_M$ *classifies* ν, *the normal bundle of* $\iota(M)$. *Let* $\alpha \in \pi_{r+1}(\psi_M)$, $2r + 1 \leqq m$, $r \leqq c - 2$. *Then we can do surgery to* M *in* L *to kill* α, *and preserve all side conditions. If* $2r = m \geqq 4$, α *determines a regular homotopy class of immersions* $S^r \to M$; *we can do surgery if and only if this contains an imbedding.*

PROOF. First suppose $m \geqq 2r + 1$. Then we can represent $\partial_*\alpha$ by an imbedding $\bar{f}: S^r \to M$. Represent α by a pair $(\bar{f}, \bar{g})$; then (as before), $\omega \circ \bar{g}$ induces a framing of $\bar{f}^*\nu$. Use the first vector of this framing (e.g., *via* the exponential map) to extend $\bar{f}$ to an imbedding $\bar{h}_0: S^r \times I \to L$, disjoint from M except at $\bar{h}_0(S^r \times 0) = \bar{f}(S^r)$. Now represent $\upsilon_*(\alpha) \in \pi_{r+1}(\omega \circ \psi_M)$ by a pair $(\bar{f}, \bar{h})$, e.g., $\bar{h} = \upsilon \circ \bar{g}$, and alter $\bar{h}$ by a homotopy so that

(i) in a collar neighborhood of the boundary, it agrees with $\bar{h}_0$,

(ii) it is a smooth imbedding, possible since L has dimension $m + c \geqq 2(r + 1) + 1$,

(iii) it is disjoint from M except at the boundary, possible since in a collar neighborhood of the boundary, it already has this property, and $m + c > m + (r + 1)$.

Now the normal bundle of $\bar{h}(D^{r+1})$ is trivial, and may be supposed framed. The last $(c - 1)$ vectors of the framing of $\bar{f}^*\nu$ induce a $(c - 1)$-frame on the boundary of this, and hence define an element of $\pi_r(V_{m+c-r-1,c-1})$, which vanishes since $m \geqq 2r + 1$. Hence we can identify them with the first $(c - 1)$ vectors of the normal frame: the others give an $(m - r)$-field over $\bar{h}(D^{r+1})$ which gives on the boundary a framing of the normal bundle of $\bar{f}(S^r)$ in M.

If $m = 2r$, we proceed as in Lemma 1.3. The homotopy class of $\bar{f}$ is well-defined and $\omega \circ \bar{g}$, $\upsilon \circ \bar{g}$ induce framings of $\bar{f}^*\nu$, $\bar{f}^*\tau_L$. Write ε for the first vector of $\bar{f}^*\nu$, identify ε witn $-\eta$, and deduce a framing of $\tau_{S^r} + \varepsilon$. So we have a framing of $\tau_{S^r} + \bar{f}^*\nu$, which we wish to identify with part of the framing of $\bar{f}^*\tau_L = \bar{f}^*\tau_M + \bar{f}^*\nu$. Now 'peel off' in turn the vector fields which frame $\bar{f}^*\nu$, and we see we have defined an imbedding of τ_{S^r} in $\bar{f}^*\tau_M$, and hence a regular homotopy class of immersions $S^r \to M$. If this contains an imbedding, we recover the situation above; again we can imbed D^{r+1}, as $\dim L = 2r + c \geqq 3r + 2 \geqq 2r + 3$ (except in the trivial case $r = 0$, $c = 2$).

Now use the $(m - r)$-frame to extend $\bar{h}$ to an imbedding $h: D^{r+1} \times D^{m-r} \to L$, where $f = h \mid S^r \times D^{m-r}$ has range M. Then our cobordism $W \subset L \times I$ can be described as obtained from $(M \times I) \cup (D^{r+1} \times D^{m-r} \times 1)$ by 'pushing down' $D^{r+1} \times \operatorname{Int}(D^{m-r}) \times 1$; or perhaps better, from $(M - \operatorname{Im} f) \times I$ by attaching a 'saddle' in $D^{r+1} \times D^{m-r} \times I$, containing $S^r \times D^{m-r} \times 0$ and $D^{r+1} \times S^{m-r-1} \times 1$, and then (again) rounding corners. (For details, cf. [8, p. 460]).

Then $\bar{g}$ defines an extension $\psi_W: W \to X$ of ψ_M, and since $\bar{h}$ was homotopic (rel S^r) to $\upsilon \circ \bar{g}$, $\upsilon \circ \psi_W$ is homotopic to the imbedding of W in $L \times I$. Also, $\omega \circ \psi_W$ classifies the normal bundle of W; again, as this holds for M, we need only check the agreement of the two induced framings on the boundary of the attached cell. But our construction ensured that these agreed (note that when we introduce angles, the first vector of $\bar{f}^*\nu$ is deformed to coincidence with the vertical vector in $L \times I$). This completes the proof.

THEOREM 1.6. *Suppose given maps $\psi_M: M^m \to X$, $\upsilon: X \to L^{m+c}$, $\omega: X \to B(O_c)$, and a smooth imbedding $\iota: M^m \to L^{m+c}$ such that $\upsilon \circ \psi_M \simeq \iota$, and $\omega \circ \psi_M$ classifies the normal bundle of $\iota(M)$. Suppose M^m compact, X^k finite, $m \geqq 2k$, and $c \geqq k + 1$. Then there is a manifold $W \subset L \times I$, formed from M by attaching a finite set of handles of dimensions $\leqq k$, and a k-connected extension $\psi_W: W \to X$ of ψ_M, such that $\upsilon \circ \psi_W \simeq$ the inclusion, and $\omega \circ \psi_W$ classifies the normal bundle of W. If $N = \partial_+ W = W \cap (L \times 1)$, $\psi_N = \psi_W \mid N$, then $\psi_N: N \to X$ is also k-connected.*

This follows immediately from 1.1 and 1.5, just as 1.4 followed from 1.1 and 1.2.

REMARK. There are analogues of all the above for PL-manifolds. Details will appear elsewhere.

2. Duality, and maps of degree 1

It is not to be expected that if $M^m (m = 2k$ or $2k + 1)$ is a closed manifold, X a finite CW-complex, and $\psi: M \to X$ a map, we should be able to perform surgery to make the map ψ $(k + 1)$-connected, since duality imposes certain restrictions upon $H_k(M)$. For example, if M is $(k-1)$-connected and $m = 2k$, then $H_k(M)$ is a free abelian group which, if k is even, has even rank. If $m = 2k+1$ and k is even, there must exist a finite abelian group B such that the torsion subgroup of $H_k(M)$ is isomorphic either to $B \oplus B$ or to $B \oplus B \oplus \mathbf{Z}_2$.

However, we do not see any way to proceed with surgery by merely imposing conditions on $H_k(X)$. More natural and useful is to consider the case when X itself satisfies duality; hence we now seek to define this.

Suppose first that M^m is a closed, connected oriented manifold, with fundamental class $[M] \in H_m(M, \mathbf{Z})$. The simplest version of Poincaré duality states that $\frown[M]$: $H^r(M) \cong H_{m-r}(M)$ is an isomorphism for all integers r. We must generalise this to other coefficient groups, possibly twisted as coefficient bundles. Suppose M is (or has the homotopy type of) a finite CW-complex, and write $\tilde{M}$ for its universal cover, Λ for the (integral) group ring of $\pi_1(M)$. If B is a Λ-module, the operations of π on B determine a bundle $\mathfrak{B}$ over M, associated to $\tilde{M} \to M$: $\mathfrak{B}$ is a bundle of abelian groups. Write C_* for the chain complex of $\tilde{M}$. The operation of π on $\tilde{M}$ induces an operation on chain groups under which they are finitely generated free Λ-modules: the cells of M determine a free basis. We write $H_*(M; \mathfrak{B})$ and $H^*(M; \mathfrak{B})$ for the homology of the complexes $C_* \otimes_\Lambda B$ and $\mathrm{Hom}_\Lambda(C_*, B)$; this is equivalent to Steenrod's definition [17] of homology and cohomology with a bundle of coefficients. We shall assume the usual properties of these groups.

Observe that $H_*(M; \Lambda)$ can be identified with $H_*(\tilde{M})$; similarly for cohomology where, if $\tilde{M}$ is non-compact, we use finite cochains (or *compact supports*). Now $\tilde{M}$ is a manifold, so $\frown[\tilde{M}]$: $H^*(\tilde{M}) \to H_*(\tilde{M})$ is an isomorphism[1]. The corresponding map of chains $\mathrm{Hom}_\Lambda(C_*, \Lambda) \to C_*$ (also induced by cap product with a fundamental cycle for M) is a map of free chain complexes of finite rank inducing homology isomorphisms, so by a result of Whitehead [28] is a chain homotopy equivalence. Hence so is the induced map

$$\mathrm{Hom}_\Lambda(C_*, B) = \big(\mathrm{Hom}_\Lambda(C_*, \Lambda)\big) \otimes_\Lambda B \longrightarrow C_* \otimes_\Lambda B\ ,$$

and we have isomorphisms $\frown[M]$: $H^*(M; \mathfrak{B}) \to H_*(M; \mathfrak{B})$. For this argument I am indebted to J. Milnor.

In the non-orientable case, there is an extra complication. If $w^1 \in H^1(M; \mathbf{Z}_2)$ is the first characteristic class, it induces a double covering of M; let $\mathbf{Z}^t$ be the associated bundle with fibre $\mathbf{Z}$, on which $\mathbf{Z}_2$ operates by change of sign. As this is a group automorphism, $\mathbf{Z}^t$ is a bundle of groups; the fundamental class $[M]$ now lies in $H_m(M, \mathbf{Z}^t)$. We write, for any bundle $\mathfrak{B}$ of groups, $\mathfrak{B}^t =$

[1] The fundamental class $[\tilde{M}]$ is represented by an infinite (locally finite) chain; however, cap product with it or with $[M]$ give the same result.

$\mathfrak{B} \otimes \mathbf{Z}^t$. Recalling the ideas of [23], we are led to the following definition.

X is a *Poincaré complex* if it is a CW-complex, dominated by a finite complex, which has a characteristic class $w^1(X) \in H^1(X; \mathbf{Z}_2)$, defining a bundle $\mathbf{Z}^t$ of twisted integer coefficients over X, and a class $[X] \in H_m(X, \mathbf{Z}^t)$ such that cap product with $[X]$ induces an isomorphism $H^*(\tilde{X}) \cong H_*(\tilde{X})$.

It follows from the argument above that for all $\mathfrak{B}$, r we have isomorphisms $\frown[X]$: $H^r(X; \mathfrak{B}) \cong H_{m-r}(X, \mathfrak{B}^t)$. We call $[X]$ the fundamental class. Since $H_m(X; \mathbf{Z}^t) \cong H^0(X; \mathbf{Z})$, a direct sum of copies of the integers, one for each component of X, and since $[X]$ must clearly determine a generator for each component, $[X]$ is determined uniquely up to a sign for each component.

Next we must consider manifolds with boundary. If N^{m+1} is compact, with boundary M^m, then the fundamental class $[N]$ lies in $H_{m+1}(N, M; \mathbf{Z}^t)$ and cap product with it, as above, induces isomorphisms (for all r, $\mathfrak{B}$)

$$H^{r+1}(N; \mathfrak{B}) \cong H_{m-r}(N, M; \mathfrak{B}^t)$$
$$H^{r+1}(N, M; \mathfrak{B}) \cong H_{m-r}(N; \mathfrak{B}^t)\ .$$

We call (Y, X) a *Poincaré pair* if Y is a CW-complex, X a subcomplex (both dominated by finite complexes), with X a Poincaré complex. Also there must be $w^1(Y) \in H^1(Y; \mathbf{Z}_2)$ (with $i^*w^1(Y) = w^1(X)$) defining a coefficient bundle $\mathbf{Z}^t$, and a class $[Y] \in H_{m+1}(Y, X; \mathbf{Z}^t)$ such that we have $\partial_*[Y] = [X]$ and

$$\frown[Y]\colon H^*(\tilde{Y}) \cong H_*(\tilde{Y}, \tilde{X})$$

(where $\tilde{Y}$ is the covering of Y which is the universal covering over each component; $\tilde{X}$ is the induced covering of X).

As above, we must then obtain isomorphisms

$$\frown[Y] : H^{r+1}(Y; \mathfrak{B}) \longrightarrow H_{m-r}(Y, X; \mathfrak{B}^t)$$

for all r, $\mathfrak{B}$. Also, as the following diagram is commutative up to sign,

$$\begin{array}{ccccccccc}
\cdots H^r(X; i^*\mathfrak{B}) & \to & H^{r+1}(Y, X; \mathfrak{B}) & \to & H^{r+1}(Y; \mathfrak{B}) & \to & H^{r+1}(X; i^*\mathfrak{B}) & \to \cdots \\
\downarrow{\scriptstyle [X]\frown} & & \downarrow{\scriptstyle [Y]\frown} & & \downarrow{\scriptstyle [Y]\frown} & & \downarrow{\scriptstyle [X]\frown} & \\
\cdots H_{m-r}(X; i^*\mathfrak{B}^t) & \to & H_{m-r}(Y; \mathfrak{B}^t) & \to & H_{m-r}(Y, X; \mathfrak{B}^t) & \to & H_{m-r-1}(X; i^*\mathfrak{B}^t) & \to \cdots,
\end{array}$$

the Five Lemma shows that we also have isomorphisms

$$\frown[Y] : H^{r+1}(Y, X; \mathfrak{B}) \longrightarrow H_{m-r}(Y; \mathfrak{B}^t)\ .$$

Again, $[Y]$ is unique up to a sign on each component of Y. I do not know whether it follows from the other conditions that $\partial_*[Y]$ is a fundamental class for X.

There are further instances of duality isomorphisms for manifolds which have analogues in this context. Suppose (Y, X) a Poincaré pair, that X is a union of subcomplexes X_+, X_- which meet in W (possibly empty), and that each of (X_+, W), (X_-, W) is also a Poincaré pair. Suppose also that in

$$H_{m+1}(Y, X; \mathbf{Z}^t) \xrightarrow{\partial_*} H_m(X; \mathbf{Z}^t) \xrightarrow{j_*} H_m(X, W; \mathbf{Z}^t) \cong H_m(X_+, W; \mathbf{Z}^t) + H_m(X_-, W; \mathbf{Z}^t)\ ,$$

we have $j_*\partial_*[Y] = [X_+] - [X_-]$. Then we call (Y, X_+, X_-) a *Poincaré triad*. The geometric model for the situation is a manifold N^{m+1} with boundary M^m; and submanifolds M^m_+, M^m_- of M^m with $M^m_+ \cup M^m_- = M^m$; $M^m_+ \cap M^m_- = L^{m-1} = \partial M^m_+ = \partial M^m_-$: an important special case is where L is empty, and then M^m_+, M^m_- form a partition of the set of components of M^m. We have the commutative diagram

$$\begin{array}{ccccccc}
\cdots H^r(X_+; i^*\mathfrak{B}) & \longrightarrow & H^{r+1}(Y, X_+; \mathfrak{B}) & \longrightarrow & H^{r+1}(Y; \mathfrak{B}) & \longrightarrow & \cdots \\
\big\downarrow [X_+]\frown & & \big\downarrow [Y]\frown & & \big\downarrow [Y]\frown & & \\
\cdots H_{m-r}(X_+, W; i^*\mathfrak{B}^t) & \longrightarrow & H_{m-r}(Y, X_-; \mathfrak{B}^t) & \longrightarrow & H_{m-r}(Y, X; \mathfrak{B}^t) & \longrightarrow & \cdots
\end{array}$$

where the rows are exact, in view of the excision isomorphism

$$H_{m-r}(X_+, W; i^*\mathfrak{B}^t) \cong H_{m-r}(X, X_-; i^*\mathfrak{B}^t)\ .$$

It follows by the Five Lemma that the middle map is an isomorphism. This argument is due to W. Browder [3].

We shall be doing surgery on a map $\psi: M \to X$ of a manifold M to a Poincaré complex X with $\psi_*[M] = [X]$. Such maps are called of degree 1. We first consider the closed case.

LEMMA 2.1. *Let M, X be Poincaré complexes, $\psi: M \to X$ of degree 1, $\mathfrak{B}$ a coefficient bundle over X. Then the diagram*

$$\begin{array}{ccc}
H^r(M; \psi^*\mathfrak{B}) & \xleftarrow{\psi^*} & H^r(X; \mathfrak{B}) \\
{[M]\frown}\big\downarrow & & {[X]\frown}\big\downarrow \\
H_{m-r}(M; \psi^*\mathfrak{B}^t) & \xrightarrow{\psi_*} & H_{m-r}(X; \mathfrak{B}^t)
\end{array}$$

is commutative, and induces an isomorphism of the cokernel, $K^r(M; \mathfrak{B})$ say, of ψ^* onto the kernel $K_{m-r}(M; \mathfrak{B}^t)$ of ψ_*. In particular, if ψ is k-connected, ψ_* and ψ^* are isomorphisms if $r < k$ or $> m - k$.

PROOF. The diagram commutes, since if $c \in H^r(X; \mathfrak{B})$,

$$\psi_*([M]\frown\psi^*c) = \psi_*[M]\frown c = [X]\frown c\ .$$

Since $[M]\frown$ and $[X]\frown$ are isomorphisms, $\psi^+ = ([M]\frown) \circ \psi^* \circ ([X]\frown)^{-1}$ is a right inverse to ψ_*, so $H_{m-r}(M; \psi^*\mathfrak{B}^t)$ splits as the direct sum of the kernel of ψ_* and the isomorphic image by ψ^+ of $H_{m-r}(X; \mathfrak{B}^t)$. Similarly the cohomology splits, $[M]\frown$ preserves the split, and we have the stated isomorphism.

If ψ is k-connected, it induces homology and cohomology isomorphisms in dimensions $< k$. Hence if $r < k$ or if $r > m - k$ three, and hence all four of the maps in the diagram above are isomorphisms.

Now we must consider the case corresponding to boundaries. Suppose $\psi\colon (N, M_+, M_-) \to (Y, X_+, X_-)$ a map of Poincaré triads with $\psi_*[N] = [Y]$. (It follows that each $\psi_*[M_\varepsilon] = [X_\varepsilon]$). Consider the diagram

$$\begin{array}{ccc} H^{r+1}(N, M_+; \psi^*\mathfrak{B}) & \xleftarrow{\psi^*} & H^{r+1}(Y, X_+; \mathfrak{B}) \\ {\scriptstyle [N]\frown}\downarrow & & \downarrow{\scriptstyle [Y]\frown} \\ H_{m-r}(N, M_-; \psi^*\mathfrak{B}^t) & \xrightarrow{\psi_*} & H_{m-r}(Y, X_-; \mathfrak{B}^t) . \end{array}$$

Precisely the same considerations apply as in the lemma, and we have rephrasing of the same results; here, the induced isomorphism is denoted by

$$[N]\frown\colon K^{r+1}(N, M_+; \mathfrak{B}) \cong K_{m-r}(N, M_-: \mathfrak{B}^t) .$$

We shall identify the K's with the direct summands of the corresponding homology and cohomology groups. The direct sum splitting is natural in several senses.

LEMMA 2.2. *Let $\psi\colon (N, M_+, M_-) \to (Y, X_+, X_-)$ be a map of degree 1 of Poincaré triads. Then the direct sum splittings above are preserved*

(a) *in an exact sequence induced by a short exact sequence of coefficient bundles,*

(b) *in any of the homology (or cohomology) exact sequences of the triad, and*

(c) *(cohomology only) under cup products by elements of* Im ψ^*

PROOF. In (a) or (b) we have a sequence of maps of square diagrams of the type above, forming 4 exact sequences. It is immediate that the sequences of subgroups defined by Ker ψ^* and Im ψ^* are subcomplexes; the result follows (cf. [24]). We point out that the exact sequences we have in mind in (b) are those of the pairs (N, M_+), (N, M_-), (M_+, L), (M_-, L), and the Mayer-Vietoris sequences (see below).

In (c), since ψ^* preserves cup products, one half of the splitting is certainly preserved. But so is the other, since if d lies in some K^r, so does $d \cdot \psi^*c$, for

$$\begin{aligned} \psi_*([N]\frown d \cdot \psi^*c) &= \psi_*\big(([N]\frown d)\frown\psi^*c\big) \\ &= \psi_*([N]\frown d)\frown c = 0\frown c = 0 \end{aligned}$$

(similarly if the appropriate fundamental class is $[M_+]$ or $[M_-]$ rather than $[N]$).

PROPOSITION 2.3. *Let* ψ: $(N, M) \to (Y, X)$ *be a* $(k+1)$*-connected map of degree* 1 *of Poincaré pairs, inducing on* $(m-k)$*-connected map of* M^m *to* X^m. *If* $m \leqq 2k$, ψ: $N \to Y$ *is a homotopy equivalence.*

PROOF. Without loss of generality, we may suppose Y (hence also N) connected. If $k \geqq 1$, ψ induces an isomorphism of fundamental groups; write $\pi = \pi_1(N)$, and identify it with $\pi_1(Y)$, and $\Lambda = \mathbf{Z}[\pi]$. Now we prove ψ $(k+2)$-connected; the result will follow by induction and Whitehead's theorem [27]. We have

$$\pi_{k+2}(\psi) \cong \pi_{k+2}(\tilde{\psi}) \cong H_{k+2}(\tilde{\psi})$$

by the relative Hurewicz theorem, and since ψ_* is onto in

$$H_{k+2}(N; \Lambda) \xrightarrow{\psi_*} H_{k+2}(Y; \Lambda) \longrightarrow H_{k+2}(\tilde{\psi}) \longrightarrow H_{k+1}(N; \Lambda) \xrightarrow{\psi_*} H_{k+1}(Y; \Lambda)\ ,$$

we have $H_{k+2}(\tilde{\psi}) \cong K_{k+1}(N; \Lambda)$. But by the relative form of (2.1), $K_{k+1}(N; \Lambda) \cong K^{m-k}(N, M; \Lambda^t)$. So, by (2.2), $\pi_{k+2}(\psi) \cong K^{m-k}(N, M; \Lambda^t)$ lies in the exact sequence

$$K^{m-k-1}(M; \Lambda^t) \longrightarrow K^{m-k}(N, M; \Lambda^t) \longrightarrow K^{m-k}(N; \Lambda^t)\ .$$

But the first group vanishes since the induced map $M \to X$ is $(m-k)$-connected; the last, since the induced map $N \to Y$ is $(k+1)$-connected, and $m-k \leqq k$.

If $k < 0$, the result is empty; if $k = 0$, the only cases arising are $m = -1, 0$, and here the result is trivial, since a Poincaré (-1)-complex is empty, and a connected Poincaré 0-complex (resp. 1-pair) has the homotopy type of a point (resp. of $(I, \partial I)$), as we will show elsewhere.

The following somewhat technical lemma will be useful later; the proof follows ideas of [23].

LEMMA 2.4. *Let* ψ: $(N, M) \to (Y, X)$ *be a* k*-connected map of degree* 1 *of connected Poincaré pairs, inducing a* $(k-1)$*-connected map* $M \to X$. *Let* $k \geqq 2$, *and* $[N]$ *have dimension* $2k$. *Then* $G = K_k(N; \Lambda)$ *is a projective* Λ*-module, and if* N *and* Y *are homotopy equivalent to finite* CW*-complexes there is a finitely generated free* Λ*-module* F *with* $F \oplus G$ *free. Similarly for* $K_k(N, M; \Lambda)$.

PROOF. Replace Y by the mapping cylinder of ψ. Then we can suppose ψ an inclusion of a subcomplex. In fact, since ψ is k-connected, we may suppose that $Y - N$ has no cells of dimension $\leqq k$. Perhaps the easiest way to see this is to note that Y and N may be assumed to have finite skeletons (by [23, Th. A]) and apply to the $(k+1)$-skeletons a result of Whitehead [28, Lem. 15].

Let C_i be the i^{th} chain group of $(\tilde{Y}, \tilde{N})$: this is a free Λ-module. All $K_i(M)$ and $K^i(M)$ vanish for $i < k - 1$, and $K_i(N)$ and $K^i(N)$ for $i < k$; by the exact sequences of 2.2, $K_i(N, M)$ and $K^i(N, M)$ vanish for $i < k$; and by duality, all six vanish for $i > k$. In particular, $K_i(N; \Lambda) = 0$ for $i \neq k$, so the sequence

$$(1) \qquad \cdots C_{k+2} \longrightarrow C_{k+1} \longrightarrow G \longrightarrow 0$$

is exact. Write $d_i: C_{i+1} \to C_i$ for the boundary operators, and $B_i = \operatorname{Im} d_i$. Then (1) gives rise to short exact sequences, which we write as

$$(2) \qquad \begin{array}{l} 0 \longrightarrow B_{k+1} \xrightarrow{b_{k+1}} C_{k+1} \longrightarrow G \longrightarrow 0 \\ 0 \longrightarrow B_i \xrightarrow{b_i} C_i \xrightarrow{l_{i-1}} B_{i-1} \longrightarrow 0 \qquad (i \geqq k+2) . \end{array}$$

We prove that the sequences (2) all split. Now B_i defines a coefficient bundle $\mathfrak{B}_i$, and $K^i(N; \mathfrak{B}_i) \cong H^{i+1}(\operatorname{Hom}_\Lambda(C_*, B_i))$. Thus we may regard $c_{i+1} (i \geqq k+1)$ as an $(i+1)$-cochain. It is a cocycle since $c_{i+1} \circ d_{i+1} = 0$. But as $i \geqq k+1$, the group $K^i(N; \mathfrak{B}_i)$ vanishes, so c_{i+1} is a coboundary, say

$$c_{i+1} = s_i \circ d_i = s_i \circ b_i \circ c_{i+1} .$$

As c_{i+1} is onto, $s_i \circ b_i = 1$, so all the sequences (2) split; in particular, $C_{k+1} = B_{k+1} \oplus G$, so G is projective.

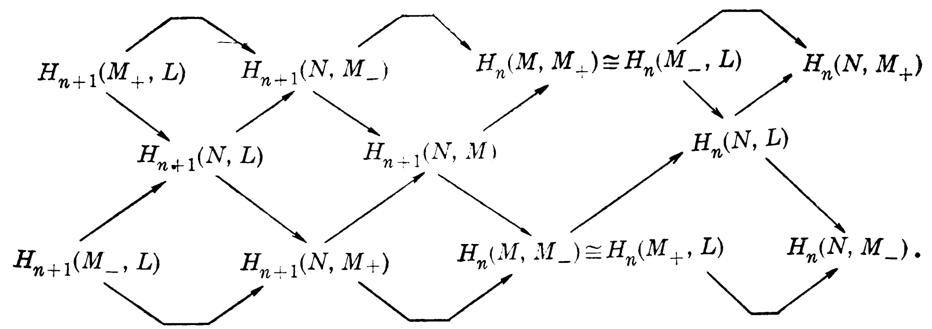
If N and Y are finite, so is the complex C_*. Using the splittings just established, we have

$$G \oplus_{i \text{ even}} C_{k+1} \cong G \oplus_j B_j \cong \oplus_{i \text{ odd}} C_{k+1} ;$$

as the C_{k+1} are free modules of finite rank, this proves the result.

The argument for $K_k(N, M)$ is similar; here we may suppose M, N, X subcomplexes of Y with $M = N \cap X$, and must use cochains of $(Y, N \cup X)$. For the last part, we assume all four complexes to be finite.

We must now discuss in detail the effect of surgery. First note that if (N, M_+, M_-) is a proper triad, with $M_+ \cup M_- = M$ and $M_+ \cap M_- = L$, we can combine the homology exact sequences (with respect to any bundle over N of coefficients) of the triples (N, M_ε, L) and (N, M, M_ε) into the diagram

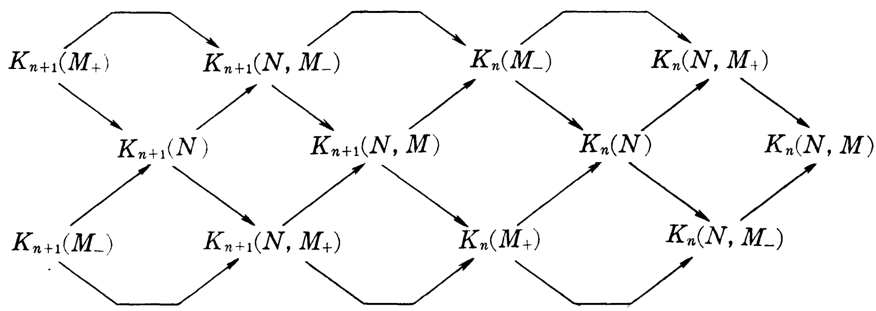


We shall frequently use diagrams of this shape, and so introduce some conventions concerning them. It will be convenient to have all squares and triangles commutative: this is easy to arrange for the triangles (re-define one map as the composite of the other two). The usual conventions give a negative sign in alternate squares: we avoid this by defining the map $H_{n+1}(N, M) \to H_n(M_-, L)$ to be minus the usual boundary map. If we again adjust signs suitably, an argument of Eilenberg and Steenrod [6, § 1.15] shows that sequences of the shape

$$\cdots H_{n+1}(N, L) \longrightarrow H_{n+1}(N, M_-) \oplus H_{n+1}(N, M_+) \longrightarrow H_{n+1}(N, M) \longrightarrow H_n(N, L) \longrightarrow \cdots$$

are exact. We shall refer to all sequences of this type as Mayer-Vietoris sequences.

Next suppose $\psi: (N, M_+, M_-) \to (Y, X_+, X_-)$ a map of degree 1 of Poincaré triples. Then for any bundle of coefficients over Y, there is a commutative exact diagram as above, but with the groups K_n, since by Lemma 2.2 the whole diagram splits as a direct sum. If, in particular, ψ induces a homotopy equivalence of L on $W = X_+ \cap X_-$, then all $K_n(L)$ vanish, and the homology exact sequences of pairs involving L show that $K_n(M_-; \mathfrak{B}) \cong K_n(M_-, L; \mathfrak{B})$, and similarly for M_+ and N. Thus the diagram becomes (again with some coefficient bundle $\mathfrak{B}$ over Y)

(4) 

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We now apply the sequence to surgery. Let M_- be a connected smooth manifold with boundary L, (X_-, W) a Poincaré pair, $\psi_-: (M_-, L) \to (X_-, W)$ a map of degree 1 inducing a homotopy equivalence of L on W. Suppose N obtained from $M_- \times I$ by attaching an $(r + 1)$-handle ($r \geqq 0$—so N is connected) and that ψ_- extends to $\psi: N \to X_-$. Then ∂N is the union of M_-, $L \times I$, and a manifold M_+ obtained from M_- by surgery. In order to fit exactly the definition of Poincaré triad, we can adjoin $L \times I$ to M_- (this does not even change homeomorphism type), and write L for $L \times 1$. It is easy to verify that

$$(Y, X_+, X_-) = (X \times I, X \times 1, X \times 0 \cup W \times I)$$

is also a Poincaré triad. Let $F: N \to I$ have $F(M_-) = 0$, $F(M_+) = 1$, e.g. a Morse function, with just one singularity (of index $r + 1$) corresponding to the handle. Then

$$\varphi = \psi \times F: (N, M_+, M_-) \longrightarrow (Y, X_+, X_-)$$

is a map of degree 1 of Poincaré triads, inducing a homotopy equivalence of L on W.

THEOREM 2.5. *Let $\psi_-: (M_-, L) \to (X_-, W)$ be a map of degree 1 of a connected smooth manifold and its boundary to a Poincaré pair, inducing a homotopy equivalence of L on W, and the map $\varphi: (N, M_+, M_-) \to (Y, X_+, X_-)$ be obtained, as above, by attaching an $(r + 1)$-handle. Then for any coefficient bundle $\mathfrak{B}$ over Y, with fibre B, we have a commutative exact diagram (4). The groups $K_n(N, M_-)$ vanish for $n \neq r + 1$; $K_{r+1}(N, M_-) \cong B$, and the map from this to $K_r(M_-)$ is induced by the class $x \in K_r(M_-; \Lambda)$ on which surgery is done. The groups $K_n(N, M_+)$ vanish for $n \neq m - r$; $K_{m-r}(N, M_+) \cong B$, and the map from $K_{m-r}(M_-)$ to this is induced by intersection with x.*

PROOF. We have already obtained the diagram (4). As X_+, X_- are deformation retracts of Y, all $H_n(Y, X_\varepsilon)$ vanish, so $K_n(N, M_\varepsilon; \mathfrak{B}) = H_n(N, M_\varepsilon; \varphi^*\mathfrak{B})$. But N is obtained from $M_- \times I$ by attaching an $(r + 1)$-handle; or, up to homotopy, an $(r + 1)$-cell. Thus $H_n(N, M_-; \varphi^*\mathfrak{B}) = 0$ for $n \neq r + 1$, and gives B for $n = r + 1$. Similarly if we replace M_- by M_+ and $r + 1$ by $m - r$.

It remains to describe the maps. Now $K_{r+1}(N, M_-) = H_{r+1}(N, M_-)$ is generated by the class of the chain represented by the attached $(r + 1)$-cell. Taking the boundary to $K_r(M_-)$ is induced, then, by taking the boundary sphere of the cell, the attaching sphere of the handle, which determines the class x. Similarly, $K_{m-r}(N, M_+)$ is generated by the class of the dual $(m - r)$-cell. Now the attaching sphere has a neighborhood $S^r \times D^{m-r}$ in M_-, to which the handle is attached; the complementary subset of M_- is isotopic (in N) to the corresponding subset of M_+. An $(m - r)$-sphere in M_- can be made to meet S^r transversely, and then meet $S_r \times D^{m-r}$ in various discs $P_i \times D^{m-r}$. So all is homotopic to M_+, except for these discs, which are homotopic to the dual $(m - r)$-cell mentioned above. And they are determined by intersections with the attaching sphere.

3. Surgery in the middle dimension: case $m = 2k$

We return to the problem of § 1, and in particular to Lemma 1.3. Here, if $M^{2k} = \partial_+ V$, $k \geqq 2$, we have a map $\psi: M \to X$, and for each element α of $G = \pi_{k+1}(\psi)$, a regular homotopy class of immersions of S^k in V; surgery on

α is possible if and only if this class contains an imbedding.

Our next task is to consider the self-intersection of spheres corresponding to α. To compute the effect of surgery, we also need to study mutual intersections between different regular homotopy classes.

Let $*$ be the base point of M. Orient the tangent space at $*$, corresponding to the homology orientation $[M]$. We shall suppose each immersed k-sphere provided with a path (defined up to homotopy) which joins it to the base point; it is equivalent to specifying a lifting of the sphere to $\tilde{M}$. Let S_1^k, S_2^k be two such spheres. Without loss of generality, we may suppose that they meet transversely in a finite set of points, none at a singularity of either S_i. We now give to each intersection point P a sign ε_P and an element g_P of π.

We have paths α_1, α_2 from $*$ to P; α_i proceeds by the characteristic path of S_i^k, then round the sphere S_i^k to P (since $k \geqq 2$, S^k is simply-connected; it is understood that the path avoids singular points, or at least does not change over branches at one). Define g_P as the homotopy class of $\alpha_1^{-1} \cdot \alpha_2$ and $\varepsilon_P = \pm 1$ according as the orientation of S_1^k at P (i.e., the image by the immersing map of the standard orientation of S^k), followed by that of S_2^k, does or does not agree with the transport along α_1 of the chosen orientation at $*$.

If S_1^k, S_2^k correspond to x, $y \in G$, we define $\varphi(x, y) = \sum_P \varepsilon_P g_P$, summed over all intersections P. Write this as $\sum_{g \in \pi} n(g)\, g$. Then $n(g)$ is the intersection number of the homology classes x, yg^{-1} in $\tilde{M}$ and so it, and $\varphi(x, y)$ are well defined.

A similar procedure gives us an estimate of the self-intersection of an immersion.

We define an anti-automorphism of Λ, $\lambda \to \lambda^-$ by

$$(\sum n(g)\, g)^- = \sum w(g)\, n(g)\, g^{-1} ,$$

where $w(g) = \pm 1$ according as g preserves or reverses orientation. Let I be the subgroup of Λ consisting of all $\nu + (-1)^{k-1} \nu^-$ for $\nu \in \Lambda$; and denote by V the quotient group Λ/I.

THEOREM 3.1. *Intersections define a bilinear map* φ: $G \times G \to \Lambda$, *satisfying*

(i) $\varphi(x, y\lambda) = \varphi(x, y)\, \lambda$,

(ii) $\varphi(y, x) = (-1)^k \varphi(x, y)^-$.

Self-intersections define a map $\mu : G \to V$ *satisfying*

(iii) $\varphi(x, x) = \mu(x) + (-1)^k \mu(x)^-$,

(iv) $\mu(x + y) - \mu(x) - \mu(y) = \varphi(x, y) \pmod I$,

(v) $\mu(x\lambda) = \lambda^- \mu(x)\, \lambda$.

If $k \geqq 3$, we can do surgery on x if and only if $\mu(x) = 0$.[2]

PROOF. We have already defined φ. Since it can be defined by intersection numbers of homology classes in $\widetilde{M}$, it is certainly bilinear over **Z**.

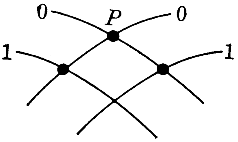
We first verify (i) when $\lambda = g$ for some $g \in \pi$. Now yg is represented by the same sphere S_2^k, but with characteristic path α_2 replaced by $\alpha_2 g$. Thus g_P becomes $g_P g$, ε_P is unaltered, and the formula holds. Since φ is bilinear, it follows at once that (i) holds in general.

To check formula (ii) we must see what happens to each ε_P and g_P when we interchange the two k-spheres. As g_P is represented by $\alpha_1^{-1} \cdot \alpha_2$, it becomes $\alpha_2^{-1} \cdot \alpha_1$, i.e., g_P^{-1}. For the sign, we first have a factor $(-1)^k$ corresponding to interchange of the spheres, and a further sign if α_1 and α_2 do not induce the same orientation; i.e., if $w(g) = -1$.

Now consider an immersed sphere S^k representing x. At each self-intersection P we make (arbitrarily) a choice of order of the two branches at P. The definition of $\varepsilon(P)$ and $g(P)$ is now as before. As in the proof of (ii) above, if we change the order, $\varepsilon(P)g(P)$ is replaced by $(-1)^k \varepsilon(P)g(P)^-$, so is altered only by an element of I. Thus S^k determines the element $\sum \varepsilon(P)g(P)$ modulo I, i.e., in V. Call it $\mu(S^k)$.

If two immersions are regularly homotopic, we can suppose the regular homotopy generic. Then the self-intersections vary continuously, except that at each of a finite number of moments, two self-intersections appear (or disappear) together. For these two we have the same element of π (at the crucial moment, they determine the same path) and opposite values of ε. Thus the invariant μ is constant throughout, and depends only on the regular homotopy class x.

To verify (iii), note that the immersion actually given by Lemma 1.3 is a framed immersion; i.e., an immersion of $S^k \times D^k$. The induced 'parallel' immersions of $S^k \times 0$, $S^k \times 1$ will then meet twice near each self-intersection P of $S^k = S^k \times 0$, and each ordering of the branches occurs once. So we have



[2] Our criterion for imbeddings contradicts a result announced by Kervaire (Comment. Math. Helv. 39 (1965), 271–280), that x is homotopic to an imbedding if $\mu(x)$ has finite order, since changing the sphere in its homotopy class can only alter $\mu(x)$ by a multiple of $1 \in G$. The mistake occurs on p. 273, line 19, where it is assumed that A and A' are distinct: they need not be, if τ has order 2.

$$\begin{aligned}\mu(x) &= \sum \varepsilon(P)g(P)\\ \varphi(x, x) &= \sum\{\varepsilon(P)g(P) + (-1)^k \varepsilon(P)g(P)^-\}\\ &= \mu(x) + (-1)^k \mu(x)^- .\end{aligned}$$

(Note that although $\mu(x)$ is only defined modulo I, if $\iota \in I$ $\iota + (-1)^k \iota^- = 0$.)

To prove (iv) we need a geometric construction for the immersion representing the sum of two elements of G (we could also have used this to give a geometric proof of bilinearity of φ). Let $\tilde{f}, \tilde{g}: S^k \to \tilde{M}$ be immersions representing x, y, and let $\tilde{h}: D^k \times I \to \tilde{M}$ be an imbedding, with $\tilde{h}(D^k \times 0)$, $\tilde{h}(D^k \times 1)$ lying on $\tilde{f}(S^k)$, $\tilde{g}(S^k)$ and with the same induced orientations. Let f, g, h, be the projections on M; we assume also $h(D^k \times \operatorname{Int} I)$ disjoint from the images of f and g. (As $k \geqq 2$, it is easy to construct $\tilde{h}$ by thickening a suitable arc). Now if from $f(S^k) \cup g(S^k)$ we delete $h(D^k \times \{0, 1\})$, and replace by $h(S^{k-1} \times I)$, and round corners, we obtain an immersed sphere which, we claim, represents $x + y$. Recall that, if we use f, x to attach D^{k+1} to M and extend ψ over it, the tangent framing of D^{k+1} at $f(S^k)$ can be used as the first $(k + 1)$ vectors of the framing of $\tau_M \oplus \varepsilon$ induced by $\omega \circ \psi$; in fact this is what defined the immersion. The same holds for g, y. If we attach both, the two discs D^{k+1} are joined by $h(D^k \times I)$; the union is a disc Δ^{k+1} whose boundary is the sphere constructed above. We now push $h(\operatorname{Int} D^k \times I)$ outside M (e.g., in a thickening of M by a collar neighbourhood of ∂M) and round corners; then the tangent framings of the discs fit together, and we have a consistent extension of ψ over Δ which satisfies the framing condition.

Now the self-intersections of the constructed sphere consist precisely of the self-intersections of f and of g, and the intersections of f with g. If at each of the latter we call $f(S^k)$ the first branch, we obtain (iv) at once.

To check (v), as for (i), we first take the case $\lambda = g$. Then each $\varepsilon(P)$, $g(P)$ become (as in (i)) $w(g)\,\varepsilon(P)$, $g^{-1}\,g(P)\,g$, and the formula holds. Now we verify, using (iv), that if (v) holds for λ_1, λ_2 it also holds for $\lambda_1 + \lambda_2$, and clearly $\mu(-x) = \mu(x)$, so it holds in general.

Finally, suppose $\mu(x) = 0$, and let S^k represent x. It is easily checked that we can make choices of order so that the intersections can be put into pairs (P_i, Q_i) with $\varepsilon(P_i) = 1$, $\varepsilon(Q_i) = -1$ and $g(P_i) = g(Q_i)$. Let α_i be an arc on S^k joining the first branches at P_i, Q_i; β_i one joining the second branches; we may suppose that (except at their ends) α_i and β_i contain no singularities. Since $g(P_i) = g(Q_i)$, the simple closed curve $\beta_i^{-1}\cdot\alpha_i$ is null-homotopic in M. Since also the two intersections have opposite signs, we can use the process of Whitney [29] to remove them both with a regular homotopy, provided $k \geqq 3$. Similarly, we can remove all the self-intersections, and so obtain an imbedded sphere (and clearly μ vanishes for an imbedding) so the result now follows from Lemma 1.3.

The proof of Theorem 3.1 is now complete.

We can prove more about (G, φ), and hence μ, in some particular cases. For if (X, W) is a Poincaré pair, and $\psi_M: (M, \partial M) \to (X, W)$, a map of degree 1, which is k-connected, and induces a $(k-1)$-connected map $\partial M \to W$, we can appeal to (2.4) to see that G is a projective Λ-module. If, in addition, X is finite, then since M (as a compact smooth manifold) is also a finite CW-complex, by (2.4) again there is a finitely generated free Λ-module F with $F \oplus G$ free. Note that it is easy to see that G is finitely generated; this follows essentially from (1.1). It is also possible to obtain some properties of φ. We sum up our results in

THEOREM 3.2. *Let M^{2k} be a compact smooth manifold, (X, W) a Poincaré pair, and $\psi_M: (M, \partial M) \to (X, W)$ a map of degree* 1, *which is k-connected, and induces a $(k-1)$-connected map $\partial M \to W$. Write $G = \pi_{k+1}(\psi_M)$. Then G is a finitely generated projective Λ-module. If X is finite, there is a finitely generated free Λ-module F with $F \oplus G$ free. If $G = 0$, ψ is a homotopy equivalence. If ψ induces a homotopy equivalence $\partial M \to W$, φ is non-singular.*

PROOF. The first two conclusions were obtained above; the third follows from (2.3). Non-singularity is to be interpreted as follows: since $y \to \varphi(x, y)$ is Λ-linear, we have a map $A\varphi: G \to \mathrm{Hom}_\Lambda(G, \Lambda)$, defined by $A\varphi(x)(y) = \varphi(x, y)$. We call φ *non-singular* if $A\varphi$ is an isomorphism.

We will obtain the result by interpreting $A\varphi$ homologically. Now $G = K_k(M; \Lambda)$, and we assert that $\mathrm{Hom}_\Lambda(G, \Lambda)$ may be identified with $K^k(M: \Lambda)$. For, with the notation of (2.4), this last is the kernel of $\mathrm{Hom}_\Lambda(C_{k+1}, \Lambda) \to \mathrm{Hom}_\Lambda(C_{k+2}, \Lambda)$, and the assertion follows from the left exactness of Hom_Λ. Now duality gives an isomorphism of $K^k(M; \Lambda)$ with $K_k(M, \partial M; \Lambda^t)$, and Λ^t is isomorphic (as Λ-module) to Λ, for $g \to w(g) \otimes g$ induces an isomorphism. We observe that duality is induced by cap product with the fundamental class; this induces the same map as cup products, or intersection numbers (cf. 2.2).

We have thus identified $A\varphi$ with α in the exact sequence

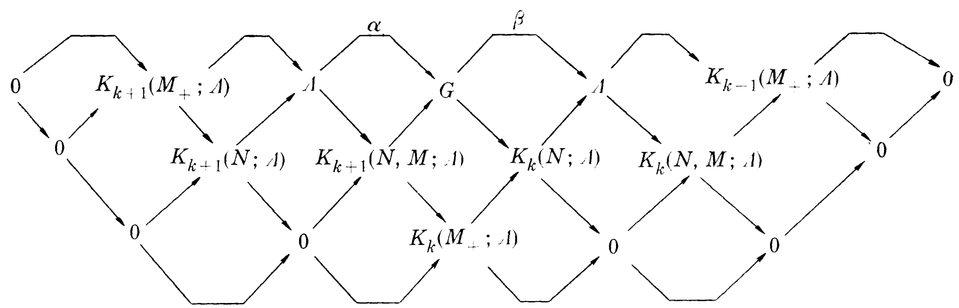
$$K_k(\partial M; \Lambda) \longrightarrow K_k(M; \Lambda) \xrightarrow{\alpha} K_k(M, \partial M; \Lambda) \longrightarrow K_{k-1}(\partial M; \Lambda) ;$$

it is now trivial that, if the $K_i(\partial M; \Lambda)$ vanish, α is an isomorphism.

We retain the general hypothesis of 3.2 for the next theorem.

THEOREM 3.3. *Suppose G has a free Λ-basis $e_1, \cdots, e_r, f_1, \cdots, f_r$ with $\varphi(e_i, e_j) = 0$, $\varphi(e_i, f_j) = \delta_{ij}$, and $\mu(e_i) = 0$. Then we can do surgery to kill G,* *and so (by 2.3) make ψ a homotopy equivalence.*

PROOF. By Theorem 3.1, we can do surgery on the class e_r. By Theorem 2.5, we have the following commutative exact diagram



where, since ψ is k-connected, it follows using (2.1) and (2.2) that all the groups not listed vanish. Also by (2.5), the map α: $\Lambda \to G$ takes the unit to e_r, and the map β: $g \to \Lambda$ is induced by intersections with e_r. More precisely, $\beta(x) = \varphi(e_r, x)$: this follows from the argument of (2.5), using the definition of φ. Clearly α is mono, and since $\beta(f_r) = 1$, β is epi; thus $K_i(M_+; \Lambda)$ vanishes for $i \neq k$, and ψ induces a k-connected map $M_+ \to X$.

Now $K_{k+1}(N, M; \Lambda)$ can be identified with Ker β, and hence with the submodule of G generated by $e_1, \cdots, e_r, f_1, \cdots, f_{r-1}$. The image of Λ is the submodule of this generated by e_r. Hence $K_k(M_+; \Lambda)$ is a free module, with basis $e_1, \cdots, e_{r-1}, f_1, \cdots, f_{r-1}$. Moreover, these basis elements can all be represented by immersed spheres in M_-, disjoint from the sphere used for surgery. Thus essentially the 'same' spheres (in $(M \times 1) \cap M_+$) continue to represent them in M_+. In particular, φ and μ are unaltered. The result now follows by induction on r.

The situation in which we are now nearest to a definitive solution is when ψ: $(M, \partial M) \to (X, W)$ is a map of degree 1 of a compact smooth manifold to a connected finite Poincaré pair, inducing a homotopy equivalence $\partial M \to W$, and with $\dim [M] = 2k \geqq 6$. Given the extra assumptions of (1.4) or (1.6), we can do surgery to make ψ_M k-connected, and then we have the (G, φ, μ) of (3.1). Also, we can always take $\alpha = 0 \in \pi_k(\psi)$ and do surgery; this replaces M by the connected sum $M \# (S^k \times S^k)$, and G by $G \oplus \Lambda \oplus \Lambda$, and by (3.2) we can do this a finite number of times to make G free. Again by (3.2), φ is non-singular.

But by (3.3), if (G, φ, μ) has a certain particular form, we can do surgery to make ψ a homotopy equivalence. We are left with the purely algebraic problem of classifying structures (G, φ, μ).

4. The algebraic problem

We first recall the notation which has been used in the preceding sections: π is a finitely presented group, Λ its integral group ring. We have a homomorphism $w: \pi \to \{\pm 1\}$ and a sign $\eta(= (-1)^k)$. The anti-automorphism bar of Λ is defined by

$$\left(\sum_{g\in\pi} n(g)\, g\right)^- = \sum_{g\in\pi} w(g)\, n(g)\, g^{-1}$$

(where $n(g)$ is an integer, zero for all but a finite number of $g \in \pi$), and

$$I = \{\nu - \eta\nu^-: \nu \in \Lambda\}\ , \qquad V = \Lambda/I\ .$$

Our data consist of a Λ-module G, a map $\varphi: G \times G \to \Lambda$, and a map $\mu: G \to V$ satisfying

(P1) G is a finitely generated free module.

(P2) φ is right Λ-linear; i.e., $\varphi(x, y\lambda + y'\lambda') = \varphi(x, y)\lambda + \varphi(x, y')\lambda'$.

Thus we can define $A\varphi: G \to \mathrm{Hom}_\Lambda(G, \Lambda)$ by $A\varphi(x)(y) = \varphi(x, y)$.

(P3) $A\varphi$ is an isomorphism.

(P4) $\varphi(y, x) = \eta\varphi(x, y)^-$.

(P5) $\varphi(x, x) = \mu(x) + \eta\mu(x)^-$.

(P6) μ is quadratic with associated bilinear map φ; i.e.,

$$\mu(x\lambda + x'\lambda') = \lambda^-\mu(x)\lambda + \lambda^-\varphi(x, x')\lambda' + \lambda'^-\mu(x')\lambda'\ .$$

(Note that though V is not a Λ-module, $\lambda^-\mu\lambda$ is well-defined for $\mu \in V$ and $\lambda \in \Lambda$.)

The first version of our problem is to find reasonable sufficient conditions that G admit a free basis $e_1, \cdots, e_r, f_1, \cdots, f_r$ with

$$\varphi(e_i, e_j) = 0 \qquad \varphi(e_i, f_j) = \delta_{ij} \qquad \mu(e_i) = 0\ .$$

We wish to reformulate this, and first make some comments on the nature of the above axioms (P1) – (P6).

REMARK 4.1. *$\mathrm{Hom}_\Lambda(G, \Lambda)$ has a natural Λ-module structure, with respect to which $A\varphi$ is an isomorphism of Λ-modules.*

For the anti-automorphism bar permits us to give G a left Λ-module structure by defining $\lambda g = g\lambda^-$. This yields a right Λ-module structure on $\mathrm{Hom}_\Lambda(G, \Lambda)$. (P4) now shows that $A\varphi$ is a morphism of right Λ-modules.

We now concentrate on (P2), (P4), (P5), and (P6).

REMARK 4.2. *Suppose $e_1, \cdots, e_r$ a free Λ-base of G. Then the values $\varphi(e_i, e_j)$ determine φ; they may be chosen arbitrarily subject to $\varphi(e_j, e_i) = \varphi(e_i, e_j)^-$, provided each $\varphi(e_i, e_i)$ has the form $\nu + \eta\nu^-$. Given φ, the $\mu(e_i)$ determine μ; they may be chosen arbitrarily subject to $\varphi(e_i, e_i) = \mu(e_i) + \eta\mu(e_i)^-$. Hence φ and μ together are uniquely determined by the* *(independent) choice of the* $\mu(e_i)$ *and* $\varphi(e_i, e_j)$ $(i < j)$ *to satisfy the axioms except* (P3).

This corresponds to the elementary remark that, with respect to a basis, a quadratic form corresponds to a symmetric matrix; the proof is essentially the same, and is left to the reader.

REMARK 4.3. *Suppose that* $e_1, \cdots, e_r$ *form a base of a free direct summand* E *of* G, *and that* $\varphi(e_i, e_j) = 0$ *for all* $1 \leqq i, j \leqq r$. *Then we can find elements such that* $e_1, \cdots, e_r, f_1, \cdots, f_r$ *form a base of a free direct summand* F *of* G, *and* $\varphi(e_i, f_j) = \delta_{ij}$. *If also each* $\mu(e_i) = 0$, *we may also take* $\varphi(f_i, f_j) = 0$ *and* $\mu(f_i) = 0$.

As $\mathrm{Hom}_\Lambda(E, \Lambda)$ is a direct summand of $\mathrm{Hom}_\Lambda(G, \Lambda)$, the map of G to it induced by $A\varphi$ is onto. For each j, let $f_j \in G$ correspond to the homomorphism $E \to \Lambda$ with $e_i \to \eta\delta_{ij}$. Thus $\varphi(e_i, f_j) = \delta_{ij}$. Let F be the free Λ-module of rank $2r$ with base $e'_1, \cdots, e'_r, f'_1, \cdots, f'_r$, $\alpha: F \to G$ the map with $\alpha(e'_i) = e_i$, $\alpha(f'_i) = f_i$. The composite

$$F \xrightarrow{\alpha} G \xrightarrow{A\varphi} \mathrm{Hom}_\Lambda(G, \Lambda) \xrightarrow{\mathrm{Hom}(\alpha, 1)} \mathrm{Hom}_\Lambda(F, \Lambda)$$

is an isomorphism of free Λ-modules, as is easily verified. So the image of α is the required free direct summand.

If now $\mu(e_i) = 0$, we choose for each i some μ_i in the coset $\mu(f_i) \subset \Lambda$, and make the substitution

$$f_i \longrightarrow f_i - \eta\{(\textstyle\sum_{j<i} e_j\varphi(f_j, f_i)) + e_i\mu_i\} .$$

REMARK 4.4. *Suppose* (G_1, φ_1, μ_1) *and* (G_2, φ_2, μ_2) both satisfy all of (P1) – (P6). We can define the (orthogonal, direct) sum as $(G_1 + G_2, \varphi, \mu)$, where

$$\varphi((x_1, x_2), (y_1, y_2)) = \varphi_1(x_1, y_1) + \varphi_2(x_2, y_2)$$
$$\mu(x_1, x_2) = \mu_1(x_1) + \mu_2(x_2) ;$$

it is trivial to verify that this again satisfies the axioms. We can now define a *Grothendieck group* $\mathfrak{G}$. Let F be the free abelian group whose basis is the set of isomorphism classes $\langle\Phi\rangle$ of all $\Phi = (G, \varphi, \mu)$; R the subgroup generated by the $\langle\Phi + \Psi\rangle - \langle\Phi\rangle - \langle\Psi\rangle$, $\mathfrak{G}$ the quotient group F/R. Thus $\mathfrak{G}$ is the universal group for additive functions on forms Φ, with value in an abelian group. It follows from the definition that, if Φ and Φ' yield the same element of $\mathfrak{G}$, there exists some Ψ with $\Phi + \Psi \cong \Phi' + \Psi$.

REMARK 4.5. We call Φ a *hyperbolic plane* if G has a free base e, f with $\varphi(e, e) = 0$, $\varphi(e, f) = 1$. Note that if $\mu(e) = 0$, by (6.3) we may suppose $\mu(f) = 0$, and by (4.2) this determines Φ up to isomorphism: we call Φ a *standard plane*, and a sum of copies of Φ (or any isomorph) a *kernel*: these are the ones we wish to characterise.

Suppose $\Phi = (G, \varphi, \mu)$ such that G is free, with base $e_1, \cdots, e_r$, say. Define $-\Phi = (G, -\varphi, -\mu)$, and denote its base by $e'_1, \cdots, e'_r$. Then it is clear that $e''_1 = e_1 + e'_1, \cdots, e''_r = e_r + e'_r$ satisfy the hypotheses of both parts of (4.3), hence we can find a kernel, the sum of r standard planes, which is a direct summand of $G + G$ with complement P, say.

Any element of P is orthogonal to each e''_i, so if expressed as $\sum e_i\lambda_i + e'_i\lambda'_i$, must satisfy $\lambda_i = \lambda'_i$ for each i (by P3), hence be a linear combination of e''_i, and so zero. Thus $\Phi + (-\Phi)$ is a kernel.

Note that it now follows from the last sentence of (4.4), by adding $-\Psi$ to each side, that if Φ and Φ' determine the same element of $\mathfrak{G}$, then they become isomorphic on adding a suitable kernel to each.

REMARK 4.6. The *rank* of a free (or even projective) Λ-module G is well-defined, for the trivial homomorphism $\pi \to \{1\}$ determines a ring homomorphism $\Lambda \to \mathbf{Z}$, and we can calculate the rank of the free abelian group $G \otimes_\Lambda \mathbf{Z}$. The rank of a hyperbolic plane is 2. Rank is additive, so determines a homomorphism $\mathfrak{G} \xrightarrow{r} \mathbf{Z}$. We shall see below that the image of this consists of the even integers. So a right inverse r' to r is defined by mapping 2 to the trivial plane. The cokernel of r' (or kernel of r) is the reduced group $\tilde{\mathfrak{G}}$. If Φ and Φ' determine the same element of $\tilde{\mathfrak{G}}$, then they become isomorphic on adding appropriate kernels.

We can now reformulate our problem quite simply as: determine $\tilde{\mathfrak{G}}$. A determination will take the form first, of finding additive functions of forms Φ, which vanish on kernels, and so define homomorphisms of $\tilde{\mathfrak{G}}$. We must find enough of these to give a monomorphism of $\tilde{\mathfrak{G}}$ to some (known) abelian group, and then determine the image of the monomorphism. The 'reasonable sufficient conditions on Φ' referred to in the first formulation of the problem will be that the additive functions listed all vanish on Φ.

We now give some examples of additive functions. They may be divided into three classes: invariants of

$$\varphi \otimes 1\colon (G \otimes_{\mathbf{Z}} \mathbf{R}) \times (G \otimes_{\mathbf{Z}} \mathbf{R}) \longrightarrow \Lambda \otimes_{\mathbf{Z}} \mathbf{R}\ ,$$

invariants of φ itself, and invariants of μ. We discuss these in the reverse order.

EXAMPLE 4.7. The trivial homomorphism $\pi \to 1$ induces a ring homomorphism $\Lambda \to \mathbf{Z}_2$ which induces a map $V \to \mathbf{Z}_2$. Write G_2 for $G \otimes_\Lambda \mathbf{Z}_2$. We assert that μ induces a non-singular quadratic map $\mu_2\colon G_2 \to \mathbf{Z}_2$; indeed, it is trivial to verify for $x \in G$ that the image of $\mu(x)$ in $\mathbf{Z}_2$ is unaltered by adding to x elements of the form $2y$ or $y(1-g)$ $(y \in G,\ g \in \pi)$. Clearly also, the associated bilinear map φ_2 to μ_2 is found by tensoring φ over Λ with $\mathbf{Z}_2$, so is non-singular.

P5 now shows that, for $x \in G_2$, $\varphi_2(x, x) = 0$. As φ_2 is non-singular, the rank of G_2 is even. Hence the rank of G is even, as we have already mentioned. In fact G_2 has a free $\mathbf{Z}_2$-basis $u_1, \cdots, u_r, v_1, \cdots, v_r$ with $\varphi_2(u_i, u_j) = \varphi_2(v_i, v_j) = 0$, $\varphi_2(u_i, v_j) = \delta_{ij}$.

Our additive function is the *Arf invariant* of μ_2,

$$c(\mu_2) = \sum \mu_2(u_i)\, \mu_2(v_i) \ .$$

It is easily seen that this is invariant: $c(\mu_2)$ is equal to that value (0 or 1) which μ_2 takes most frequently on the finite set G_2. Our form of the definition shows that $c(\mu_2)$ is additive and zero on kernels, so defines a homomorphism $c: \tilde{\mathfrak{G}} \to \mathbf{Z}_2$

EXAMPLE 4.8. The simplest invariant of a quadratic form is its determinant. Here, non-commutativity forces us to be a little more circumspect; an automorphism of a free Λ-module of finite rank induces one of the free Λ-module M of countable rank. The group $K_1(\Lambda)$ is defined (cf. [28], [1]) as the commutator quotient group of the group of automorphisms of finite type (i.e., the difference from the identity has finite rank) of M.

Take a free basis for G. This determines one for $\mathrm{Hom}_\Lambda(G, \Lambda)$, so $A\varphi$ determines a matrix A. If the free basis is changed by a matrix B, A becomes $B\,A\,B^*$, where B^* is obtained by transposing B and replacing each element by its conjugate under bar. (P4) shows that $A = \eta A^*$.

The map $B \to B^*$ is an anti-automorphism of the group of automorphisms of the free Λ-module. So we also have an automorphism $*$ of $K_1(\Lambda)$. The matrix A determines an element $x \in K_1(\Lambda)$ with $x^* = x$, since G has even rank so even if $\eta = -1$, the scalar matrix η is a commutator. The element $x \in K_1(\Lambda)$ is determined up to addition of an arbitrary element of the form $y + y^*$.

Let us write $\Delta(\Lambda)$ for the quotient of the group of symmetric elements $(x^* = x)$ of $K_1(\Lambda)$ by its subgroup of traces $(y^* + y)$. Then the matrix A above determines a class $\Delta(\Phi) \in \Delta(\Lambda)$, which is an invariant of φ. This invariant is clearly additive; however, it need not vanish on kernels: a standard plane has the matrix $\begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix}$ which does not have determinant 1 even if $\pi = \{1\}$ and $\eta = 1$. We write Δ_0 for its class in $\Delta(\Lambda)$. Then if $\Phi = (G, \varphi, \mu)$ has G free of rank $2r$, we define the *discriminant* $\Delta(\Phi) = \Delta(\varphi) - r\Delta_0$. This is additive and vanishes on kernels, so gives rise to a homomorphism $\Delta: \tilde{\mathfrak{G}} \to \Delta(\Lambda)$.

EXAMPLE 4.9. Real quadratic forms are entirely classified by the signature (the number of positive minus that of negative terms when in diagonal form). This again gives rise to invariants in our case. The involution bar on Λ extends in a natural way to $\Lambda \otimes_Z \mathbf{R}$, which is a semi-simple real associative algebra with involution. Let us suppose also that π is finite. Then $\Lambda \times_{\mathbf{Z}} \mathbf{R}$ is finite-dimensional, and so decomposes into a direct sum of simple algebras, which are matrix rings over **R**, **C** or **H**. The involution may interchange these in pairs or leave them fixed. The various cases were classified by Weil [26], there are 10 essentially distinct. In each case, the Grothendieck group of modules with non-singular hermitian form is isomorphic either to **Z** or to $\mathbf{Z} + \mathbf{Z}$; in the $\mathbf{Z} + \mathbf{Z}$ case, an invariant analogous to the signature is needed for classification, in addition to the rank of the module.

Hence for each direct summand of the algebra-with-involution $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$, of certain types, we have a signature invariant of φ, with integer values; the collection of all these defines the multi-signature $\Sigma(\Phi) \in \alpha\mathbf{Z}$ (where α is the number of relevant summands of $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$). This is additive and vanishes on kernels, so defines a homomorphism

$$\Sigma : \tilde{\mathfrak{S}} \longrightarrow \alpha \mathbf{Z} .$$

The reader might well expect us at this point to define, or at least indicate definitions for, a generalisation of the Hasse-Minkowski invariant from ordinary quadratic forms to our present problem. However, for a unimodular quadratic form over the integers, all the Hasse-Minkowski invariants for odd primes are trivial; the ones for $p = 2$ and $p = \infty$ are then equal, and determined by the signature. We suspect that an analogous result holds in general.

It seems to us that the correct method of attack on the problem, at least when π is finite, would be as follows. We must use Hasse's idea of proceeding from local to global results, and from results over fields to results over rings. The notion of extension fields does not quite apply (Λ is not a field), but the obvious substitute concept seems to work; namely, to consider the tensor product of Λ over **Z** with the reals, p-adic rationals, rationals, and p-adic integers in turn as the ground ring of the problem. So the ground ring is always semi-simple, though not simple; or, we have an order (not a maximal one) in a semi-simple algebra. Note that the invariant Σ decides the problem over the reals. We conjecture that the problem over p-adic rationals, p prime to $|\pi|$, is trivial.

Alternatively, we can consider the algebraic group (over **Q**) of automorphisms of a kernel (preserving φ, μ), and seek to determine its Galois cohomology. The classification of principal homogeneous spaces by the first cohomology group will give our rational classification; then we must use the validity of the strong approximation theorem in its connected and simply-connected covering group.

When we have settled the classification of (G, φ), we must turn to μ.

Here, we can easily give a theory which is complete in some cases.

LEMMA 4.10. *Let G be a sum of hyperbolic planes, and suppose $c(\mu_2) = 0$. Then we can write G as a sum of hyperbolic planes with $c(\mu_2) = 0$ for each.*

PROOF. We shall write μ_2 also for the composite map $G \to G_2 \to \mathbf{Z}_2$. Let e_i, f_i be the standard base of the i^{th} hyperbolic plane, so $\varphi(e_i, e_i) = 0$, $\varphi(e_i, f_i) = 1$. First make the substitution $f_i \to f_i - \eta e_i \mu(f_i)$, which reduces $\varphi(f_i, f_i)$ to 0.

By hypothesis, the number of values of i for which the corresponding hyperbolic plane has $c = 1$ (i.e., for which $\mu_2(e_i) = \mu_2(f_i) = 1$) is even. Group these in pairs, and suppose for convenience that $i = 1$, $i = 2$ is such a pair. Then

$$\begin{aligned} e_1' &= e_1 + e_2 \qquad & f_1' &= f_1 \\ e_2' &= e_2 & f_2' &= f_2 - f_1 \end{aligned}$$

gives a new decomposition of these two planes into two planes each with zero Arf invariant. This proves the lemma.

Now I is spanned by the $g - \eta w(g)\, g^{-1} (g \in \pi)$; in fact, g and g^{-1} give the same (up to sign), and these form a $\mathbf{Z}$-basis of I. So V is a sum of copies of $\mathbf{Z}$ and of $\mathbf{Z}_2$. A $\mathbf{Z}_2$ occurs whenever $g - \eta w(g)\, g^{-1} = 2g$; i.e., $g^2 = 1$ and $w(g) = -\eta$. Now $\mu + \eta\mu^- = 0$ implies that the components of μ in the infinite cyclic summands vanish. Hence if V has only one summand $\mathbf{Z}_2$ and this holds, μ vanishes if and only if its image in $V \otimes \mathbf{Z}_2 = \mathbf{Z}_2$ does.

LEMMA 4.11. *If V has no summand $\mathbf{Z}_2$, every hyperbolic plane is standard. If V has one summand $\mathbf{Z}_2$, every hyperbolic plane with zero Arf invariant is standard.*

PROOF. In the first case, $\varphi(e, e) = 0$ implies $\mu(e) = 0$. In the second, either $\mu_2(e) = 0$ or $\mu_2(f) = 0$; in the latter case, make the substitution $e \to f$, $f \to \eta e$. But now, by the remark above, $\varphi(e, e) = 0$ and $\mu_2(e) = 0$ imply $\mu(e) = 0$. The result now follows in either case by the last clause of 4.3.

We list a few examples of the above,

Cases where V has no summand $\mathbf{Z}_2$, $\eta = 1$, and $w = 1$ (e.g., $|\pi|$ odd).

Cases where V has one summand $\mathbf{Z}_2$, $\eta = -1$ and $|\pi|$ is odd, or, π has order 2 and w is non-trivial.

Case where V has two summands $\mathbf{Z}_2$, $\eta = -1$, $w \equiv 1$ and π has order 2. In this case we have the

Complement to Lemma 4.11. *If π has order* 2, $w \equiv 1$, *and* $\eta = -1$, *every hyperbolic plane with zero Arf invariant is standard.*

PROOF. Let (e, f) be the standard basis, so $\varphi(e, e) = \varphi(f, f) = 0, \varphi(e, f) = 1$. Since $c = 0$, we may suppose that μ_2 vanishes on the mod 2 reduction of e, so $\mu(e) = 0$ or $1 + T$. In the former case we are home by (4.3); the substitution $e \to e + f(1 + T)$ reduces the latter case to the former.

We make one further preliminary remark of a general nature before turning to particular cases.

LEMMA 4.12. *Suppose π finite or free. Let α denote a collection of invariants of (G, φ), additive, vanishing for hyperbolic planes, and such that $\alpha(G, \varphi) = 0$ implies that G has a free basis $e_1, \cdots, e_r$ with $\varphi(e_1, e_1) = 0$. Then $\alpha(G, \varphi) = 0$ implies that (G, φ) is a sum of hyperbolic planes.*

PROOF. By 4.3, G has a hyperbolic plane as direct summand: let P be the complement. Then P is a projective Λ-module, and the direct sum of P with a free module of rank 2 is free. Now if π is free, any projective Λ-module is free by Bass [2]. If π is finite, any projective Λ-module is the direct sum of a free Λ-module and an ideal of finite index in Λ, according to Swan [18], so if P is non-zero, its rank (which is even) must be at least 2. By another result of Swan [19], since the rank of P is at least 2, and P plus a free module is free, P is itself free.

Hence we can apply the above argument again to P (if $P \neq 0$), and continue splitting off hyperbolic planes till we reduce the complement to zero. The lemma follows.

PROPOSITION 4.13. *Suppose $\pi = \{1\}$. Then if $\eta = 1$, we have an isomorphism $\sigma/8$: $\tilde{\mathfrak{G}} \cong \mathbf{Z}$; if $\eta = -1$, an isomorphism c: $\tilde{\mathfrak{G}} \cong \mathbf{Z}_2$.*

This result is well-known; we include it for completeness.

PROOF. If $\eta = -1$, $\varphi(e, e) = - \varphi(e, e) = 0$ for any e. If $\eta = 1$, the hypothesis $\sigma(\varphi) = 0$ implies that the unimodular quadratic form $\varphi(x, x)$ represents zero (Milnor [12]); hence certainly we can find an indivisible e with $\varphi(e, e) = 0$; but, any indivisible element is part of a free basis. By (4.12) we have a direct sum of hyperbolic planes (provided $\sigma = 0$ if $\eta = 1$).

Now if $\eta = 1$, V has no summand $\mathbf{Z}_2$, so every hyperbolic plane is standard (4.11). If $\eta = -1$, V is isomorphic to $\mathbf{Z}_2$, so every hyperbolic plane with zero Arf invariant is standard (4.11). But by (4.10), if $c = 0$, G is a sum of hyperbolic planes with zero Arf invariant.

This shows that if $\eta = 1$, $\sigma : \tilde{\mathfrak{G}} \to \mathbf{Z}$, and if $\eta = -1$, $c : \tilde{\mathfrak{G}} \to \mathbf{Z}_2$, is monomorphic. When $\eta = -1$, c is onto as $\mu(e) = \mu(f) = 1$, $\varphi(e, f) = 1$ determine (by Remark 4.2) φ and μ, and it is trivial that φ is unimodular. When $\eta = 1$, φ is an even quadratic form so by [15, 106.1] $\sigma(\varphi)$ is divisible by 8. Moreover, [11] there exists an even quadratic form (G, φ) with signature 8 and determinant 1, and if we define $\mu(x) = \frac{1}{2}\varphi(x, x)$ we can easily check (P1) — (P6). This concludes the proof.

Complement. When $\eta = 1$, $c \equiv 0$ *and* $\Delta \equiv 1 \in \{\pm 1\}$. *When* $\eta = -1$, Σ *belongs to the zero group, and* $\Delta \equiv 1 \in \{\pm 1\}$. We need merely verify these facts on the examples given above.

We now propose to determine $\tilde{\mathfrak{G}}$ when π is cyclic of prime order p. Let T denote a generator of π. Then $\Lambda \otimes_Z \mathbf{Q} = \mathbf{Q}[T/T^p = 1]$ is a direct sum of two fields, isomorphic to $\mathbf{Q}$ and to $\mathbf{Q}[\zeta]$, where $\zeta = \exp(2\pi i/p)$. The ring Λ does not split accordingly (it is not a maximal order), but we can use the splitting as follows. Write $\Lambda_0 = \mathbf{Z}$, $\Lambda_1 = \mathbf{Z}[\zeta]$. Define homomorphisms $\alpha_0 : \Lambda \rightarrow \Lambda_0$ and $\alpha_1 : \Lambda \rightarrow \Lambda_1$ by $T \rightarrow 1$ and by $T \rightarrow \zeta$. Then we have an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\alpha} \Lambda_0 \oplus \Lambda_1 \xrightarrow{\beta} \mathbf{Z}_p \longrightarrow 0$$

where $\beta\,(m, \sum_0^{p-1} n_i\,\zeta^i) = \sum_0^{p-1} n_i - m \pmod p$.

G is a free Λ-module. Denote its tensor products (over Λ) with Λ_0, Λ_1 and $\mathbf{Z}_p$ by G_0, G_1, G_p respectively. ($\mathbf{Z}_p$ is made a Λ-module by the homomorphism $\beta\alpha_1 = -\beta\alpha_0$.) So we have an exact sequence

$$0 \longrightarrow G \xrightarrow{\alpha} G_0 \oplus G_1 \xrightarrow{\beta} G_p \longrightarrow 0\ .$$

If $w \equiv 1$, the involution 'bar' on Λ induces the identity on Λ_0 and complex conjugation (also denoted by a bar) on Λ_1. Tensor product then defines

$$\varphi_0 \colon G_0 \times G_0 \longrightarrow \Lambda_0 \qquad \varphi_1 \colon G_1 \times G_1 \longrightarrow \Lambda_1$$
$$\mu_0 \colon G_0 \longrightarrow V_0 \qquad \mu_1 \colon G_1 \longrightarrow V_1$$

(where the definitions of V_0 and V_1 from Λ_0 and Λ_1 are the same as of V from Λ) and of course, if $p = 2$, $\mu_2 \colon G_2 \rightarrow \mathbf{Z}_2$. Invariants σ and Δ for φ_1 (and for φ_0) can be constructed just as before. In this case, $\Delta(\varphi_1)$ lies in the group $\Delta(\Lambda_1)$ of real units of $\mathbf{Z}[\zeta]$, modulo norms of complex units.

THEOREM 4.14. *If* $p = 2$, $\eta = 1$, $\varepsilon \equiv 1$, *then* $(\frac{1}{8}\sigma(\varphi_0), \frac{1}{8}\sigma(\varphi_1)) \colon \tilde{\mathfrak{G}} \cong \mathbf{Z} + \mathbf{Z}$. *If* $p = 2$, *otherwise*, $c \colon \tilde{\mathfrak{G}} \cong \mathbf{Z}_2$. *If p is odd*, Φ *determines zero in* $\tilde{\mathfrak{G}}$ *if and only if* $\sigma(\varphi_1) = 0$, $\Delta(\varphi_1) = 1$, *and* (*if* $\eta = 1$) $\sigma(\varphi_0) = 0$, (*if* $\eta = -1$) $c = 0$.

PROOF. Let us first of all suppose $\varepsilon \equiv 1$ (necessarily true when $p \neq 2$). Then Theorem (4.13) tells us that G_0 is a sum of hyperbolic planes, provided (if $\eta = 1$) that $\sigma(\varphi_0) = 0$. If $p = 2$, the same applies to G_1. Hence we turn to G_1 when p is odd. The ground work here was done by Shimura [16].

First we consider equivalence over $\mathbf{Q}[\zeta]$. By [16, 5.8] $\sigma(\varphi_1) = 0$ and $\Delta(\varphi_1) = 1$ imply that φ_1 is isomorphic over $\mathbf{Q}[\zeta]$ to a kernel. Shimura proves this in the hermitan case, but we may convert the skew-hermitian case to it by multiplying values of φ_1 by the purely imaginary $\zeta - \zeta^{-1}$. This is not a unit, so the resulting form is modular, but not unimodular.

Next we consider equivalence over the local rings $\mathbf{Z}_{(q)}[\zeta]$ (where $\mathbf{Z}_{(q)}$ denotes the q-adic integers). Since these are principal ideal rings, torsion-free modules are free. We have a vector space $V = G_1 \otimes_Z \mathbf{Q}_{(q)}$ over $\mathbf{Q}_{(q)}[\zeta]$ with a free base $e'_1, \cdots, e'_r, f'_1, \cdots, f'_r$ such that $\varphi(e'_i, e'_j) = \varphi(f'_i, f'_j) = 0$, $\varphi(e'_i, f'_j) = \delta_{ij}$. Let K be the subspace spanned by $e'_1, \cdots, e'_r$. We regard $G_1 \otimes_Z \mathbf{Z}_{(q)} = G'_1$ as a lattice in V. Let $H_1 = G'_1 \cap K$. Then G'_1/H_1 is torsion-free, and so free, so H_1 is a direct summand of G'_1. Hence G'_1 has a free $\Lambda_1 \otimes_Z \mathbf{Z}_{(q)}$-basis $e_1, \cdots, e_r, f_1, \cdots, f_r$ with the e_i in K and so $\varphi(e_i, e_j) = 0$. The same proof as in (4.3) now shows that G'_1 is a sum of hyperbolic planes and (as in 4.10) it follows that each is standard for φ. (We have given our own proof since, in the ramified case $q = p$, our lattice is not maximal.)

By [16, 5.24], each genus g contains $[\mathfrak{C} : \mathfrak{C}_0] \times [\mathfrak{E}(g) : f_g(E_0)]$ classes of lattices over $\mathbf{Z}[\zeta] = \Lambda_1$. Here, $\mathfrak{C}$ is the ideal class group of $\mathbf{Q}[\zeta]$. It is seen from the proof of the theorem and from [16, 2.15.1] that the first obstruction vanishes when we have a free Λ_1-module G_1. As to the second, we assert $\mathfrak{E}(g) = f_g(E_0)$. For by [16, 4.18] since the only ramified prime—$(\zeta - 1)$—is odd, $|\mathfrak{E}(g)| = 1$ or 2. By the proof of [16, 4.16], $\mathfrak{E}(g)$ is generated by the image of $(\zeta - 1)$ under $a \to a^{-1} a^{-}$, i.e., by $- \zeta^{-1}$. But this is in E_0, so the index is 1 as asserted.

Thus our hypotheses imply G_1 a kernel over Λ_1, as well as G_0 over Λ_0. We must now return to G. By Lemma 4.12, to show that G is a sum of hyperbolic planes, it will suffice to prove that G has a free basis $e_1, \cdots, e_r$ with $\varphi(e_1, e_1) = 0$, or even that $e_1 \in G$ generates a free direct summand. It will then follow from 4.10, 4.11, and its complement that (still assuming $w \equiv 1$), our hypotheses imply that G is a kernel. Let us call an element which generates a free direct summand *primitive*, and if $\varphi(e, e) = 0$, call e *isotropic*.

We assert that an element of G is primitive if and only if its images in G_0 and G_1 both are; moreover, our exact sequence shows that a pair of elements, one from G_0 and one from G_1, have a common antecedent in G if and only if they have a common image in G_p. To prove our assertion, let $e \in G$ determine $e_0 \in G_0$, $e_1 \in G_1$, which generate free direct summands $\Lambda_0 e_0$, $\Lambda_1 e_1$. If we choose retractions, we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & G_0 \oplus G_1 & \longrightarrow & G_p & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda_0 \oplus \Lambda_1 & \longrightarrow & \Lambda_p & \longrightarrow & 0 \, ; \end{array}$$

if the composite map $G \to \Lambda_p$ vanishes, we have an induced map $G \to \Lambda$ with $e \to 1$, so e is primitive. So we choose a retraction $G_1 \to \Lambda_1$; the composite

$G_1 \to \Lambda_1 \to \Lambda_p \cong \mathbf{Z}_p$ necessarily factors through $G_1/(\zeta - 1)G_1 = G_p$ (since $(\zeta-1)^p/p$ is a unit), so also defines a map $G_0 \to G_p \to \mathbf{Z}_p$; and $e_0 \to 1$. As G_0 is free abelian with e_0 primitive, we can lift this to a map $G_0 \to \mathbf{Z}$ with $e_0 \to 1$, as required.

Now G_0 and G_1, being kernels, certainly contain primitive isotropic vectors; these reduce to non-zero isotropic elements of G_p. We assert that, conversely, if the rank r of G is 2, every non-zero isotropic element of G_p arises from a primitive isotropic element of G_1; if $r \geqq 4$, it arises from a primitive isotropic element of G_0. Thus in any case, we can find a pair (e_0, e_1) of primitive isotropic vectors in G_0, G_1 with common image in G_p, and so defining (as required) a primitive $e \in G$ which is clearly isotropic.

First let $r = 2$, and (e, f) be a standard base for G_1. Any isotropic vector of G_p is the reduction of either re or rf for some r. Hence it is also that reduction of $e(\sum_0^{r-1} \zeta^i)$ which, as $\sum_0^{r-1} \zeta^i$ is a unit, is primitive. Now let $r \geqq 4$. Then the group of automorphs of G_p is transitive on isotropic vectors, and each automorph of G_p is the mod p reduction of one of G_0: our assertion follows at once from these two facts. The proof is virtually identical with that of [21, Th. 1] (in the case $r = 4$): the formulas for α, β, γ, δ, δ', ε define automorphs of G_0 and of G_p, and the group on G_p is transitive on isotropic vectors, hence (by the proof of the Corollary) the whole group of automorphs. The extension to higher values of r is immediate.

We must now return to the case when $w = -1$, and so $p = 2$. Here, bar does not induce involutions of Λ_0 and Λ_1; instead, it interchanges them. Hence φ induces, not forms on G_0 and G_1, but a non-singular pairing $\psi\colon G_0 \times G_1 \to \Lambda_1$. For if $\alpha_0(x) = \alpha_0(x')$ and $\alpha_1(y) = \alpha_1(y')$, we can check that $\varphi(x', y') - \varphi(x, y)$ is a multiple of $T + 1$, so $\alpha_1\varphi(x, y) = \alpha_1\, \varphi(x', y')$. As before, the mod 2 reduction of ψ is the bilinear map $\varphi_2\colon G_2 \times G_2 \to \Lambda_2$ associated to μ_2. Choose a basis $h_1, \cdots, h_r, h'_1, \cdots, h'_r$ of G_2 with $\varphi_2(h_i, h_j) = \varphi_2(h'_i, h'_j) = 0$, $\varphi_2(h_i, h'_j) = \delta_{ij}$. Lift to a free basis $g_1, \cdots, g_r, g'_1, \cdots, g'_r$ of G_0. Let the dual (with respect to ψ) basis of G_1 be $f_1, \cdots, f_r, f'_1, \cdots, f'_r$. These elements have mod 2 reductions $h'_1, \cdots, h'_r, h_1, \cdots, h_r$. Hence G has a free basis $e_1, \cdots, e_r$, $e'_1, \cdots, e'_r$ with $\alpha(e_i) = (g_i, f'_i)$, $\alpha(e'_i) = (g'_i, f_i)$. Then $\alpha_1\, \varphi(e_i, e'_j) = \psi(g_i, f_j) = \delta_{ij}$, and

$$\alpha_0\, \varphi(e_i, e'_j) = \alpha_1\, \varphi(e'_j, e_i)^- = \psi(g'_j, f'_i)^- = \delta_{ij}^- = \delta_{ij}\ ,$$

so $\varphi(e_i, e'_j) = \delta_{ij}$. Similarly, $\varphi(e_i, e_j) = \varphi(e'_i, e'_j) = 0$. So, without any assumptions, G is a sum of hyperbolic planes. Lemmas 4.10 and 4.11 now imply that if $c = 0$, G is a kernel.

It remains to give examples which show that our invariants can be non-trivial. That c can be non-zero in the stated cases follows from (4.2), which shows that the $\mathbf{Z}_2$-components of the values of μ on the basis elements can be chosen independently of φ. Now let $p = 2$, $\eta = 1$, $w = 1$. That $\sigma(\varphi_0)$ and $\sigma(\varphi_1)$ are both divisible by 8 follows as in (4.13). Examples with $\sigma = 8$ were constructed there.

Now suppose given $\varphi_0\colon G_0 \times G_0 \to \Lambda_0$ and $\varphi_1\colon H_1 \times H_1 \to \Lambda_1$, satisfying the (modified) (P1) – (P4). These induce (*via* tensor product)

$$\varphi_p\colon G_p \times G_p \longrightarrow \Lambda_p \qquad \varphi'_p\colon H_p \times H_p \longrightarrow \Lambda_p\ ,$$

which are just non-singular symmetric or skew-symmetric forms over $\mathbf{Z}_p$, of the same ranks as G_0, H_1 which we may suppose equal and even. Then in the skew case, φ_p and φ'_p are necessarily isomorphic; in the symmetric case there are two types, distinguished (if p is odd) by discriminant, or (if $p = 2$) by the existence or not of a corresponding μ. If we assume an isomorphism, we can define G as the kernel of the difference map $G_0 + H_1 \to G_p$, and the restriction of $\varphi_0 + \varphi_1$ to G will then necessarily take values in Λ and satisfy (P1) – (P4).

5. Surgery in the middle dimension: case $m = 2k - 1$

We return again to the problem of § 1, where we are doing surgery on a map $\psi\colon M^m \to X$, and $m = 2k - 1$. By the results of § 1, we can do surgery to make ψ $(k-1)$-connected, but to go further needs a more detailed argument. For although given an element x of $G = \pi_k(\psi)$, we can imbed a sphere representing x and do surgery, this will not necessarily simplify G: the exact result will be discussed below. In fact, by the remark preceding Theorem (1.4), x determines a regular homotopy class of immersions, and we can use any imbedding in this class, but different imbeddings will in general lead to different results.

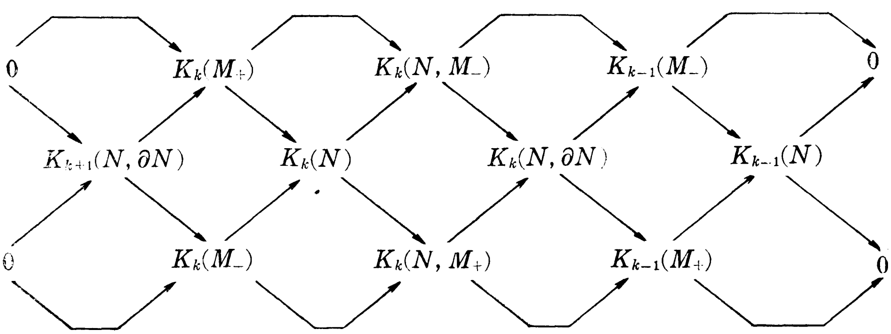
Consider a regular homotopy $H\colon S^{k-1} \times I \to M^{2k-1} \times I$ between two imbeddings. We may suppose H in 'general position', so that its image has (at most) isolated double points, with transverse intersections. To count these, we adopt the same conventions as in § 3 for spheres. Then the self-intersections of H will be measured by an invariant μ in the value group $V = \Lambda/I$.

If μ vanishes and $k \geqq 3$, the same technique as in § 3 can be used to remove the self-intersections of H, keeping the ends fixed, and also keeping H level-preserving, so that in this case the spheres are isotopic. We omit the details as we shall have no need to use this result. We do not assert that if $\mu \neq 0$ the spheres cannot be isotopic (this is in general false).

However, if $k \geqq 2$, given an imbedding $f\colon S^{k-1} \to M$ and $\mu \in V$, we assert the existence of a regular homotopy of f with self-intersection μ. For since $V = \Lambda/I$, it is sufficient to be able to obtain any element of Λ: since we can always follow one homotopy by another, and μ is then additive, it is sufficient to be able to obtain any $\pm g$, $g \in \pi$. An example, with $k = 2$, of a regular homotopy with self-intersection ± 1 is given if we have a knot of S^1 in $\mathbf{R}^3$, described in the usual way [5] *via* a projection on $\mathbf{R}^2$, with some double points where one branch crosses over another. For our regular homotopy, we seize the lower branch at such a point and move it upwards, cutting through the upper branch as we do so. Similarly, for any value of k, given two $(k-1)$-discs in $\mathbf{R}^{2k-1}$, we can isotope one to lie 'underneath' the other, and then move it upwards cutting through the other at one point. Now, to obtain our general regular homotopy, we take two discs near the base-point of S^{k-1}, move one round a loop in M, representing g, and disjoint (except at the ends) from S^{k-1} (here is where we need $k \geqq 2$), and then perform the above construction in a neighbourhood of the base point.

We observe that the imbeddings and immersions under discussion are all, strictly speaking, based and framed. For example, although any two $(2l-1)$-spheres in $\mathbf{R}^{4l-1}$ are isotopic, if we take a regular homotopy with self-intersection $\mu \in V \cong \mathbf{Z}$ from a $(2l-1)$-sphere to itself, we return with a framing altered by the image of μ under $\mathbf{Z} \cong \pi_{2l}(S^{2l}) \xrightarrow{\partial_*} \pi_{2l-1}(SO_{2l})$. This explains the relation of our μ to the framing used by Kervaire and Milnor [10].

Now that we know which surgeries are possible, we must check what effect they have on the group G which we are trying to kill. We shall use the notation introduced for (2.1) and (2.5). Also, let $f_-\colon S^{k-1} \times D^k \to M_-$ be the map used for surgery; $f_+\colon D^k \times S^{k-1} \to M_+$ the complementary map. We have the commutative exact diagram



where the coefficient module Λ is understood. Also, $f_+(D^k \times 1)$ represents the generator of $K_k(N, M_-) \cong \Lambda$, and $f_-(1 \times D^k)$ that of $K_k(N, M_+) \cong \Lambda$; we write ε_-, ε_+ for their images in $K_k(N, \partial N)$; x_-, x_+ for their images in $K_{k-1}(M_-) = G_-$ and $K_{k-1}(M_+) = G_+$ respectively. Note that ε_- generates the kernel of $K_k(N, \partial N) \to G_+$; similarly for ε_+. The map $K_k(M_+) \to K_k(N, M_-) \cong \Lambda$ is induced by intersection numbers with x_+: $y \to x_+ \frown y$. There are duality relations, such as $K_k(M_+) \cong K^{k-1}(M_+) \cong \mathrm{Hom}_\Lambda(K_{k-1}(M_+), \Lambda) = \mathrm{Hom}_\Lambda(G_+, \Lambda)$. All these results follow from § 2, particularly (2.5). Also, again by [23, Th. A], G_- and G_+ are finitely generated: this is almost all we can say about them as modules.

Any $x \in G_-$ determines two ideals of Λ, namely

$$A(x) = \{\lambda \in \Lambda : x\lambda = 0\}$$
$$B(x) = \{f(x) \mid f \in \mathrm{Hom}_\Lambda(G_-, \Lambda)\}:$$

$A(x)$ is a right ideal, $B(x)$ a left ideal.

We have just observed that $K_k(M_-) \cong \mathrm{Hom}_\Lambda(G_-, \Lambda)$, so $B(x)$ consists of intersection numbers $y \frown x$. Its conjugate $B(x)^-$, consisting of all $x \frown y$, is the image of $K_k(M_-) \to \Lambda$, or the kernel of $K_k(N, M_+) \to K_k(N, \partial N)$ above (if we do surgery starting with x), hence $B(x_-)^- = A(\varepsilon_+)$.

We call x (provisionally) a *torsion element* of G_- if $B(x) = 0$. The torsion elements of G_- form a subgroup G_-^*; the intersection of the kernels of all homomorphisms $G_- \to \Lambda$.

Now assume π finite. Then $B(x) = 0$ if and only if $A(x)$ contains some non-divisor of zero, and hence some non-zero integer. Write $\bar{\Lambda} = \mathbf{Q} \otimes_Z \Lambda = \mathbf{Q}[\pi]$; then $\bar{\Lambda}$ is an injective Λ-module. (Analogous results can be obtained when π is the infinite cyclic group generated by t by writing $\bar{\Lambda} = \mathbf{Q}(t)$, but the general case is not yet clear: we think that it should be sufficient to take $\bar{\Lambda}$ as the injective envelope of Λ (as Λ-module) when Λ is noetherian.)

For our detailed discussion, we shall need to use linking numbers. Write $J = \{\nu + (-1)^k \bar{\nu} : \nu \in \Lambda\}$, in analogy to I. We make the

HYPOTHESIS 5.1. *M^m is a connected compact manifold; (X, W) a Poincaré pair, $\psi: (M, \partial M) \to (X, W)$ is of degree 1 and induces a homotopy equivalence $\partial M \to W$ and a $(k-1)$-connected map $M \to X$, $m = 2k - 1$; there is a map ω as in §1. Also, $k \geqq 2$ and π is finite.*

THEOREM 5.2. *Under (5.1), linking numbers define a non-singular, $(-1)^k$-hermitian bilinear form $b: G^* \times G^* \to \bar{\Lambda}/\Lambda$. Self-linking defines a homogeneous quadratic map $q: G^* \to \bar{\Lambda}/J$, with associated bilinear map $b + (-1)^k b^-$. Moreover, if we do surgery on $x \in G^*$, and $\lambda \in A(x)$, $\varepsilon_-\lambda = \varepsilon_+(q(x)\lambda)$, where different determinations of $q(x)$ correspond to different spheres representing x. Also, if $\bar{y} \in K_k(N, \partial N)$ maps to $y \in G^*$, and $\lambda_0 \in A(y)$, we have $\bar{y}\lambda_0 = \varepsilon_+(b(y, x)\lambda_0)$, where the different determinations of $b(y, x)$ correspond to the different choices of $\bar{y}$ given y.*

PROOF. The exact sequence $0 \to \Lambda \to \bar{\Lambda} \to \bar{\Lambda}/\Lambda \to 0$ of coefficient modules induces exact sequences for M in homology and cohomology:

$$\begin{array}{ccccccccccccc} 0 \to & K_k(\Lambda) & \to & K_k(\bar\Lambda) & \xrightarrow{\alpha} & K_k(\bar\Lambda/\Lambda) & \xrightarrow{\beta} & K_{k-1}(\Lambda) & \xrightarrow{\gamma} & K_{k-1}(\bar\Lambda) & \to & K_{k-1}(\bar\Lambda/\Lambda) & \to 0 \\ & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & \\ 0 \to & K^{k-1}(\Lambda) & \to & K^{k-1}(\bar\Lambda) & \xrightarrow{\alpha'} & K^{k-1}(\bar\Lambda/\Lambda) & \xrightarrow{\beta'} & K^k(\Lambda) & \xrightarrow{\gamma'} & K^k(\bar\Lambda) & \to & K^k(\bar\Lambda/\Lambda) & \to 0\,. \end{array}$$

We can identify G^* with the kernel of γ, hence with the image of β, or cokernel of α, or of α'. But we have already remarked in §2 that, (by left exactness of Hom_Λ) $K^{k-1}(B) \cong \mathrm{Hom}_\Lambda(G, B)$. Now we also have an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(G, \Lambda) \to \mathrm{Hom}_\Lambda(G, \bar\Lambda) \xrightarrow{\alpha'} \mathrm{Hom}_\Lambda(G, \bar\Lambda/\Lambda) \to \mathrm{Ext}^1_\Lambda(G, \Lambda) \to \mathrm{Ext}^1_\Lambda(G, \bar\Lambda)$$

where the last term vanishes since $\bar\Lambda$ is injective, so the cokernel of α' can also be identified with $\mathrm{Ext}^1_\Lambda(G, \Lambda)$. We shall make yet a third identification. Let δ: $\mathrm{Hom}_\Lambda(G, \bar\Lambda/\Lambda) \to \mathrm{Hom}_\Lambda(G^*, \bar\Lambda/\Lambda)$ be the obvious map; since G^* is torsion, any homomorphism to $\bar\Lambda$ (or to Λ) is zero, so $\delta\alpha' = 0$, and δ factorises through the cokernel, inducing a map $G^* \to \mathrm{Hom}_\Lambda(G^*, \bar\Lambda/\Lambda)$. This defines the required bilinear map b. To prove it an isomorphism, and so b non-singular, we use the two exact sequences

$$\mathrm{Hom}_\Lambda(G^*, \bar\Lambda) = 0 \longrightarrow \mathrm{Hom}_\Lambda(G^*, \bar\Lambda/\Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(G^*, \Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(G^*, \bar\Lambda) = 0\,.$$

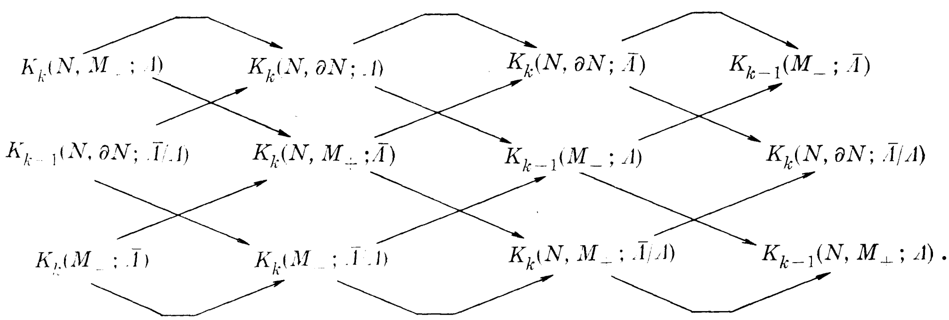
$$\mathrm{Ext}^1_\Lambda(G/G^*, \Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(G, \Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(G^*, \Lambda) \longrightarrow \mathrm{Ext}^2_\Lambda(G/G^*, \Lambda)\,.$$

The extreme terms of the second sequence vanish since G/G^* is torsion-free, hence **Z**-projective and ([4, XII, 1.1]) Λ is weakly injective, so we can appeal to [4, X, 8.2a]. Putting the results together gives isomorphisms

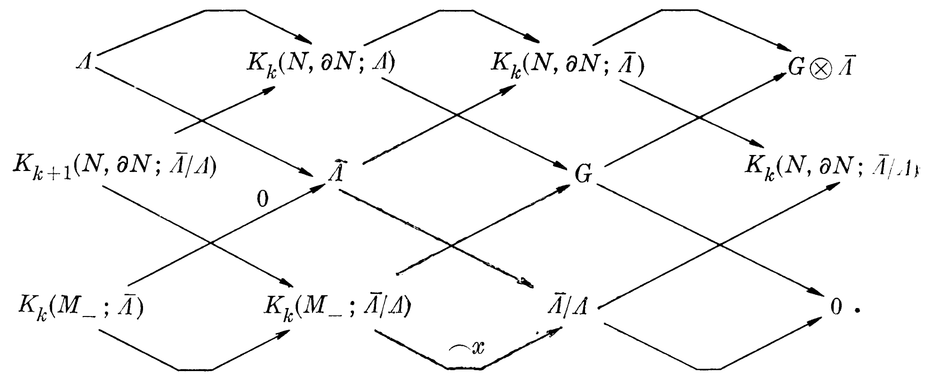
$$G^* \longrightarrow \mathrm{Ext}^1_\Lambda(G, \Lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(G^*, \Lambda) \longleftarrow \mathrm{Hom}_\Lambda(G^*, \bar\Lambda/\Lambda)$$

whose composite is the map constructed above.

We next relate b to surgery. For this we combine the surgery exact sequence arising from $M_- \to (N, M_+) \to (N, \partial N)$ with the short exact sequence of coefficient modules $\Lambda \to \bar\Lambda \to \bar\Lambda/\Lambda$: we obtain the diagram



Since N is obtained from M_+ by attaching a single k-handle, we know the relative groups of (N, M_+). Also, the homomorphisms into these are given (2.5) by intersection numbers with x; recall that in our case, x is torsion, so the map to $\bar{\Lambda}$ is zero. Hence we can rewrite the diagram as



Now given $y \in G^*$, $y \to 0$ in $G \otimes \bar{\Lambda}$, and so lifts to some element $t \in K_k(M_-; \bar{\Lambda}/\Lambda)$. By definition, $b(y, x) = t \frown x$, the image of t in $\bar{\Lambda}/\Lambda$. But y also lifts to $\bar{y} \in K_k(N, \partial N; \Lambda)$ which has image, say, $y \in K_k(N, \partial N; \bar{\Lambda})$. By commutativity, the image of $\dot{y}$ in $G \otimes \bar{\Lambda}$ is the same as that of y, i.e., 0, so y lifts to a unique element $\beta \in \bar{\Lambda}$. We assert that β has image $b(y, x) \in \bar{\Lambda}/\Lambda$; this can be seen by a prolonged diagram chase.

Now $y\lambda_0 = 0$, so $\bar{y}\lambda_0$ lifts to some element of Λ; since the maps $\Lambda \to \bar{\Lambda} \to K_k(N, \partial N; \bar{\Lambda})$ are both monomorphisms, this element must be just $\beta\lambda_0$. This proves the last assertion of the theorem, since the choices of $\bar{y}$ and of β, given y, can only vary by the image of some element of Λ.

The form b is related to geometry as follows. Let $x, y \in G$, $\lambda \in A(x)$. We can represent x and y by spheres ξ and η imbedded disjoint in $\tilde{M}$. As $x\lambda = 0$, $\xi\lambda$ is a boundary; say $\xi\lambda = \partial\zeta$: we can choose the k-chain ζ to meet η transversely, so the intersection invariant $\iota_\lambda = \eta \frown \zeta$ is defined (as in §3). If ζ is changed, ι_λ is altered by an element of $B(y)$, so if y is torsion, ι_λ depends only on ξ, η, and λ. Also, clearly, $\iota_{\lambda\rho} = \iota_\lambda \rho$, so $\lambda \to \iota_\lambda$ is a module homomorphism $A(x) \to \Lambda$, and if x is also torsion, this extends uniquely to $\Lambda \to \bar{\Lambda}$; the image of the unit can be defined (for example) as $(1/r)\iota_r$, where $r \in \mathbf{Z} \cap A(x)$.

If ξ is changed, say to $\xi + \partial\zeta_0$, we can change ζ_0 to $\zeta + \zeta_0\lambda$, so ι_λ is changed to $\iota_\lambda + (\eta \frown \zeta_0)\lambda$; similarly if η is changed to $\eta + \partial\tau_0$, the change is $\partial\tau_0 \frown \zeta = \partial(\tau_0 \frown \zeta) + (-1)^k \tau_0 \frown \partial\zeta = 0 + (-1)^k(\tau_0 \frown \xi)\lambda$. In either case, the resulting $(1/r)\iota_r \in \bar{\Lambda}$ is changed only by an element of Λ. It is now clear that its class in $\bar{\Lambda}/\Lambda$ is precisely $b(x, y)$. We have only to observe that the intersection numbers used here are equivalent to (and essentially the same as) the duality isomorphisms in our first version of the definition.

To obtain the symmetry property of b, write $\xi r = \partial\zeta$, $\eta s = \partial\tau$; then using (as in the preceding paragraph) the fact that the boundary of a 1-chain always has zero Kronecker index, we have

$$\begin{aligned} 0 = \partial(\tau \frown \zeta) &= \partial\tau \frown \zeta + (-1)^{k-1}\tau \frown \partial\zeta \\ &= \partial\tau \frown \zeta + (-1)^{k-1}(\partial\zeta \frown \tau)^- \\ &= \eta s \frown \zeta + (-1)^{k-1}(\xi r \frown \tau)^- \\ &= s(\eta \frown \zeta) + (-1)^{k-1} r(\xi \frown \tau)^- \\ &= rs\{b(x, y) + (-1)^{k-1} b(y, x)^-\} \qquad \bmod rs\,\Lambda\,, \end{aligned}$$

so b is $(-1)^k$-hermitian, as stated.

To obtain q, we must be more precise. If $x \in G$ is represented by ξ, ξ is a framed immersion (or imbedding) of S^{k-1}. We obtain ξ' from ξ by moving a small distance along the first vector of the framing; then ξ' and ξ are disjoint. Taking ξ' for ξ and ξ for η above we obtain, if $x\lambda = 0$, an element ι_λ of Λ. Now any other representation of x is regularly homotopic to ξ; we have moreover analysed regular homotopies. Note that since the track of a regular homotopy with point self-intersections is homotopy equivalent to ξ with 1-cells attached, we may suppose the track disjoint from ζ. Write η for the track of the regular homotopy, η' for its deformation along the first vector of the framing. Then if ξ_0, ξ_1 are the initial and final positions of ξ, we have

$$\partial\zeta = \xi_0'\lambda\,, \qquad \iota_\lambda^0 = \xi_0 \frown \zeta\,, \qquad \partial\eta' = \xi_1' - \xi_0',$$

so $\partial(\zeta + \eta'\lambda) = \xi_1'\lambda$,

$$\iota_\lambda^1 = \xi_1 \frown (\zeta + \eta'\lambda) = \xi_0 \frown \zeta + (\xi_1 - \xi_0) \frown \zeta + (\xi_1 \frown \eta')\lambda\,.$$

Here, the first term is ι_λ^0; the second is

$$\partial\eta \frown \zeta = (-1)^k\, \eta \frown \partial\zeta = (-1)^k\, \eta \frown \xi_0'\lambda\,.$$

So $\iota_\lambda^1 - \iota_\lambda^0 = \gamma\lambda$, where $\gamma = \xi_1 \frown \eta' + (-1)^k(\eta \frown \xi_0')$. But here we have been considering only the track of the homotopy; now recall that self-intersections were originally computed in $\widetilde{M} \times I$. Let $\bar{\eta}$, $\bar{\eta}'$ denote the regular homotopies in $\widetilde{M} \times I$, projecting on η and η'. Let μ be the invariant of the homotopy, i.e., the self-intersection invariant of $\bar{\eta}$ (as in § 3), and $\bar{\eta} \frown \bar{\eta}' = \mu + (-1)^k\bar{\mu}$. But since $\bar{\eta}'$ is isotopic (relative to its boundary) to $\xi_0' \times I \cup \eta' \times 1$,

$$\begin{aligned} \bar{\eta} \frown \bar{\eta}' &= \bar{\eta} \frown (\xi_0' \times I \cup \eta' \times 1) \\ &= \pm(\eta \frown \xi_0') \pm (\xi_1 \frown \eta') \end{aligned}$$

where considerations of orientation show that the signs $(-1)^k$, 1 must be taken. Thus, finally,

$$\iota_\lambda^1 - \iota_\lambda^0 = \left(\mu + (-1)^k\bar{\mu}\right)\lambda$$

showing that, in particular, $(1/r)\iota_r$ is changed by exactly $\mu + (-1)^k\mu^-$, a typical element of J. Hence we are at liberty to write $q(x)$ for its class modulo J.

We pause to mention that of course $q(x)$ cannot be an arbitrary element of $\bar{\Lambda}/J$: it must in fact be a torsion element. For if $rx = 0$, we have seen that $0 = \iota_r + (-1)^{k-1}\iota_r^-$, so $2\iota_r \in J$, and $2r\,q(x) = 0$. Also, it is clear from the definition that modulo Λ, $q(x)$ reduces to $b(x, x)$.

It is evident that $q(-x) = q(x)$. To establish the quadratic character of q, we represent x and y by imbedded spheres ξ and η, and join by a tube to find a sphere representing $x + y$. We also have a parallel tube joining ξ' and η'. Choose r such that $rx = 0 = ry$, and write $r\xi' = \partial\zeta$, $r\eta' = \partial\tau$, where ζ and τ may be supposed disjoint from the tubes, and meeting ξ and η transversely. Then for the sum of ξ' and η', we use $\zeta + \tau$ plus r times a solid tube, disjoint from the tube joining ξ and η. Then $r\,q(x + y)$ is represented by the intersection of this with the 'sum' of ξ and η, i.e., by

$$(\xi + \eta)\frown(\zeta + \tau) = \xi\frown\zeta + \eta\frown\zeta + \xi\frown\tau + \eta\frown\tau \,.$$

Here, the first and last terms represent $rq(x)$ and $rq(y)$. The second represents $rb(x, y)$; moreover, we saw above that

$$r\,\{\eta\frown\zeta + (-1)^{k-1}(\xi\frown\tau)^-\} = 0 \,,$$

so the sum of the two middle terms represents $r(b(x, y) + (-1)^k\, b(x, y)^-)$ modulo rJ, and not merely modulo $r\Lambda$. Hence dividing by r, we obtain, modulo J, the equation

$$q(x + y) = q(x) + q(y) + b(x, y) + (-1)^k\, b(x, y)^- \,,$$

as asserted.

It remains only to relate q to surgery. We will first give a geometrical account of the relation of b to surgery. Let $y\lambda_0 = 0$, and let the disjoint imbedded spheres, ξ, η represent x, y. We form N from $M \times I$ by attaching a disc δ to $\xi \times 1$, and thickening, so $M_- = M \times 0$. Thus y lifts to $\bar{y} \in K_k(N, \partial N)$ represented by $\eta \times I$. As $y\lambda_0 = 0$, we can write $\eta\lambda_0 = \partial\tau$. And $\bar{y}\lambda_0$ lifts to a class $y \in K_k(N, M_+)$, represented by $(\eta \times I)\,\lambda_0 + \tau \times 0$. But $K_k(N, M_+) \cong \Lambda$, the isomorphism being given by intersection numbers with δ. So $\bar{y}\lambda_0 = \varepsilon_+\,\alpha$, where

$$\alpha = \delta\frown\{(\eta \times I)\lambda_0 + (\tau \times 0)\} = \delta\frown(\tau \times 0) = \xi\frown\tau = b(y, x)\lambda_0 \,.$$

Now recall that ε_- is the class of $D^k \times 1$, or δ', in $K_k(N, \partial N)$. The boundary $S^{k-1} \times 1$ is just the sphere we have called ξ' (or $\xi' \times 0$). We have $x\lambda = 0$, and write $\xi'\lambda = \partial\zeta$. Then $\varepsilon_-\lambda$ is the image of the class $\bar{\varepsilon}$ in $K_k(N, M_+)$ represented by $(-\delta'\lambda + \zeta)$; and $\bar{\varepsilon}$ is β times the generator, or $\varepsilon_-\lambda = \varepsilon_+\beta$, where

$$\beta = \delta\frown(\zeta - \delta'\lambda) = \delta\frown\zeta = \xi\frown\zeta = q(x)\lambda \ .$$

(Note the sign $-\delta'\lambda$; the map $K_k(N, \partial N) \to K_{k-1}(M_-)$ was defined as minus the boundary map.)

This completes the proof of the theorem.

We have now described our linking numbers and shown how they influence surgery. Next we will try to formulate necessary and sufficient conditions for killing the group G. First suppose that it is possible to do a sequence of surgeries on $M = M_-$, giving a cobordism N of M_- to M_+, and extending ψ_M to a map $N \to X$, whose restriction to M_+ is a homotopy equivalence. Choose also a map $N \to I$ with $M_- \to 0$ and $M_+ \to 1$. The product gives a map of degree 1 of Poincaré triads (as in § 2).

$$\psi: (N, M_-, M_+, \partial M \times I) \longrightarrow (X \times I, X \times 0, X \times 1, W \times I)$$

where $\partial M \times I$ may be adjoined to M_- or to M_+ (similarly for $W \times I$). As $\dim N = 2k$, we can perform surgery until ψ_N is k-connected. The exact sequences (which hold by (2.2)) now show that all K-groups of N, of (N, M_-), of (N, M_+) and of $(N, \partial N)$ vanish except in dimension k. Now by (2.4), these groups (with coefficient module Λ) are all projective Λ-modules, and if X and W are finite, we can add finitely generated free modules to make them free. We perform surgery on $(k - 1)$-spheres imbedded trivially on N a number of times (in fact once is enough, if π is finite): this adds $\Lambda \oplus \Lambda$ to each module, so we may suppose them free.

Thus all that is left is an exact sequence

$$0 \longrightarrow K_kM \longrightarrow K_kN \longrightarrow K_k(N, M) \longrightarrow K_{k-1}M \longrightarrow 0 \ ,$$

where $K_kN \cong K_k(N, M_+)$ and $K_k(N, M) \cong K_k(N, \partial N)$ are dual to each other, and hence are free modules of the same rank. The map $K_k(N) \to K_k(N, M)$ corresponds to a bilinear form on $K_k(N)$ with values in Λ; this, of course, is none other than the pairing φ which we studied in § 3. Here, of course, it is no longer non-singular, but the proof of (3.1) includes this case.

LEMMA 5.3. *In the situation above, the form φ on $K_k(N)$ determines b and q on G^* ($G = K_{k-1}(M)$) as follows: Let $x, y \in G^*$ lift to $\bar{x}, \bar{y} \in K_k(N, M)$. If $x\lambda = 0$, lift $\bar{x}\lambda$ to $\bar{\bar{x}} \in K_k(N)$, and write $\iota_\lambda = \langle \bar{y}, \bar{\bar{x}} \rangle$. Then the ι_λ determine b and q as in Theorem* 5.2; i.e. *if $rx = 0$,*

$$b(x, y) = (1/r)\iota_r \ (\text{mod } \Lambda), \quad q(x) = (1/r)\langle \bar{x}, \bar{\bar{x}} \rangle \ (\text{mod } J) \ .$$

PROOF. Let ξ be a sphere representing x, and $\bar{\xi}$ a k-chain of (N, M) (or rather, of the universal cover) representing $\bar{x}$, with $\partial\bar{\xi} = -\xi$. Similarly, define η and $\bar{\eta}$. Now $x\lambda = 0$, so we can write $\xi\lambda = \partial\zeta$ for some k-chain ζ of $\tilde{M}$. Then $\bar{\bar{x}}$ may be taken as the class of $\bar{\xi}\lambda + \zeta$, and $\langle \bar{y}, \bar{\bar{x}} \rangle = \bar{\eta} \frown (\bar{\xi}\lambda + \zeta) = (\bar{\eta} \frown \bar{\xi})\,\lambda + \bar{\eta} \frown \zeta$ so that $(1/r)\iota_r$ calculated as above differs from our previous value only by $(\bar{\eta} \frown \bar{\xi}) \in \Lambda$. Thus the assertion about b is correct.

As to q, we first note that since $x \longrightarrow 0$ in $K_{k-1}(N)$, ξ is regularly null-homotopic in N, and bounds an immersed disc $\bar{\xi}$ where $-\bar{\xi}$ represents some choice of $\bar{x} \in K_k(N, M)$. Moving this parallel to itself (as usual), we obtain $\bar{\xi}'$, with boundary ξ'. Set $\partial\zeta = \xi'\lambda$. Then $-\bar{\xi}$ represents $\bar{x}$, $(-\bar{\xi}'\,\lambda + \zeta)$ represents some $\bar{\bar{x}}$, and the intersection is $-(\bar{\xi} \frown \bar{\xi}')\,\lambda + (\bar{\xi} \frown \zeta) = -(\bar{\xi} \frown \bar{\xi}')\,\lambda + \xi \frown \zeta$. But here again, $\bar{\xi} \frown \bar{\xi}'$ is (as in § 3) an element of J, and $\xi \frown \zeta$ is the same ι_λ as we used before. Thus we obtain the correct value of $q(x)$.

To complete the lemma, we show that altering the choices of $\bar{x}$ and $\bar{\bar{x}}$ will only alter $\langle \bar{x}, \bar{\bar{x}} \rangle$ by an element of $J\lambda$. For if we add to $\bar{x}$ the image of $\bar{\bar{z}}$ (and hence to $\bar{x}\lambda$ the image of $\bar{\bar{z}}\lambda$), and to $\bar{\bar{x}}$ add $\bar{\bar{z}}\lambda$ plus $\bar{\bar{w}}$ (where $\bar{\bar{w}}$ goes to 0 in $K_k(N, M)$), we replace $\langle \bar{x}, \bar{\bar{x}} \rangle$ by

$$\langle \bar{x} + A\varphi(\bar{\bar{z}}),\ \bar{\bar{x}} + \bar{\bar{z}}\lambda + \bar{\bar{w}} \rangle = \langle \bar{x}, \bar{\bar{x}} \rangle + \langle \bar{x}, \bar{\bar{z}} \rangle\lambda + \varphi(\bar{\bar{z}}, \bar{\bar{x}}) + \varphi(\bar{\bar{z}}, \bar{\bar{z}})\lambda\ ,$$

since $\bar{\bar{w}}$ is orthogonal to $K_k(N)$. Now $\varphi(\bar{\bar{z}}, \bar{\bar{z}}) \in J$, $\varphi(\bar{\bar{z}}, \bar{\bar{x}}) = (\bar{\bar{z}} \frown \bar{x})\lambda$, and $(\bar{x} \frown \bar{\bar{z}}) + (\bar{\bar{z}} \frown \bar{x}) \in J$ by the symmetry property of intersections.

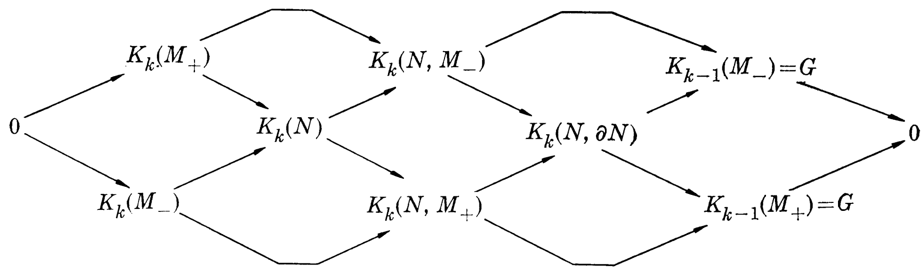
The main result of this chapter is a partial converse of the above lemma. Before we go on to prove this, however, we give another result, similar to the above, which arises as follows. We are developing a sort of cobordism theory for the manifold M. In usual cobordism theory, the cobordism classes form a group, and the inverse of a manifold is given by the same manifold with the opposite orientation.

LEMMA 5.4. *There exists a finitely generated projective Λ-module P, which may be supposed free if X is finite, and an exact sequence*

$$0 \longrightarrow K_k(M) \oplus K_k(M) \longrightarrow P \xrightarrow{A\varphi} \mathrm{Hom}\,(P, \Lambda) \longrightarrow G \oplus G \longrightarrow 0\ ,$$

where (as in § 3) $A\varphi$ is associated to a $(-1)^k$-hermitian bilinear form φ. On $G^* \oplus G^*$, φ induces $b \oplus -b$ and $q \oplus -q$.

PROOF. The proof is almost the same as that of 5.3. However, instead of starting with a cobordism of M to a manifold M_+ with ψ a homotopy equivalence, we start with the product $M \times I$ (for which ψ is already $(k-1)$-connected) and do surgery to make ψ k-connected. We then have the exact commutative diagram



in which, by (2.4), the middle four modules are all projective and, as in (5.3), can be made free if X is finite. Also, $K_k(N, \partial N)$ is dual to $K_k(N)$: we write $P = K_k(N)$; then $K_k(N, \partial N) \cong \mathrm{Hom}_\Lambda(P, \Lambda)$.

We observe that the sequence

$$0 \longrightarrow K_k(M_-) \longrightarrow K_k(N) \longrightarrow K_k(N, M_-) \longrightarrow G \longrightarrow 0$$

resembles closely the sequence in (5.3), but here the two middle modules, although free, are not dual to each other.

From the above diagram, however, we can extract the Mayer-Vietoris sequence

$$0 \longrightarrow K_k(M_-) \oplus K_k(M_+) \longrightarrow K_k(N) \longrightarrow K_k(N, \partial N) \longrightarrow G \oplus G \longrightarrow 0\ ,$$

which is the desired sequence. It remains to identify the bilinear and quadratic forms on $G^* \oplus G^*$. The identification proceeds as in the previous lemma, if we note two points. Firstly, the two copies of G^* are orthogonal for the induced b. For, if $x \in K_{k-1}(M_-)^*$ and $y \in K_{k-1}(M_+)^*$, we can lift x to $\bar{x} \in K_k(N, M_-)$ and y to $\bar{y} \in K_k(N, M_+)$. If $x\lambda = 0$, $\bar{x}\lambda$ lifts to $\tilde{x} \in K_k(N)$. Now ι_λ is the intersection of the image of $\bar{y}$ in in $K_k(N, M)$ with $\tilde{x}$, which is the same as the intersection of $\bar{y}$ with the image $\bar{x}\lambda$ of $\tilde{x}$, so $\iota_\lambda = (\bar{y} \frown \bar{x})\lambda$. Thus $b(x, y) = 0 \pmod{\Lambda}$.

Secondly, the induced b and q on $K_{k-1}(M_+)^*$ have the opposite signs from what we had before, in view of our sign conventions about the maps $K_{k-1}(M_+) \leftarrow K_k(N, M) \rightarrow K_{k-1}(M_-)$.

We observe that (in the case when G is a torsion module) the existence of an exact sequence $0 \rightarrow P \rightarrow F \rightarrow G \rightarrow 0$ with P and F projective is equivalent to G being a cohomologically trivial Λ-module. It is not clear whether this is significant in the present context.

We are now ready for the partial converse to (5.3), which is the main result of this chapter.

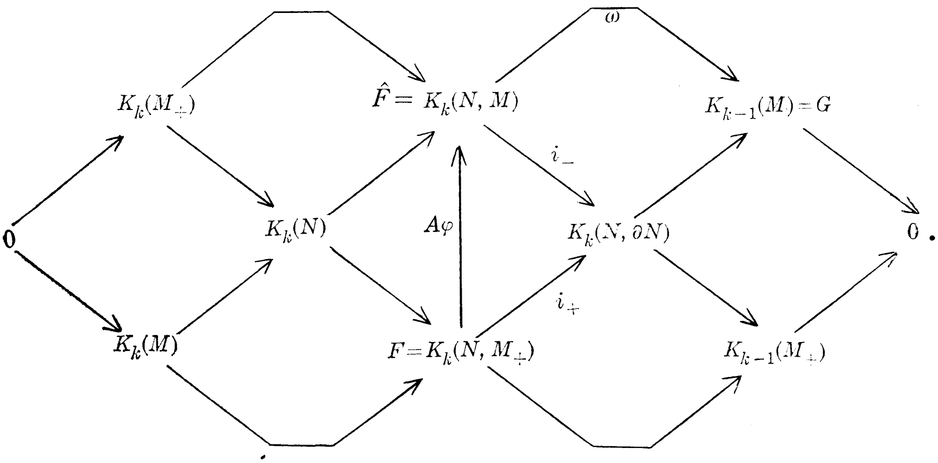
THEOREM 5.6. *Assume* (5.1) *and* $k \geqq 3$; *suppose* G *finite. Then surgery to kill* $G = K_{k-1}(M)$, *and so make* ψ_M *a homotopy equivalence, is possible if* *and only if there is a finitely generated free* Λ*-module* F, *and a non-degenerate* $(-1)^k$*-hermitian bilinear map* φ: $F \times F \to \Lambda$ *and* μ *as in* (3.1), *such that there is an exact sequence* (*of* Λ*-modules*)

$$0 \longrightarrow F \xrightarrow{A\varphi} \operatorname{Hom}_\Lambda(F, \Lambda) \xrightarrow{\omega} K_{k-1}(M) \longrightarrow 0$$

such that φ *and* μ *induce* b *and* q *on* G (*as in* (5.3)).

PROOF. Necessity of the condition was established in (5.3); suppose then that it is satisfied. Write $\hat{F}$ for $\operatorname{Hom}_\Lambda(F, \Lambda)$; choose a basis $e_1, \cdots, e_r$ of F and the dual basis $\hat{e}_1, \cdots, \hat{e}_r$ of $\hat{F}$. If we are to use the above in analogy with (5.3), we must do surgeries using the classes $\omega(\hat{e}_i)$. It is more convenient to perform them all together than separately.

Since it is not clear which isotopy class of imbeddings in the regular homotopy class determined by $\omega(\hat{e}_i)$ ought to be used, we start with a completely arbitrary such imbedding ξ_i for each i, and perform surgery. Let us compute the result. We shall temporarily denote the resulting manifold by N, and write $M = \partial_- N = M_-$, $M_+ = \partial_+ N$. We shall have the diagram of exact sequences, which would be commutative if the map $A\varphi$ were omitted,



Note that if in this diagram $i_- \circ (A\varphi) = i_+$, it follows (since by the Mayer-Vietoris sequence $i_- \oplus i_+$ is onto) that i_- is onto, so $K_{k-1}(M_+) = 0$, and we have attained our objective. Thus we must investigate the deviation of the triangle from commutativity, and then (if necessary) *re*-choose the spheres ξ_i to force the diagram to be commutative.

Write $A\varphi(e_i) = \hat{e}_j \lambda_{ji}$ (with the dummy suffix convention). Since this is sent to 0 by ω, the cycle $\xi_j \lambda_{ji}$ in $\tilde{M}$ is a boundary. Write $\partial \zeta_i = \xi_j \lambda_{ji}$, where ζ_i is a finite k-chain in $\tilde{M}$. Now also we have $-\xi_i = \partial \delta_i$ where the δ_i are the cores of the attached handles. Thus $A\varphi(e_i)$ lifts to the class f_i in $K_k(N)$ represented by the cycle (in $\tilde{N}$) $\delta_j \lambda_{ji} + \zeta_i$. If we could choose this to have image e_i in F, our triangle would be commutative (for e_i and $A\varphi(e_i)$ would then both have the same image as f_i in $K_k(N, \partial N)$). Now the image in F is determined by intersection numbers with the generators of $\hat{F}$. We must use δ'_i rather than δ_i as chains representing the generators, so that intersection numbers shall be defined. Then our image in F is $e_k\mu_{ki}$, where

$$\mu_{ki} = \delta'_k \frown (\delta_j\lambda_{ji} + \zeta_i) = \delta'_k \frown \zeta_i = -\xi'_k \frown \zeta_i ,$$

since the δ_i and δ'_i are all disjoint in N (so all their lifts are in $\tilde{N}$). We observe that this description of μ_{ki} is closely related to linking numbers in M.

Next we recall how to change the ξ_i. Each can be subjected to a regular homotopy with arbitrary self-intersection invariant ν_i in Λ/J. In addition, these homotopies may meet each other; we assert that the mutual intersection numbers $\rho_{ij} \in \Lambda$ $(i < j)$ may be chosen independently of each other and of the ν_i. Indeed, this follows at once from the same argument as was used at the beginning of § 5 to establish that the ν_i could take arbitrary values. The other intersections are determined by the formulas $\rho_{ji} = (-1)^k\bar{\rho}_{ij}$, $\rho_{ii} = \nu_i + (-1)^k\bar{\nu}_i$.

Now perform regular homotopies with tracks $\bar{\eta}_i$ in $\tilde{M} \times I$, projecting to η_i in $\tilde{M}$. Then we can write $\partial\eta_i = \xi^1_i - \xi^0_i$. Set $\zeta^1_i = \zeta^0_i + \eta_j\lambda_{ji}$. This has the correct boundary $\xi^1_j\lambda_{ji}$. The change in the μ's is given by

$$\mu^1_{ki} - \mu^0_{ki} = \xi^{1\prime}_k \frown \zeta^0_i + (\xi^{1\prime}_k \frown \eta_j)\lambda_{ji} - \xi^{0\prime}_k \frown \zeta^0_i .$$

The sum of the first and third terms here is

$$(\xi^{1\prime}_k - \xi^{0\prime}_k) \frown \zeta^0_i = \partial\eta'_k \frown \zeta^0_i = (-1)^k\, \eta'_k \frown \partial\zeta^0_i = (-1)^k\, (\eta'_k \frown \xi^0_j)\lambda_{ji} ,$$

so

$$\begin{aligned} \mu^1_{ki} - \mu^0_{ki} &= (\xi^{1\prime}_k \frown \eta_j + (-1)^k\, \eta'_k \frown \xi^0_j)\lambda_{ji} \\ &= (\bar{\eta}'_k \frown \bar{\eta}_j)\, \lambda_{ji} = \rho_{kj}\lambda_{ji} \end{aligned}$$

by an argument above. We wish to make our choices so that $\mu^1_{ki} = \delta_{ki}$ for then, as noted earlier, $i_- \circ A\varphi = i_+$, and the surgery kills the group G.

Now by hypothesis, G is a finite group, hence for some positive integer r, $rG = 0$. We can then write $r\xi_i = \partial\bar{\xi}_i$. Now $r\zeta_i - \bar{\xi}_j\lambda_{ji}$ is a k-cycle, and so has zero intersection numbers with the ξ'_k, which represent torsion classes. Thus

$$r\mu_{ki} = r\xi'_k \frown \zeta_i = \xi'_k \frown \bar{\xi}_j\lambda_{ji} = \sigma_{kj}\lambda_{ji}$$

if we write $\sigma_{kj} = \xi'_k \frown \bar{\xi}_j$. Note that the μ_{ki} are independent of the choice of the ζ's in this case. Also, $(1/r)\sigma_{kj} \equiv b(x_j, x_k) \pmod \Lambda$, by the geometrical definition of b. On the other hand, by hypothesis, if $r\hat{e}_j = A\varphi(e_k\kappa_{kj})$ we have $b(x_j, x_k) = (1/r)\kappa_{kj} \pmod \Lambda$. So $(\sigma_{kj} - \kappa_{kj}) \in r\Lambda$, and $\lambda_{ij}\kappa_{jk} = r\delta_{ik}$, so we have inverse matrices.

We now choose $\rho_{ij} = (1/r)(\kappa_{ij} - \sigma_{ij})$. Then

$$r\mu^1_{ki} = r\mu^0_{ki} + r\rho_{kj}\lambda_{ji} = r\mu^0_{ki} + (\kappa_{kj} - \sigma_{kj})\lambda_{ji} = \kappa_{kj}\lambda_{ji} = r\delta_{ki} ,$$

and $\mu^1_{ki} = \delta_{ki}$ as required. To justify this choice, we note that, since the matrix λ_{ij} is $(-1)^k$-hermitian, so is κ_{ij}, and σ_{ij} is also, for

$$r\sigma_{ji} = r\xi_j \frown \bar{\xi}_i = \partial\bar{\xi}_j \frown \bar{\xi}_i = (-1)^k \bar{\xi}_j \frown \partial\bar{\xi}_i = (-1)^k \{\partial\bar{\xi}_i \frown \bar{\xi}_j\}^- = (-1)^k r\bar{\sigma}_{ij} .$$

Thus ρ_{ij} has the requisite symmetry property. Finally, we must verify that $\rho_{ii} \in J$. But this is immediate since $q(x_i) = (1/r)\kappa_{ii} \pmod J$ and also $= (1/r)\sigma_{ii}$ $\pmod J$.

It is not quite clear how to generalise the above to the case when G is no longer torsion. Perhaps some more refined linking numbers would be necessary.

We can deduce some corollaries from the above argument, regarding surgery of torsion groups. Let Γ denote a triple (G, b, q) where G is a finite Λ-module, and b and q are as in (5.2). We can define the sum $\Gamma + \Gamma'$ of two triples by taking the direct sum of the groups and quadratic maps, and the orthogonal direct sum of the bilinear maps; and the negative $-\Gamma$ of a triple by changing the signs of b and q. Write $\Gamma \sim 0$ if Γ satisfies the condition of (5.6). Then (if X is finite), (5.4) shows that we are only interested in triples Γ with $\Gamma + -\Gamma \sim 0$.

We form the Grothendieck group of all triples Γ with $\Gamma + -\Gamma \sim 0$, modulo triples $\Gamma \sim 0$. Then Γ determines zero in the Grothendieck group if and only if for some triple $\Gamma' \sim 0$, we have $\Gamma + \Gamma' \sim 0$; this follows by an immediate induction from the definition.

LEMMA 5.7. *If* $\Gamma' \sim 0$ *and* $\Gamma + \Gamma' \sim 0$, *then* $\Gamma \sim 0$. *Thus the obstruction to surgery presented by* Γ *depends only on the class of* Γ *in the Grothendieck group.*

PROOF. Let the given exact sequences be

$$0 \longrightarrow A \xrightarrow{\Phi_1} \hat{A} \xrightarrow{\psi_1} G \oplus H \longrightarrow 0$$
$$0 \longrightarrow \hat{B} \xrightarrow{\Phi_2} \hat{B} \xrightarrow{\psi_2} H \longrightarrow 0 ,$$

where Φ_1 and Φ_2 are associated to φ_1 and φ_2. We shall regard Φ_1 and Φ_2 as inclusion maps, so the dual pairing of $\hat{A}$ and A extends the pairing φ_1; we extend it in turn (in the obvious way) to a pairing of $\hat{A}$ with itself into $\bar{\Lambda}$, and similarly for B, except that we alter the sign of these pairings. Thus for X, $Y \in \hat{A}$ we have

$$b(\psi_1 X, \psi_1 Y) = \langle X, Y\rangle \qquad (\mathrm{mod}\ \Lambda)$$
$$q(\psi_1 X) = \langle X, X\rangle \qquad (\mathrm{mod}\ J)$$

and similarly for $\hat{B}$ (with minus signs).

Define C by the exact sequence

$$0 \longrightarrow C \longrightarrow \hat{A} \oplus \hat{B} \xrightarrow{\psi_1 - \psi_2} G \oplus H \longrightarrow 0\ ;$$

so $A \oplus B \subset C$, and the image of the quotient in $\hat{A} \oplus \hat{B}/A \oplus B \cong G \oplus H \oplus H$ consists of the elements $(0, h, h)$, $h \in H$. Taking duals, we find also

$$A \oplus B \subset \hat{C} \subset \hat{A} \oplus \hat{B}\ ,$$

and $\hat{C}/A \oplus B$ is the annihilator of $C/A \oplus B$, so consists of elements (g, h, h) $h \in H$. Thus we have an exact sequence

$$0 \longrightarrow C \xrightarrow{\Phi_3} \hat{C} \xrightarrow{\psi_3} G \longrightarrow 0\ .$$

From the hermitian (or skew-hermitian) character of φ_1, φ_2 follows also that of φ_3. The projection of C on $\hat{B}$ is surjective with kernel A, hence C is free.

Let $X = (X_1, X_2) \in \hat{C}$, so $\psi_1(X_1) = (\psi_3(X), \psi_2(X_2))$; similarly for Y. Then

$$\langle X, Y\rangle = \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle = b(\psi_1 X_1, \psi_1 Y_1) - b(\psi_2 X_2, \psi_2 Y_2) \quad (\mathrm{mod}\ \Lambda)$$
$$= b(\psi_3 X, \psi_3 Y)$$
$$\langle X, X\rangle = \langle X_1, X_1\rangle + \langle X_2, X_2\rangle = q(\psi_1 X_1) - q(\psi_2 X_2) = q(\psi_3 X) \quad (\mathrm{mod}\ J)\ .$$

So φ_3 induces b and q as required, and in consequence, if $X \in C$, $0 = \langle X, X\rangle = \varphi_3(X, X)\ (\mathrm{mod}\ J)$ so φ_3 admits a form μ. This completes the proof.

6. The case: $m = 2k - 1$ and π of order 2

The results of § 5 present algebraic problems apparently even more formidable than the one studied in § 4, and even so, (5.6) is not a complete result. The only case previously settled in the literature is where X is simply connected, and so π trivial; geometric expositions may be found in Wall [20] or Kervaire and Milnor [10]; the algebraic problem is explicitly dealt with in [22]. All the proofs are quite long; they seem closely related, and involve an induction on order, and a discussion of special cases.

Since the cases when π is trivial is already so involved, we will now confine ourselves to the case when π has order 2. The pattern of the proof resembles the case when π has order 1; the details, naturally, are more complicated. We give a summary of results at the end of the chapter.

We first deal with the torsion free part of G. Recall that $\psi: M^{2k-1} \to X$ has degree 1 and is k-connected; $G = K_k(M; \Lambda)$. The effect of surgery is described by (2.5) and (5.1), which will be used repeatedly below without further reference. Throughout this chapter we make the hypothesis that (5.1) holds and π has order 2; we write T for the non-unit element.

LEMMA 6.1. *If k is odd and M is orientable, the parity of the rank of G is a cobordism invariant. Assume (in this case) that the rank is even. Then (in all cases) we can do surgery to make G a torsion group.*

PROOF. Write G^* for the torsion subgroup of G, $G^0 = G/G^*$. Then G^0 is a torsion-free Λ-module, hence by a well-known result (see e.g. [30]), is isomorphic to a direct sum of copies of the modules Λ, $\mathbf{Z}_+$, $\mathbf{Z}_-$; where $\mathbf{Z}_+$, $\mathbf{Z}_-$ denote the group $\mathbf{Z}$ with T operating by the trivial resp. sign-changing automorphism.

Suppose Λ appears. Choose a generator, and lift to $x \in G$. Now do surgery on x. Then $A(x) = 0$ and $B(x) = \Lambda$, so $K_k(N, \partial N) \cong G$, and $K_{k-1}(M_+)$ is the quotient of G by the submodule generated by x; in particular, the rank is lowered by 2.

If there are no Λ's, but $\mathbf{Z}_+$ and $\mathbf{Z}_-$ both appear, we choose generators y and z of these, and do surgery on $x = y + z$. This time $A(x) = 0$ and $B(x)$ has index 2 in Λ; $K_k(N, \partial N)$ is an extension of $\mathbf{Z}_2$ by G, and again the rank of $K_{k-1}(M_+)$ is 2 less.

It remains to look at the case when G^0 is a sum of copies of $\mathbf{Z}_+$($\mathbf{Z}_-$ is treated similarly). Here, we must distinguish 3 cases.

If M and X are non-orientable, we use the Euler characteristic. Since X (also M) has odd dimension, we know that the Euler characteristic $\chi(X)$ vanishes if ∂X is empty; in general, $\chi(X) = \frac{1}{2}\chi(\partial X)$. But the map $\partial M \to \partial X$ is a homotopy equivalence, so $\chi(\partial M) = \chi(\partial X)$, and $\chi(M) = \chi(X)$. Thus the Euler characteristic of the groups K is zero: $K_{k-1}(M; \mathbf{Z}_+)$ and $K_k(M; \mathbf{Z}_+)$ have the same rank. Using the duality isomorphism $K_k(M; \mathbf{Z}_+) \cong K^{k-1}(M; \mathbf{Z}_-)$, and the universal coefficient theorem, we deduce that $K_{k-1}(M; \mathbf{Z}_-)$ also has the same rank. In the situation above, one of these ranks is zero, hence so is the other, and G is already a torsion group.

If M and X are orientable, choose a generator of one of the summands $\mathbf{Z}_+$, and lift to $x \in G$. Then $B(x)$ is the set of multiples (in Λ) of $1 + T$, so $K_k(N, \partial N)$ is an extension of $\mathbf{Z}_-$ by G, and contains ε_- which projects to x, and hence generates a submodule of rank 1 or 2. In the second case, the net result of surgery is to reduce the rank of G by 1; in the first, the rank of G is unaltered, but one of the copies of $\mathbf{Z}_+$ has been replaced by a $\mathbf{Z}_-$. Thus in this case, if there were at least two copies of $\mathbf{Z}_+$ originally, we return to a situation dealt with above, where we can reduce the rank of G by a further surgery.

By induction, then, we may suppose that either $G^0 = 0$, when we are done; or, $G^0 \cong \mathbf{Z}_+$. If now k is even, we can deal with the latter case by being more careful about our surgery. As $x - xT$ has finite order a, we have a relation

$$\varepsilon_-(a - aT) = \varepsilon_+(b + cT) ,$$

and as $(1 + T) \in B(x)$, $T\varepsilon_+ = -\varepsilon_+$, so we may suppose $c = 0$. Now if $b \neq 0$, x_+ is a torsion element and the resulting G is finite. If $b = 0$ we move the sphere representing x by a regular homotopy with self-intersection 1. Then (as in (5.2)), this has the effect of replacing ε_- by $\varepsilon'_- = \varepsilon_- + 2\varepsilon_+$ and hence b by $b + 4a$. So we can ensure $b \neq 0$.

Finally we check that when k is odd, the parity of the rank of G is a cobordism invariant. If N is a cobordism of M_- to M_+, we may suppose ψ_N to be k-connected. We have the Mayer-Vietoris sequence

$$K_kN \xrightarrow{\alpha} K_k(N, \partial N) \longrightarrow K_{k-1}(M_-) \oplus K_{k-1}(M_+) \longrightarrow 0 ,$$

where by (2.4) we may assume the first two modules free, and they are dual to each other. The matrix of α is (by (3.2)) skew-symmetric, so α has even rank. As $K_k(N, \partial N)$ is free, it too has even rank. The result follows.

We now assume G a torsion group; we shall use induction on the order of G. We first cover the cases where it is easy to effect a reduction in the order.

LEMMA 6.2. *Let $x \in G$ be such that*

(i) *if k is odd and M orientable,* $|x\Lambda| > 2$,

(ii) *if M is non-orientable,* $b(x, x) \neq 0$,

(iii) *if k is even and M orientable,* $b(x, x) \neq 0$ *and* $q(x)$ *is not divisible by either of* $1 \pm T$.

Then we can perform a surgery starting with x, and in case (i), *a second surgery, to reduce the order of G.*

PROOF. Since G is finite, x is a torsion element, so $B(x) = 0$, and ε_+ generates a free submodule of $K_k(N, \partial N)$ for the first surgery. If $x\lambda = 0$, then, $\varepsilon_-\lambda = \varepsilon_+\mu$ where μ is determined by $q(x) \in \bar{\Lambda}/J$.

Now in case (i), $J = 0$. Since $q(x)$ is a torsion element of $\bar{\Lambda}$, it vanishes. Hence $\mu = 0$. In case (ii), if k is even, J consists of the even integers: $\bar{\Lambda}/J = \mathbf{Q}/2\mathbf{Z}1 + \mathbf{Q}T$. Now the coefficient of T in μ vanishes; similarly if k is odd, the coefficient of 1 vanishes. By hypothesis, if $xr = 0$, $\varepsilon_-r = \varepsilon_+s$ with $r \nmid s$. In case (iii) we have $\varepsilon_-r = \varepsilon_+(a + bT)$ where by hypothesis $r \nmid (a + bT)$, and $2r \nmid (a \pm b)$.

In case (i) our surgery first gives an extension of Λ by G, then kills the torsion element ε_-. The resulting G_1 has a submodule Λ generated by x_+; G_1^* maps monomorphically to $G_1/x_+\Lambda \cong G/x\Lambda$. We have $G_1^0 \cong \Lambda$ or to $\mathbf{Z}_+ + \mathbf{Z}_-$.

In the first case, choose a generator, lift to G_1, and perform surgery. As in (6.1), the net result is to kill G_1^0 and keep the torsion subgroup unaltered, so $G_2 \cong G_1^*$, isomorphic to a subgroup of $G/x\Lambda$, hence if $x \neq 0$, G_2 has lower order than G. If $G_1^0 \cong \mathbf{Z}_+ + \mathbf{Z}_-$, then the order of $G_1/x_+\Lambda$ must be at least twice the order of G_1^*, so $|G_1^*| \leqq |G|/2|x\Lambda|$. As in (6.1), perform surgery on the sum $y + z$ of the generators; we first obtain an extension of $\mathbf{Z}_2$ by G_1, then kill $(y + z)\Lambda$, whose image in G_1^0 has index 2. The resulting group G_2 is finite, and

$$|G_2| \leqq 4|G_1^*| \leqq \frac{2|G|}{|x\Lambda|} < |G|$$

by hypothesis.

In cases (ii) and (iii), we use (5.2), which says that in the equation $\varepsilon_-\lambda = \varepsilon_+\{q(x)\lambda\}$, we can use any lift to $\bar{\Lambda}$ of $q(x) \in \bar{\Lambda}/J$. Thus in $\varepsilon_- r = \varepsilon_+ s$, resp. $\varepsilon_- r = \varepsilon_+(a + bT)$, we can alter s resp. a and b modulo $2r$. Since, by hypothesis, they are not divisible by r, we may suppose $0 < |s| < r$ (or similarly for a, b). Also $a \neq \pm b$, hence $a + bT$ generates an ideal I of index $|a^2 - b^2|$ in Λ; and $0 < |a^2 - b^2| < r^2$ (or in case (ii), we have index $s^2 < r^2$).

Now perform surgery. We assert that the order of G is multiplied by s^2/r^2, resp. $|a^2 - b^2|/r^2$; hence is decreased. For if $H = K_k(N, \partial N)/(r\varepsilon_-\Lambda)$, the maps of $K_k(N, \partial N)$ onto G and G_+ factor through H; the kernel of the former is generated by the image of ε_+, hence has order s^2, resp. $|a^2 - b^2|$, whereas the other kernel is generated by the image of ε_-, so has order r^2.

This lemma is not strong enough to permit us to complete an induction, and further progress seems to depend on the following trick, which is valid for an arbitrary ring Λ.

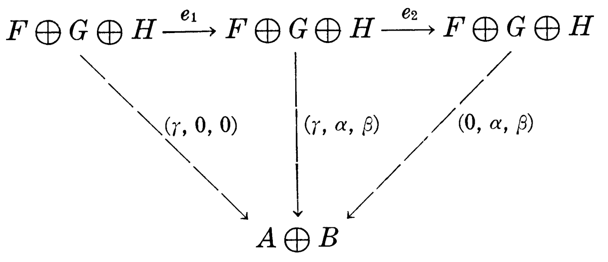
LEMMA 6.3. *Suppose given exact sequences of Λ-modules*

$$0 \longrightarrow F' \longrightarrow F \xrightarrow{\gamma} A \oplus B \longrightarrow 0$$

$$0 \longrightarrow G' \longrightarrow G \xrightarrow{\alpha} A \longrightarrow 0$$

with F, F', G projective. Then G' is projective.

PROOF. Choose an epimorphism $\beta : H \to B$ with H projective; let H' be its kernel. We shall construct automorphisms e_1, e_2 of $F \oplus G \oplus H$ such that



is commutative. Then e_2e_1 induces an isomorphism of the projective module $F' \oplus G \oplus H$ onto $F \oplus G' \oplus H'$, so G' and H' are projective.

To construct e_2, we lift (since F is projective) γ to a homomorphism, $f: F \to G \oplus H$, inducing $f': F \oplus G \oplus H \to F \oplus G \oplus H$ which is 0 on $G \oplus H$. Then set $e_2 = 1 + f'$, with inverse $1 - f'$. The construction of e_1 is similar.

In applying this result, we shall use (5.5), which shows that G, if a torsion group, and of the form $A \oplus B$, satisfies the hypothesis. Note that the direct sum decomposition is merely additive. Also note that for the ring Λ in question now, every projective module is free.

COROLLARY 6.4. *Suppose G a torsion group and $x \in G$ such that $x\Lambda$ is an (additive) direct summand. Then $A(x)$ is isomorphic (as a module) to Λ. In particular, $A(x) \neq \langle 1 - T, 1 + T \rangle$.*

For take $0 \to A(x) \to \Lambda \to x\Lambda \to 0$ as the second sequence in the lemma; note that since x is torsion, $A(x)$ has rank 2.

We are now ready to settle the easy cases.

THEOREM 6.5. *Suppose G a torsion group. Then we can do surgery to kill G altogether, in cases* (i) *and* (ii).

PROOF. By induction on the order of G. First take case (i). By (6.2) and induction, we can assume that for any $x \in G$, $|x\Lambda| \leqq 2$. So for $x \neq 0$, $A(x)$ has index 2, hence is $\langle 1 - T, 1 + T \rangle$. This ideal annihilates G, which is therefore a module over the quotient ring, which is a field. Any non-zero x thus generates a direct summand: but this contradicts (6.4). Hence G is zero.

Next consider case (ii). We suppose k even. The case when k is odd gives an isomorphic problem, as we see by multiplying b and q by T. By (6.2) and induction, we can assume $b(x, x) \equiv 0$. For any x, y, $b(x, y) + b(y, x) = b(x + y, x + y) - b(x, x) - b(y, y) = 0$. Let $b(x, y) = \lambda + \mu T$; then $b(y, x) = b(x, y)^- = \lambda - \mu T$, so $2\lambda = 0$. As $\lambda \in \mathbf{Q}/\mathbf{Z}$, this implies $\lambda = 0$ or $\frac{1}{2}$. Now $b(x, yT) = \mu + \lambda T$, hence also $\mu = 0$ or $\frac{1}{2}$. Thus $b(x, 2y) = 0$ for all x, y; as b is non-singular, $2y = 0$ for all y, so $2G = 0$. Thus we can consider G as a vector space over $\mathbf{Z}_2$, with an action of the group $\{1, T\}$. But representations of this group over $\mathbf{Z}_2$ are well known; G is a direct sum of modules generated by single elements, whose annihilators are $\langle 2 \rangle$ or $\langle 1 - T, 1 + T \rangle$. We may exclude the second case by (6.4).

Now let $x \neq 0$ in G. As $b(x, x) = 0$, we have $q(x) = 0$ or 1. First suppose $q(x) = 0$. Then on surgery, we have $2\varepsilon_- = 0$; $K_k(N, \partial N)$ is an extension of Λ by G, and for $G_1 = K_k(M_+)$ we kill ε_-, which generates a submodule of order 4. The same proof as for case (i) of (6.2) now shows that we can do a further surgery with the net effect of reducing the order of G.

Now suppose $q(x) = 1$, and so $2\varepsilon_- = 2\varepsilon_+$. Here we use the fact that b is non-singular, so for some y, $b(x, y) = \frac{1}{2}$. Then y lifts to an element $\eta \in K_k(N, \partial N)$ with $2\eta = \varepsilon_+$. Thus although the surgery, after setting $\varepsilon_- = 0$, gives a new group with the same order as G, we have a class η of order 4 and so, by the argument above, can apply (6.2) again to reduce the order of G. Hence by induction we can reduce G to zero.

We observe that the two cases above are fairly similar to the two cases arising when π is trivial. The third case, however, is considerably more troublesome, and we need further lemmas.

LEMMA 6.6. *Suppose $G^0 \cong \mathbf{Z}_+$ and $|G^*| = d$. Then we can perform a sequence of surgeries to make G a finite group of order at most $2d$.*

PROOF. Choose a generator of G^0 and lift to $x \in G$. Then $B(x) = \langle 1 + T\rangle$. Let $A(x) = \langle b - bT\rangle$. Then we have $\varepsilon_+(1 + T) = 0$, and can write

$$\varepsilon_-(b - bT) = c\varepsilon_+ .$$

We can perform surgery on different spheres in the regular homotopy class determined by x; this will alter ε_- by an arbitrary multiple of $2\varepsilon_+$, and hence alter c modulo $4b$. We may thus suppose $-2b < c \leqq 2b$. If $c \neq 0$, we perform surgery, and the resulting group has order $c|G^*|/b \leqq 2d$. If $c = 0$ and $b \neq 1$, we obtain a group H with $|H^*| = |G^*|/b$ and $H^0 \cong \mathbf{Z}_-$; we can then apply the above procedure to H.

If $c = 0$ and $b = 1$, we try lifting to a different x. The only case when $b = 1$ for all choices of x is when $(T - 1)$ annihilates G. Then G is a direct sum of cyclic groups, each a submodule: let y_i be their generators and n_i their orders. On surgery, these lift to classes $\eta_i \in K_k(N, \partial N)$, and $\eta_i(1 - T)$ and $n\eta_i$ are multiples of ε_+. If any of these multiples are non-zero, surgery gives a group H as above; if all are zero, we simply have $H^* = G^*$, so can again reduce unless $(1 + T)$ also annihilates G^*.

In this final case, we change c to 4 and perform surgery. We end with the direct sum of G^* and a module A of order 4, generated (say) by z. Since G^* is annihilated by $\langle 1 - T, 1 + T\rangle$, (6.4) now shows that $G^* = 0$. But the annihilator of the generator of A is $\langle 1 + T, 4\rangle = \langle 1 + T, 2 - 2T\rangle$, and this also is prohibited by (6.4). The lemma is proved.

We return to our induction on $|G|$, and now give a more careful analysis of cases when the order can be reduced.

LEMMA 6.7. (i) *Suppose G contains an element x which does not satisfy*

(A) *Either $xT = \varepsilon x$ and $q(x) = V(1 + \varepsilon T)$ with $\varepsilon = \pm 1$ and*

$$|V| = 1/2\ ,\ 2/3 \quad or \quad 3/4; \quad or,\ b(x, x) = 0\ .$$

Then we can perform a sequence of surgeries to reduce $|G|$.

(ii) *If every* $x \in G$ *satisfies* (*A*), *then for every* $x \in G$, $6x = 0$.

PROOF. We have already (in (6.2)) dealt with the cases when $q(x)$ is not divisible by either of $1 \pm T$; hence we may now assume it divisible by $1 + T$. Let $A(x) = \langle \alpha + \beta T \rangle$ with $\alpha > |\beta| > 0$, or $A(x) = \langle \gamma + \gamma T, \delta - \delta T \rangle$ with $\gamma, \delta > 0$ (this is a complete list of all ideals in Λ of rank 2). Then $|\Lambda : A(x)| = \alpha^2 - \beta^2$ or $2\gamma\delta$.

On surgery, we have

$$\varepsilon_-(\alpha + \beta T) = \varepsilon_+(a + aT)$$

or

$$\varepsilon_-(\gamma + \gamma T) = \varepsilon_+(c + cT) , \qquad \varepsilon_-(\delta - \delta T) = 0$$

since $q(x)$ is divisible by $1 + T$. Changing ε_- by a suitable multiple of $2\varepsilon_+$ we may suppose, since $b(x, x) \neq 0$,

$$0 < |a| < \alpha + \beta \quad \text{or} \quad 0 < |c| < 2\gamma .$$

When we do surgery, we acquire a subgroup $\mathbf{Z}_-$, and the order of the torsion subgroup is multiplied by at most $|a|/(\alpha^2 - \beta^2)$ or $|c|/2\gamma\delta$; then by (6.6) we can get rid of the $\mathbf{Z}_-$, at most doubling the order of the torsion subgroup. Thus the order is certainly decreased, except perhaps if $\alpha - \beta = 1$ or $\delta = 1$. We note that these are precisely the cases with $Tx = x$.

Even in these cases, we have reduced the order half the time: i.e., when $|a| < \alpha$ or $|c| < \gamma$. For the other cases we need new procedures. We have $q(x) = v(1 + T)$ with $|v| \leqq 1$, and the above gives a simplification if $|v| < \frac{1}{2}$.

If $\frac{1}{2} < |v| < 5/8$, we consider $2x$. We have $q(2x) = 4q(x) = 4v(1 + T)$; reducing by $2 + 2T$, we have replaced $|v|$ by $4|v| - 2$, which satisfies $0 < 4|v| - 2 < \frac{1}{2}$. The preceding paragraph guarantees a simplification in this case.

If $5/8 \leqq |v| \leqq 11/16$, we similarly consider $3x$. Here, $|v|$ is replaced by $9|v| - 6$ which lies between $-3/8$ and $3/16$, so we can simplify unless $|v| = 2/3$.

If $11/16 < |v| < 3/4$, we use $2x$, and have $3/4 < 4|v| - 2 < 1$, which reduces us to the final case $3/4 < |v| < 1$. Here we use a different technique. Let $nx = 0$, so on surgery $n\varepsilon_- = \varepsilon_+ nv(1 + T)$. Suppose for definiteness $v > 0$. Decrease ε_- by $2\varepsilon_+$; then

$$n\varepsilon_- = \varepsilon_+\{(nv - 2n) + nvT\} .$$

Now perform surgery. The order of G is multiplied by

$$\{(nv)^2 - (nv - 2n)^2\}/n^2 = v^2 - (v - 2)^2 = 4(1 - v) < 1 .$$

Thus in all cases claimed, we can decrease the order of G. This completes the proof of (i).

As to (ii), we first observe that for every $x \in G$ we have

$$q(x)\,(6 + 6T) = 0 = q(x)\,(6 - 6T)$$

(note that if $b(x, x) = 0$, we already have $2q(x) = 0$). Hence if a denotes either of $6 \pm 6T$, we have, for $x, y \in G$,

$$2b(x, y)\,a = \{b(x, y) + b(y, x)\}\,a = \{q(x + y) - q(x) - q(y)\}\,a = 0$$

modulo 2, so $0 = b(x, y)\,a = b(x, ya)$. As b is non-singular, $ya = 0$. In particular, $12y = 0$, so G only has 2-torsion and 3-torsion; these clearly form an orthogonal direct sum in G.

Now suppose $xT = x$ and $q(x) = (3/4)(1 + T)$. By the previous paragraph, we may suppose x an element of 2-torsion, and that $x(2 + 2T) = 0$. Hence for any $y \in G$, $b(y, x)$ is annihilated by $\langle 1 - T, 2 + 2T\rangle$, hence is of the form $\lambda + \lambda T$, with λ divisible by 1/4. The map $y \rightarrow -(4\lambda)x$ retracts G onto the submodule generated by x, which is therefore a direct summand. By (6.4), this case cannot occur.

We now know that for x in G, $6q(x) = 0$. We deduce, as in the last paragraph but one, that 6 annihilates G. This completes the proof of the lemma.

THEOREM 6.8. *If k is even and M orientable, the Grothendieck group of* (5.7) *has order* 2.

PROOF. We first deal with the Sylow 2-subgroup of G. By (6.7), this is a vector space over the field with two elements, hence (as in an earlier case) a direct sum of modules on one generator x, with annihilator $\langle 2\rangle$ or $\langle 1 + T, 1 - T\rangle$; by (6.4) the second case cannot occur. Since $xT \neq \pm x$, by (i) of (6.7) we have $b(x, x) = 0$. Choose y with $b(x, y) = \frac{1}{2}$ (by non-singularity).

Now perform surgery starting with x. We can lift y to $\eta \in K_k(N, \partial N)$ such that $2\eta = \varepsilon_+$; we also have $2\varepsilon_- = \varepsilon_+(2q(x))$, where $q(x)$ may be 0, 1, T or $1 + T$.

If $q(x)$ is zero, the order of the torsion subgroup is decreased on surgery by a factor of 16 (neither x nor y continues to contribute), so it is easy to decrease $|G|$ by a further surgery.

If $q(x)$ is 1 or T, the order of G is unchanged by surgery, but we acquire an element of order 4. By (6.7), we can perform further surgeries to reduce $|G|$.

If $q(x) = 1 + T$, the order of the torsion subgroup is decreased on surgery by a factor of 4 (by considering y), and we acquire a subgroup $\mathbf{Z}_-$. By (6.6), we can perform further surgeries to return to a finite group of order less than $|G|$.

We shall use (5.7) to deal with the 3-torsion. Suppose $xT = x$ and $q(x) = \{(2/3)(1 + T)\}$; also (by (ii) of (5.7)) we may suppose $3x = 0$. For any $y \in G$, $b(y, x)$ is annihilated by $\langle 3, T - 1\rangle$, hence is a multiple $\{(v/3)(1 + T)\}$. The map $y \to -vx$ retracts G onto the submodule X generated by x; the complement is orthogonal to X, so we have an orthogonal split as in (5.7). Split off as many such summands as possible; then in the residue, $b(x, x)$ vanishes, so the residue is annihilated by 2, hence vanishes.

Thus G is an orthogonal direct sum of submodules of one of four types; determined by a generator x with

$$3x = 0, \qquad xT = x, \qquad q(x) = \pm \frac{2}{3}(1 + T)$$

or

$$3x = 0, \qquad xT = -x, \qquad q(x) = \pm \frac{2}{3}(1 - T)\,.$$

We assert that the sum of any two such modules is always equivalent to zero. There are essentially three cases to verify. If we have one submodule with $xT = x$ and one with $yT = -y$, we perform surgery on $z = x + y$, with $A(z) = 3$ and $q(z) = \pm 2/3$ or $\pm (2/3)T$; say $q(z) = (2/3)U$, where U is a unit in Λ. On surgery, we have $3\varepsilon_- = 2\varepsilon_+ U$, so obtain a module generated by x_+, with $A(x_+) = 2$. A further surgery gives $2\varepsilon_- = 3\varepsilon_+ U$; replacing ε_- by $\varepsilon_- - 2\varepsilon_+ U$ we have $2\varepsilon_- = -\varepsilon_+ U$, and so G is killed altogether.

This shows that the first two types listed above determine some element $\mathfrak{X}$ in the Grothendieck group, and the second two each determine $-\mathfrak{X}$. We next prove $2\mathfrak{X} = 0$; for this it is enough to show that the sum of the first two modules listed above is null-equivalent. But this follows from (5.4). We could also easily give a direct proof. Finally, we assert $\mathfrak{X} \neq 0$.

The proof is by contradiction. If $\mathfrak{X} = 0$, there is a free Λ-module F, and a symmetric bilinear form φ on F (with values in Λ) with each $\varphi(x, x) \in J$, i.e., divisible by 2, such that the cokernel of the associated map $A\varphi : F \to \hat{F}$ is the module above. Hence the determinant of φ is (up to a unit in Λ) the generator of the annihilator of x, i.e., $1 + 2T$. Thus there is a matrix (a_{ij}) with $a_{ji} = a_{ij}$ and $a_{ii} = 2b_i$, say, with determinant $\pm (1 + 2T)$ or $\pm (T + 2)$. This, we shall see, does not exist.

First calculate modulo 2Λ. Then we can replace a_{ii} by zero and a_{ji}(if $i < j$) by $-a_{ij}$, so making the matrix skew. If it has odd order, the determinant then vanishes. If even order, the determinant is the square of the pfaffian form, which has, say, value $r + sT$. The square is $(r^2 + s^2) + 2sT$. Thus r and s have opposite parity, and the determinant can never be $\pm (T + 2)$.

Now calculate modulo 4Λ. The order of the matrix is even; the cofactor of a_{ii}, by the above, lies in 2Λ. Thus we can replace $a_{ii} = 2b_i$ by 0. We now replace a_{ji}(if $i < j$) by $-a_{ij}$, and calculate the effect on the terms in turn. First let t be a term involving some a_{ij} without the corresponding a_{ji}. There is a symmetric term t' obtained by reflecting in the diagonal: by symmetry, $t' = t$. Now t is a product of (say) $2n$ terms. If j of these have the sign changed, the terms of t' whose signs are changed are the reflections of the remaining $(2n - j)$. So $2t$ becomes $(-1)^j\, 2t$, which is the same (modulo 4) as $2t$—or as $(-1)^n\, 2t$. The remaining terms are products of n factors $a_{ij}a_{ji}$; these have sign multiplied by $(-1)^n$.

Thus we obtain a skew-symmetric matrix, whose determinant agrees (modulo 4Λ) with $(-1)^n$ times the given determinant, hence with $\pm(1 + 2T)$. So we have

$$r^2 + s^2 + 2rsT = \pm(1 + 2T) \qquad (\text{mod. } 4)\,.$$

Since $r^2 + s^2$ is odd, one of r and s is even. But then $2rs$ is divisible by 4. We have obtained a contradiction; this completes the proof of the theorem.

Summary 6.9. Recall that we assume that π is of order 2.

If M is non-orientable, we can perform surgery to kill G, and obtain a homotopy equivalence.

If M is orientable and k is odd, the parity of the rank of G is a surgery invariant: if the rank is even, we can perform surgery to kill G.

If M is orientable and k is even, we can always perform surgery to make G finite, and even of order at most 3. There is a mod 2 obstruction to completing the surgery.

We do not know how to describe this mod 2 obstruction in general. However, suppose G finite and of odd order. Then (cf. [13 pp. 106–7]) the *determinant* of the Λ-module G is defined, as an element of Λ which is a unit in $\bar{\Lambda}$. We can multiply by a unit to normalise this determinant as $r + sT$, where $r \equiv 1$ (mod 4) and s is even. Then the parity of $\frac{1}{2}s$ is our mod 2 obstruction.

7. Relative surgery

In the preceding chapters, we have concentrated on one manifold at a time, with the typical hypothesis: ψ induces a homotopy equivalence of ∂M on ∂X. We shall now (following [7] and [24]) consider the situation where we are allowed to change by surgery both M and ∂M. For all the results in this chapter, we make the following hypothesis and notation.

Hypothesis (7.0). *$(X, \partial X)$ is a Poincaré pair, with X connected. M is a compact smooth manifold of dimension m. $\psi : (M, \partial M) \to (X, \partial X)$ is a map of degree* 1. *$\omega : X \to BO$ is such that $\omega \circ \psi$ is a classifying map for the stable*

tangent bundle of M.
We can also, of course, alter the last sentence of the hypothesis to suit the relative situation of (1.5) and (1.6).

To illustrate the kind of thing we have in mind, we start with a comparatively straightforward result.

THEOREM 7.1. *Suppose that X is finite, that the pair $(X, \partial X)$ is 1-connected, and that $m = 2k \geqq 6$. Then we can do surgery to make ψ a homotopy equivalence $M \to X$ and, if $\pi_1(\partial X) = \pi_1(X)$, also of the pairs $(M, \partial M) \to (X, \partial X)$.*

PROOF. We may suppose X connected. By Theorem 1.4, we can do surgery to ∂M to make the induced map $(k-1)$-connected. Note that such surgery involves constructing a cobordism N with $\partial_- N = \partial M$. If we attach N to M along ∂M, giving M', and extend ψ in the obvious way from M over N (mapping N to ∂X) we obtain $\psi\colon (M', \partial M') \to (X, \partial X)$, as above, but now inducing a $(k-1)$-connected map $\partial M' \to \partial X$. In future, we shall always understand this construction when we speak of *doing surgery on ∂M.*

Again by (1.4), we can suppose the map $M \to X$ to be k-connected. As $k \geqq 3$, and $(X, \partial X)$ is 1-connected, it now follows that all four of ∂X, X, ∂M, M are connected; the maps $\partial M \to \partial X$ and $M \to X$ induce isomorphisms of fundamental groups, and the map $\partial M \to M$ an epimorphism. Write (as usual) Λ for the integral group ring of π, the fundamental group of M. The only non-vanishing groups K_i (coefficients Λ understood) are

$$0 \longrightarrow K_k(\partial M) \longrightarrow K_k(M) \longrightarrow K_k(M, \partial M) \longrightarrow K_{k-1}(\partial M) \longrightarrow 0\ ;$$

by (2.4), as X is finite, we may suppose the middle two modules free. Choose a basis $\{e_i\}$ of $K_k(M, \partial M)$.

We assert that the elements e_i can be represented by disjoint imbeddings $f_i\colon (D^k, \partial D^k) \to (M, \partial M)$. Granted this, we first deform ψ to map a neighborhood of each $f_i(D^k)$ to the base point (possible since f_i represents $e_i \in K_k(M, \partial M)$); then take disjoint neighborhoods $N_i \cong D^k \times D^k$ of the $f_i(D^k)$, and delete their interiors from M, leaving M_0. Then ψ induces a map $\psi\colon (M_0, \partial M_0) \to (X, \partial X)$ with the same properties; indeed, our construction is the inverse of that of the first paragraph of the proof: write $N = \bigcup N_i$. As N is mapped to the base point, we have an exact sequence

$$K_{k+1}(M, \partial M) \to K_{k+1}(M, N) \to H_k(N, \partial M) \to K_k(M, \partial M) \to K_k(M, N) \to H_{k-1}(N, \partial M).$$

Here, the extreme terms evidently vanish; the central map is an isomorphism, since $H_k(N, \partial M)$ also has a free basis represented by the f_i. So, for $i = k$ or $k+1$ (and it is clear for other values of i)

$$0 = K_i(M, N) = K_i(M_0, \partial M_0) \qquad \text{by excision.}$$

Now in the exact sequence for $(M_0, \partial M_0)$, all terms $K_i(M_0, \partial M_0)$ vanish; by duality, so do the $K_i(M_0)$, and by exactness also the $K_i(\partial M_0)$. By (2.3), the map $M_0 \to X$ is a homotopy equivalence; the same applies to ∂M_0 if it has the same fundamental group.

It remains to prove our assertion. Write Φ for the quadruple

$$\Phi: \begin{array}{ccc} \partial M & \longrightarrow & M \\ \downarrow{\scriptstyle \psi_{\partial M}} & & \downarrow{\scriptstyle \psi_M} \\ \partial X & \longrightarrow & X \end{array}$$

and consider the commutative exact diagram

$$\begin{array}{ccccccccc} \pi_{k+1}(\psi_{\partial M}) & \longrightarrow & \pi_{k+1}(\psi_M) & \longrightarrow & \pi_{k+1}(\Phi) & \longrightarrow & \pi_k(\psi_{\partial M}) & \longrightarrow & \pi_k(\psi_M) = 0 \\ \downarrow & & \downarrow \wr\!\wr & & \downarrow & & \downarrow & & \downarrow \\ K_k(\partial M) & \longrightarrow & K_k(M) & \longrightarrow & K_k(M, \partial M) & \longrightarrow & K_{k-1}(\partial M) & \longrightarrow & K_{k-1}(M) = 0 \end{array}$$

where the vertical maps are Hurewicz homomorphisms. Let $\pi' = \pi_1(\partial M)$, and write Λ' for the integral group ring of π'. Then the penultimate vertical map is a composite $\pi_k(\psi_{\partial M}) \cong K_{k-1}(\partial M; \Lambda') \to K_{k-1}(\partial M; \Lambda)$ with the first map a Hurewicz isomorphism. By hypothesis, π' maps onto π, hence Λ' maps onto Λ, so by the universal coefficient theorem (here we use $(k-1)$-connectivity of $\psi_{\partial M}$), $K_{k-1}(\partial M; \Lambda')$ maps onto $K_{k-1}(\partial M; \Lambda') \otimes_{\Lambda'} \Lambda \cong K_{k-1}(\partial M; \Lambda)$. Thus the penultimate vertical map in the diagram is surjective; by the Five Lemma, so is the preceding map $\pi_{k+1}(\Phi) \to K_k(M, \partial M)$. We lift e_i to an element of $\pi_{k+1}(\Phi)$; it is then represented by a map f_i: $(D^k, \partial D^k) \to (M, \partial M)$, and a null-homotopy of $\psi \circ f_i$.

Put the maps f_i in general position. They are then imbeddings, except for isolated self-intersections (and mutual intersections). For each such intersection P, join it by an arc α to the boundary along each branch of D^k (or each of the two D^k's) which meet at P. These two arcs α, α' define a (necessarily zero) element of $\pi_1(M, \partial M)$, so we can find a (singular) disc D^2 joining $\alpha \cup \alpha'$ to ∂M. But as $k \geqq 3$, if we put D^2 in general position, it becomes imbedded disjointly from our D^k's. We can now get rid of the intersection point P by deforming a neighborhood of α across D^2, to end up disjoint from α' (*piping the singularity across the boundary*) in the usual way. We can thus get rid of all intersections and self-intersections, which justifies our assertion and completes the proof.

We could enunciate and prove a similar result for the odd-dimensional case, by the same method. This would be somewhat weaker, for we need $(X, \partial X)$

to be 2-connected. We can obtain stronger theorems by combining relative surgery with the results of previous chapters. In fact, the first point to observe in the odd-dimensional case is that we can kill one further group before we need to consider duality.

THEOREM 7.2. *Suppose* (7.0) *and that* $m = 2k + 1$. *Then we can do surgery such that* ψ_M *and* $\psi_{\partial M}$ *are k-connected, and also* $K_k(M, \partial M) = 0$.

PROOF. All but the last assertion follow from (1.4). We note that $K_k(M)$ is finitely generated (e.g. by (1.1)), hence so is its quotient group $K_k(M, \partial M)$. We kill each of a set of generators in turn.

To kill $x \in K_k(M, \partial M)$, first lift to $y \in K_k(M)$. Add a k-handle to M, by forming the boundary-connected sum with $S^k \times D^{k+1}$; we can extend ψ by mapping the latter to the base point. The effect on $K_k(M)$ is to form the direct sum with a copy of Λ, generated (say) by e; $K_k(M, \partial M)$ is unaltered. We will do surgery on $z = e + y$. Note that projection on the new copy of Λ sends z to e; hence $z\Lambda$ is a free direct summand of the new $K_k(M)$, and z has image $x \in K_k(M, \partial M)$.

In order to compute the effect of surgery, we really need an extension of (2.5) to the case when ψ does not induce a homotopy equivalence of L on W; this we shall leave to the reader, as no new principle is involved. Write N for the cobordism; M_+ for $\partial_+ N$; P for the quadruple

$$P: \begin{array}{ccc} \partial M & \longrightarrow & M \\ \downarrow & & \downarrow \\ M_+ & \longrightarrow & N \end{array} .$$

Then we have, first, an exact sequence

$$K_{k+1}(M, \partial M) \longrightarrow K_{k+1}(N, M_+) \longrightarrow K_{k+1}(P) \longrightarrow K_k(M, \partial M) \longrightarrow 0 ,$$

where the first map $K_k(M, \partial M) \to \Lambda$ is induced by intersection numbers with z, hence is surjective (by duality, since z generates a free direct summand). So $K_{k+1}(P) \cong K_k(M, \partial M)$. We have a dual exact sequence

$$\Lambda \cong K_{k+1}(N, M) \longrightarrow K_{k+1}(P) \longrightarrow K_k(M_+, \partial M) \longrightarrow 0 ,$$

and the commutative square

$$\begin{array}{ccc} \Lambda \cong K_{k+1}(N, M) & \longrightarrow & K_{k+1}(P) \\ \downarrow & & \downarrow \cong \\ K_k(M) & \longrightarrow & K_k(M, \partial M) \end{array}$$

(where the map $\Lambda \to K_k(M)$ is induced by z) shows that the kernel of the induced epimorphism

$$K_k(M, \partial M) \cong K_{k+1}(P) \longrightarrow K_k(M_+, \partial M)$$

is generated by the image of z, i.e., by x, as required. This completes the proof.

Note that the above makes no assumption about the connectivity or the fundamental groups of the spaces involved. By (2.4), we may now suppose, if X is finite (at the expense of adding a few handles $S^k \times D^{k+1}$ to M), that $K_k(M)$ and $K_{k+1}(M, \partial M)$, hence also $K_k(\partial M)$, are free Λ-modules.

Now recall the Grothendieck group of § 4; as we wish here to emphasise its dependence on π, we write it as $\tilde{\mathfrak{S}}(\pi)$. (Strictly speaking, it depends not only on π, but also on $w: \pi \to \{\pm 1\}$ and on $\eta = (-1)^k$). We assert that $\tilde{\mathfrak{S}}(\pi)$ is a covariant functor of π; a homomorphism $f: \pi \to \pi'$, such that $w(\pi') \circ f = w(\pi)$ induces $f_*: \tilde{\mathfrak{S}}(\pi) \to \tilde{\mathfrak{S}}(\pi')$. For an element of $\tilde{\mathfrak{S}}(\pi)$ can be represented by a matrix over $\mathbf{Z}[\pi]$; we take the image of the matrix under f. The desired properties are then easily checked. Geometrically, if $\Phi = (G, \varphi, \mu)$ represents the element of $\tilde{\mathfrak{S}}(\pi)$, we replace G by $G \otimes_{\mathbf{Z}[\pi]} \mathbf{Z}[\pi']$, and take the maps induced by φ and μ (note that $f(I_\pi) \subset I_{\pi'}$, so f induces a map $V_\pi \to V_{\pi'}$; this defines μ on the new basis, and hence, by (4.2), everywhere).

From the theorem, we can now make deductions about the obstructions to surgery on boundary components. Let ∂X have components W_i, with fundamental groups π_i, and inclusions $i_i: \pi_i \to \pi$; let L_i be the corresponding components of ∂M, and let $\theta_i \in \mathfrak{S}(\pi_i)$ be the surgery obstruction for $\psi_i: L_i \to W_i$.

LEMMA 7.3. *Assume* (7.0), *that X is finite, and that $m = 2k + 1 \geqq 5$. Perform surgery as in* (7.2). *Then*

$$\textstyle\sum_i i_{i*}(\theta_i) = 0 \in \tilde{\mathfrak{S}}(\pi) .$$

PROOF. Let $\Lambda_i = \mathbf{Z}[\pi_i]$. Then $K_k(L_i; \Lambda) = K_k(L_i; \Lambda_i) \otimes_{\Lambda_i} \Lambda$, and as $K_k(L_i; \Lambda_i)$ (with the induced φ and μ) represents θ_i, $K_k(L_i; \Lambda)$ represents $i_{i*}(\theta_i)$. And $K_k(\partial M; \Lambda)$ is the direct sum of all these, and so represents $\sum_i i_{i*}(\theta_i)$. On the other hand, by the theorem, we have

$$0 \longrightarrow K_{k+1}(M, \partial M) \xrightarrow{\alpha} K_k(\partial M) \longrightarrow K_k(M) \longrightarrow 0 \quad ,$$

where each term is a free Λ-module and the extreme terms are dual, hence of the same rank. By (4.3) it will be enough to show that φ and μ vanish identically on the image of α.

Now, as in the proof of (7.1), consider the ladder

$$\begin{array}{ccccccc}
\pi_{k+2}(\psi_M) & \longrightarrow & \pi_{k+2}(\Phi) & \longrightarrow & \pi_{k+1}(\psi_{\partial M}) & \longrightarrow & \pi_{k+1}(\psi_M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \wr\!\!\mid\!\mid \\
K_{k+1}(M) = 0 & \longrightarrow & K_{k+1}(M, \partial M) & \longrightarrow & K_k(\partial M) & \longrightarrow & K_k(M) \, .
\end{array}$$

Here, the vertical maps need not be epimorphisms: ∂M need not even be connected! However, the images of the $\pi_{k+1}(\psi_{L_i})$ generate $K_k(\partial M)$ as a Λ-module, so any element of this group may be represented as the sum of spheres S^k in various components of ∂M, joined to the base point by paths in M (and null-homotopies in the W_i). If we had an element of Im α, it becomes zero in $K_k(M)$, thus the maps of spheres S^k in M extend to a mapping of a $(k+1)$-sphere, with several k-discs removed (the S^k's being the boundaries of these discs); say T^{k+1}. Also, as $\pi_{k+1}(\psi_M) \cong K_k(M)$, we may choose the map of T to be null-homotopic in X. But then, using ω, we have a framing of the bundle over T induced from τ_M, so as T is parallelisable, an injection of τ_T in τ_M, and now (as in (1.3)), we can use Hirsch's theorem to say that the mapping of T can be taken as an immersion, unique up to regular homotopy.

Now consider μ; to define this, we consider self-intersections of spheres. But if the immersion of T is in general position, as $k \geqq 2$ there are no triple points, so the double lines are a non-singular 1-chain, with boundary the self-intersections of the spheres. Similarly for mutual intersections φ; the intersections of the spheres form the boundary of the 1-chain given by the transverse intersection of the corresponding manifolds T.

We now observe that the Kronecker index of the boundary of a 1-chain vanishes (there is no ambiguity about elements of π, since T is simply-connected); similarly for self-intersections, provided that along each double line of T we make a choice of order of the two branches. Hence φ and μ vanish identically on the image of α, as asserted.

COROLLARY 7.4. *With the hypothesis of* (7.3), *suppose ∂X connected, and with the same fundamental group as X. Then if $k \geqq 3$, we can perform surgery to make $\partial M \longrightarrow \partial X$ a homotopy equivalence.*

The above lemma gives a sort of cobordism criterion, in that the obstruction for a bounding manifold ∂M satisfies a restriction; we note that the vanishing of the signature of a boundary is closely related to this.

COROLLARY 7.5. *Suppose* (7.0), *that $m = 2k \geqq 4$, that X is finite, that $\psi_{\partial M}$ is a homotopy equivalence, and that ψ_M is k-connected. Then the surgery obstruction $\theta \in \widetilde{\mathfrak{S}}(\pi)$ is not altered by any surgery which leaves ∂M fixed.*

PROOF. Suppose N a cobordism of $(M, \partial M)$ to $(M_+, \partial M)$, with ψ_{M_+} k-connected. After surgery on N, we may suppose ψ_N k-connected; then M, N, and M_+ have the same fundamental group. If ∂M is empty, (7.3) gives $\theta(M) + \theta(M_+) = 0$; but we conventionally change the orientation of M, so $\theta(M) = \theta(M_+)$ (using (4.5)). If ∂M is non-empty, we are not interested in $\theta(\partial N)$ but in $i_*\theta(\partial N)$ which, since $K_k(\partial N) = K_k(M) \oplus K_k(M_+)$, is just

$\theta(M_+) - \theta(M)$, and vanishes.

We next note that, even if $\psi_{\partial M}$ is a homotopy equivalence, it may be advantageous to alter ∂M in our efforts to make ψ_M a homotopy equivalence: this might happen, for example, in the proof of (7.1).

THEOREM 7.6. *Assume* (7.0) *with* $m = 2k \geqq 4$. *Let* W *be a component of* ∂X, *with fundamental group* π', *and inclusion* $i: \pi' \to \pi$. *Let* G' *be a free* π'-*module, and* $\varphi': G' \times G' \to \Lambda'$, $\mu': G' \to I'$ *satisfy the conditions of* (3.1); *suppose* G, φ, μ *formed by tensoring* (*over* Λ') *with* Λ, *as above. Then we can perform relative surgery, to replace* $K_k(M)$ *by* $K_k(M) \oplus G$, *with the maps* φ *and* μ *of the orthogonal direct sum.*

PROOF. Choose a basis $e'_1, \cdots, e'_r$, for G' over Λ'; this induces a free basis $e_1, \cdots, e_r$ for G over Λ. Select disjoint $(k+1)$-discs $\zeta_1 \cdots, \zeta_r$ in the component L of M with $\psi(L) = W$: write $\xi_i^0 = \partial\zeta_i$ for the boundary k-spheres. Now (as in §5) we will perform simultaneous regular homotopies of all the ξ_i^0. If

$$\varphi'(e'_i, e'_j) = \varphi'_{ij}, \text{ and } \mu'(e'_i) = \mu'_i,$$

we choose regular homotopies, with tracks η_i in L and $\bar{\eta}_i$ in $L \times I$, such that $\partial\eta_i = \xi_i^1 - \xi_i^0$ with the ξ_i^1 all disjoint, and the self-intersection invariant of $\bar{\eta}_i$ is μ'_i; for $i < j$, $\bar{\eta}_i \frown \bar{\eta}_j = \varphi'_{ij}$.

Now perform relative surgeries (as in the first paragraph of (7.1)) using the spheres ξ'_i. This has the effect of adding r k-handles to L by homotopically trivial maps, and hence replacing $K_k(M)$ by its direct sum with a free module of rank r, which we may identify with G (by identifying the i^{th} generator with e_i). Imbedded k-spheres in the resulting manifold, representing the e_i, may be constructed as follows.

It is more convenient to use as a model the manifold obtained from M by attaching a collar $L \times I$ before attaching the handles. Let δ_i be the core of the i^{th} handle. Then $\bar{\eta}_i$ is a cylinder in $L \times I$, with $\partial\bar{\eta}_i = \xi_i^1 \times 1 - \xi_i^0 \times 0$, hence $(\delta_i \cup \bar{\eta}_i \cup \zeta_i)$ is a sphere σ_i imbedded with some corners which can easily be smoothed. This sphere σ_i represents e_i. Note that σ_i is disjoint from the interior of M, hence e_i is orthogonal to $K_k(M)$.

It remains to compute $\varphi(e_i, e_j)$ and $\mu(e_i)$. As the δ_i and ζ_i are all disjointly imbedded, the only intersections and self-intersections that arise are those of the $\bar{\eta}_i$. But, by definition, the self-intersection in $L \times I$ of $\bar{\eta}_i$ is μ'_i; the intersection of $\bar{\eta}_i$ and $\bar{\eta}_j$ is φ'_{ij} (for $i < j$, hence also by symmetry for $i > j$). The intersections and signs are exactly the same in M, but elements of π' are replaced by their images in π. Thus the values of φ and μ on G are as asserted.

COROLLARY 7.7. *Suppose in* (7.6) *that* $\psi_{\partial M}$ *is a homotopy equivalence. Then, by relative surgery preserving this property, we can add to* θ_M *any element of the form* $\sum i_{i*}\theta_i$, *with* $\theta_i \in \tilde{\mathfrak{S}}(\pi_i)$, *where* π_i *are the fundamental groups of the components of* ∂X, i_i *the inclusions.*

For, on applying the theorem, $K_{k-1}(L)$ is replaced by its direct sum with the cokernel of $A\varphi$, the map associated to φ. Thus if we choose φ non-singular, ψ_L remains a homotopy equivalence.

COROLLARY 7.8. *Suppose in* (7.6) *that* $\psi_{\partial M}$ *is a homotopy equivalence, and that for some component* W *of* ∂X, $\pi_1 W = \pi_1 X$. *Let* $k \geqq 3$. *Then by relative surgery affecting only the corresponding component* L *of* ∂M, *we can make* ψ_M *a homotopy equivalence too.*

By the preceding corollary, we can reduce the surgery obstruction for M to zero; then by § 3, since $k \geqq 3$, we can make ψ_M a homotopy equivalence without further alternation to ∂M.

For further development, we ought to prove two results analogous to (3.3) and (5.6). However, it is perhaps even more desirable that (5.6) itself should first be strengthened.

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(Received November 22, 1965)