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1. Introduction and statement of results

According to A. Haefliger and M. W. Hirsch in (6), if M and M' are homeomorphic smooth *n*-manifolds and if M admits a tangent *k*-field with $k \leq \frac{1}{2}(n-1)$, then so does M'. Hirsch has asked whether 'homeomorphic' can be replaced by 'homotopy-equivalent' in the above assertion. This note gives a partial answer.

Let the term 'manifold' be restricted to mean one which is compact, connected, arientable, differentiable of class C^{∞} , and without boundary. Let SO(r) denote the rotation group in r dimensions. Let \oplus denote Whitney addition of vector bundles. According to context, the integer mwill denote either itself or the trivial m-plane bundle over the appropriate space. Throughout, let M, M' be homotopy-equivalent n-manifolds with tangent vector bundles τ , τ' , let w_i denote the *i*th mod 2 Stiefel-Whitney class of M, and suppose that $k \leq \frac{1}{2}(n-1)$. The assertion that there exists a continuous field of tangent k-frames to M will be abbreviated to 'M admits a k-field'. Let S^n denote the standard n-sphere.

THEOREM (1.1). If the group of $\tau \oplus 1$ is reducible to SO(n-k), then so is the group of $\tau' \oplus 1$.

THEOREM (1.2). Suppose that M admits a k-field and that one of the following holds:

- (b) S^n admits a k-field;
- (c) $n = 2^r(2m+1)-1$ for some integers $r, m \ge 1$; and $w_i = 0$ for $1 \le i \le 2^r$ and, if r > 3, for $i = 2^{r+1}m$.

Then M' admits a k-field.

The proofs take as point of departure Theorem (3.6) of (1). In fact Hirsch's question could be answered affirmatively if the fibre homotopy type of the tangent sphere-bundle were known to be a homotopy invariant of manifolds. The truth of this in special cases is used in (1.2).

There is a conjecture of W. Hurewicz that any two simply connected homotopy-equivalent manifolds are homeomorphic, and I am not aware of any two homotopy-equivalent *n*-manifolds with n > 3 which are known *Proc. London Math. Soc.* (3) 15 (1965) 543-56

⁽a) n is even;

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not to be homeomorphic. It is therefore difficult to say to what extent the above results are new, or how sharp (1.2) is.

Preliminaries and notation are contained in §2. We prove (1.1) in §3, and (1.2) in §4 and §5. Related topics are discussed in §6; for example the tangent sphere-bundle of a π -manifold M is shown to be fibre-homotopy trivial if and only if M is parallelizable (cf. Theorem 2 of (14)).

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2. Preliminaries and notation

The *n*-manifold M has the homotopy type of a CW-complex $N \cup_{\theta} e$, where N is a subcomplex of dimension less than n, and e is an *n*-cell attached by a map $\theta: S^{n-1} \to N$ (see for example (15)). Such a CWcomplex will be called a *principal decomposition* of M. The Puppe sequence of θ (§1 of (17)) will be written:

 $S^{n-1} \xrightarrow{\theta} N \xrightarrow{P\theta} M \xrightarrow{Q\theta} S^n \xrightarrow{S\theta} SN \to \cdots.$

Let the basepoint in S^n be the 'north pole' when S^n is identified with the unreduced suspension of an 'equatorial' S^{n-1} . Let H(n) denote the space of homotopy equivalences of S^{n-1} , with the compact-open topology. Let SH(n) be the component of H(n) consisting of maps of degree +1. There is a natural inclusion of SO(n) in SH(n). Let F(n) be the subspace of SH(n+1) consisting of basepoint-preserving maps. Unreduced suspension of maps defines an injection of SH(n) into F(n).

Since H(n), SH(n), F(n) are associative H-spaces, with multiplication defined by composition of maps, the results of (5) give corresponding 'classifying spaces' BH(n), BSH(n), BF(n), and maps between them induced by H-maps of the H-spaces. A map induced in this way by a canonical injection will be referred to as 'the natural map' between the appropriate classifying spaces.

Given spaces A, B, with basepoints, let $[A, B]^*$ denote the set of homotopy classes of maps from A to B, and let [A, B] denote the corresponding set in which maps and homotopies preserve basepoints. The basepoint in any CW-complex is assumed to be a vertex. When A is a CW-complex, and B is connected and simply connected, $[A, B]^*$ may be replaced by [A, B].

The following can be proved by the methods of (19).

(a) If K is a CW-complex, then [K, BSH(n)] is in natural one-to-one correspondence with the set of oriented fibre homotopy equivalence

classes of oriented Hurewicz fibrings over K in which the fibres are homotopy (n-1)-spheres. The analogous unoriented result is a special case of the classification theorem in (19).

(b) For even *n*, there is an Euler class *E* in the integral cohomology group $H^n(BSH(n), Z)$ such that if $i: BSO(n) \to BSH(n)$ is the natural map then $i^*(E)$ is the universal Euler class in $H^n(BSO(n), Z)$.

Observe that (a) can be twisted around to give a definition of 'oriented' as applied to fibrings and fibre homotopy equivalence.

LEMMA (2.1). Let K be an n-dimensional CW-complex and let $j: BSH(r) \rightarrow BSH(r+1)$ be the natural map. Then

 $j_* : [K, BSH(r)] \rightarrow [K, BSH(r+1)]$

is surjective for r = n and bijective for r > n.

Proof. Replace j by a Hurewicz fibre map with fibre Q. (See for example p. 241 of (19).) When $r \ge 3$, it follows from Corollary (2.2) of (9) and the exactness of the homotopy sequence of the triple (SH(r+1), F(r), SH(r)) that Q is (r-1)-connected. A short special argument gives the same result when r = 2. For $n \ge 2$ the lemma now follows by obstruction theory, and the case n = 1 is trivial.

Let BH denote the direct limit of the BH(n), with the weak topology, and define BSO, BSH similarly. The natural maps of BSH(n) to BH(n)induce a map $i: BSH \rightarrow BH$.

LEMMA (2.2). Let K be a connected, finite-dimensional CW-complex. Then $i_*: [K, BSH] \rightarrow [K, BH]^*$ is injective.

Proof. When considering maps of a finite-dimensional complex into BSH or BH, one may, in view of (2.1) and the results of (19), regard these spaces as H-spaces and i as an H-map. Observe that $i_*: \pi_r(BSH) \to \pi_r(BH)$ is injective for r = 1 and bijective for r > 1. Replace i by a Hurewicz fibre map with fibre W. From the homotopy sequence of i it follows that $\pi_r(W) = 0$ for r > 0. Let W_0 denote the path-component of the basepoint in W. Since K is connected, $[K, W] = [K, W_0]$. The latter set consists of a single element, by obstruction theory. Hence $i_*: [K, BSH] \to [K, BH]$ has trivial kernel, and since it is a loop homomorphism (Theorem (1.1) of (10)) it is therefore injective. Let g, $h: K \to BSH$ be basepoint-preserving maps such that $i \circ g$ and $i \circ h$ represent the same element of $[K, BH]^*$. Then by Corollary (4.4) of (10), $i \circ g$ and $i \circ f$ represent the same element of [K, BH]. Since $i_*: [K, BSH] \to [K, BH]$ is injective the result follows.

The same name will often be given to a map and its homotopy class, to a bundle or fibring and the appropriate classifying map.

Let K be an n-dimensional CW-complex, and let ξ , η be oriented r-plane bundles over K. Let $J: BSO(r) \to BF(r)$, $G: BSO(r) \to BSH(r)$ be the natural maps. If $J_*(\xi) = J_*(\eta)$ ($G_*(\xi) = G_*(\eta)$), then ξ and η will be said to be *J*-equivalent (*G*-equivalent). Thus by (a) above, *G*-equivalence of ξ and η coincides with oriented fibre homotopy equivalence of the associated sphere-bundles. If ξ , η are *G*-equivalent they are clearly *J*-equivalent; when $n \leq 2r - 4$, the converse is true by Corollary (2.2) of (9) and obstruction theory. If the stable forms of ξ , η are *J*-equivalent (*G*-equivalent) then ξ , η will be said to be stably *J*-equivalent (stably *G*-equivalent). Analogous terminology will be used for homotopy sphere fibrings.

3. A key lemma, and the stable case

The following key lemma depends on consequences of (21) deduced in (9).

LEMMA (3.1). Let ξ , η be J-equivalent oriented r-plane bundles over an n-dimensional CW-complex. Suppose that $\xi = \xi' \oplus (r-s)$ for some oriented s-plane bundle ξ' , with $r \ge s \ge \frac{1}{2}(n+1)$. Then $\eta = \eta' \oplus (r-s)$ for some oriented s-plane bundle η' . If $s > \frac{1}{2}(n+1)$, then this η' may be chosen J-equivalent to ξ' .

The proof will be given later in this section.

In the next lemma, all maps and homotopies are to preserve basepoints unless the contrary is stated. Suppose given a commutative square of maps

$$\begin{array}{c} A \xrightarrow{g} B \\ \downarrow^i & \downarrow^j \\ X \xrightarrow{h} Y \end{array}$$

in which each space is pathwise-connected. For $r \ge 1$, let $\pi_r(i)$ denote the *r*th relative homotopy set (group, if r > 1) of the mapping cylinder of *i*, mod *A*, and define $\pi_r(j)$ similarly. Together *h* and *g* induce a function

$$(h,g)_*: \pi_r(i) \to \pi_r(j).$$

LEMMA (3.2). With the above notation, suppose that $(h,g)_*$ is bijective for $1 \le r \le n$. Let K be an n-dimensional CW-complex, and let b in [K, B] and x in [K, X] be such that $j_*(b) = h_*(x)$. Then there exists an a in [K, A] with $i_*(a) = x$. If moreover $(h,g)_*$ is surjective when r = n+1, then this a may be chosen so that $g_*(a) = b$.

Proof. Let I denote the closed interval [0,1] on the real line. Given spaces $P \supseteq Q$, R, let $\Omega_P(Q, R)$ denote the space of paths in P which begin in Q and end in R. (The points of such a space are not required to be basepoint-preserving maps.) Let Z denote the (n-1)-skeleton of K. Using mapping cylinders, we may suppose that g, h, i, j are inclusions.

Suppose that the hypotheses of the first part of the lemma hold. Let $F: (K \times I, K \times 1) \to (Y, B)$ be a homotopy from $h \circ x$ to $j \circ b$. Define $\varphi: K \to \Omega_Y(X, B)$ by $\varphi(k)(t) = F(k, t)$. Consider the fibrings of $\Omega_X(X, A)$, $\Omega_Y(X, B)$ over X defined by taking the initial point of each path. The 'five lemma' applied to the homotopy sequences of these fibrings shows that the inclusion of $\Omega_X(X, A)$ in $\Omega_Y(X, B)$ induces a bijection of rth homotopy sets for $0 \leq r \leq n-1$ (injectivity in dimension n-1 will not be used). Hence, by obstruction theory (cf. Appendix to (18)), there is a deformation $\Gamma: Z \times I \to \Omega_Y(X, B)$ of the restriction $\varphi | Z$ to a map into $\Omega_X(X, A)$. Define $G: Z \times I \times I \to Y$ by $G(z, s, t) = \Gamma(z, s)(t)$. (The reader may find it helpful to sketch the associated map of $I \times I$ into the space of maps from Z to Y.) Define $H: (Z \times I, Z \times 1) \to (X, A)$ by

$$H(z,t) = \begin{cases} G(z, 2t, 0) & \text{if } 0 \leq 2t \leq 1, \\ G(z, 1, 2t-1) & \text{if } 1 \leq 2t \leq 2, \end{cases}$$

and $E: Z \times I \to B$ by E(z,t) = G(z,t,1). Then H is a homotopy of x | Zthrough X to the map $a': Z \to A$ defined by a'(z) = G(z,1,1); while Eis a homotopy of b | Z through B to a'. It is easy to see that $h \circ H: (Z \times I, Z \times 1) \to (Y, B)$ may be deformed to $F | (Z \times I, Z \times 1)$ by a homotopy of pairs leaving $Z \times 0$ pointwise fixed. It follows that if $P: (Z \times I \cup K \times 0, Z \times 1) \to (X, A)$ is defined by

$$\begin{cases} P(z,t) = H(z,t) & \text{if } z \in Z, \\ P(k,0) = x(k), \end{cases}$$

then $h \circ P$ is homotopic to $Q = F | (Z \times I \cup K \times 0, Z \times 1)$ by a homotopy of pairs. Consider the obstruction to extending P to a map $R: (K \times I, K \times 1) \to (X, A)$. For each closed *n*-cell \tilde{e} of K, the obstruction c to extending P over $(\tilde{e} \times I, \tilde{e} \times 1)$ lies in $\pi_n(X, A)$, and since $h \circ P$ is homotopic to Q, $(h, g)_*(c) = 0$. But $(h, g)_*$ is injective in dimension n, so an extension R exists. The map $a: K \to A$ required for the first part of the lemma may be defined by a(k) = R(k, 1).

Now suppose that in addition $(h,g)_*$ is surjective in dimension n+1. Then Z may be replaced by K in the first part of the above argument, and the analogue of a' will have the properties required for the second part of the lemma.

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Proof of (3.1). Apply (3.2) with X = BSO(r), A = BSO(s), Y = BF(r), B = BF(s), g, h, i, j the natural maps, $x = \eta$, and $b = J_*(\xi')$. The hypotheses on $(h, g)_*$ follow from Theorem (3.2) of (9).

Proof of (1.1). Let $f: M' \to M$ be a homotopy equivalence, and choose orientations so that f preserves orientation. By Theorem (3.6) of (1), and (2.1) and (2.2) above, $f^*(\tau) \oplus 1$ and $\tau' \oplus 1$ are *J*-equivalent. By hypothesis the group of $\tau \oplus 1$ is reducible to SO(n-k); hence so is the group of $f^*(\tau) \oplus 1$. Theorem (1.1) now follows from the first part of (3.1), with r = n+1 and s = n-k.

The second part of (3.1) yields

THEOREM (3.3). Let $f: M' \to M$ be an orientation-preserving homotopy equivalence, and suppose that $k < \frac{1}{2}(n-1)$. If $\tau \oplus 1 = \sigma \oplus (k+1)$ then $\tau' \oplus 1 = \sigma' \oplus (k+1)$, where σ' may be chosen J-equivalent to $f^*(\sigma)$.

We note explicitly the following corollary of (3.1).

COROLLARY (3.4). If τ and τ' are J-equivalent and if M admits a k-field with $k \leq \frac{1}{2}(n-1)$, then so does M'.

4. The even-dimensional case

Throughout this section let n be even. The following theorem will be proved later in the section.

THEOREM (4.1). If $f: M' \to M$ is an orientation-preserving homotopy equivalence of even-dimensional manifolds, then τ' and $f^*(\tau)$ are *G*-equivalent.

Theorem (1.2) (a) and the unstable analogue of (3.3) for even n follow from (4.1) and (3.1).

Let $K = L \cup_{\theta} e$ be a CW-complex, where L is a subcomplex of dimension less than n, and e is an n-cell attached by the map $\theta \colon S^{n-1} \to L$. Suppose that the integral homology group $H_n(K, Z)$ is non-zero. Let the result of acting, as in (4.3) of (17), by a in $\pi_n(BSH(n))$ on x in [K, BSH(n)], be written a.x. Recall that E denotes the Euler class.

LEMMA (4.2). With the above notation and hypotheses, if E(a.x) = E(x) then E(a) = 0.

Proof. Let Y be an Eilenberg-MacLane space of type (Z, n), and write BSH(n) = B. Let \lor denote disjoint union with basepoints identified.

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Here Δ is the diagonal map, ∇ the 'folding' map, *m* the multiplication, *i*, *j*, *k* the inclusions, $Q\theta$ as in §2, and η the 'pinching' map used for defining *a.x.* The diagram yields

$$E(a.x) = Q\theta^*(E(a)) + E(x).$$

Since $H_n(K, Z) \neq 0$, $Q\theta^*$ is injective, and the lemma follows.

LEMMA (4.3). Let $x \in \pi_n(BSH(n))$, with n even. Then the stable class and the Euler class of x together determine x uniquely.

Proof. Since $\pi_n(BSH(n))$ is a group, on which the Euler class is additive (special case of (4.2)), it is sufficient to suppose that x has stable class and Euler class both zero, and prove that x is zero. Let y in $\pi_{n-1}(SH(n))$ be the image of such an x under transgression in the universal principal quasifibration for SH(n) (see (5)). The following diagram commutes up to sign:

$$\begin{split} \pi_n(S^n) & \stackrel{\Delta}{\longrightarrow} \pi_{n-1}(SO(n)) \stackrel{i}{\longrightarrow} \pi_{n-1}(SH(n)) \stackrel{j}{\longrightarrow} \pi_{n-1}(F(n)) \stackrel{I}{\longrightarrow} \pi_{2n-1}(S^n) \\ & \downarrow^v \qquad \qquad \downarrow^k \qquad \qquad \downarrow^S \\ \pi_{n-1}(SH(n+2)) & \stackrel{u}{\longleftarrow} \pi_{n-1}(F(n+1)) \stackrel{I}{\longrightarrow} \pi_{2n}(S^{n+1}). \end{split}$$

Here *i*, *j*, *k*, *u*, *v* are induced by inclusions, Δ is transgression in the principal tangent bundle of S^n , *S* denotes suspension, and *I* is the isomorphism defined in (2.10) of (20). From the homotopy sequence of the standard projection of SH(n+2) on S^{n+1} it follows that *u* is injective, and hence kj(y) = 0 since v(y) = 0. Thus SIj(y) = 0, and, by (21), $Ij(y) = q[\iota, \iota]$ for some integer *q*, where $[\iota, \iota]$ is the Whitehead square of a generator ι of $\pi_n(S^n)$. Hence $Ij(y) = Iji\Delta(q\iota)$, by ((20) (22)). Now *I* is bijective, and by Corollary (2.2) of (9) *j* is injective (a short special argument is used when n = 2). Hence $y = i\Delta(q\iota)$. Thus *y* characterizes a stably trivial *n*-plane bundle (over S^n) whose Euler class is zero. The Thom complex of this bundle consists of a 2*n*-cell attached to S^n by $\pm q[\iota, \iota]$ (§ 1.3 and § 3 of (11)). Since $[\iota, \iota]$ has Hopf invariant ± 2 , the square of a generator of $H^n(T, Z)$ is $\pm 2q$ times a generator of $H^{2n}(T, Z)$.

The definition of E by means of the Thom isomorphism (p. 41 of (13)) shows that q = 0. Hence x = 0, as required.

LEMMA (4.4). Let K be the CW-complex considered in (4.2). Suppose that x, y are stably G-equivalent oriented Hurewicz homotopy (n-1)-sphere fibrings over K, with E(x) = E(y). Then x and y are G-equivalent.

Proof. Write $BSH(n) = B_n$. Let B_n^L be the space of maps from L to B_n , with the compact-open topology. Let u, in B_n^L , be a map which extends to K. There is a homomorphism $\alpha_u : \pi_1(B_n^L, u) \to \pi_n(B_n)$ such that for any extension v of u to K, a.v = v if and only if a is in the image of α_u (see the proof of (3.3) in (3)). In the following commutative diagram, j_* and j_* are induced by the natural map $j: B_n \to B_{n+1}$:

$$\begin{array}{cccc} \pi_1(B_n{}^L, u) & \stackrel{\alpha_u}{\longrightarrow} & \pi_n(B_n) \\ & & & \downarrow^{j_{\sharp}} & & \downarrow^{j_{\bullet}} \\ \pi_1(B_{n+1}{}^L, ju) \stackrel{\alpha_{j_u}}{\longrightarrow} & \pi_n(B_{n+1}). \end{array}$$

First note that a is in the image of α_u if and only if both E(a) = 0 and $j_*(a)$ is in the image of α_{ju} . For suppose the latter conditions hold. Since $S^1 \times L$ has dimension at most n, j_* is surjective, by an obvious relativization of (2.1). Thus $j_*(a) = j_*\alpha_u(x)$ for some z in $\pi_1(B_n{}^L, u)$. By (4.2), $E\alpha_u(z) = 0$; hence, by (4.3), $a = \alpha_u(z)$. Necessity of the conditions is immediate.

Now consider the following commutative diagram:

$$\begin{array}{cccc} \pi_n(B_n) & \xrightarrow{Q\theta^*} & [K, B_n] & \xrightarrow{P\theta^*} & [L, B_n] \\ & & & & \downarrow^{j_*} & & \downarrow^{j_*} \\ \pi_n(B_{n+1}) & \xrightarrow{Q\theta^*} & [K, B_{n+1}] & \xrightarrow{P\theta^*} & [L, B_{n+1}], \end{array}$$

where $P\theta$, $Q\theta$ are as in §2. Suppose that x, y, in $[K, B_n]$, are as in the hypotheses of the lemma. Then $j_*(x) = j_*(y)$, so $j_*P\theta^*(x) = j_*P\theta^*(y)$, and by (2.1), $P\theta^*(x) = P\theta^*(y)$. Hence y = a.x for some a in $\pi_n(B_n)$, by (4.5) on p. 326 of (17). Then $j_*(x) = j_*(y) = j_*(a).j_*(x)$ by naturality; i.e. $j_*(a)$ is in the image of α_{ju} , where $u = P\theta^*(x)$. Also E(a) = 0 by (4.2); hence a is in the image of α_u and x = a.x = y, as required.

Proof of (4.1). From the hypotheses of the theorem, τ' and $f^*(\tau)$ are stably G-equivalent as in §3, and they have the same Euler class. The conclusion now follows by (4.4), if we use a principal decomposition of M'.

5. The odd-dimensional case

Recall that M, M' are homotopy equivalent *n*-manifolds with tangent vector bundles τ , τ' , that w_i denotes the *i*th mod-2 Stiefel-Whitney class of M, and that $k \leq \frac{1}{2}(n-1)$. All cohomology groups in this section will be taken with mod-2 coefficients.

Theorem (1.2)(b) follows immediately from (1.1) and the following lemma.

LEMMA (5.1). Suppose that the group of $\tau \oplus 1$ is reducible to SO(n-r), and that S^n admits an r-field, for some $r \leq n$. Then M admits an r-field.

Proof. Assume n odd, since otherwise there is nothing to prove. Write $BSO(n) = B_n$, BSO = B. In the following commutative diagram, g, h, i, j are induced by natural maps, and $Q\theta$ is as in §2:

By hypothesis, $j(\tau) = ji(x)$ for some x in $[M, B_{n-r}]$, so $\tau = a.i(x)$ for some a in $\pi_n(B_n)$, as in the proof of (4.4). By naturality it is therefore sufficient to prove g surjective. Since the kernel of h is generated by the tangent bundle of S^n , which is by hypothesis in the image of g, it is sufficient to prove $h \circ g$ surjective. When $n \not\equiv 1 \mod 8$, $\pi_n(B) = 0$; while if $n \equiv 1 \mod 8$ then $r \leq 1$ since S^n admits an r-field, and the homotopy sequence of the standard projection of SO(n+1) on SO(n+1)/SO(n-1) shows that $h \circ g$ is surjective.

THEOREM (5.2). Let $f: M' \to M$ be an orientation-preserving homotopy equivalence. Suppose that $n = 2^{r}(2m+1) - 1$ for some integers $r, m \ge 1$. Suppose that $w_i = 0$ for $1 \le i \le 2^{r}$ and for $i = 2^{r+1}m$. Then τ' and $f^*(\tau)$ are G-equivalent.

Proof. Let *n* have the above form, and let Φ_n be the operation defined in (4) to detect the Whitehead square of a generator of $\pi_n(S^n)$; Φ_n may be taken to be an unstable secondary cohomology operation associated with an Adem relation

$$Sq^{n+1} = Sq^{2'}Sq^{2'+1m} + \sum b_{j}Sq^{n+1-j}Sq^{j},$$
(5.3)

where b_j is a binomial coefficient taken mod 2, and the sum is over $1 \leq j \leq 2^{r-1}$. Thus Φ_n is of degree *n*, and is defined on those *n*-dimensional cohomology classes on which Sq^j vanishes for $1 \leq j \leq 2^{r-1}$ and for $j = 2^{r+1}m$. In particular, Φ_n is defined on the fundamental class l' in

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the Thom complex T of τ , and (cf. §1 of (2)) a secondary characteristic class of τ is defined by $\Phi_n(\tau) = \psi^{-1}\Phi_n(U)$, where $\psi \colon H^*(M) \to \tilde{H}^*(T)$ is the Thom isomorphism. Two lemmas are required for the proof of (5.2). Let \mathscr{A} denote the mod-2 Steenrod algebra.

LEMMA (5.4). Let n be as in (5.2), and let $1 \leq j \leq 2^{r-1}$. Then Sq^{n+1-j} is in the right ideal of \mathscr{A} generated by all Sq^i with $1 \leq i \leq 2^{r-1}$.

The other lemma is a secondary analogue of (4.2). Let α_i , β_i be elements of positive degree in \mathscr{A} such that $\sum_i \alpha_i \beta_i$ vanishes on all *n*-dimensional cohomology classes, and let Φ be a secondary characteristic class associated with this unstable relation in the manner indicated above. Suppose that $K = L \cup e$ is a CW-complex in which L is a subcomplex of dimension less than n, and e is an n-cell attached by a map $\theta: S^{n-1} \to L$. Suppose that x in [K, BSH(n)] classifies a fibring for which Φ is defined and consists of a single element. Suppose that $a \in \pi_n(BSH(n))$. Let $H^n(K) \neq 0$.

LEMMA (5.5). With the above notation and hypotheses, $\Phi(a.x)$ is defined and consists of a single element. Moreover $\Phi(a.x) = \Phi(x)$ if and only if $\Phi(a) = 0$. (For the application, the first two conclusions could be assumed.)

Assuming these lemmas, and freely using the naturality of Φ_n , we proceed with the proof of (5.2). Let E be the total space of τ , and let E_0 be the complement of the zero cross-section in E. Since $\tilde{H}^*(T) \rightarrow H^*(M \times M)$ isomorphic with $H^*(E, E_0)$, there is an injection $\lambda \colon \tilde{H}^*(T) \rightarrow H^*(M \times M)$ as on p. 47 of (13) (cf. §4 of (2)). Let $\mathbf{U} = \lambda(U)$. Define U', λ', U' similarly. By Theorem (4.10) of (2), $\Phi_n(\mathbf{U})$ is defined. By (5.3), (5.4), and (4), Φ_n can be chosen so that the indeterminacy of $\Phi_n(\mathbf{U})$ is contained in $\sum Sq^j H^{2n-j}(M \times M)$, where the sum is over $1 \leq j \leq 2^r$. Now

$$Sq^{j}H^{2n-j}(M \times M) = v_{i}.H^{2n-j}(M \times M),$$

where v_j is the *j*th Wu class of $M \times M$. It is easy to check that $v_j = 0$ for $1 \leq j \leq 2^r$. Thus $\Phi_n(\mathbf{U})$ has zero indeterminacy. Hence, since λ is injective, $\Phi_n(U)$, or equivalently $\Phi_n(\tau)$, has zero indeterminacy. Similarly $\Phi_n(\mathbf{U}')$, $\Phi_n(\tau')$ are defined and have zero indeterminacy. The same is true of $\Phi_n(f^*(\tau))$, since $f^* \colon H^*(M) \to H^*(M')$ is bijective. Since \mathbf{U} is a homotopy invariant of M (see e.g. Theorem 15 of (13)), $\Phi_n(\mathbf{U}') = 0$ if and only if $\Phi_n(\mathbf{U}) = 0$. Thus, since λ , λ' are injective, $\Phi_n(\tau) = 0$ if and only if $\Phi_n(\tau') = 0$. But $f^* \colon H^n(M) \to H^n(M')$ is an isomorphism of groups of order 2, so $\Phi_n(\tau') = f^*\Phi_n(\tau) = \Phi_n(f^*(\tau))$. Also, τ' and $f^*(\tau)$ are stably G-equivalent as in §3. Theorem (5.2) now follows by replacing E by Φ_n and (4.2) by (5.5), in the proofs of (4.3) and (4.4).

Proof of (1.2) (c). First let $n \equiv 1 \mod 4$. Each of M, M' admits a 1-field. If $w_{n-1} \neq 0$ then neither admits a 2-field. If $w_{n-1} = 0$, then the conclusion follows by (5.2) and (3.1). Next let $n \equiv 3 \mod 8$. If M admits a k-field with $k \leq 3$, then so does M' by (1.2) (b). If $w_{n-3} \neq 0$, then neither M nor M' admits a 4-field. If $w_{n-3} = 0$, apply (5.2) and (3.1). The argument when $n \equiv 7 \mod 16$ is similar. For other odd n I am unable to make the transition from using (1.2) (b) to using (5.2), and only the latter method is employed.

Proof of (5.4). The proof is by induction on r. For r = 1, it is sufficient to observe that $Sq^n = Sq^1Sq^{n-1}$ since n is odd. Suppose the lemma holds for r < s, let $n = 2^s(2m+1)-1$, and let $1 \le j \le 2^{s-1}$. When n+1-j is odd, $Sq^{n+1-j} = Sq^1Sq^{n-j}$. When n+1-j is even the result follows by (5.3) and the inductive hypothesis, since 2^s does not divide n+1-j.

Proof of (5.5). Let

$$S^n \xrightarrow{r_1} S^n \lor K \xrightarrow{r_2} K$$

be the canonical retractions and inclusions. Let

$$\nabla : BSH(n) \lor BSH(n) \to BSH(n)$$

be the 'folding' map, and let $j_3: K \to S^n \vee K$ be the 'pinching' map used for defining a.x. Then $r_2 \circ j_3$ is homotopic to the identity map of K, and $r_1 \circ j_3 = Q\theta$, where the latter is as in §2. Let T_m $(1 \le m \le 4)$ denote the Thom complexes of a, x, a.x, and $\nabla \circ (a \vee x)$. Let U_m be the fundamental cohomology class of T_m , and ψ_m the appropriate Thom isomorphism. All the fibrings considered are induced from the universal fibring analogous to that on p. 243 of (19), so there are maps $t_m: T_m \to T_4$ $(1 \le m \le 3)$ such that $t_m^*\psi_4 = \psi_m j_m^*$. It follows that the map

$$t_1^* \oplus t_2^* \colon H^*(T_4) \to H^*(T_1) \oplus H^*(T_2)$$

is injective.

Since $\Phi(x)$ is defined, $\beta_i(U_2) = 0$ for each i; and $\beta_i(U_1) = 0$ for dimensional reasons. Hence, by injectivity of $t_1^* \oplus t_2^*$, and naturality, $\beta_i(U_4) = 0$. Hence $\beta_i(U_3) = 0$, for each i, and $\Phi(U_3)$ is defined.

To see that $\Phi(U_3)$ has zero indeterminacy, let $F \to P \to K(Z_2, n)$ be the fibring and φ the element of $H^{2n}(P)$, used for defining Φ (see (4)). Let $V_4: T_4 \to P$ be a lift of U_4 . Then $V_m = V_4 \circ t_m$ is a lift of U_m ($1 \le m \le 3$). Let Δ_m be the diagonal map of T_m , and $\mu: F \times P \to P$ the action as in (16). By (16), we require to show that for any map $g_3: T_3 \to F$, the following holds with m = 3:

$$\Delta_m^*(g_m \times V_m)^* \mu^*(\varphi) = V_m^*(\varphi). \tag{5.6}_m$$

Since $t_3^*: [T_4, F] \rightarrow [T_3, F]$ is bijective, it is sufficient to show that $(5.6)_4$

holds for any $g_4: T_4 \to F$. Since $\Phi(a), \Phi(x)$ have zero indeterminacy, and, by naturality, both sides of $(5.6)_4$ have the same image under t_m^* (m = 1, 2). Hence $(5.6)_4$ holds since $t_1^* \oplus t_2^*$ is injective.

Finally it will be shown that $\Phi(a.x) = Q\theta^*\Phi(a) + \Phi(x)$. Since $Q\theta^*$ is injective, (5.5) will follow. By naturality,

$$j_m^* \psi_4^{-1} \Phi(U_4) = \psi_m^{-1} \Phi(U_m) \quad (1 \le m \le 3).$$

Thus

$$\begin{split} \Phi(a.x) &= j_3^* \psi_4^{-1} \Phi(U_4) \\ &= j_3^* (r_1^* j_1^* + r_2^* j_2^*) \psi_4^{-1} \Phi(U_4) \\ &= j_3^* r_1^* \Phi(a) + j_3^* r_2^* \Phi(x) \\ &= Q \theta^* \Phi(a) + \Phi(x), \end{split}$$

as required.

6. Further remarks

Lemma (3.1) has implications for differentiable immersions and embeddings. I shall mention only cases where something new seems to emerge. Let M be an *n*-dimensional homotopy π -manifold (i.e. the tangent sphere-bundle of M is stably fibre-homotopy trivial). Suppose 2m > 3n. As a corollary of work of W. Browder, R. V. Desapio, and A. Haefliger it follows that M immerses in Euclidean space \mathbb{R}^m , and if $\pi_i(M) = 0$ for $4i \leq n$ then M embeds in \mathbb{R}^{m+1} —with a restriction when $n \equiv 2 \mod 4$. The same is true without the restriction, by (3.1), Theorem (3.6) of (1), Theorem (6.4) of (7), and Theorem 1 of (12).

With the notation of §1, suppose that $2(\tau \oplus 1)$ is trivial. Then it is known that M immerses in \mathbb{R}^{2n-k} (k < n) if and only if the group of $\tau \oplus 1$ is reducible to SO(n-k). By §3, if $k \leq \frac{1}{2}(n-1)$ then 'trivial' here can be replaced by '*J*-equivalent to the trivial bundle'.

Using (3.1), partial results on fields of tangent planes are obtained as follows. The notation of §1 will be used, except that k is now allowed to be any integer between 1 and n.

THEOREM (6.1). If the group of $\tau \oplus 1$ is reducible to $SO(k) \times SO(n+1-k)$ then so is the group of $\tau' \oplus 1$.

THEOREM (6.2). If either n is even or the hypotheses of (5.2) hold, and if M admits a continuous field of tangent k-planes, then so does M', except possibly in the following cases: n even and 2k = n; n odd and 2k = n - 1 or n + 1.

Nothing was claimed in (3.1) about uniqueness of η' . An obstruction theory relating to this might be interesting (cf. (8)). The following relies on the vanishing of the first obstruction, for a homotopy π -manifold.

THEOREM (6.3). The tangent sphere-bundle of a π -manifold M is fibrehomotopy trivial if and only if M is parallelizable.

The proof requires a lemma.

LEMMA (6.4). Let M be a c-connected (c > 0), n-dimensional homotopy π -manifold, and suppose that $r \ge n-1-2c$. Let $Q\theta$ be as in §2. Then $Q\theta^*: \pi_n(BF(r)) \rightarrow [M, BF(r)]$ has trivial kernel.

Proof. Let $N \cup_{\theta} e$ be a principal decomposition of M (§2). It is sufficient to show that $S\theta^* \colon [SN, BF(r)] \to \pi_n(BF(r))$, or equivalently $\theta^* \colon [N, F(r)] \to \pi_{n-1}(F(r))$, is zero. By the isomorphisms in (2.10) of (20) the latter transforms into $(S^r\theta)^* \colon [S^rN, S^r] \to \tau_{n+r-1}(S^r)$. Since M is a homotopy π -manifold, θ is stably trivial. The result follows since $S^r\theta$ is stable when $r \ge n-1-2c$.

Proof of (6.3). Sufficiency is obvious. Suppose that the tangent sphere-bundle of M is fibre-homotopy trivial. By Lemma (3.2) of (10) and the method of proof of (2.2), τ is *J*-equivalent to the trivial bundle. Write $BSO(n) = B_n$, BSO = B, BSH(n) = B'. In the following commutative diagram, *i* and *J* are induced by natural maps.

$$\begin{array}{cccc} \pi_n(B) & \xleftarrow{i} & \pi_n(B_n) \xrightarrow{J} & \pi_n(B') \\ & & & & \downarrow^{Q\theta^*} & & \downarrow^{Q\theta^*} \\ [M,B] & \xleftarrow{i} & [M,B_n] \xrightarrow{J} & [M,B']. \end{array}$$

Since *M* is a π -manifold, $i(\tau) = 0$. Thus $\tau = Q\theta^*(x)$ for some *x* in $\pi_n(B_n)$. Since $Q\theta^*i(x) = 0$ and *M* is a spin manifold, i(x) = 0. Since $Q\theta^*J(x) = 0$, J(x) = 0 by (6.4). It follows quickly that x = 0, so $\tau = 0$, as required.

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