THE BROWDER-DUPONT INVARIANT

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1. Introduction

Let M be a smooth closed manifold of *odd* dimension q. Consider all q-plane bundles over M which are stably bundle equivalent to the tangent bundle τ of M. By a special case of Theorem 1.6 in [**7**], these fall into either one or two bundle equivalence classes, and there is a method for deciding, given M, which case occurs. When there are two classes, we may ask for an invariant to distinguish between them. Similar questions are posed by, for example, (a) fibre homotopy equivalence classes of $(q-1)$-sphere bundles or fibrings over M which are stably fibre homotopy equivalent to the tangent sphere bundle and (b) the analogue for spherical fibrings over a Poincaré complex.

These questions have a feature in common with the Kervaire–Arf invariant question for $2q$-manifolds: to tackle them, it is convenient to have an operation or characteristic class which is widely defined, has small indeterminacy, and detects the tangent bundle of the q-sphere S^q (q odd, $q \neq 1, 3, 7$). In [**5**], Dupont exploited this similarity, adapting Browder's technique from [**3**] to construct a mod 2 number $b(\xi)$ for certain bundles ξ over M. He concentrated on case (a) above, and gave a homotopy-theoretic proof of a result of Benlian, Hirsch, and Wagoner (see [**2**]).

Concentrating on the bundle equivalence case, we construct a version $b_B(\xi)$ of $b(\xi)$ which has less indeterminacy than the original one. (Dr Dupont has confirmed the observation that the type II indeterminacy in § 4 of [**5**] is non-zero whenever the type I indeterminacy is zero; but he points out that this does not invalidate the proof of the main Theorem 5.1 of [**5**], since consistent lifts may be chosen there. For further remarks concerning [**5**] see § 7 below.) We show that if there are two distinct classes of q-plane bundles over M stably equivalent to τ, then b_B distinguishes between them, and $b_B(\tau)$ is the Kervaire mod 2 semi-characteristic of M. As an application, we strengthen some known results about fields of tangent k-frames on a q-manifold.

One could more generally consider the q-plane bundles in any fixed stable class over any q-dimensional complex, again say for q odd. However, although Theorem 1.6 of [**7**] enumerates such bundles, the invariant b_B is not obviously susceptible to generalization.

The reason we restrict to odd q is that the Euler class provides a suitable invariant for answering the analogous questions when q is even, at least for oriented bundles.

In § 2 we state the main results; these are slightly sharper than is indicated above. In §§ 3 and 4 we prove some auxiliary results about Wu orientations. The main results are proved in §§ 5 and 6. An essential ingredient in the evaluation of b_B on tangent bundles is the technique used in [4] to extend the definition of Kervaire invariants. Indeed, Ed Brown has shown (private communication) how to describe much of the present work in the framework of [4]. We retain the language of [3] in order to make clear how this work relates to [5]. In § 7 we comment on cases (a) and (b) above, and in § 8 we give the application to vector fields.

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2. Construction of b_B and main results

In this section we recall Dupont's construction of $b(\xi)$, and describe how to refine it. We then state the main results.

All homology and cohomology groups will be taken with mod 2 coefficients. Throughout, q will denote a fixed odd integer. An Eilenberg–Maclane space $K(\mathbf{Z}_2, r)$ will be denoted by K_r. The product m-plane bundle over any space will be denoted by m.

Let B be a classifying space for the orthogonal group $O(n)$, with $n \gg q$, and let γ be the canonical n-plane bundle over B. From § 4 of [3] recall that if $\pi: E \to B$ is any fibring over B and if $\bar{\gamma} = \pi^*(\gamma)$, then an *$E$-orientation* of an n-plane bundle η is a bundle map from η to $\bar{\gamma}$. If an E-orientation of η exists we say that η is *E-orientable*. Two E-orientations of η are *equivalent* if they are homotopic through bundle maps. In particular, still following [3], let $\pi: B\langle v_{q+1}\rangle \to B$ be the principal fibring classified by a representative map $v_{q+1}: B \to K_{q+1}$ for the universal Wu class v_{q+1}. When the value of q is understood, we refer to $B\langle v_{q+1}\rangle$-orientations as *Wu orientations*. In this case, the Thom complex $T(\bar{\gamma})$ is part of a Wu spectrum, in the terminology of [3]; in particular $H^n(T(\bar{\gamma})) \approx \mathbf{Z}_2$ and χSq^{q+1} is zero on $H^n(T(\bar{\gamma}))$, where χ is the canonical anti-automorphism of the mod 2 Steenrod algebra. An S-dual X of (a finite skeleton of) $T(\bar{\gamma})$ is then part of the corresponding Wu cospectrum; if dimensions are chosen so that $H^{n+2q}(X)$ is dual to $H_n(T(\bar{\gamma}))$, then $H^{n+2q}(X) \approx \mathbf{Z}_2$ and Sq^{q+1} is zero on $H^{n+q-1}(X)$.

Now following [5], we let M be a smooth closed connected q-manifold with tangent bundle τ and stable normal bundle ν. Then 2ν is Wu orientable; let $a\colon 2\nu \to \bar{\gamma}$ be a Wu orientation. Since a is a bundle map, it gives rise to a map of Thom complexes,

$$T(a)\colon T(2\nu) \to T(\bar{\gamma}).$$

Suppose that ξ is a q-plane bundle over M which is stably bundle equivalent to τ. By choosing a trivialization of $\xi \oplus \nu$ we specify an S-duality between $T(2\nu)$ and $\Sigma^n T(\xi)$, where Σ denotes suspension (cf. [1]). We choose also a fixed S-duality between X and (a finite skeleton of) $T(\bar{\gamma})$. Then the S-dual of $T(a)$ is a map

$$g\colon X \to \Sigma^n T(\xi),$$

called an *X-orientation* of ξ.

Next let $U_\xi\colon T(\xi) \to K_q$ represent the Thom class of ξ, let $f\colon X \to \Sigma^n K_q$ denote the composition $\Sigma^n U_\xi \circ g$, and let ι denote the fundamental class in $H^q(K_q)$. Then (cf. [5, § 4]) the functional Steenrod square Sq_f^{q+1} is defined on $\Sigma^n \iota$ and $Sq_f^{q+1}(\Sigma^n \iota)$ lies in $H^{n+2q}(X) \approx \mathbf{Z}_2$.

In [5] Dupont described the analogue of this construction for ξ a $(q-1)$-sphere fibring which is stably fibre homotopy equivalent to the tangent sphere bundle τ_S, and denoted $Sq_f^{q+1}(\Sigma^n\iota)$ by $b(\xi)$ in that case. He pointed out the choices involved in the construction of $b(\xi)$, and proved the beautiful result that the choice of (fibre homotopy) trivialization of the Whitney join $\xi + \tau_S$ causes indeterminacy in the value of $b(\xi)$ only if all such ξ are fibre homotopy equivalent to τ_S. However (cf. § 1 above), the choice of Wu orientation of 2ν also causes indeterminacy. We therefore restrict this choice by imposing a symmetry condition.

Given a vector bundle ζ over a space A, let $t\colon \zeta \times \zeta \to \zeta \times \zeta$ denote the bundle map which switches the factors in the product bundle $\zeta \times \zeta$. Reverting to the context of general E-orientations as in § 4 of [3], we say that an E-orientation $a\colon \zeta \times \zeta \to \bar{\gamma}$ is *symmetric* if a and $a \circ t$ are equivalent. (Recall that equivalent means homotopic through bundle maps.) Let $\bar{\Delta}\colon \zeta \oplus \zeta \to \zeta \times \zeta$ denote the natural bundle map covering the diagonal map of A. The following proposition will be proved in § 4.

PROPOSITION 2.1. *Let $\pi\colon E \to B$ be the principal fibring over the classifying space B induced by a map*

$$v\colon B \to \prod_{i=1}^{r} K_{q_i+1},$$

where the q_i are odd integers. Let A be a space having the homotopy type of a countable connected CW-complex, and let ζ be a vector bundle over A. Then *for any two symmetric E-orientations* a_1, a_2 *of* $\zeta \times \zeta$, *the E-orientations* $a_1 \circ \bar{\Delta}$ *and* $a_2 \circ \bar{\Delta}$ *of* $\zeta \oplus \zeta$ *are equivalent.*

PROPOSITION 2.2. *Let* ν *be the stable normal bundle of a q-manifold (q odd). Then there exists a symmetric Wu orientation of* $\nu \times \nu$.

With these propositions in mind, we define $b_B(\xi)$ by going through the bundle equivalence case of Dupont's construction, except that we restrict ourselves to using a Wu orientation of 2ν of the form $a \circ \bar{\Delta}$, where a is a symmetric Wu orientation of $\nu \times \nu$. An X-orientation of ξ arising from such a Wu orientation of 2ν (and some trivialization of $\xi \oplus \nu$) will be called *allowable.*

Before stating the main results, we introduce an alternative point of view on the choices involved in the above construction which allows sharper statements to be made. I am grateful to J. Morgan and F. Quinn for drawing my attention to this. Instead of considering q-plane bundles which are stably equivalent to τ, we consider reductions of the stable tangent bundle to a q-plane bundle. One way of defining such a reduction is as an equivalence class of pairs (ξ, θ), where ξ is a q-plane bundle over M, $\theta\colon \xi \oplus 2 \to \tau \oplus 2$ is a bundle equivalence, and the pairs (ξ_1, θ_1), (ξ_2, θ_2) are equivalent if there exists a bundle equivalence $\varphi\colon \xi_1 \to \xi_2$ such that θ_1 and $\theta_2 \circ (\varphi \oplus 1)$ are homotopic through bundle equivalences. Now there is a canonical trivialization of $\tau \oplus \nu$, obtained by embedding M in a high-dimensional euclidean space. Using this, we may associate with any pair (ξ, θ) as above a distinguished homotopy class of trivializations of $\xi \oplus \nu$. The allowable X-orientation of ξ arising from a trivialization in this distinguished class is thus unique up to homotopy. We define $b_B(\xi, \theta)$ by going through the construction of $b_B(\xi)$, using at the appropriate stage an allowable X-orientation in this unique class. In § 5 we shall prove the following sharp version of Corollary 4.5 of [**5**] (stated here in the bundle equivalence case).

THEOREM 2.3. *The above* $b_B(\xi, \theta)$ *is well defined on equivalence classes of pairs.*

When stating results in the $b_B(\xi)$ terminology, it will be convenient to say that M has *James–Thomas number i* if there are precisely i bundle equivalence classes of q-plane bundles over M which are stably bundle equivalent to τ.

COROLLARY 2.4. *If the James–Thomas number of* M *is* 2, *then* $b_B(\xi)$ *is well defined for any q-plane bundle* ξ *which is stably bundle equivalent to* τ.

In § 5 we shall also check that b_B does the job it is designed for.

THEOREM 2.5. *Suppose that* $(\xi_1, \theta_1), (\xi_2, \theta_2)$ *are inequivalent pairs. Then* $b_B(\xi_1, \theta_1) \neq b_B(\xi_2, \theta_2)$.

COROLLARY 2.6. *If* ξ *is a q-plane bundle over* M *which is stably equivalent but not equivalent to* τ, *then* $b_B(\xi) \neq b_B(\tau)$.

The next result, which will be proved in § 6, is a more precise form of the evaluation result mentioned in § 1. We recall that the Kervaire mod 2 semi-characteristic $k(M; \mathbf{Z}_2)$ of M is the congruence class mod 2 of $\sum_i \operatorname{rank} H_{2i}(M; \mathbf{Z}_2)$.

THEOREM 2.7. *Let* 1 *denote the identity map of* $\tau \oplus 2$. *Then*

$$b_B(\tau, 1) = k(M; \mathbf{Z}_2).$$

COROLLARY 2.8. *If the James–Thomas number of* M *is* 2, *then* $b_B(\tau) = k(M; \mathbf{Z}_2)$.

EXAMPLE 2.9. Let τ_q denote $\tau(S^q)$. Then $b_B(\tau_q) = 1$ for $q \neq 1, 3, 7$.

3. Homotopy-symmetric lifts

We now establish some facts about homotopy-symmetric lifts which will be used in the study of symmetric orientations in § 4. Throughout this section let A be a space having the homotopy type of a countable connected CW-complex and let B be any space. The maps in this section and the next will generally not be required to preserve basepoints, but in this section we need to compare basepoint-preserving and free homotopy sets. For any spaces X, Y with basepoints, let $[X, Y]_0$, $[X, Y]$ denote the sets of homotopy classes of maps from X to Y, where the maps and homotopies are basepoint-preserving in the first set, free in the second set. Let $j\colon [X, Y]_0 \to [X, Y]$ denote the function which forgets about basepoints.

Let $t\colon A \times A \to A \times A$ denote the factor-switching map. We shall say that a map $f\colon A \times A \to B$ is *homotopy-symmetric* if f is homotopic to $f \circ t$. Suppose that f is homotopy-symmetric and let $H\colon A \times A \times I \to B$ be a homotopy from f to $f \circ t$. We shall say that a map $\Phi\colon A \times A \times I^2 \to B$ is an *H_2-structure extending* H if

$$\begin{aligned}
\Phi(a_1, a_2, 0, s) &= f(a_1, a_2),\\
\Phi(a_1, a_2, 1, s) &= f(a_2, a_1),\\
\Phi(a_1, a_2, r, 0) &= H(a_1, a_2, r),\\
\Phi(a_1, a_2, r, 1) &= H(a_2, a_1, 1-r),
\end{aligned}$$

for all a_1, a_2 in A and r, s in I.

Let $K = \prod_{i=1}^{r} K_{q_i+1}$, where the q_i are odd integers, and let $\pi: E \to B$ be the principal fibring classified by some map $v: B \to K$. For convenience we shall use $\bar{H}^i(X)$ to denote $H^i(X)$ if $i \neq 1$, and $[X, K_1]$ (free homotopy classes) if $i = 1$.

PROPOSITION 3.1. (a) *Suppose that $f: A \times A \to B$ is a map which lifts to E, and that H is a homotopy from f to $f \circ t$ which extends to an H_2-structure. Then there exists a lift $f': A \times A \to E$ of f such that f' is homotopic to $f' \circ t$ by a homotopy covering H.*

(b) *Suppose that $f: A \times A \to B$ is a map and that H_0, H_1 are homotopies from f to $f \circ t$ which are homotopic (through homotopies from f to $f \circ t$). Suppose also that for $i = 0, 1$, $f'_i: A \times A \to E$ is a lift of f such that f'_i is homotopic to $f'_i \circ t$ by a homotopy covering H_i. Then the homotopy classes of f'_0 and f'_1 (as lifts of f) differ by the action of a symmetric element of $\sum_{i=1}^{r} \bar{H}^{q_i}(A \times A)$.*

For simplicity, we make an extra assumption in proving this proposition: that if $q_i = 1$ for some i, then $H^1(A) = 0$. (In Remark 3.4 we shall indicate how to proceed in general.) Even assuming this extra condition, we need to take further notice of basepoints. As basepoint in a product space $Y = \prod_i Y_i$ we take the point $\{y_i\}$, where y_i is the basepoint in Y_i. We make the identifications $[X, Y]_0 = \prod_i [X, Y_i]_0$, $[X, Y] = \prod_i [X, Y_i]$. We now consider the forgetful function

$$j_i: [A \times A, \Omega K_{q_i+1}]_0 \to [A \times A, \Omega K_{q_i+1}].$$

Since $A \times A$ is a CW-complex (with basepoint a vertex, say), and ΩK_{q_i+1} is path-connected for $q_i \geqslant 1$, it follows that j_i is onto for $q_i \geqslant 1$. For $q_i > 1$, ΩK_{q_i+1} is simply-connected and j_i is one-to-one. For $q_i = 1$, the extra assumption ensures that $[A \times A, \Omega K_{q_i+1}]_0 \approx H^1(A \times A) = 0$, and that j_i is trivially one-to-one. From the previous remarks about the case where Y is a product space, it follows that

$$j: [A \times A, \Omega K]_0 \to [A \times A, \Omega K]$$

is an isomorphism, and we may therefore identify $[A \times A, \Omega K]$ with $\sum_{i=1}^{r} H^{q_i}(A \times A)$.

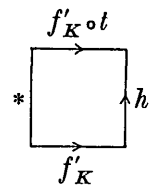
Proof of 3.1(a). The function spaces in the proof will all be supposed to have the compact-open topology. We shall use the same notation for a map and its homotopy class.

We are given that f lifts to E. For some (provisional) choice of a lift f', let α denote the obstruction to the existence of a homotopy from f' to $f' \circ t$ covering H. Then $\alpha \in [A \times A, \Omega K]$ and as above we may identify α with an element of $\sum_{i=1}^{r} H^{q_i}(A \times A)$. We now show that α is symmetric.

LEMMA 3.2. *With the above notation, $t^*\alpha = \alpha$.*

Proof. Let $*$ denote the basepoint in K, $\mathscr{L}K$ the space of paths in K beginning at $*$, and $p\colon \mathscr{L}K \to K$ the endpoint map. There is a one-to-one correspondence between maps g into E and pairs of maps (g_K, g_B) into $\mathscr{L}K, B$ such that $p \circ g_K = v \circ g_B$.

We are given $g_B = H\colon A \times A \times I \to B$. Let $g'_K\colon A \times A \times \partial I \to \mathscr{L}K$ be the map given by f'_K on $A \times A \times \{0\}$ and by $f'_K \circ t$ on $A \times A \times \{1\}$, where f' is the provisional choice of lift of f. Then α is the obstruction to the extension of g'_K to a map $g_K\colon A \times A \times I \to \mathscr{L}K$ such that $p \circ g_K = v \circ g_B$. Let K^I denote the space of (free) paths in K, and let $h\colon A \times A \to K^I$ denote the adjoint of $v \circ g_B$. As a map of $A \times A$ into the loop space ΩK, $\alpha = f'_K.h.(f'_K \circ t)^{-1}$, where the dot denotes composition of paths and the inverse means that the path runs backwards. It is convenient to describe α by means of a diagram showing the adjoint map of the boundary ∂I^2 into the function space $K^{A \times A}$.



The labels indicate suitable adjoints of the maps named.

Now since H extends to an H_2-structure, it follows that α is homotopic to β, where

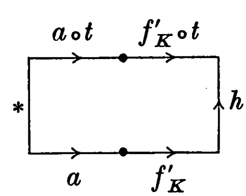
$$\beta = f'_K.(h \circ t)^{-1}.(f'_K \circ t)^{-1}.$$

On the other hand,

$$t^*\alpha = (f'_K \circ t).(h \circ t).(f'_K)^{-1} = \beta^{-1}.$$

But each factor in the product K is a $K(\mathbf{Z}_2, q_i + 1)$, so $\beta^{-1} = \beta$, and hence $t^*\alpha = \alpha$ as required.

We now return to the proof of 3.1(a). Using the Künneth formula, Lemma 3.2, and the oddness of q_i, we may write $\alpha = a + t^*a$ for some (not necessarily unique) a in $\sum_{i=1}^{r} H^{q_i}(A \times A)$. We now alter f' by the action of a, where this action is defined as usual by the principal fibre space structure of $\pi\colon E \to B$. This alters the obstruction diagram to the following:



The obstruction to a suitable homotopy covering H is now

$$a+\alpha+t^*a = 0.$$

Proof of 3.1(b). This may be summed up by a diagram as follows:

$$\begin{array}{c|c|c|c|c}
 & f'_{0,K}\circ t & v\circ f\circ t & f'_{1,K}\circ t & \\ \hline
* & H'_{0,K} & \Psi & H'_{1,K} & * \\ \hline
 & f'_{0,K} & v\circ f & f'_{1,K} &
\end{array}$$

Here Ψ corresponds to a homotopy from H_0 to H_1 through homotopies from f to $f\circ t$, and H'_i is a homotopy from f'_i to $f'_i\circ t$ covering H_i. The difference d between the lifts f'_0 and f'_1 is represented by the bottom edge of the diagram, and the whole diagram represents a homotopy from d to $d\circ t$.

REMARK 3.3. We have presented the above proofs as if $[A\times A,\Omega K]_0$ and $[A\times A,\Omega K]$ were identical. This is justified since the isomorphism $j\colon [A\times A,\Omega K]_0 \to [A\times A,\Omega K]$ is natural with respect to t^* and u_*, where $u\colon \Omega K\to\Omega K$ denotes inversion of loops.

REMARK 3.4. Suppose that $q_i = 1$ for some i, and we do not assume that $H^1(A) = 0$. It is clear that in proving 3.1(a), we may consider separately the obstruction corresponding to each factor K_{q_i+1}. Suppose that the obstruction α lies in $[A\times A,\Omega K_2]$. The latter group may no longer be isomorphic to $H^1(A\times A)$, but a 'Künneth' formula still holds: there is an isomorphism $\varphi\colon [A,\Omega K_2]\times[A,\Omega K_2]\to[A\times A,\Omega K_2]$ defined by

$$\varphi(\alpha_1,\alpha_2) = m\circ(\alpha_1\times\alpha_2),$$

where m denotes the multiplication (loop composition) in ΩK_2. Moreover, $t^*\circ\varphi = \varphi\circ t$, where t switches the factors in $[A,\Omega K_2]\times[A,\Omega K_2]$. Using this, and the fact that $t^*\alpha = \alpha$, we get that $\alpha = \varphi(\alpha_1,\alpha_1)$ for some α_1 in $[A,\Omega K_2]$. We alter the provisional choice of lift by the action of $\alpha_1\circ p_1$, where $p_1\colon A\times A\to A$ denotes projection on the first factor, and check that the new obstruction class is zero.

The proof of 3.1(b) is unaffected when $H^1(A)\neq 0$, and we have only to note that $\bar{H}^1(A\times A)$ may no longer be equal to $H^1(A\times A)$ in this case.

4. Symmetric orientations

Throughout this section we use the notation and the methods of § 4 of [3]. In particular, $\rho\colon\tilde{\gamma}\to\gamma$ denotes the natural bundle map covering the fibre map $\pi\colon E\to B$, where B is a classifying space as in § 2 above. We shall deal with vector bundles over CW-complexes but, as observed by Browder, these could be replaced by 'bundles' in some other suitable category.

We begin by recalling an alternative description of E-orientations. Let η be a vector bundle with base space X (a CW-complex). Then there is a canonical one-to-one correspondence between E-orientations of η and pairs (c,f') such that $c\colon \eta \to \gamma$ is a bundle map and $f'\colon X \to E$ is a lift of the map $f\colon X \to B$ covered by c: to the E-orientation $a\colon \eta \to \bar{\gamma}$ we associate the pair (c,f') where $c = \rho \circ a$ and $f'\colon X \to E$ is the map covered by a. It is easily checked that this does set up a one-to-one correspondence, since $\bar{\gamma} = \pi^*(\gamma)$. If the E-orientation a corresponds in this way to the pair (c,f'), then in the terminology of [3], a is *canonical* (*with respect to* c). If for $i = 0, 1$ the E-orientation a_i of η corresponds to the pair (c_i,f'_i), then there is a similar one-to-one correspondence between equivalences from a_0 to a_1 and pairs (θ, H') where θ is a homotopy through bundle maps from c_0 to c_1, and $H'\colon X \times I \to E$ is a homotopy from f'_0 to f'_1 such that $H = \pi \circ H'$ is the homotopy covered by θ. Suppose in particular that $c_0 = c_1 = c$, and that $f\colon X \to B$ is the map covered by c; thus a_0 and a_1 are both canonical (with respect to c) and $\pi \circ f'_0 = \pi \circ f'_1 = f$. Then a canonical equivalence between a_0 and a_1, as defined by Browder, corresponds to a pair (θ, H') where θ is the constant homotopy of c and H' is a homotopy from f'_0 to f'_1 through lifts of f.

Proof of Proposition 2.1. Suppose that $a_1, a_2\colon \zeta \times \zeta \to \bar{\gamma}$ are symmetric E-orientations as in the hypotheses of Proposition 2.1. Then by the 'onto' part of Lemma 4.1 in [3], a_2 is equivalent to an E-orientation a_0 such that $c_0 = c_1 = c$, say, where (c_i,f'_i) is the pair corresponding to a_i in the way described above. Since a_2 is symmetric, so is a_0. For $i = 0, 1$, let (θ_i, H'_i) be a pair corresponding to an equivalence from a_i to $a_i \circ t$. Thus θ_0 and θ_1 are homotopies from c to $c \circ t$ through bundle maps. Let $f\colon A \times A \to B$ denote the map covered by c, and let $H_i\colon A \times A \times I \to B$ denote the homotopy covered by θ_i. We shall show that H_0 and H_1 are homotopic (through homotopies from f to $f \circ t$). For let us define a bundle map h from $\zeta \times \zeta \times I^2 | A \times A \times \partial I^2$ to γ as follows. For x in the total space of $\zeta \times \zeta$ and r, s in I, let

$$h(x, 0, s) = c(x),$$

$$h(x, 1, s) = c(t(x)),$$

$$h(x, r, i) = \theta_i(x, r) \quad (i = 0, 1).$$

Since γ is universal, h extends to a bundle map $\bar{h}\colon \zeta \times \zeta \times I^2 \to \gamma$. The map from $A \times A \times I^2$ to B covered by $\bar{h}$ is a homotopy from H_0 to H_1 through homotopies from f to $f \circ t$.

The hypotheses of Proposition 3.1(b) are now satisfied, and we conclude that the homotopy classes of f_0' and f_1' (as lifts of f) differ by the action of a symmetric element of $\sum_{i=1}^{r} \bar{H}^{q_i}(A \times A)$. As in the proof of 3.1(a), any such symmetric element may be written $\delta + t^*\delta$ for some δ. If $\Delta\colon A \to A \times A$ denotes the diagonal map then

$$\Delta^*(\delta + t^*\delta) = 2\Delta^*(\delta) = 0,$$

the latter equality holding because the coefficients are mod 2. Thus $f_0' \circ \Delta$ and $f_1' \circ \Delta$ are homotopic through lifts of $f \circ \Delta$. This implies that the E-orientations $a_0 \circ \bar{\Delta}$ and $a_1 \circ \bar{\Delta}$ are (canonically) equivalent. Thus $a_1 \circ \bar{\Delta}$ and $a_2 \circ \bar{\Delta}$ are equivalent, since a_0 and a_2 are equivalent.

REMARK 4.1. Remarks 3.3 and 3.4 apply, with appropriate modifications, to the above proof.

Proof of Proposition 2.2. The proof is by universal example. Let B be a classifying space for $O(2n)$, and let $\gamma, \bar{\gamma}$ be as before. Let B_1 be the n_1-skeleton of a classifying space B_2 for $O(n_2)$, where $n \gg n_1 \gg n_2 \gg q$. Let γ_1'' be the restriction to B_1 of the canonical n_2-plane bundle over B_2, and let $\gamma_1 = \gamma_1'' \oplus (n - n_2)$. Then the product bundle $\gamma_1 \times \gamma_1$ is classified by a bundle map $c\colon \gamma_1 \times \gamma_1 \to \gamma$. Let $f\colon B_1 \times B_1 \to B$ denote the map covered by c. Using the universality of γ as in the proof of Proposition 2.1, we first get a homotopy θ (through bundle maps) from c to $c \circ t$, and then a bundle map $\Psi\colon \gamma_1 \times \gamma_1 \times I^2 \to \gamma$, satisfying

$$\begin{aligned}
\Psi(x, r, 0) &= \theta(x, r),\\
\Psi(x, r, 1) &= \theta(t(x), 1-r),\\
\Psi(x, 0, s) &= c(x),\\
\Psi(x, 1, s) &= c(t(x)),
\end{aligned}$$

for r, s in I and x in the total space of $\gamma_1 \times \gamma_1$. At the base-space level, θ covers a homotopy H from f to $f \circ t$, and Ψ covers an H_2-structure Φ extending H.

Next, following an idea used by Dupont in [5], we let B_1' be the space formed from B_1 by killing off the Wu classes v_i for $q+1 \leqslant 2i \leqslant 2q+2$. Thus there is a principal fibring $\pi_1\colon B_1' \to B_1$ with fibre $\prod_{2i=q+1}^{2q+2} K_{i-1}$. Let $\bar{\gamma}_1 = \pi_1^*(\gamma_1)$, and let $\rho_1\colon \bar{\gamma}_1 \to \gamma_1$ be the natural bundle map covering π_1. As observed in [5], $f \circ (\pi_1 \times \pi_1)\colon B_1' \times B_1' \to B$ lifts to $B\langle v_{q+1}\rangle$. For any such lift f', the pair $(c \circ (\rho_1 \times \rho_1), f')$ corresponds to a Wu orientation of $\bar{\gamma}_1 \times \bar{\gamma}_1$. Moreover, $\Phi \circ (\pi_1 \times \pi_1 \times 1)$ is an H_2-structure extending the homotopy $H \circ (\pi_1 \times \pi_1 \times 1)$. Hence by Proposition 3.1(a), there is a lift

$f'\colon B_1'\times B_1'\to B\langle v_{q+1}\rangle$ of $f\circ(\pi_1\times\pi_1)$ such that f' is homotopic to $f'\circ t$ by a homotopy H' covering $H\circ(\pi_1\times\pi_1\times 1)$. Let $a\colon \bar\gamma_1\times\bar\gamma_1\to\bar\gamma$ be the Wu orientation corresponding to the pair $(c\circ(\rho_1\times\rho_1),f')$, where f' is such a lift. Then the pair $(\theta\circ(\rho_1\times\rho_1\times 1),H')$ corresponds to an equivalence between a and $a\circ t$. Hence a is a symmetric Wu orientation of $\bar\gamma_1\times\bar\gamma_1$.

Now on dimensional grounds, ν is B_1'-orientable. Let $\alpha\colon \nu\to\bar\gamma_1$ be a B_1'-orientation of ν. Then $a\circ(\alpha\times\alpha)\colon \nu\times\nu\to\bar\gamma$ is a symmetric Wu orientation of $\nu\times\nu$, as required.

5. Proofs of Theorems 2.3 and 2.5

We first observe that Corollaries 2.4 and 2.6 follow immediately from Corollary 4.5 and Lemma 5.2 of [5], together with Proposition 2.1 above. The aim of this section is to prove the sharper versions of these results stated in Theorems 2.3 and 2.5.

Proof of Theorem 2.3. Suppose that (ξ_1,θ_1) and (ξ_2,θ_2) are equivalent pairs as in the hypotheses of Theorem 2.3. Thus there exists a bundle equivalence $\varphi\colon \xi_1\to\xi_2$ such that θ_1 and $\theta_2\circ(\varphi\oplus 1)$ are homotopic through bundle equivalences. Let ψ_i denote the trivialization of $\xi_i\oplus\nu$ associated with θ_i as described in § 2. It follows easily that ψ_1 and $\psi_2\circ(\varphi\oplus 1)$ are homotopic through bundle trivializations, where 1 denotes the identity map of ν, and hence that the following diagram commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "g_2"] \arrow[d, "g_1"'] & \Sigma^n T(\xi_2) \arrow[d, "\Sigma^n U_2"] \\ \Sigma^n T(\xi_1) \arrow[r, "\Sigma^n U_1"'] \arrow[ur, "\Sigma^n T(\varphi)"] & \Sigma^n K_q \end{tikzcd}$$

Here g_i is the allowable X-orientation, constructed with the use of the trivialization ψ_i, and U_i is the Thom class of ξ_i. The conclusion of Theorem 2.3 now follows by homotopy invariance of the functional Steenrod square.

Because of the remark at the beginning of this section, we omit the deduction of Corollary 2.4 from Theorem 2.3.

Proof of Theorem 2.5. As observed at the beginning of this section, the result is known when ξ_1 and ξ_2 are not equivalent as bundles. Suppose that there is a bundle equivalence $\varphi\colon \xi_1\to\xi_2$. Then by definition of the equivalence of pairs, (ξ_2,θ_2) is equivalent to (ξ_1,θ_0) where $\theta_0=\theta_2\circ(\varphi\oplus 1)$. It is therefore sufficient to show that $b_B(\xi,\theta_0)\neq b_B(\xi,\theta_1)$ when the pairs (ξ,θ_0) and (ξ,θ_1) are not equivalent. This is implicit in [5] and [7], and we shall only sketch the method.

We recall from § 2 of [5] that given a self-equivalence α of a vector bundle η over a (paracompact) space X, we may form the mapping-torus bundle η_α over $X \times S^1$, and it makes sense to write $w_{q+1}(\alpha)$ when we mean $w_{q+1}(\eta_\alpha)$.

LEMMA 5.1. *Suppose that ξ is a q-plane bundle over a q-dimensional CW-complex (q odd) and that α is a self-equivalence of $\xi \oplus 2$. Then α is homotopic (through bundle equivalences) to $\varphi \oplus 1$ for some self-equivalence φ of ξ if and only if $w_{q+1}(\alpha) = 0$.*

The proof is straightforward; for example, one can use the methods of § 4 in [7] and of § 4 above. We omit the details.

COROLLARY 5.2. *Pairs (ξ, θ_0), (ξ, θ_1) of the type described in § 2 are equivalent if and only if $w_{q+1}(\theta_0^{-1} \circ \theta_1) = 0$.*

Returning to the sketch proof of Theorem 2.5, we have inequivalent pairs (ξ, θ_0) and (ξ, θ_1), so by Corollary 5.2, $w_{q+1}(\theta_0^{-1} \circ \theta_1) \neq 0$. As in the proof of Theorem 2.3, if $g\colon X \to \Sigma^n T(\xi)$ is the allowable X-orientation corresponding to (ξ, θ_1), then the allowable X-orientation corresponding to (ξ, θ_0) is $T(\theta_0^{-1} \circ \theta_1) \circ g$, and the required result now follows by Theorem 4.4 of [5].

6. Evaluation of $b_B(\tau)$

This section is devoted to the proof of the evaluation Theorem 2.7. An essential ingredient is Brown's technique for generalizing the Kervaire invariant (see [4], particularly Example 1.27). For a $2q$-manifold Q, Brown defined $\varphi_h\colon H^q(Q) \to \mathbf{Z}_4$ satisfying

$$\varphi_h(u+v) = \varphi_h(u) + \varphi_h(v) + j(u \cup v[Q]),$$

where $j\colon \mathbf{Z}_2 \to \mathbf{Z}_4$ is the non-zero homomorphism and $[Q]$ is the fundamental homology class of Q. The construction of φ_h is by means of a composition

$$H^q(Q) \longrightarrow \{Q, K_q\} \xrightarrow{D} \{S^{n+2q}, T(\nu(Q)) \wedge K_q\} \xrightarrow{W}$$
$$\longrightarrow \{S^{n+2q}, T(\bar{\gamma}) \wedge K_q\} \xrightarrow{h} \mathbf{Z}_4,$$

where the braces denote stable homotopy classes of stable maps. The first function assigns to u in $H^q(Q)$ the S-class of a representative map $u\colon Q \to K_q$; in the notation of [4], $u \mapsto F(u, 0)$. The other functions are homomorphisms: D is given by S-duality, and W is induced by a choice of Wu orientation, say a, of $\nu(Q)$. Finally, h satisfies $h(\lambda) = 2$ in the notation of [**4**]; for convenience we shall call such an h *admissible.*

Now let ψ denote Browder's quadratic function (see [**3**]), constructed using the same Wu orientation a of $\nu(Q)$. Thus ψ takes values in $\mathbf{Z}_2$, and does not depend on a choice of admissible h, but the domain of ψ may be a proper subset of $H^q(Q)$. The following proposition is implicit in [**4**], and we omit the proof.

PROPOSITION 6.1. *Let φ_h and ψ as above be constructed using the same Wu orientation of $\nu(Q)$ (and some admissible h in the case of φ_h). If $\psi(u)$ is defined, then $\varphi_h(u) = j(\psi(u))$.*

Proof of Theorem 2.7. The technique of this proof is similar to that used by Dupont in [**5**] and by Thomas in [**11**].

We construct ψ and φ_h for $M \times M$ using a symmetric Wu orientation a of $\nu(M \times M) = \nu \times \nu$ (where as before ν denotes the stable normal bundle of M). The S-dual of the corresponding map $T(a)$ of Thom complexes, with respect to the canonical S-duality between $T(\nu \times \nu)$ and $\Sigma^n((M \times M)^+)$, is an X-orientation $g: X \to \Sigma^n((M \times M)^+)$. The S-dual of

$$T(\Delta): T(2\nu) \to T(\nu \times \nu)$$

with respect to the S-duality between $T(2\nu)$ and $\Sigma^n T(\tau)$ corresponding to the identity bundle equivalence of $\tau \oplus 2$ is well known to be $\Sigma^n f$, where $f: (M \times M)^+ \to T(\tau)$ collapses the complement of a tubular neighbourhood of the diagonal in $M \times M$ to the compactification point in $T(\tau)$. Hence the allowable X-orientation of $\Sigma^n T(\tau)$ corresponding to the pair $(\tau, 1)$ is $\Sigma^n f \circ g$, the S-dual of $T(a \circ \bar{\Delta})$. It follows by naturality of the functional Steenrod square that $b_B(\tau, 1) = \psi(f^* U_\tau) = \psi(U)$, where $U = f^* U_\tau$. Hence by Proposition 6.1, $j(b_B(\tau, 1)) = \varphi_h(U)$. Now there exists A in $H^q(M \times M)$ such that $U = A + t^* A$ and $(A \cup t^* A)[M \times M] = k(M; \mathbf{Z}_2)$ (see [**10**, §4]). The point of introducing φ_h is that $\varphi_h(A)$ is defined, whereas $\psi(A)$ may not be. We now have

$$\begin{aligned} j(b_B(\tau, 1)) &= \varphi_h(U) \\ &= \varphi_h(A) + \varphi_h(t^* A) + j((A \cup t^* A)[M \times M]) \\ &= \varphi_h(A) + \varphi_h(t^* A) + j(k(M; \mathbf{Z}_2)). \end{aligned}$$

To complete the proof of Theorem 2.7 it is sufficient to show that $\varphi_h(A)$ and $\varphi_h(t^* A)$ cancel each other out. We shall prove this using the symmetry of the Wu orientation a. In fact signs are not important, since $x \cup x = 0$ for all x in $H^q(M \times M)$, so (cf. [**4**]) $\varphi_h(H^q(M \times M)) \subset 2\mathbf{Z}_4$.

Let us consider the diagram

$$\begin{tikzcd}
H^q(M\times M) \arrow[r] \arrow[d, "t"'] & \{M\times M, K_q\} \arrow[r, "D"] \arrow[d, "t"'] & \{S^{n+2q}, T(\nu\times\nu)\wedge K_q\} \arrow[r, "W"] \arrow[d, "(T(t)\wedge 1)_*"'] & \{S^{n+2q}, T(\bar{\gamma})\wedge K_q\} \\
H^q(M\times M) \arrow[r] & \{M\times M, K_q\} \arrow[r, "D"] & \{S^{n+2q}, T(\nu\times\nu)\wedge K_q\} \arrow[ur, "W"'] &
\end{tikzcd} \tag{6.2}$$

where the horizontal maps are those used in the construction of φ_h. If (6.2) commutes, then $\varphi_h \circ t^* = \varphi_h$ as required. The first square in (6.2) obviously commutes. The triangle commutes since the symmetry of a implies that $T(a)$ and $T(a)\circ T(t)$ are homotopic. It remains to show that the second square commutes. Since $2\{M\times M, K_q\} = 0$, we shall ignore signs. For u in $\{M\times M, K_q\}$ consider the diagram

$$\begin{tikzcd}
S^{n+2q} \arrow[r, "\beta\wedge\beta"] \arrow[dr, "\beta\wedge\beta"'] & T(\nu)\wedge T(\nu) \arrow[r, "T(d)"] \arrow[d, "t"] & T(\nu)\wedge T(\nu)\wedge(M\times M)^+ \arrow[r, "1\wedge u"] \arrow[d, "t\wedge t"] & T(\nu)\wedge T(\nu)\wedge K_q \arrow[d, "t\wedge 1"] \\
& T(\nu)\wedge T(\nu) \arrow[r, "T(d)"] & T(\nu)\wedge T(\nu)\wedge(M\times M)^+ \arrow[r, "1\wedge(u\circ t)"] & T(\nu)\wedge T(\nu)\wedge K_q
\end{tikzcd}$$

where β is the normal invariant of M (so $\beta\wedge\beta$ is the normal invariant of $M\times M$), $d\colon M\times M \to M^4$ is the diagonal map, and each t represents a factor-switching map. The triangle homotopy-commutes up to sign, and the squares clearly commute. The upper composition represents $D(u)$ and the lower composition represents $D(t^*u)$, hence $(t\wedge 1)_*D(u) = D(t^*u)$ as required. This completes the proof of Theorem 2.7.

7. The fibre homotopy case

It is fairly clear how the analogous theory proceeds to define $b_F(\xi)$ or $b_F(\xi, \theta)$ when

(a) ξ is a $(q-1)$-sphere bundle or fibring over M which is stably fibre homotopy equivalent to the tangent sphere-bundle τ_S, and θ is a stable fibre homotopy equivalence of ξ with τ_S, or

(b) M is a Poincaré complex and ξ is as in (a), where τ_S now denotes the negative of the Spivak normal fibring of M.

The step from (a) to (b) merely enlarges the domain of definition of b_F. To examine the relation between b_B and b_F, let ξ_S denote the sphere-bundle associated with (some choice of riemannian metric on) a vector bundle ξ. If $b_F(\xi_S)$ is well defined then so is $b_B(\xi)$ and the two are equal. However, $b_B(\xi)$ may be well defined when $b_F(\xi_S)$ is not. In other words, it can happen that ξ is stably bundle equivalent to τ and ξ_S is fibre homotopy equivalent to τ_S but ξ is not bundle equivalent to τ. Using [**7**] (see also [**10**]) we may show that there exists such a bundle ξ over $S^{13} \times P$, where P is the real projective plane.

This is perhaps an appropriate place for a comment which has been communicated to me by J. L. Dupont, concerning Theorem 5.4 of [**5**]. The statement of that theorem refers to a homotopy equivalence $f\colon M \to M'$ of smooth closed odd-dimensional manifolds such that τ and $f^*(\tau')$ are stably bundle equivalent. The conclusion asserts that τ and $f^*(\tau')$ are bundle equivalent. Dupont notes that the proof contains a gap: the snag is that $b_B(\tau)$ and $b_B(f^*(\tau'))$ may not be equal, for by [**1**], f determines a unique stable fibre homotopy equivalence φ of $\tau + 1$ with $f^*(\tau') + 1$ (called the homotopy differential of f) and one can prove that

$$b_B(\tau) - b_B(f^*(\tau')) = w_{q+1}(\psi^{-1} \circ \varphi)$$

in the notation of §5 above, where ψ is a stable bundle equivalence between τ and $f^*(\tau')$. Dupont points out that there are three possibilities:

(i) the James–Thomas number of M is 1, in particular $f^*(\tau') \equiv \tau$;

(ii) all stable bundle equivalences ψ satisfy

$$w_{q+1}(\psi^{-1} \circ \varphi) = 0, \quad f^*(\tau') \equiv \tau;$$

(iii) all such ψ satisfy $w_{q+1}(\psi^{-1} \circ \varphi) \neq 0$, $f^*(\tau') \not\equiv \tau$.

We do not know whether case (iii) can actually occur; if it occurs for $f\colon M \to M'$, then M must be a manifold, such as $S^{13} \times P$ above, for which the indeterminacy in $b_B(\tau)$ is zero and the indeterminacy in $b_F(\tau_S)$ is non-zero.

8. An application

In this section, we apply the foregoing theory to strengthen slightly a theorem of Frank and Thomas.

THEOREM 8.1 (cf. [**6**, Theorem 1]). *Suppose*

(a) $q+1 = 2^r(2m+1)$ *for integers* r, m *with* $r > 0$, $m > 0$, *and* M *is a* q*-manifold with* $w_i = 0$ *for* $0 < i \leqslant 2^r$,

(b) M *admits a tangent* 2^r*-field.*

Then $k(M; \mathbf{Z}_2) = 0$.

As Thomas has observed in [**12**], the converse is not true in general. However, the following stable-to-unstable result holds.

PROPOSITION 8.2. *Suppose that* M *satisfies Theorem* 8.1(a), *that* $k(M; \mathbf{Z}_2) = 0$, *and that* M *stably admits a tangent* 2^r*-field* (*that is, the geometric dimension of* τ *is at most* $q-2^r$). *Then* M *admits a tangent* 2^r*-field.*

As a preliminary to clarify the situation, we note that a manifold satisfying the hypotheses of Theorem 8.1 or of Proposition 8.2 has James–Thomas number 2. This follows, for example, from [**7**]; we omit the details, since the result is not needed for the proofs of Theorem 8.1 or Proposition 8.2.

We next modify the construction of b_B to suit manifolds satisfying Theorem 8.1(a). For orientable manifolds, the previous theory may be carried through in an oriented context, with appropriate changes. (The cohomology coefficients are still $\mathbf{Z}_2$, however.) By [**7**, 7.2], if ξ, η are oriented q-plane bundles (q odd) which are bundle equivalent then they are oriented bundle equivalent. Because of this, and since the results in § 2 hold in the oriented case also, we may compute $b_B(\xi)$ in the oriented context.

Let $\pi\colon B\langle v_{q+1}\rangle \to B$, $\rho\colon \bar{\gamma} \to \gamma$ be as in § 2, but taken in the oriented context. Let $w'_i = \pi^*(w_i)$. Let $\pi'\colon E \to B\langle v_{q+1}\rangle$ be the principal fibring obtained by killing off in $B\langle v_{q+1}\rangle$ the w'_i for $i = 2^s$, $1 \leqslant s \leqslant r$. Let $\pi_E\colon E \to B$ be the composite fibring $\pi \circ \pi'$, and let $\gamma_E = \pi_E^*(\gamma) = \pi'^*(\bar{\gamma})$. Let ρ', ρ_E be the natural bundle maps covering π', π_E. Thus $\rho_E = \rho \circ \rho'$.

Suppose that M satisfies Theorem 8.1(a). Just as in the proof of Proposition 2.2 we may show that $\nu \times \nu$ admits a symmetric E-orientation $a\colon \nu \times \nu \to \gamma_E$. Then $\rho' \circ a$ is a symmetric Wu orientation of $\nu \times \nu$. Suppose that ξ is an oriented q-plane bundle over M which is stably bundle equivalent to τ. By passing to the Thom map $T(\rho' \circ a)$ and then to S-duals we get an allowable X-orientation of ξ of the form

$$X \xrightarrow{h} X' \xrightarrow{g'} \Sigma^n T(\xi),$$

where g' is S-dual to $T(a \circ \bar{\Delta})$ and h is S-dual to $T(\rho')$.

Since ξ is oriented, we have an integral Thom class $\tilde{U}_\xi$ for ξ, and then $U_\xi = r \circ \tilde{U}_\xi \colon T(\xi) \to K_q$, where $r\colon K(\mathbf{Z}, q) \to K_q$ denotes mod 2 reduction. Let $\tilde{\iota}$ denote the mod 2 reduction of the fundamental class of $K(\mathbf{Z}, q)$ and let

$$f' = \Sigma^n \tilde{U}_\xi \circ g' \colon X' \to \Sigma^n K(\mathbf{Z}, q).$$

Then $f'^*(\Sigma^n \tilde{\iota}) = g'^*(\Sigma^n U_\xi) = 0$, where the second equality follows by S-duality since a covers a homotopy-symmetric map $h\colon M \times M \to E$, so $h \circ \Delta$ induces the zero homomorphism of $H^q(E)$ as in the proof of Proposition 2.1. As in [3] it follows that $f'^* H^{n+2q}(\Sigma^n K(\mathbf{Z}, q)) = 0$ and $Sq^{q+1} H^{n+q-1}(X') = 0$. Hence $Sq^{q+1}_{f'}(\Sigma^n \tilde{\iota})$ is defined and takes a value in $H^{n+2q}(X') \approx \mathbf{Z}_2$, with zero indeterminacy. Since $T(\rho')$ has degree 1 (mod 2) in dimension n, h^* has degree 1 in dimension $n+2q$. Hence

$$b_B(\xi) = Sq^{q+1}_{f'}(\Sigma^n \tilde{\iota}).$$

Proof of Theorem 8.1. Now assume that 8.1(b) also holds, and write $2^r = s$. Thus $\tau = \zeta \oplus s$ for some oriented $(q-s)$-plane bundle ζ. By [1] there is a natural homeomorphism

$$\alpha\colon T(\tau) \to \Sigma^s T(\zeta).$$

Let $\tilde{U}_\zeta\colon T(\zeta) \to K(\mathbf{Z}, q-s)$ represent the Thom class of ζ. Then $\tilde{U}_\tau$ is represented by the composition

$$T(\tau) \xrightarrow{\alpha} \Sigma^s T(\zeta) \xrightarrow{\Sigma^s \tilde{U}_\zeta} \Sigma^s K(\mathbf{Z}, q-s) \xrightarrow{\Sigma^s \iota} K(\mathbf{Z}, q),$$

where ι is the (integral) fundamental class of $K(\mathbf{Z}, q-s)$. Thus $f' = \Sigma^{n+s}\iota \circ f$, where $f = \Sigma^{n+s}\tilde{U}_\zeta \circ \Sigma^n \alpha \circ g' \colon X' \to \Sigma^{n+s} K(\mathbf{Z}, q-s)$. Hence $Sq^{q+1}_{f'}(\Sigma^n \tilde{\iota}_q) = Sq^{q+1}_f(\Sigma^{n+s}\tilde{\iota}_{q-s})$ modulo the maximum indeterminacy involved, which is $Sq^{q+1} H^{n+q-1}(X') + f^* H^{n+2q}(\Sigma^{n+s} K(\mathbf{Z}, q-s))$, and this is zero.

For $q+1$ as in Theorem 8.1(a) we may write

$$Sq^{q+1} = Sq^s Sq^{q+1-s} + \sum b_j Sq^{q+1-j} Sq^j, \tag{8.3}$$

where the b_j are mod 2 binomial coefficients and the sum is over $1 \leqslant j \leqslant \frac{1}{2}s = 2^{r-1}$. Each term $a_s = Sq^s Sq^{q+1-s}$, $a_j = Sq^{q+1-j} Sq^j$ on the right-hand side of (8.3) annihilates $\Sigma^{n+s}\tilde{\iota}_{q-s}$ for dimensional reasons, so the functional operations $(a_s)_f$, $(a_j)_f$ are defined on $\Sigma^{n+s}\tilde{\iota}_{q-s}$. Moreover, each Sq^{q+1-j} in (8.3) is in the right ideal of the mod 2 Steenrod algebra generated by the Sq^i with $1 \leqslant i \leqslant \frac{1}{2}s$ (see [6] or [9, 5.4]), and for $1 \leqslant i \leqslant s$,

$Sq^i H^{n+2q-i}(X') = 0$ by S-duality, since each such Sq^i (and hence each such χSq^i) is zero on $H^n(T(\gamma_E))$. Hence $(a_s)_f$ and each $(a_j)_f$ is defined on $\Sigma^{n+s}\tilde{\iota}_{q-s}$ with zero indeterminacy. We may therefore calculate $Sq_f^{q+1}(\Sigma^{n+s}\tilde{\iota}_{q-s})$ by calculating separately $(a_s)_f$ and $(a_j)_f$ on $\Sigma^{n+s}\tilde{\iota}_{q-s}$ and summing.

Now $(a_s)_f(\Sigma^{n+s}\tilde{\iota}_{q-s}) = Sq_f^s(Sq^{q+1-s}\Sigma^{n+s}\tilde{\iota}_{q-s}) = 0$ modulo the maximum indeterminacy involved, which is

$$Sq^s H^{n+2q-s}(X') + f^* H^{n+2q}(\Sigma^{n+s}K(\mathbf{Z}, q-s)) = 0.$$

Similarly for each j with $1 \leqslant j \leqslant \frac{1}{2}s$,

$$(a_j)_f(\Sigma^{n+s}\tilde{\iota}_{q-s}) = Sq_f^{q+1-j}(Sq^j\Sigma^{n+s}\tilde{\iota}_{q-s})$$

modulo maximum indeterminacy, which is again zero.

Let $t\colon K(\mathbf{Z}, q-s) \to K_{q-s+j}$ represent $Sq^j\tilde{\iota}_{q-s}$, and let $e = \Sigma^{n+s}t$. Then $e^*(\Sigma^{n+s}\iota_{q-s+j}) = Sq^j(\Sigma^{n+s}\tilde{\iota}_{q-s})$, and $Sq_{e\circ f}^{q+1-j}(\Sigma^{n+s}\iota_{q-s+j})$ is defined, since $Sq^{q+1-j}(\Sigma^{n+s}\iota_{q-s+j})$ is zero on dimensional grounds. Hence

$$Sq_f^{q+1-j}(Sq^j\Sigma^{n+s}\tilde{\iota}) = Sq_{e\circ f}^{q+1-j}(\Sigma^{n+s}\iota_{q-s+j})$$

modulo maximum indeterminacy, which is again zero. But

$$t\circ\tilde{U}_\zeta\colon T(\zeta) \to K_{q-s+j}$$

represents $Sq^j\tilde{U}_\zeta$, which is zero since $w_j(\zeta) = w_j(\tau) = 0$. Thus $t\circ\tilde{U}_\zeta$ is null-homotopic, and hence $e\circ f = e\circ\Sigma^{n+s}\tilde{U}_\zeta\circ\Sigma^{n+s}\alpha\circ g'$ is null-homotopic. Hence $Sq_{e\circ f}^{q+1-j}(\Sigma^{n+s}\iota_{q-s+j}) = 0$. This completes the proof that $b_B(\tau) = 0$. Hence by Theorem 2.7, $k(M; \mathbf{Z}_2) = 0$.

Proof of Proposition 8.2. Suppose that $\tau \oplus 1 = \zeta \oplus (2^r + 1)$, where ζ is some $(q-2^r)$-plane bundle over M. As in the proof of Theorem 8.1, $b_B(\zeta \oplus 2^r) = 0$. By hypothesis and Theorem 2.7, $b_B(\tau) = k(M; \mathbf{Z}_2) = 0$. Hence by Corollary 2.6, τ is bundle equivalent to $\zeta \oplus 2^r$, as required.

Remarks. (a) When $r > 1$, the hypotheses of Theorem 8.1 do not imply that the Kervaire semi-characteristic with rational coefficients $k(M; \mathbf{Q})$ is zero. For example, the Stiefel manifold M of 2-frames in $2^{r-1}(2m+1)+1$-space is parallelizable yet $k(M; \mathbf{Q}) = 1$.

(b) The connection between this section and the unstable secondary operations used in [6] and in § 5 of [9] is as follows. Let Φ denote one of these secondary operations and suppose that ξ is a q-plane bundle over M which is stably equivalent to τ, where q and M are as in the hypotheses of Theorem 8.1 or Proposition 8.2. Then Φ is defined on the Thom class U_ξ, and with the aid of the second Peterson–Stein formula, it can be proved that $\varphi^{-1}(\Phi(U_\xi))[M] = b_B(\xi)$, where φ is the Thom isomorphism for ξ.

(c) When $m = 0$, the analogues of Theorems 8.1 and 8.2 are the results of Kervaire relating parallelizability of odd-dimensional π-manifolds with vanishing of the Kervaire mod 2 semi-characteristic (see [8]).

(d) It is possible that comparison of the methods of this section with forthcoming work of M.-L. Welland would be interesting.

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