ON TANGENT BUNDLES OF FIBRE SPACES AND QUOTIENT SPACES.

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Introduction. One of the most important invariants associated with a differentiable manifold is its tangent bundle. Indeed, the tangent bundle has been involved, directly or indirectly, in virtually all of the recent progress in differential topology. In spite of this, relatively little has been done toward the determination of the tangent bundle of a given manifold or class of manifolds. (Two exceptions to this statement are Borel and Hirzebruch, [2], Proposition 7.5 and Wu, [12], p. 86.) In this paper, we investigate the tangent bundles of manifolds occurring in differentiable fiber bundles.

The first section of the paper is devoted to the statements of the main theorems. Theorem 1.1 gives information about the tangent bundle of a quotient manifold X/G in terms of a G-equivariant embedding of the manifold X in Euclidean space. The second result, Theorem 1.2 describes the tangent bundle of the total space of a bundle with fiber F and group G in terms of a G-equivariant embedding of F in Euclidean space and Theorem 1.3 is a combination of Theorems 1.1 and 1.2 applying to bundles with fibres X/G.

The next two sections give applications. In Section 2, we apply Theorem 1.1 to quaternionic projective spaces and, in Section 3, to manifolds of constant positive curvature. The remaining four sections give proofs of results stated in the first three sections.

Finally, I would like to express my gratitude to W. S. Massey and L. Auslander for many stimulating and informative conversations during the preparation of this paper.

1. The main theorem. Let ξ be a fibre bundle [1] and suppose a group H acts on E_ξ and B_ξ with $\pi_\xi(hx) = h\pi_\xi(x)$ for $x \in E_\xi$, $h \in H$. Then π_ξ induces a map $\pi_\xi'\colon E_\xi/H \to B_\xi/H$ and, under suitable circumstances, the triple $(E_\xi/H, \pi_\xi', B_\xi/H)$ is again a fiber bundle with fiber F_ξ and group G_ξ (see

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[1] We use the term fiber bundle to mean "espace fibré," as defined in Cartan [3], exposé 6. Our notation is that of Borel-Hirzebruch [2].

Lemma 4.1). Under these circumstances, we say that H acts on the bundle ξ and denote the quotient bundle by ξ/H.

Suppose M is a differentiable manifold. We will denote the tangent bundle of M by $\tau(M)$ and the trivial r-plane bundle over M by $\theta^r = \theta^r(M)$. If ξ is a differentiable fiber bundle, we denote by $\tau_F(\xi)$ the bundle of vectors in $\tau(E_\xi)$ tangent to fibers and by $\tau_\perp(\xi)$ the bundle of vectors in $\tau(E_\xi)$ orthogonal to fibers (in some Riemannian metric).

Clearly $\tau(E_\xi)$ is equivalent to the Whitney sum $\tau_F(\xi) \oplus \tau_\perp(\xi)$ and $\tau_\perp(\xi) = \pi_\xi{}^{\#}\tau(B_\xi)$, the bundle over E_ξ induced by π_ξ from $\tau(B_\xi)$.

For the remainder of this section, all bundles will be differentiable.

Suppose ξ is a principal bundle with E_ξ compact and let $\phi\colon E_\xi \to R^n$ be an embedding which is equivariant relative to a representation $\alpha\colon G_\xi \to 0(n)$.[2] Giving E_ξ the Riemannian metric induced by ϕ and letting ν_ϕ be the normal bundle of the embedding ϕ, we see that G_ξ acts on both ν_ϕ and $\tau_F(\xi)$ as described above. Now if $\alpha(\xi)$ denotes the n-plane bundle associated with the α-extension of ξ (see [2], p. 477), we have

THEOREM 1.1.

$$\tau(B_\xi) \oplus \tau_F(\xi)/G_\xi \oplus \nu_\phi/G_\xi = \alpha(\xi).$$

For example, suppose $\xi = (S^n, \pi, P_n(R))$ is the standard Z_2-bundle where S^n is the n-sphere and $P_n(R)$ is n-dimensional real projective space. Then, the usual embedding $\phi\colon S^n \to R^{n+1}$ is equivariant relative to the representation $\alpha\colon Z_2 \to 0(n+1)$ which takes the non-zero element of Z_2 into the negative of the identity in $0(n+1)$. Thus by Theorem 1.1,

$$\tau(P_n(R)) \oplus \nu_\phi/Z_2 = \alpha(\xi).$$

Now, $\alpha(\xi)$ is easily seen to be the $(n+1)$-fold Whitney sum of the line bundle $\hat{\xi}$ associated with ξ and ν_ϕ/Z_2 is trivial since ν_ϕ has a Z_2-equivariant cross section. Therefore, we have the well known result[3]

$$\tau(P_n(R)) \oplus \theta^1 = (n+1)\hat{\xi}.$$

The analogous result for complex projective space follows in exactly the same way.

Next, let ξ be a principal bundle and suppose G_ξ acts (on the left) on a manifold F. Let ζ be the associated bundle with fiber F and suppose

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[2] Mostow [8] has shown that, if E_ξ and G_ξ are compact, such embeddings always exist. See also Palais [9].

[3] This result and the analogue for complex projective space are proved by Milnor in his notes on characteristic classes pp. 10-12 and 74-75. See also Atiyah [1], Lemma 4.5.

$\psi\colon F \to R^n$ is embedding which is equivariant relative to a representation $\beta\colon G_\xi \to 0(n)$. Now, since $E_\zeta = E_\xi \times_{G_\xi} F$ and $E_{\beta(\xi)} = E_\xi \times_{G_\xi} R^n$, ψ induces an embedding $\phi\colon E_\xi \to E_{\beta(\xi)}$.

THEOREM 1.2.

$$\tau_F(\zeta) \oplus \nu_\phi = \pi_\zeta{}^{\#}\beta(\xi).$$

To illustrate, suppose ξ is a principal $0(n)$-bundle, $\hat{\xi}$ the associated vector bundle, and ζ the associated sphere bundle. The usual embedding $\psi\colon S^{n-1} \subset R^n$ is $0(n)$-equivariant and, since ν_ψ has an $0(n)$-equivariant cross section, the normal bundle of the induced embedding $\phi\colon E_\zeta \to E_{\hat{\xi}}$ is easily seen to be trivial. Thus,

$$\tau_F(\zeta) \oplus \theta^1 = \pi_\zeta{}^{\#}\hat{\xi}$$

where θ^1 is the trivial line bundle over E_ζ (see Wu [12], p. 86).

Now suppose ξ and η are principal bundles and suppose G_ξ acts on E_η with $(gx)g' = g(xg')$ for $g \in G_\xi$, $g' \in G_\eta$, and $x \in E_\eta$. Then, G_ξ acts on B_η and we can form the bundles ξ_1 with fiber E_η, ξ_2 with fiber B_η associated with ξ. Furthermore, the map $1 \times \pi\colon E_\xi \times E \to E_\xi \times B$ induces maps

$$\pi_1\colon E_\xi \times E_\eta \longrightarrow E_\xi \times_{G_\xi} B_\eta = E_{\xi_2},$$

$$\pi_2\colon E_\xi \times_{G_\xi} E_\eta = E_{\xi_1} \to E_\xi \times_{G_\xi} B_\eta = B_{\xi_2}$$

and the triples $\zeta_1 = (E_\xi \times E_\eta, \pi_1, E_{\xi_2})$, $\zeta_2 = (E_{\xi_1}, \pi_2, E_{\xi_2})$ are principal bundles. (ζ_1 is a principal $G_\xi \times G_\eta$ bundle and ζ_2 a principal G_η bundle. See Lemma 4.2 below.)

Let $G_\xi \times G_\eta$ act on E_η by $(g_1, g_2)x = g_1 x g_2{}^{-1}$ and let $\psi\colon E_\eta \to R^n$ be an embedding equivariant relative to a representation $\gamma\colon G_\xi \times G_\eta \to 0(n)$. Then $1 \times \psi\colon E_\xi \times E_\eta \to E_\xi \times R^n$ induces an embedding $\phi\colon E_{\xi_1} \to E_{\alpha(\xi)}$ where $\alpha\colon G_\xi \to 0(n)$ is the restriction of γ to $G_\xi \times 1$. (The diagram of Section 5.1 should help clarify the situation here.)

THEOREM 1.3. *With notation as above,*

$$\tau_F(\xi_2) \oplus \tau_F(\zeta_2)/G_\eta \oplus \nu_\phi/G_\eta = \gamma(\zeta_1).$$

For example, let ξ be a principal $0(n+1)$-bundle and $\eta = (S^n, \pi, P_n(R))$ the standard principal Z_2-bundle. Denote by ξ_1 the S^n-bundle associated with ξ, by ξ_2 the $P_n(R)$-bundle associated with ξ, by ζ_1 the principal $Z_2 \times 0(n+1)$-bundle $\pi_1\colon E_\xi \times S^n \to E_{\xi_2}$, and by ζ_2 the principal Z_2-bundle $\pi_2\colon E_{\xi_1} \to E_{\xi_2}$. If $\psi\colon S^n \to R^{n+1}$ is the usual embedding and $\gamma\colon Z_2 \times 0(n+1) \to 0(n+1)$ the multiplication map (identifying Z_2 with the center of $0(n+1)$), it is easily seen that ψ is equivariant relative to the representation γ so, by Theorem 1.3, we have $\tau_F(\xi_2) \oplus \nu_\phi/Z_2 = \gamma(\zeta_1)$.

Now, if $\gamma_1 : Z_2 \times 0(n+1) \to Z_2 = 0(1)$ and $\gamma_2 : Z_2 \times 0(n+1) \to 0(n+1)$ are the projection maps, one sees easily that $\gamma(\zeta_1) = \gamma_1(\zeta_1) \otimes \gamma_2(\zeta_1)$, $\gamma_1(\zeta_1) = \hat{\zeta}_2$, and $\gamma_2(\zeta_1) = \pi_{\xi_2}{}^{\#}\hat{\xi}$. (We use $\hat{\zeta}_2$ to denote the line bundle associated with ζ_2 and $\hat{\xi}$ the $(n+1)$-plane bundle associated with ξ.) Furthermore, since the embedding $\psi: S^n \to R^{n+1}$ has a $Z_2 \times 0(n+1)$ equivariant normal field, it follows that ν_ϕ/Z_2 is trivial and we have

$$\tau_F(\xi_2) \oplus \theta^1 = \zeta_2 \otimes \pi_{\xi_2}{}^*\hat{\xi}.$$

As a consequence, we have the formula of [2], p. 517 (see [13], Lemma 5.1).

$$\omega(\tau_F(\xi_2)) = \sum_{i+j=n+1} (1 + \omega_1(\hat{\zeta}))^i \pi_{\xi_2}{}^* \omega_j(\hat{\xi}).$$

In exactly the same way, we can treat bundles with fiber $P_n(C)$ and group $U(n+1)$ and bundles with lens spaces as fiber and suitably restricted groups.

2. The tangent bundle of quaternionic projective space. Let α: $Sp(1) \to SO(4n+4)$ be the composite.

$$Sp(1) \to Sp(1) \times \cdots \times Sp(1) \subset Sp(n+1) \subset SO(4n+4)$$

where the first map is the diagonal map. Then $Sp(1)$ acts on $S^{4n+3} \subset R^{4n+4}$ via α and defines a principal $Sp(1)$-bundle ξ with $E_\xi = S^{4n+3}$ and $B_\xi = P_n(H)$, the n-dimensional quaternionic projective space. Then, by Theorem 1.1, we have

$$\tau(P_n(H)) \oplus \tau_F(\xi)/Sp(1) \oplus \nu/Sp(1) = \alpha(\xi)$$

where ν is the normal bundle to the embedding $S^{4n+3} \subset R^{4n+4}$. In fact, $\alpha(\xi)$ is the $(n+1)$-fold Whitney sum $(n+1)\hat{\xi}$ where $\hat{\xi}$ is the 4-plane bundle associated with ξ and $\nu/Sp(1)$ is trivial. Further, if β: $Sp(1) \to SO(3)$ is the 2-fold covering (see Steenrod [11], p. 115), we prove in Section 6 that $\tau_F(\xi)/Sp(1) = \beta(\xi)$. Thus we have [4] (compare Lemma 4.5 of [1]).

THEOREM 2.1. $\tau(P_n(H)) \oplus \beta(\xi) \oplus \theta^1 = (n+1)\hat{\xi}$.

For any vector bundle ζ, we denote the Stiefel-Whitney class by $w(\zeta) = \sum w_i(\zeta)$ and the Pontrjagin class by $p(\zeta) = \sum p_i(\zeta)$. If M is a manifold, $w(M) = w(\tau(M))$ and $p(M) = p(\tau(M))$.

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[4] In [7] the author and W. C. Hsiang prove an analogue of Theorem 2.1 for real, complex, and quaternionic Grassmann manifolds.

To compute the characteristic classes of $P_n(H)$, we need the following.

LEMMA 2.2. *Let v be a generator of $H^4(P_n(H);Z)$ and $u \in H^4(P_n(H);Z_2)$ its reduction* mod 2. *Then*

$$w(\beta(\xi)) = 1, \qquad p(\beta(\xi)) = 1 + 4v,$$
$$w(\hat{\xi}) = 1 + u, \qquad p(\hat{\xi}) = (1 + v)^2.$$

We will prove this lemma in Section 7. As an immediate consequence, we have (see Hirzebruch [5] and Borel and Hirzebruch [2], pp. 517-520)

COROLLARY 2.3. *Let $v \in H^4(P_n(H);Z)$ be a generator and $u \in H^4(P_n(H);Z)$ its reduction* mod 2. *Then*

$$w(P_n(H)) = (1 + u)^{n+1},$$
$$p(P_n(H)) = (1 + v)^{2n+2}(1 + 4v)^{-1}.$$

3. Manifolds of constant positive curvature. Let M be a compact orientable Riemannian n-manifold of constant positive curvature. As is well known (see Hopf [6]), M is finitely covered by the n-sphere S^n where the group of covering transformations G acts on S^n via a representation α: $G \to 0(n+1)$. Let $\xi = (S^n, \pi, M)$ denote the covering and ν the normal bundle of the embedding $S^n \subset R^{n+1}$. Since ν clearly has a G-equivariant cross section, ν/G is trivial and we have

THEOREM 3.1. $\tau(M) \oplus \theta^1 = \alpha(\xi)$.

We now consider some special cases.

Let Z_m be the cyclic group of order m with generator g. For any integer q relatively prime to m, we define a representation $\alpha(q): Z_m \to SO(2)$ by

$$\alpha(q)(g) = \begin{vmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{vmatrix}$$

where $\gamma = 2\pi q/m$. If $q_1, \cdots, q_n$ are relatively prime to m, we let Z_m act on $S^{2n+1} \subset R^{2n+2}$ via the direct sum $\alpha(1) \oplus \alpha(q_1) \oplus \cdots \oplus \alpha(q_n)$. This action defines a principal Z_m bundle ξ with $E_\xi = S^{2n+1}$ and B_ξ is the lens space $L(m; q_1, \cdots, q_n)$.

COROLLARY 3.2. *Let $L = L(m; q_1, \cdots, q_n)$ and ξ be the principal Z_m-bundle over L. Then*

$$\tau(L) \oplus \theta^1 = \xi_0 \oplus \xi_1 \oplus \cdots \oplus \xi_n$$

where ξ_0 is the 2-plane bundle associated with ξ and ξ_j is the 2-plane bundle *associated with the $\alpha(q_j)$-extension of ξ. Furthermore, if v is a generator of $H^2(L;Z)$ and u its reduction* mod 2, *then*

$$p(L)=(1+v^2)\prod_{j=1}^{n}(1+q_j^2v^2),$$
$$w(L)=(1+u)\prod_{j=1}^{n}(1+q_ju).$$

The first part of the corollary is an immediate consequence of Theorem 3.1 whereas the expressions for the characteristic classes follow easily from the fact that the $\alpha(q)$-extension of ξ has classifying map

$$L\subset B_{Z_m}\xrightarrow{\lambda}B_{SO(2)}.$$

Here L is considered the $(2n+1)$-skeleton of B_{Z_m} and λ is induced by the homomorphism $\alpha(q):Z_m\to SO(2)$.

Let H_m denote the generalized quaternion group with generators a and b and relations $aba=b$ and $a^r=b^2$ where $r=2^{m-1}$. For any odd integer q, let $\beta(q):H_m\to SO(4)$ be the representation defined by

$$\beta(q)(a)=\left|\begin{array}{cccc}\cos\gamma & \sin\gamma & & 0\\ -\sin\gamma & \cos\gamma & & \\ & & \cos\gamma & -\sin\gamma\\ & 0 & \sin\gamma & \cos\gamma\end{array}\right|,$$

$$\beta(q)(b)=\left|\begin{array}{cc}0 & I\\ -I & 0\end{array}\right|$$

where I is the 2×2 identity matrix and $\gamma=q\pi/2^{m-1}$.

If $q_1,\cdots,q_n$ is a sequence of odd integers, we let H_m act on $S^{4n+3}\subset R^{4n+4}$ via the direct sum $\beta(1)\oplus\beta(q_1)\oplus\cdots\oplus\beta(q_n)$. The action defines a principal H_m-bundle ξ with $E_\xi=S^{4n+3}$ and $B_\xi=N(m;q_1,\cdots,q_n)$.

COROLLARY 3.3. *Let $N=N(m;q_1,\cdots,q_n)$ and ξ be the principal H_m-bundle over N. Then*

$$\tau(N)\oplus\theta^1=\xi_0\oplus\xi_1\oplus\cdots\oplus\xi_n$$

where $\xi_0=\beta(1)(\xi)$ and $\xi_j=\beta(q_j)(\xi)$.

Furthermore, if u is the non-zero element of $H^4(N;Z_2)\approx Z_2$,

$$w(N)=(1+u)^{n+1}. \tag{3.1}$$

The first part of Corollary 3.3 follows immediately from Theorem 3.1. We will prove equation (3.1) in Section 7.

Remark. Note that, since the homomorphism $\beta(q): H(m) \to SO(4)$ factors through $Sp(1) \subset SO(4)$, the group of the bundle $\tau(N) \oplus \theta^1$ admits a reduction to the sympletic group. Thus we can say that N has a generalized almost quaternionic structure.

We return now to the general case. Let M be a compact orientable manifold of constant positive curvature and let S be a p-sylow subgroup of the fundamental group G of M. Then S is cyclic for $p > 2$ and either cyclic or a generalized quaternion group for $p = 2$ (see P. Smith, [10]). Considering M as S^n/G, we see that the inclusion may $S \subset G$ induces a map of S^n/S onto S^n/G and S^n/S is either a real projective space, a a lens space, or an $N(m, q_1, \cdots, q_n)$ defined above. (Here $n = 4r + 3$ and 2^{m+1} is the order of S.) Furthermore, the map of S^n/S onto M induces a monomorphism on $\bmod p$ cohomology (see, for example, Cartan-Eilenberg, [4], p. 259.) Thus, since we know the characteristic classes of S^n/S, we can compute the Stiefel-Whitney classes of M as well as the $\bmod p$ components of the Pontrjagin classes for any prime p.

4. Two preliminary lemmas.

LEMMA 4.1. *Let ξ be a principal bundle with E_ξ compact and suppose a group H acts (on the left) on E_ξ such that*

(4.1) $(hx)g = h(xg)$ *for* $h \in H$, $x \in E_\xi$, *and* $g \in G_\xi$,

(4.2) *the induced action of H on B_ξ is without fixed points,*

(4.3) *the spaces E_ξ/H and B_ξ/H are Hausdorff.*

Then the triple $(E_\xi/H, \pi_\xi', B_\xi/H)$ is a principal G_ξ-bundle where π_ξ' is induced by π_ξ.

Proof. All we need to show is that, for any $x, y \in E_\xi/H$ with $\pi_\xi' x = \pi_\xi' y$, there is a unique $g \in G_\xi$ with $xg = y$ and that the resulting map of

$$A' = \{(x, y) \in E_\xi/H \times E_\xi/H \ni \pi_\xi' x = \pi_\xi' y\}$$

into G_ξ is continuous.

The fact that there is a unique $g \in G_\xi$ with $xg = y$ for $(x, y) \in A'$ follows from the corresponding property for ξ. To see that the resulting map

$f'\colon A' \to G_\xi$ is continuous, we let $A = \{(x,y) \in E_\xi \times E_\xi \ni \pi_\xi x = \pi_\xi y\}$ and let $f\colon A \to G_\xi$ be the map with $xf(x,y) = y$. Now H acts on A by $h(x,y) = (hx, hy)$ and $f(hx,hy) = f(x,y)$ (by (4.1)) so f induces a map $f_1\colon A/H \to G_\xi$. Furthermore, the identity map of $E_\xi \times E_\xi$ onto itself induces a map $\gamma\colon A/H \to A'$ which is clearly continuous, onto, and, by (4.2), one-one. In fact, since E_ξ is compact and A closed in $E_\xi \times E_\xi$, A/H is compact so γ is a homomorphism. Letting $f'\colon A' \to G_\xi$ be the composite $f_1\gamma^{-1}$, we see that $xf'(x,y) = y$ for $(x,y) \in A'$. This completes the proof of Lemma 4.1.

Now, let ξ and H be as in the lemma and suppose G_ξ acts on F. Then, we can form the F-bundle associated with both ξ and ξ/H. In fact, H acts on $E_\xi \times_{G_\xi} F$ (on the first factor) and one sees easily that $(E_\xi \times_{G_\xi} F)/H = (E_\xi/H) \times_{G_\xi} F$. Thus the F-bundle associated with ξ/H is the quotient by H of the F-bundle associated with ξ.

Lemma 4.2. *Let ξ and ζ be principal bundles and suppose G_ξ acts on E_ζ with $g(xh) = g(xh)$, $g \in G_\xi$, $x \in E_\zeta$, and $h \in G_\zeta$. Then G_ξ acts on B_ζ and the triple $(E_\xi \times_{G_\xi} E_\xi, \pi, E_\xi \times_{G_\xi} B_\xi)$ is a principal G_ξ-bundle where π is induced by the map $1 \times \pi_\zeta\colon E_\xi \times E \to E_\xi \times B_\zeta$.*

Proof. Let

$$A = \{(z,z') \in E_\xi \times_{G_\xi} E_\zeta \times E_\xi \times_{G_\xi} E_\zeta \ni \pi z = \pi z'\},$$
$$A_1 = \{(x,y) \in E_\xi \times E_\xi \ni \pi_\xi x = \pi_\xi y\},$$
$$A_2 = \{(x,y) \in E_\zeta \times E_\zeta \ni \pi_\zeta x = \pi_\zeta y\},$$

and let $f_1\colon A_1 \to G_\xi$, $f_2\colon A_2 \to G_\zeta$ be the maps with $xf_i(x,y) = y$ for $(x,y) \in A_i$, $i = 1, 2$. Then we define $f\colon A \to G_\zeta$ by the formula $f(z,z') = f_2(f_1(x,x')y, y')$ where (z,z') in A is represented by (x,y,x',y') in $E_\xi \times E_\zeta \times E_\xi \times E_\zeta$. The verification that f is well defined and has the required properties is left to the reader.

5. The proofs of the main theorems.

Proof of Theorem 1.1. Since $\tau(B_\xi) = \tau_\perp(\xi)/G_\xi$ and $\tau_\perp(\xi) \oplus \tau_F(\xi) = \tau(E_\xi)$, it suffices to show that $(\tau(E_\xi) \oplus \nu_\phi)/G_\xi = \alpha(\xi)$. Now, $\tau(E_\xi) \oplus \nu_\phi$ is trivial and can be considered as the bundle over E_ξ induced by ϕ from the trivial n-plane bundle over R^n. Thus, we have an equivalence $F\colon E \to E_\xi \times R^n$ where E is the total space of $\tau(E_\xi) \oplus \nu_\phi$. In fact, this equivalence is G_ξ-equivariant so induces an equivalence between E/G_ξ and $E_\xi \times_{G_\xi} R^n$. However, E/G_ξ is the total space of $(\tau(E_\xi) \oplus \nu_\phi)/G_\xi$ and $E_\xi \times_{G_\xi} R^n = E_{\alpha(\xi)}$ so Theorem 1.1 is proved.

Proof of Theorem 1.2. We first prove the following lemma.

LEMMA 5.1. *Let η be a differentiable n-plane bundle. Then* $\tau_F(\eta) = \pi_\eta{}^{\#}\eta$.

Proof. Suppose η is associated with the $0(n)$-bundle ζ and $\tau = \tau(R^n)$. Then the total space of $\tau_F(\eta)$ is simply $E_\zeta \times_{0(n)} E_\tau$. (See [2], p. 478). The lemma now follows easily from the fact that $E_\tau = R^n \times R^n$.

To prove Theorem 1.2, we first note that the bundle $\tau_F(\zeta) \oplus \nu_\phi$ over E_ζ is induced by $\phi: E_\zeta \to E_{\beta(\xi)}$ from the bundle $\tau_F(\beta(\xi))$. Now, by Lemma 5.1, $\tau_F(\beta(\xi))$ is induced by $\pi_{\beta(\xi)}: E_{\beta(\xi)} \to B_\xi$ from $\beta(\xi)$ and, since $\pi_{\beta(\xi)}\phi = \pi_\xi$, the theorem is proved.

Proof of Theorem 1.3. The following diagram should help the reader keep track of the bundles involved in this proof.

$$\begin{tikzcd}
E_\xi \times E_\eta \arrow[r, "1 \times \psi"] \arrow[d, "p_1"'] & E_\xi \times R^n \arrow[d, "p_2"] \\
E_\xi \times_{G_\xi} E_\eta \arrow[r, "\phi"] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_{\xi_1}"] & E_\xi \times_{G_\xi} R^n \arrow[d, "\pi_{\alpha(\xi)}"] \\
E_\xi \times_{G_\xi} B_\eta \arrow[r, "\pi_{\xi_2}"'] & B_\xi
\end{tikzcd}$$

Note, first of all that the map $\pi_2: E_\xi \times_{G_\xi} E_\eta \to E_\xi \times_{G_\xi} B_\eta$ induces a bundle epimorphism $\pi_2: \tau_F(\xi_1) \to \tau_F(\xi_2)$ with kernel $\tau_F(\zeta_2)$. Choosing a Riemannian metric on $E_\xi \times_{G_\xi} E_\eta$, we see that $\tau_F(\xi_1)$ is equivalent to $\tau_F(\zeta_2) \oplus \pi_2{}^{\#}\tau_F(\zeta_2)$ so that $\tau_F(\zeta_1)/G = [\tau_F(\zeta_2) \oplus \pi_2{}^{\#}\tau_F(\xi_2)]/G_\eta$. In fact, if we choose the Riemannian metric on $E_\xi \times_{G_\xi} E_\eta$ to be invariant under the action of G_η, we have

$$[\tau_F(\zeta_2) \oplus \pi_2{}^{\#}\tau_F(\xi_2)]/G_\eta = \tau_F(\zeta_2)/G_\eta \oplus \pi_2{}^{\#}\tau_F(\xi_2)]/G_\eta.$$

Now $\pi_2{}^{\#}\tau_F(\xi_2)/G_\eta$ is easily seen to be equivalent to $\tau_F(\xi_2)$ so, to prove the theorem, we need only show that $[\tau_F(\xi_1) \oplus \nu_\phi]/G_\eta = \gamma(\xi)$.

Let $p: E_\xi \to E_\eta$ be the projection map. It is easily seen that $\tau_F(\xi_1) = p^{\#}\tau(E_\eta)/G_\xi$ and that $\nu_\phi = p_{\#}\nu_\psi/G_\xi$ (see [2], p. 478) so that

$$[\tau_F(\xi_1) \oplus \nu_\phi]/G_\eta = [p^\# \tau(E_\eta)/G_\xi \oplus p_\# \nu_\psi/G_\xi]/G_\eta$$
$$= p^\# [\tau(E_\eta) \oplus \nu_\psi]/G_\xi \times G_\eta.$$

Now $\tau(E_\eta) \oplus \nu_\psi$ is trivial so the total space of $[\tau_F(\xi_1) \oplus \nu_\phi]/G_\eta$ is $E_\xi \times E_\eta \times R^n/G_\xi \times G_\eta$ and, since $\psi\colon E_\eta \to R^n$ is equivariant, the action of $G_\xi \times G_\eta$ on $E_\xi \times E_\eta \times R^n$ is exactly the action which defines $\gamma(\zeta_1)$. This completes the proof of the theorem.

6. The proof of Theorem 2.1. We will need the following lemma.

LEMMA 6.1. *Suppose ξ and η are principal bundles with $E_\eta = B_\xi$ and suppose G_η acts (on the right) on ξ with $(g'x)g = g'(xg)$ for $g \in G_\eta$, $x \in E_\xi$, and $g' \in G_\xi$. Let $A = \{(x, y) \in E_\xi \times E_\xi$ with $\pi_\xi x = \pi_\xi y\}$, $f\colon A \to G_\xi$ the map with $x = f(x, y)y$, and suppose $s\colon B_\xi \to E_\xi$ is a cross section with the property that $f(s(xg^{-1})g, s(x))$ is independent of x for all $g \in G_\eta$. Then the map $\beta\colon G_\eta \to G_\xi$ defined by $\beta(g) = f(s(xg^{-1})g, s(x))$ is a homomorphism and ξ/G_η is equivalent to the β-extension of η.*

Proof. First of all,

$$\beta(g_1)\beta(g_2) = f(s(xg_1^{-1})g_1, s(x))f(s(xg_2^{-1})g_2, s(x))$$
$$= f(s(xg_2^{-1}g_1^{-1})g_1, s(xg_2^{-1}))f(s(xg_2^{-1})g_2, s(x))$$

since $f(s(xg^{-1})g, s(x))$ is independent of x. But then $(\beta g_1)\beta(g_2))s(x) = s(xg_2^{-1}g_1^{-1})g_1g_2$ so

$$\beta(g_1)\beta(g_2) = f(s(xg_2^{-1}g_1^{-1})g_1g_2, s(x))$$
$$= \beta(g_1g_2)$$

and β is a homomorphism.

Now define $\psi\colon E_\xi \to E_\eta \times G_\xi$ by $\psi(x) = (\pi_\xi(x), f(s\pi_\xi(x), x))$. Then, for $g \in G_\eta$,

$$\begin{aligned}
\psi(xg) &= (\pi_\xi(xg), f(s\pi_\xi(xg), xg)\\
&= (\pi_\xi(x)g, f(s(\pi_\xi(x)g), xg))\\
&= (zg, f(s(zg)g^{-1}, x))\\
&= (zg, f(s(zg)g^{-1}, s(z))f(s(z), x))\\
&= (zg, \beta(g)^{-1}f(s(z), x))
\end{aligned}$$

where $z = \pi_\xi(x)$. Thus ψ defines a bundle map $\psi_0\colon E_\xi/G_\eta \to E_\eta \times_{G_\eta} G_\xi$ (where G_η acts on G_ξ via β) which is easily seen to be an equivalence.

Theorem 2.1 is an immediate consequence of the following.

COROLLARY 6.2. *Let η be a principal bundle, ξ the principal $0(m)$-bundle associated with $\tau_F(\eta)$, and* Ad: $G_\eta \rightarrow 0(m)$ *the adjoint representation. Then ξ/G_η is equivalent to the* Ad-*extension of η.*

Proof. Let $T(G_\eta)_x$ denote the tangent space to G_η at $x \in G_\eta$, R_g: $T(G_\eta)_x \rightarrow T(G_\eta)_{xg}$ the map induced by right translation, and L_g: $T(G_\eta)_x \rightarrow T(G_\eta)_{gx}$ the map induced by left translation. Then

$$\mathrm{Ad}(g) = R_g L_{g^{-1}}\colon T(G_\eta)_e \rightarrow T(G_\eta)_e.$$

Let $\mu\colon E_\eta \times G_\eta \rightarrow E_\eta$ be the principal map and $d\mu_x\colon T(G_\eta)_e \rightarrow T(E_\eta)_x$ the map induced by $\mu_x\colon G_\eta \rightarrow E_\eta$, $\mu_x(g) = \mu(x, g)$. Clearly $d\mu_x$ takes $T(G_\eta)_e$ isomorphically onto the fiber of $\tau_F(\xi)$ containing x. We define a cross section s in ξ by

$$s(x) = [d\mu_x(v_1), \cdots, d\mu_x(v_m)]$$

where $v_1, \cdots, v_m$ is a base for $T(G_\eta)_e$ and $[d\mu_x(v_1), \cdots, d\mu_x(v_m)]$ denotes the frame determined by $d\mu_x(v_1), \cdots, d\mu_x(v_m)$.

Now,

$$\begin{aligned} s(xg^{-1})g &= [R_g d\mu_{xg^{-1}}(v_1), \cdots, R_g d\mu_{xg^{-1}}(v_m)] \\ &= [R_g L_{g^{-1}} d\mu_x(v_1), \cdots, R_g L_{g^{-1}} d\mu_x(v_m)] \end{aligned}$$

since $\mu(\mu(x, g^{-1}), g') = \mu(x, g^{-1}g')$. Therefore, the map $\beta\colon G_\eta \rightarrow 0(m)$ defined in Lemma 6.1 can be identified with the adjoint representation and the corollary is proved.

7. The proofs of Lemma 2.2 and Corollary 3.3.

Let

$$w = \sum w_j \in H^*(B_{O(n)}; Z_2)$$

be the universal Stiefel-Whitney class and $p = \sum p_j \in H^*(B_{SO(n)}; \mathrm{Z})$ the universal Pontrjagin class. We will need the following three lemmas.

LEMMA 7.1. *Let $j\colon Sp(1) \subset SO(4)$ be the standard inclusion and $\lambda(j)\colon B_{Sp(1)} \rightarrow B_{SO(4)}$ the induced map. Then*

$$\lambda(j)^* p = (1 + v)^2, \tag{7.1}$$

$$\lambda(j)^* w = 1 + u, \tag{7.2}$$

where v is a generator of $H^4(B_{Sp(1)}; Z)$ and u its reduction mod 2.

Proof. Let $S^1 \subset Sp(1)$ be unit quaternions of the form $a + bi$. Then

S^1 is a maximal torus. If $S^1 \times S^1$ is the usual maximal torus in $SO(4)$ and $d: S^1 \to S^1 \times S^1$ the diagonal map, then the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{d} & S^1 \times S^1 \\ \downarrow & & \downarrow \\ Sp(1) & \xrightarrow[j]{} & SO(4) \end{array}$$

is commutative. Passing to classifying spaces, we obtain a corresponding diagram in cohomology. Equation (7.1) is an immediate consequence of this diagram (see [2], p. 487) and equation (7.2) follows from the fact that p_i reduced mod 2 is w_{2i}^2.

LEMMA 7.2. *Let* $\lambda(\beta): B_{Sp(1)} \to B_{SO(3)}$ *be induced by the two fold covering* β: $Sp(1) \to SO(3)$. *Then*

$$\lambda(\beta)^* p = 1 + 4u$$

where u *is a generator of* $H^4(B_{Sp(1)}; Z)$.

Proof. The short exact sequence

$$1 \to Z_2 \to Sp(1) \longrightarrow SO(3) \to 1$$

defines a fiber map $\lambda(\beta): B_{Sp(1)} \to B_{SO(3)}$ with fiber B_{Z_2}. If (E_r, d_r) denotes the integral cohohomolgy spectral sequence of this fiber space, it is not difficult to show that $E_\infty^{0,4} \approx E_\infty^{2,2} \approx Z_2$, $E_\infty^{4,0} \approx Z$, and $E_\infty^{1,3} \approx E_\infty^{1,3} = 0$. From this, it is immediate that $H^4(B_{Sp(1)}; Z)/\lambda(\beta)^* H^4(B_{SO(3)}; Z) \approx Z_4$ and the lemma follows.

LEMMA 7.3. *Let* $\lambda(i): B_{H_m} \to B_{Sp(1)}$ *be induced by the inclusion* i: $H_m \subset Sp(1)$. *Then* $\lambda(i)^*: H^*(B_{Sp(1)}; Z_2) \to H^*(B_{H_m}; Z_2)$ *is an isomorphism into.*

Proof. In fact, $\lambda(i): B_{H_m} \to B_{Sp(1)}$ is a fiber map with fiber $Su(1)/H_m$. Using the fact that $H^3(B_{H_m}; Z_2)$ is non zero (see [4], p. 254), we see that the mod 2 cohomology spectral sequence is trivial. This proves the lemma.

Now, Lemma 2.2 follows immediately from Lemmas 7.1 and 7.2. To prove equation (3.1) we notice that the map of H_m into $SO(4)$ factors into the composite $H_m \xrightarrow{i} Sp(1) \xrightarrow{j} SO(4)$. The result now follows from Lemmas 7.1 and 7.3.

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REFERENCES.

[1] M. F. Atiyah, "Thom complexes," *Proceedings of the London Mathematical Society* (3), vol. 11 (1961), pp. 291-310.

[2] A. Borel and F. Hirzebruch, "Characteristic classes of homogeneous spaces I," *American Journal of Mathematics*, vol. 80 (1958), pp. 458-538.

[3] H. Cartan, *Séminaire Henri Cartan 1949-50*, Paris, 1956.

[4] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press.

[5] F. Hirzebruch, "Über die Quaternionalen Projektiven Räume," *Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematik-Naturwissenschaftliche Klasse*, 1954, 301-312 (1954).

[6] H. Hopf, "Zum Clifford-Kleinschen Räum problem," *Mathematische Annalen*, vol. 95 (1926), pp. 313-339.

[7] W. C. Hsiang and R. H. Szczarba, "On the tangent bundle of a Grassman manifold," *American Journal of Mathematics*, vol. 86 (1964), pp. 698-704.

[8] G. D. Mostow, "Equivariant embeddings in Euclidean space," *Annals of Mathematics*, vol. 65 (1957), pp. 432-446.

[9] R. S. Palais, "Imbedding of compact, differentiable transformation groups in orthogonal representations," *Journal of Mathematics and Mechanics*, vol. 6 (1957), pp. 673-678.

[10] P. Smith, "Permutable periodic transformations," *Proceedings of the National Academy of Sciences, U. S. A.*, vol 30 (1944), pp. 105-108.

[11] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951.

[12] W. Wu, "Sur les Classes Caracteristiques des Structures Fibres Spheriques," *Actualites Scientifiques et Industrielles*, no. 1183, Hermann, Paris, 1952.

[13] Emery Thomas, "On the tensor product of n-plane bundles," *Archiv der Mathematik*, vol. 10 (1959), pp. 174-179.