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NOTE ON DOUBLE POINTS OF IMMERSIONS

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ABSTRACT. We show that not any smooth manifold can be double point manifold of a smooth self-transverse immersion of a closed n-manifold into the euclidean space R^{n+k}. We investigate also the double covers which can (or can not) occur in this situation.

0. Notation

In this note M^n will denote a closed smooth n-manifold and $f : M^n \to R^{n+k}$ shall be its selftransverse immersion. (n and k will be sometimes specified). $\theta_2(f)$ will denote the $(n-k)$ dimensional closed manifold formed by the double points in R^{n+k}. If f has no triple points then $\theta_2(f)$ is an embedded submanifold of R^{n+k}. Its preimage $f^{-1}(\theta_2(f)) \subset M^n$ will be denoted by $D_2(f)$. (The manifolds $D_2(f)$ and $\theta_2(f)$ can be defined in a standard way even if f has multiple points, although in this case they are not embedded in M^n and R^{n+k} but immersed there.)

1. Immersions $M^n \to R^{2n-1}$

Let k be $n-1$, i.e. we consider immersions $M^n \to R^{2n-1}$. Then the double point set $\theta_2(f)$ is a 1-manifold. Call a component C of $\theta_2(f)$ nontrivial if the double cover $f^{-1}(C) \to C$ is nontrivial, i.e. $f^{-1}(C)$ is connected. Let $\delta(f) \in Z_2$ be the parity of nontrivial components of the curve $\theta_2(f)$.

Question (M. Rost): Given M^n what are the possible values of $\delta(f)$?

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Theorem 1:

a) If n is odd then for any n-manifold M^n there are immersions $f_0, f_1 : M^n \to R^{2n-1}$ with different values of δ: $\delta(f_0) = 0$ and $\delta(f_1) = 1$.

b) If n is even then for any immersion $f : M^n \to R^{2n-1}$ the following holds:

$$\delta(f) = \langle W_1 \overline{W}_{n-1}(M), [M] \rangle.$$

Proof: (This proof uses ideas of Uwe Kaiser and Li Bang He [6]).

a) For n odd any immersion $g : M^n \to R^{2n}$ has a nonzero vectorfield and so by a theorem of Hirsch it is regularly homotopic to an immersion $\bar{g} : M^n \to R^{2n-1}$. By Whitney's theorem there are immersions $M^n \to R^{2n}$ with arbitrary number of double points. Let $g_0 : M^n \to R^{2n}$ and $g_1 : M^n \to R^{2n}$ be immersions with 0 and 1 double point respectively. Now it follows from the lemma below that the corresponding maps $\bar{g}_0 : M^n \to R^{2n-1}$ and $\bar{g}_1 : M^n \to R^{2n-1}$ have δ invariants 0 and 1 respectively. (Compare with [7]).

Lemma: Let $\bar{g} : M^n \to R^{2n-1}$ be a generic immersion and let $h : M \to R^1$ be a function such that the map $g = (\bar{g}, h) : M \to R^{2n} = R^{2n-1} \times R^1$, $g(x) = (\bar{g}(x), h(x))$ is a generic immersion. Then the number of double points of $g \equiv \delta(\bar{g}) \mod 2$. (*)

Proof: Let τ be the free involution on the double point set $D(\bar{g})$ such that $\bar{g}(x) = \bar{g}(\tau x)$. Denote by ℓ_τ the line bundle over $\theta(\bar{g}) = D(\bar{g})/\tau$ for which the sphere bundle is $D(\bar{g})$.

Then both sides of (*) coincide $\mod 2$ with $\langle W_1(\ell_\tau), [\theta(\bar{g})] \rangle$. For the right side (i.e. for $\delta(\bar{g})$) this is so by definition. For the left side (for the double points of g) it is also easy. Namely let $\varphi(x)$ be $h(x) - h(\tau x)$. Then $\varphi : D(\bar{g}) \to R'$ is an antisymmetric function (i.e. $\varphi(\tau x) = -\varphi(x)$) and so ist defines a section s of the line bundle ℓ_τ.

The double points of g correspond to the pairs $(x, \tau x)$ such that $x \in D(\bar{g})$ and $h(x) = h(\tau x)$. These pairs are precisely the zeros of φ and the latters are in $1-1$-correspondence with the zeros of s, and the pairity of the zeros of s is $\langle W_1(\ell_\tau), [\theta(\bar{g})] \rangle$. The lemma and thus part a) of Theorem 1 are proven.

b) Let ν denote the normal bundle of $D_2(f)$ in M and let E^1 be the trivial line bundle. Then $\nu \oplus \varepsilon^1 = TM|_{D_2(f)}$. Hence $W_1(\nu) = W_1(M)|_{D_2(f)}$. Then

$$\langle W_1(\nu), [D_2(f)] \rangle = \langle W_1 \overline{W}_{n-1}(M), [M] \rangle$$

because the homology class in $H_1(M; Z_2)$ represented by $D_2(f)$ is dual to $\overline{W}_{n-1}(M)$ (see [3]). On the other hand if n is even then $\delta(f) = \langle W_1(\nu), [D_2(f)] \rangle$. Indeed, both sides coincide $\mod 2$ with the number of those components of the double point set $D_2(f)$ which have nonorientable normal bundles in M. ∎

2. Immersions $M^3 \to R^4$

In the next theorem we deal with immersions of closed 3-manifolds in R^4 having no triple points. The double point surface $\theta_2(f)$ can be any surface of even Euler characteristic (see [8]). What sort of double covers can arise over $\theta_2(f)$ as the double cover $D_2(f) \to \theta_2(f)$?

Theorem 2: Let $f : M^3 \to R^4$ be a selftransverse immersion having no triple points and let F be a nonorientable component of the double point surface $\theta_2(f)$ ($\subset R^4$). Then the double cover $f^{-1}(F) \to F$ can not be trivial.

Proof: Suppose that the cover $f^{-1}(F) \to F$ is trivial. Then $f^{-1}(F) = F_1 \cup F_2$, where F_1 and F_2 are disjoint and both homeomorphic to F.

Let us denote by ℓ_1 and ℓ_2 the normal line bundles of the embeddings $F_1 \subset M^3$ and $F_2 \subset M^3$. Identifying F with F_1 and F_2 we can say that the normal bundle of F in R^4 is the sum $\ell_1 \oplus \ell_2$. Let $S(\ell_1)$ and $S(\ell_2)$ be the S^0-sphere bundles over F associated to ℓ_1 and ℓ_2. If T is a small tubular neighbourhood of F in R^4, then $\partial T \cap f(M^3) = S(\ell_1) \cup S(\ell_2)$.

We are going to show that

Claim 1: $S = S(\ell_1) \cup S(\ell_2)$ is "linked" with F in R^4.

Claim 2: S and F are not "linked" with F in R^4.

Then the contradiction between these two claims will prove the theorem. Of course the expression "linked" in R^4 should be defined since S and F are not linked in R^4 in the usual sense, they do not have the right dimension for that. ($\dim S = \dim F = 2$).

Definition: Let σ be any 3 dimensional singular chain in R^4 such that $\partial\sigma = S$. One can suppose that each simplex of σ intersects F transversally. (Especially the simplices of σ of dimension 0 and 1 do not intersect F at all.) Then $\sigma \cap F$ is a 1-cycle in F and its homology class in $H_1(F; Z_2)$ will be called *the linking class* of S and F and will be denoted by $\ell(S, F)$. We say that S and F are *linked* if $\ell(S, F) \neq 0$.

Lemma: $\ell(S, F)$ is well defined.

Proof: Let $\bar{\sigma}$ be another singular chain in R^4 such that $\partial\bar{\sigma} = S$. Then $\sigma \underset{S}{\cup} \bar{\sigma}$ is a 3-cycle in R^4 and therefore there is a 4-chain τ in R^4 such that $\partial\tau = \sigma \underset{S}{\cup} \bar{\sigma}$. We can suppose that each simplex of τ intersects F transversally. Then $\tau \cap F$ is a 2-chain with bundary $\partial(\tau \cap F) = (\sigma \cap F) \cup (\bar{\sigma} \cap F)$. ∎

Proof of Claim 2: $f(M^3) \setminus T$ forms a 3 chain σ such that $\partial\sigma = S$ and $\sigma \cap F = \varnothing$. Hence $\ell(S, F) = 0$. ∎

Proof of Claim 1: Let $D(\ell_1)$ and $D(\ell_2)$ be the disc bundles of ℓ_1 and ℓ_2 with boundaries $S(\ell_1)$ and $S(\ell_2)$. Then $\sigma = D(\ell_1) \cup D(\ell_2)$ is also a 3-chain such that $\partial\sigma = S$. By small perturbation can be achieved that $D(\ell_1)$ intersect transversally the surface F in a cycle having homology class dual to $W_1(\ell_2)$. Similarly, perturbing $D(\ell_2)$ we can have that the homology class of $D(\ell_2) \cap F$ is dual to $W_1(\ell_1)$.
Hence the homomology class of $\sigma \cap F$ is dual to $W_1(\ell_1) + W_1(\ell_2) = \bar{W}_1(F) \neq 0$.
■

3. Cases when $n - k \geq 4$

In this last section we consider two cases when $\theta_2(f)$ has higher dimension.

Theorem 3: Let $f : M^8 \to R^{12}$ be a selftransverse immersion without triple points. Then M^8 is cobordant either to zero or to $P_2 \times P_6$. The double point set $\theta_2(f)$ ($\subset R^{12}$) is a 4-manifold, which is cobordant either to zero or to $P_2 \times P_2$.

Proof: The eight dimensional cobordism group is $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$. The generators are the classes of the following manifolds: P_8, $P_6 \times P_2$, $P_4 \times P_4$, $P_2 \times P_2 \times P_4$, $(P_2)^4$. Computing the normal Stiefel-Whitney numbers of these manifolds we get the following:

	P_8	$P_6 \times P_2$	$P_4 \times P_4$	$P_2 \times P_2 \times P_4$	$P_2 \times P_2 \times P_2 \times P_2$
$\overline{W}_1\overline{W}_7$	1	0	0	0	0
$\overline{W}_2\overline{W}_6$	1	0	1	0	0
$\overline{W}_3\overline{W}_5$	1	0	0	1	0
$\overline{W}_4^2$	1	0	0	1	1

Since for our manifold M^8 all these characteristic numbers must vanish M^8 must be either null-cobordant or cobordant to $P_6 \times P_2$. Note that P_6 is cobordant to such a manifold V^6, which embeds in R^{10} and the normal bundle has three independent sections. (See [8].) Therefore $V^6 \times P_2$ embeds in R^{12}.
Now we finish the proof using the arguments of [8] in Claim 1. Let W^9 be a manifold such that either

i) $\partial W^9 = M^8$ or ii) $\partial W^9 = M^8 \cup V_6 \times P_2$.

There exists a generic map $h : W^9 \to R^{12} \times [0,1]$ such that $h(M^8)$ goes into $R^{12} \times 0$ and h restricted to M is f. In the case (ii) we require also that

$h(V^6 \times P_2) \subset R^{12} \times 1$ and h restricted to V^6 is an embedding. The map h has only $\Sigma^{1,0}$ singularities (i.e. Whitney umbrellas). The normal bundle of the singular manifold $\Sigma(h)$ has the form: $\varepsilon \oplus \nu^8$, where ε is the trivial line bundle and ν^8 admits as structure group the following group of matrices:

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{ and } \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \middle| A \in O(4) \right\}.$$

Hence $\Sigma(h)$ is an orientable 4-manifold. The double points of h form a 5-manifold $\theta_2(h)$ with boundary consisting of the double points of f and $\Sigma(h)$. Therefore the double point set $\theta_2(f)$ of f is cobordant to an orientable 4-manifold (to $\Sigma(h)$). Therefore its cobordism class is zero or $[P_2 \times P_2]$. ∎

I do not know the answer to the following questions:

Question 1: Can the double point set be cobordant to $P_2 \times P_2$?

Question 2: What can be the cobordism class of the double point set if f has triple points?

The following theorem can be proven in the same way as the previous one.

Theorem 4: Let $f : M^n \to R^{n+k}$ be a selftransverse immersion of the closed n-manifold M^n, where k is even and $n \leq 2k$. Let $[\theta_2(f)] \in \mathfrak{N}_{n-k}$ denote the cobordism class of the double point set $\theta_2(f)$. Then the image of $[\theta_2(f)]$ in the quotient group $\mathfrak{N}_{n-k}$/image Ω_{n-k} depends only on the cobordism class $[M^n] \in \mathfrak{N}_n$.

Especially if M^n is 0-cobordant or $\alpha(n) > n - k$ then $\theta_2(f)$ is cobordant to an orientable manifold. ($\alpha(n)$ denotes the number of digits 1 in the binary decomposition of n.)

Proof: In the case $\alpha(n) > n - k$ we use R. Brown's theorem saying that M^n is cobordant to a manifold, which embeds in $R^{2n-\alpha(n)+1}$. ∎

For example if $k = n - 6$ then we get a homomorphism into $\mathfrak{N}_6 = Z_2 \oplus Z_2 \oplus Z_2$ (since $\Omega_6 = 0$.) defined on those elements of $\mathfrak{N}_n$, which contain a manifold that embeds in R^{2n-6}.

Question 3: What is the image of this homomorpism? ∎

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