Knots of Ten or Fewer Crossings of Algebraic Order Two

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A knot K is an S^1 embedded in S^3. If K is the boundary of a D^2 properly embedded in D^4, we call that knot *slice*. The set of knots modulo slice knots is called the *knot concordance group*: this is a group under connected sums, with the orientation–reversed mirror of a knot being its inverse. Levine defined a group called the *algebraic concordance group* of Witt classes of Seifert matrices of knots, and proved that this group is isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$. He also showed that there is a surjective homomorphism from the knot concordance group to the algebraic concordance group [5]. Casson and Gordon proved that the kernel of this map was non-trivial [1]. In this paper we investigate knots in the knot concordance group that represent elements of order two in the algebraic concordance group. We prove that many are not of knot concordance order two.

The orders of many knots in the algebraic concordance group have been calculated. Morita [9] and Kawauchi [3] have published tables of all knots of ten or fewer crossings listing, among other things, their algebraic orders, and their concordance orders if known. All the knots of ten or fewer crossings that are of algebraic order one (algebraically slice) are known to be order one in the concordance group (slice). All the knots of ten or fewer crossings which are of algebraic order four are known to be of infinite order in the knot concordance group [8]. However, many of the concordance orders of knots of algebraic order two are not known. This paper fills all but one of the gaps in the table: these knots of algebraic order two are shown not to have concordance order two, and two of them are shown to have infinite concordance order. One knot's concordance order remains a mystery.

In Kawauchi's tables, the knots (useing Rolfsen's [11] numbering) 8_1,

8_{13}, 9_{14}, 9_{19}, 9_{30}, 9_{33}, 9_{44}, 10_1, 10_{10}, 10_{13}, 10_{26}, 10_{28}, 10_{34}, 10_{58}, 10_{60}, 10_{91}, 10_{102}, 10_{119}, 10_{135}, 10_{158}, and 10_{165} are listed as algebraic order two but with unknown concordance order. With the exception of the knot 10_{158}, we will show that none of these are of concordance order two. In the first section, we consider k-twisted doubles of the unknot for $4k + 1$ prime, $k \geq 3$, and prove that all such knots are in fact of infinite concordance order. Since the knots 8_1 and 10_1 are twisted doubles of the unknot, we have that they are of infinite order. We also present corollaries that show that, for $4k + 1$ prime, $k \geq 3$, the set of k–twisted doubles of the unknot, and the set of k–twisted doubles of any algebraically slice knot, are linearly independant in the knot concordance group. This result is of wider interest. Casson and Gordon showed that all but two twisted doubles of the unknot of algebraic order one are not of concordance order one [1], and Livingston and Naik gave a family of twisted doubles of the unknot, all of algebraic order four, which are linearly independant in the knot concordance group [8].

The rest of the knots are dealt with in the second section. A *twisted Alexander polynomial* was defined by Kirk and Livingston, and was proven to be an invariant that can be used to detect slice knots [4]. By calculating these polynomials for the knots 8_{13}, 9_{14}, 9_{19}, 9_{30}, 9_{33}, 9_{44}, 10_{10}, 10_{13}, 10_{26}, 10_{28}, 10_{34}, 10_{58}, 10_{60}, 10_{91}, 10_{102}, 10_{119}, 10_{135}, and 10_{165}, we show that none are of concordance order two.

We note an error in Kawauchi's tables. The following knots are listed there as algebraic order two and concordance order two: 8_{17}, 10_{79}, 10_{81}, 10_{88}, 10_{109}, 10_{115}, 10_{118}. They are listed as such because they are amphichiral, i.e. equal to their mirror image. If a knot is oriented, and equal to its orientation–reversed mirror, then it is of concordance order two, because the reversed mirror of a knot is its inverse in the knot concordance group. The knot listed above are equal to their mirror with the same orientation, but not equal to their orientation–reversed mirror. They are also not concordant to their orientation–reversed mirror [13]. Thus they are not of concordance order two.

I take this opportunity to thank Charles Livingston for all his help.

1   Twisted Doubles of the Unknot

We begin with a formal definition. A *knot* is a homeomorphism class of a pair of manifolds, (S, K), with $S \cong S^3$, $K \cong S^1$, with a smooth embedding

$K \hookrightarrow S$. A knot is called *slice* if there is a manifold pair (B, D), $B \cong B^4$, $D \cong B^2$, with a proper smooth embedding $D \hookrightarrow B$, such that $\partial(B, D) \cong (S, K)$. We will usually abuse notation and refer to K as a knot.

In [1], Casson and Gordon define an invariant $\tau(K, \chi)$ that vanishes if a knot is slice. This invariant depends on the knot K and a character χ. Let K_n be the n–fold cyclic cover of a knot complement $S \setminus K$, and let $\bar{K}_n$ be the n–fold branched cyclic cover of $S \setminus K$. The character χ is a homomorphism $\chi \colon H_1(\bar{K}_n) \to \mathbb{Z}/d$. If a knot is algebraically slice and n is a prime power, there is a *metabolizer* of $H_1(\bar{K}_n)$: a half–rank subgroup $H \subset H_1(\bar{K}_n)$ such that H is self–annihilating under the linking form on $H_1(\bar{K}_n)$. If a knot is slice, then Casson and Gordon proved that there exists a metabolizer H such that for all characters $\chi \colon H_1(\bar{K}_n) \to \mathbb{Z}/d$, d a prime power, that are trivial on H, $\tau(K, \chi)$ vanishes. Casson and Gordon also defined another related invariant $\sigma(K, \chi)$, and a signature of the τ invariant $\sigma_1(\tau(K, \chi))$, and then proved that the two invariants are related by $|\sigma(K, \chi) - \sigma_1(\tau(K, \chi))| \leq 1$.

Let T_k be the k–twisted double of the unknot (Figure 1). T_k is algebraically slice for $k = 0, 2, 6, 12, \ldots$, i.e. for $k = u(u - 1)$. For any other k, T_k is of algebraic order two or of algebraic order four (i.e. either $T_k \# T_k$ or $T_k \# T_k \# T_k \# T_k$ is algebraically slice). According to Levine [5], T_k is of order 4 exactly when $4k + 1 = p^\alpha m$, for some prime p, $p \equiv 3 \mod 4$, $\gcd(p, m) = 1$, and some odd α. Specifically, any T_k such that $4k + 1$ is prime fails to be of algebraic order one or four, and thus is of algebraic order two.

In the concordance group, the knot T_0 is the unknot, and therefore slice; the knot T_2, popularly called the stevedore's knot, is also slice. The figure–eight–knot, T_1, is of order two in the knot concordance group. Casson and Gordon [1] proved that, except for $k = 0, 2$, no other algebraically slice twisted double of the unknot is slice. Livingston and Naik considered the knots T_k, with $k = (pq - 1)/4$, p, q primes congruent to 3 mod 4. These are of algebraic order four; they were shown to be of infinite concordance order [8]. We will show that any T_k where $k \geq 3$ and $4k + 1$ is a prime, which is
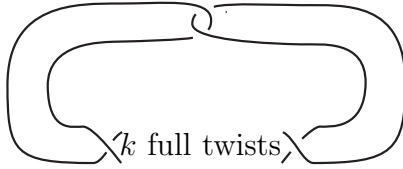

Figure 1: k–twisted double of the unknot

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of algebraic order two, is of infinite concordance order. As corollaries, the set of T_k, $k > 3$, $4k + 1$ prime, are linearly independent in the concordance group, and for any algebraically slice knot K, the set of k–twisted doubles of K are linearly independant in the knot concordance group.

The two–fold branched cover of T_k is the lens space $L(4k + 1, 2)$. As lens spaces are well understood, this is one of the few cases where it is not too difficult to calculate Casson–Gordon invariants. In fact, some of the Casson–Gordon invariants for these knots were calculated in [1].

Theorem 1.1. *Let T_k be the k–twisted double of the unknot. Then for all $k \geq 3$, if $4k + 1$ is a prime, T_k is of infinite concordance order.*

Note that this theorem applies to $8_1 = T_3$ and $10_1 = T_4$.

Proof. We are considering the knot $\#_n T_k$. Because T_k is algebraic order 2, we assume n is even. Let $M_{k,n}$ be the two–fold branched cover of $\#_n T_k$. Let $m = 4k + 1$. By [1], we have

$$\sigma(T_k, \chi^{2r}) = 4 \left(\text{area} \triangle \left(r, \frac{2r}{m} \right) - \text{int} \triangle \left(r, \frac{2r}{m} \right) \right).$$

Here $\triangle(x, y)$ is the triangle on the x, y–plane with vertices at $(0, 0)$, $(x, 0)$, and (x, y). The notation int refers to a count of integral points of the triangle as follows: count 1 for every integral interior point, $1/2$ for every non–vertex integral boundary point, and $1/4$ for every integral vertex point except $(0, 0)$. The character χ^{2r} is defined as follows. The character $\chi \colon H_1(L(m, 2)) \to \mathbb{Z}/m$ is the character of [1], pg. 26; let χ^{2r} be the $2r$–power of χ. The character is of order m, thus r takes values in the integers modulo m.

We can write $\chi(\cdot) = \text{lk}(x, \cdot)$ for some $x \in H_1(L(m, 2))$. Since $H_1(L(m, 2))$ is isomorphic to $\mathbb{Z}/m$ and m is prime, we may consider x to be the generator of $H_1(L(m, 2))$, written as 1. Then we have $\chi^{2r}(\cdot) = \text{lk}(2r, \cdot)$.

By considering triangles, Casson and Gordon [1] found that

$$\sigma(T_k, \chi^{2r}) = \frac{4r^2}{m} - 2r + 1 \text{ for } 0 < 2r < m,$$

we can further find that

$$\sigma(T_k, \chi^{2r}) = \frac{4r^2}{m} - 6r + 2m + 1 \text{ for } m < 2r < 2m.$$

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We will abbreviate the first of these polynomials as $\sigma_1(r)$ and the second as $\sigma_2(r)$. Elementary algebra reveals that the minimum for $\sigma_1(r)$ over $\mathbb{R}$ occurs at $m/4$; the closest integer point to that is $(m-1)/4 = k$, and

$$\sigma_1(k) = \left(\frac{1}{4m}\right)\left(-m^2 + 4m + 1\right),$$

which is negative for all $m \geq 5$. The only positive value of either σ_1 or σ_2 (for $k \geq 3$) occurs at $(m-1)/2$ and $(m+1)/2$ respectively, and

$$\sigma_1\left(\frac{m-1}{2}\right) = \sigma_2\left(\frac{m+1}{2}\right) = \frac{1}{m}.$$

We now need to find an appropriate metabolizer and character. A basis for the vector space $H_1(M_{k,n}) \cong (\mathbb{Z}/m)^n$ comes from the isomorphism

$$H_1(M_{k,n}) \cong \bigoplus_n H_1\big(L(m,2)\big).$$

Let M be a metabolizer of $H_1(M_{k,n})$; M is isomorphic to $(\mathbb{Z}/m)^{n/2}$. Linear algebra gives us a basis

$$(1, 0, \ldots, 0, a_{1,1}, \ldots, a_{1,n/2})$$
$$(0, 1, \ldots, 0, a_{2,1}, \ldots, a_{2,n/2})$$
$$\vdots$$
$$(0, 0, \ldots, 1, a_{n/2,1}, \ldots, a_{n/2,n/2})$$

for M, written in the given basis for $H_1(M_{k,n})$. Taking the sum of these basis elements, every metabolizer contains the element

$$(1, 1, \ldots, 1, b_1, \ldots, b_{n/2}).$$

Multiplying the above element by $k = (m-1)/4$, we see that M also contains, for some k_i's,

$$(k, k, \ldots, k, k_1, \ldots, k_{n/2}).$$

Let

$$\bar{\chi} = (\chi^k, \chi^k, \ldots, \chi^k, \chi^{k_1}, \ldots, \chi^{k_{n/2}}).$$

Note that $\bar{\chi}$ is element–wise linking with an element in the metabolizer M, and thus is trivial on M. We have

$$\sigma(\#_n T_k, \bar{\chi}) = \sum_{n/2} \sigma(T_k, \chi^k) + \sum_{i=1}^{n/2} \sigma(T_k, \chi^{k_i})$$

$$= \frac{n}{8m}(-m^2 + 4m + 1) + \sum_{i=1}^{n/2} \sigma(T_k, \chi^{k_i})$$

$$\leq \frac{n}{8m}(-m^2 + 4m + 1) + \frac{n}{2}\left(\frac{1}{m}\right)$$

$$= \frac{n}{8m}\left(-m^2 + 4m + 5\right)$$

We apply Theorem 3 from [1] and the triangle inequality to calculate

$$|\sigma(\#_n T_k, \bar{\chi}) - \sigma_1(\tau(\#_n T_k, \bar{\chi}))| = \left|\sum_i \sigma(T_k, \chi^i) - \sum_i \sigma_1(\tau(T_k, \chi^i))\right|$$

$$\leq \sum_i |\sigma(T_k, \chi^i) - \sigma_1(\tau(T_k, \chi^i))| \leq n$$

Thus we get that

$$\sigma_1(\tau(\#_n T_k, \bar{\chi})) \leq \sigma(\#_n T_k, \bar{\chi}) + n \leq \frac{n(-m^2 + 12m + 5)}{8m}$$

The expression on the right is strictly negative for all $n \geq 2, m \geq 13$. Therefore $\sigma_1(\tau(K, \chi))$ is non–zero for all $n \geq 2, k \geq 3$, so T_k is of infinite order in $\mathcal{C}$. $\qquad\square$

Corollary 1.2. *The knots $\{T_k \,|\, k = 3, 4, 5, \ldots; 4k + 1 \text{ prime}\}$ are linearly independent in the knot concordance group.*

Proof. We use the notation that M^n for the connected sum of n copies of M, or the direct product of n copies of M, depending on whether M is a space or a module.

Take a linear combination of twisted doubles of the unknot, $\#_{i=1}^N (T_{k_i})^{n_i}$, $k_i \neq k_j$ when $i \neq j$. The Alexander polynomial of this knot is $\prod(k_i t^2 -$

$(2k_i + 1)t + k_i)^{n_i}$. If $\#_{i=1}^N (T_{k_i})^{n_i}$ is slice, its Alexader polynomial must factor as $f(t)f(t^{-1})$. Thus in order that our knot to be slice, we must have that each of the n_i are even.

Let $m_i = 4k_i + 1$. The two–fold branched cover of the knot $\#_{i=1}^N (T_{k_i})^{n_i}$ is $\#_{i=1}^N L(m_i, 2)^{n_i}$. Let χ be a character

$$\chi \colon H_1(\#_{i=1}^N L(m_i, 2)^{n_i}) \cong \bigoplus_{i=1}^N H_1(L(m_i, 2))^{n_i} \to \mathbb{Z}/m_1.$$

Since there is no non–trivial map $\mathbb{Z}/m_i \to \mathbb{Z}/m_1$ for $i \neq 1$, the character χ is zero on $\bigoplus_{i=2}^N H_1(L(m_i, 2))^{n_i}$. We see by the proof of Theorem 3 in [1], that if χ is zero, then $\sigma_1(\tau(K, \chi)) = \sigma(K, \chi)$. Therefore

$$|\sigma(\#_{i=1}^N (\#_{n_i} T_{k_i}), \chi) - \sigma_1(\tau(\#_{i=1}^N (\#_{n_i} T_{k_i}), \chi))| =$$
$$|\sigma(\#_{n_1} T_{k_1}, \chi) - \sigma_1(\tau(\#_{n_1} T_{k_1}, \chi))|.$$

Thus by the proof of Theorem 1.1 above, there is a $\bar{\chi}$ such that

$$|\sigma(\#_{i=1}^N (T_{k_i})^{n_i}, \bar{\chi}) - \sigma_1(\tau(\#_{i=1}^N (T_{k_i})^{n_i}, \bar{\chi}))| \leq n_1$$

so

$$\sigma_1(\tau(\#_{i=1}^N (T_{k_i})^{n_i}, \bar{\chi})) \leq \frac{n_1(-m_1^2 + 12m_1 + 5)}{8m_1} < 0$$

and so $\#_{i=1}^N (T_{k_i})^{n_i}$ is not slice. □

This corollary, together with the work of Litherland concerning satellite knots [6], allows us to conclude that, for $k \geq 3$, $4k + 1$ prime, the set of k–twisted doubles of any given algebraically slice knot is linearly independant in the knot concordance group.

To form a k–twisted double of an arbitrary knot K, take a k–twisted double of the unknot, embedded in a solid torus. Remove a tubular neighborhood of K from S and glue in that solid torus, gluing a meridian/longitude pair of the torus to a meridian/longitude pair of the knot K. We write $T_k(K)$ for the k–twisted double of K.

Corollary 1.3. *Let K be an algebraically slice knot, and let $\{k_i\}$ be the set of positive integers such that $k_i \geq 3$ and $4k_i + 1$ is prime. Then the set $\{T_{k_i}(K)\}$ is linearly independant in the knot concordance group.*

Proof. The Seifert surface and Seifert form of a k–twisted double of a knot are identical to those of the k–twisted double of the unknot, thus for $4k + 1$ prime, a k–twisted double of any knot is of algebraic order 2.

Let $\overline{T_k(K)}_n$ the n–fold branched cover of $T_k(K)$. A trivial consequence of [6], Corollary 2, is that for any character $\chi \colon H_1(\overline{T_k(K)}_n) \to \mathbb{Z}/d$, we have

$$\tau(T_k(K), \chi) = \tau(T_k, \chi) + \sum_{i=1}^{n} \sigma_K[\chi(x_i)]$$

It is shown in [6] that characters $H_1(\overline{T_k(K)}_n) \to \mathbb{Z}/d$ are in a one–to–one correspondance with characters $H_1((\overline{T_k})_n) \to \mathbb{Z}/d$, thus we abuse notation and use χ to denote both these characters. The invariant σ_K of a knot is defined in [6]; it measures the algebraic sliceness of a knot.

The consequence of Litherland's formula is that if we assume that K is algebraically slice, then σ_K and $\sigma_K[\chi(x_i)]$ are zero, and thus

$$\tau(T_k(K), \chi) = \tau(T_k, \chi),$$

and so Corollary 1.2 suffices to prove the result. $\qquad\square$

2 Twisted Alexander Polynomials

In general, Casson–Gordon invariants are difficult to calculate, though much work has been done to develop algorithms to calculate them in special cases ([1], [2], [7], [10]). One can often overcome this difficulty by using a related invariant, the twisted Alexander polynomial. The twisted Alexander polynomial is related to the determinant of the τ Casson–Gordon invariant [4]. It is easier to calculate, in so far as there is an algorithm for calculating it [14].

The (non–twisted) Alexander polynomial is a very well–known knot invariant. Let $\tilde{K}$ be the infinite cyclic cover of K. Then $H_1(\tilde{K}; \mathbb{Q}[t, t^{-1}])$ is a torsion $\mathbb{Q}[t, t^{-1}]$ module. Since $\mathbb{Q}[t, t^{-1}]$ is a p.i.d., we can write

$$H_1(\tilde{K}; \mathbb{Q}[t, t^{-1}]) \cong \frac{\mathbb{Q}[t, t^{-1}]}{\langle p_1(t) \rangle} \oplus \ldots \oplus \frac{\mathbb{Q}[t, t^{-1}]}{\langle p_k(t) \rangle}.$$

We define the Alexander polynomial $\Delta_K(t) = \prod p_i(t)$. The product $\prod p_i(t)$ is also known as the *order* of the module $H_1(\tilde{K}; \mathbb{Q}[t, t^{-1}])$. The polynomial

$\Delta_K(t)$ is well–defined up to units in $\mathbb{Q}[t, t^{-1}]$. It is well–known that if a knot is slice, the Alexander polynomial factors as $\Delta_K(t) = f(t)f(t^{-1})$ [11]. This is actually a consequence of being algebraically slice.

A twisted Alexander polynomial of a knot is an extension of the above concept. The general definition of a twisted Alexander polynomial can be found in [4]; we restrict ourselves to the specific case needed.

Let K_n be the n–fold cyclic cover of the a knot complement $S \setminus K$. Take the map on fundamental groups induced by the covering map $K_n \to S \setminus K$, and post-compose it with the Hurewicz homomorphism. Call the composition $\eta \colon \pi_1(K_n) \to \mathbb{Z}$. As the image of η is isomorphic to $\mathbb{Z}$, we take η to be onto.

Define a map ρ as follows. Let $\bar{K}_n$ be the branched n–fold cyclic cover of K. Choose a character $\chi \colon H_1(\bar{K}_n) \to \mathbb{Z}/d$. Precompose this map with the map on homology arising from the inclusion $K_n \hookrightarrow \bar{K}_n$ and with the Hurewicz homomorphism to get a map $\pi_1(K_n) \to \mathbb{Z}/d$. Let $\mathbb{Q}(\zeta_d)$ be the extension of the rationals by a dth root of unity; $\mathbb{Z}/d$ maps into $\mathbb{Q}(\zeta_d)$ by $i \mapsto \zeta_d^i$. Thus we can define a map $\mathbb{Z}/d \to \mathbb{Q}(\zeta_d)^* = \mathbb{Q}(\zeta_d) \setminus \{0\}$. Composing, we get

$$\rho \colon \pi_1(K_n) \to \mathbb{Q}(\zeta_d)^*.$$

Let $\tilde{K}_n$ be the universal cover of K_n. The fundamental group $\pi_1(K_n)$ acts on chains $C_*(\tilde{K}_n; \mathbb{Q}(\zeta_d))$. There is also an action of $\pi_1(K_n)$ on $\mathbb{Q}(\zeta_d)[t, t^{-1}]$ via ρ and η, defined by $\gamma \cdot p(t, t^{-1}) = t^{\eta(\gamma)} \rho(\gamma) p(t, t^{-1})$. Thus ρ and η define a twisted homology $H_*(K_n; \mathbb{Q}(\zeta_d)[t, t^{-1}]_{\rho, \eta})$. This homology is a module over $\mathbb{Q}(\zeta_d)[t, t^{-1}]$, which is a p.i.d., so $H_1(K_n; \mathbb{Q}(\zeta_d)[t, t^{-1}]_{\rho, \eta})$ has a well–defined order. We call this order a twisted Alexander polynomial of the knot K. Since the only choice that was made in the definition was a choice of a character $\chi \colon H_1(\bar{K}_n) \to \mathbb{Z}/d$, we will write the twisted Alexander polynomial as $\Delta_\chi(K)$.

Just as the untwisted Alexander polynomial factors if a knot is slice, so does the twisted Alexander polynomial.

Theorem 2.1 (Theorem 6.2 [4]). *Let K be a knot, and let $\bar{K}_n$ be the n– fold branched cyclic cover of K, with n a power of a prime. If K is slice, then there is a metabolizer $H \subset H_1(\bar{K}_n)$ with the following property. For all characters $\chi \colon H_1(\bar{K}_n) \to \mathbb{Z}/d$, with d a prime–power and $\chi(H) = 0$, $\Delta_\chi(K)$ factors as $a \cdot f(t) \cdot \bar{f}(t^{-1}) \cdot (t-1)^s$, where $a \in \mathbb{Q}(\zeta_d)$, and $s = 0$ or $s = 1$ if χ is trivial or non–trivial, respectivly.*

2.1   Calculating the Twisted Alexander Polynomial

Just as the twisted Alexander polynomial is defined as an extension of the Alexander polynomial, so the calculation of the twisted polynomial is a variation on the calculation of the non–twisted polynomial. We will recall the method for calculating the former before going on to the latter.

Recall the definition of a *Fox derivative*. Let $F(x_1, \ldots, x_n)$ be a free group, and v and w be words in that group. The Fox derivative

$$\frac{\partial}{\partial x_i} w$$

is defined recursively as follows:

$$\frac{\partial}{\partial x_i} x_i = 1$$
$$\frac{\partial}{\partial x_i} (vw) = \frac{\partial}{\partial x_i} v + v \frac{\partial}{\partial x_i} w.$$

The same definition can be extended to any factor group of a free group.

Let

$$\Pi = \langle x_1, x_2, \ldots, x_m \,|\, r_1, r_2, \ldots, r_{m-1} \rangle$$

be a presentation for the fundamental group of a knot complement, i.e. $\Pi = \pi_1(S \setminus K)$. Form a matrix whose i, jth entry is the Fox derivative $\partial r_i / \partial x_j$. Delete any one column to obtain an $m - 1$ by $m - 1$ matrix. Replace all the occurrences of x_i in the entries of the matrix M with t. This matrix is actually a presentation matrix for the first homology of the infinite cyclic cover of the knot complement as a $\mathbb{Z}[t, t^{-1}]$ module, where the action of t on homology arises from the deck transformation action on the space. Take a determinant of M; this determinant is the Alexander polynomial of the knot [11].

The calculation of the twisted Alexander polynomial is analogous. This algorithm is due to Wada [14], applied to our specific case. We need a presentation of the fundamental group Π_n of the n–fold cyclic branched cover of the knot; this can be found using a Reidemeister–Schreier rewriting algorithm. We form as before the matrix M of Fox derivatives of the relations of Π_n with respect to the generators of Π_n. We again delete a column, say column j, being careful to delete a column corresponding to a generator whose

images under η is non–trivial. The matrix is now $(m-1)n \times (m-1)n$. We replace the occurrences of x_i in the entries of M by $\rho(x_i)t^{\eta(x_i)}$, and take the determinant of the resulting matrix. The resulting polynomial is divided by $(\rho(x_j)t^{\eta(x_j)} - 1)$, and then multiplied by $(t-1)$ if χ is trivial. The resulting polynomial is $\Delta_\chi(K)$ [4].

This process, though algorithmic, is tedious and difficult. For example, take the calculation of a twisted Alexander polynomial for the knot 10_{84}. The fundamental group of this knot has 3 generators and 2 relations. One needs to consider a 13–fold cover of the knot; that cover has 26 relations in its fundamental group. In this case, the image of ρ is $\mathbb{Z}/53$, so in order to calculate the twisted Alexander polynomial, one needs to take a determinant of a 25 by 25 matrix with entries which are polynomials in t with rational numbers multiplied by 53rd roots of unity as coefficients. Luckily, the algorithm lends itself to computer implementation. The polynomials for the knots mentioned here have calculated using Maple [12].

2.2 Example: The Knot 8_{13}

We will show that the knot 8_{13} is not of concordance order two, i.e. that $8_{13}\#8_{13}$ is not slice. This method is useable for all the knots of 10 or fewer crossings which are algebraic order two, but whose concordance order is unknown, except for the twisted doubles of the unknot, and 10_{158}.

The homology of the 2–fold branched cover $(\overline{8_{13}})_2$ of 8_{13} is isomorphic to $\mathbb{Z}/29$; fix an isomorphism $H_1\big((\overline{8_{13}})_2\big) \cong \mathbb{Z}/29$. For $n \in \mathbb{Z}/29$, let $\bar{n}$ be a class in $H_1\big((\overline{8_{13}})_2\big)$ representing n. Then we can define $\chi_n \colon H_1\big((\overline{8_{13}})_2\big) \to \mathbb{Z}/29$ to be the map $\chi_n(x) = \mathrm{lk}(x, \bar{n})$. We find the twisted Alexander polynomial of 8_{13} and χ_1 to be

$$
\begin{aligned}
\Delta_{\chi_1}(8_{13}) = (t-1)\Big(t^2 + \\
\big(11\,\zeta^{28} + 2\,\zeta^{27} + 11\,\zeta^{26} + \zeta^{25} + 11\,\zeta^{24} + 2\,\zeta^{23} + 10\,\zeta^{22} + 3\,\zeta^{21} + 9\,\zeta^{20} \\
+ 4\zeta^{19} + 9\,\zeta^{18} + 5\,\zeta^{17} + 7\,\zeta^{16} + 6\,\zeta^{15} + 6\,\zeta^{14} + 7\,\zeta^{13} + 5\,\zeta^{12} + 9\,\zeta^{11} + 4\,\zeta^{10} \\
+ 9\,\zeta^9 + 3\,\zeta^8 + 10\,\zeta^7 + 2\,\zeta^6 + 11\,\zeta^5 + \zeta^4 + 11\,\zeta^3 + 2\,\zeta^2 + 11\,\zeta\big)\,t + 1\Big).
\end{aligned}
$$

Here ζ is a primitive 29th root of unity. Using the Maple computer algebra program, we find that this polynomial is irreducible in $\mathbb{Q}(\zeta)[t, t^{-1}]$.

To find twisted Alexander polynomials for other characters χ_n, we note

$$\Delta(8_{13}, \chi_n) = \sigma_n(\Delta(8_{13}, \chi_1)),$$

where σ_n is the Galois automorphism that sends ζ to ζ^n.

To apply [4], Theorem 6.2, we need to find a metabolizer in

$$H_1\big((\overline{8_{13}\#8_{13}})_2\big).$$

We have that

$$H_1\big((\overline{8_{13}\#8_{13}})_2\big) \cong \mathbb{Z}/29 \oplus \mathbb{Z}/29.$$

Any metabolizer $H \subseteq H_1\big((\overline{8_{13}\#8_{13}})_2\big)$ is rank one and so is generated by a single element, either $(1,0)$, $(0,1)$, or $(1,a)$ with $a \neq 0$. As a metabolizer is self–annihilating under the linking form, and the linking form is non–singular, the first two possibilities are ruled out: neither $\mathrm{lk}((1,0),(1,0))$ nor $\mathrm{lk}((0,1),(0,1))$ can be zero. In the third case, a metabolizer is generated by $(1,a)$. We have

$$\mathrm{lk}_{8_{13}\#8_{13}}((1,a),(1,a)) = 0.$$

Now

$$\mathrm{lk}_{8_{13}\#8_{13}}((1,a),(1,a)) = (1+a^2)\,\mathrm{lk}_{8_{13}}(1,1).$$

Since the linking form is non–singular, $\mathrm{lk}_{8_{13}}(1,1)$ is non–zero, hence

$$1 + a^2 \equiv 0 \mod 29,$$

and so

$$a = 12 \text{ or } a = 17.$$

Thus we have two possible metabolizers in $H_1\big((\overline{8_{13}\#8_{13}})_2\big)$. Let

$$M_{12} = <(1,12)>, M_{17} = <(1,17)> .$$

The character $\chi_1 \oplus \chi_{12}$ disappears on M_{12} and $\chi_1 \oplus \chi_{17}$ disappears on M_{17}.

Neither

$$\Delta(8_{13}\#8_{13}, \chi_1 \oplus \chi_{12}) = (t-1)^{-1}\Delta(8_{13}, \chi_1) \cdot \Delta(8_{13}, \chi_{12})$$

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nor

$$\Delta(8_{13}\#8_{13}, \chi_1 \oplus \chi_{17}) = (t-1)^{-1}\Delta(8_{13}, \chi_1) \cdot \Delta(8_{13}, \chi_{17})$$

factor as

$$a(t-1)f(t)\bar{f}(t^{-1}).$$

This is because $\Delta(8_{13}, \chi_n)$ is irreducible, so in order for $\Delta(8_{13}\#8_{13}, \chi_1 \oplus \chi_{12})$ to factor we need, w.l.o.g.,

$$f(t) = (t-1)^{-1}\Delta(8_{13}, \chi_1)$$

and

$$t^2 \cdot \bar{f}(t^{-1}) = (t-1)^{-1}\Delta(8_{13}, \chi_{12}).$$

We assume that the constant coefficient of $f(t)$ is normalized to 1, and multiply $\bar{f}(t^{-1})$ by t^2 in order to make all the powers of t positive in $t^2 \cdot \bar{f}(t^{-1})$.

If $f(t) = (t-1)^{-1}\Delta(8_{13}, \chi_1)$, then the coefficient of t in $f(t)$ is

$$11\,\zeta^{28} + 2\,\zeta^{27} + 11\,\zeta^{26} + \zeta^{25} + 11\,\zeta^{24} + 2\,\zeta^{23} + 10\,\zeta^{22} + 3\,\zeta^{21} + 9\,\zeta^{20}$$
$$+ 4\zeta^{19} + 9\,\zeta^{18} + 5\,\zeta^{17} + 7\,\zeta^{16} + 6\,\zeta^{15} + 6\,\zeta^{14} + 7\,\zeta^{13} + 5\,\zeta^{12} + 9\,\zeta^{11} + 4\,\zeta^{10}$$
$$+ 9\,\zeta^9 + 3\,\zeta^8 + 10\,\zeta^7 + 2\,\zeta^6 + 11\,\zeta^5 + \zeta^4 + 11\,\zeta^3 + 2\,\zeta^2 + 11\,\zeta$$

Note that then the coefficient of t in $t^2 \cdot \bar{f}(t^{-1})$ is the same as the coefficient of t in $f(t)$. On the other hand, the t coefficient of $(t-1)^{-1}\Delta(8_{13}, \chi_{12})$ (normalized to have all positive powers of t and a constant coefficient 1) is

$$5\,\zeta^{28} + 11\,\zeta^{27} + 10\,\zeta^{26} + 4\,\zeta^{25} + 2\,\zeta^{24} + 6\,\zeta^{23} + 11\,\zeta^{22} + 9\,\zeta^{21} + 3\,\zeta^{20}$$
$$+ \zeta^{19} + 7\,\zeta^{18} + 11\,\zeta^{17} + 9\,\zeta^{16} + 2\,\zeta^{15} + 2\,\zeta^{14} + 9\,\zeta^{13} + 11\,\zeta^{12} + 7\,\zeta^{11} + \zeta^{10}$$
$$+ 3\,\zeta^9 + 9\,\zeta^8 + 11\,\zeta^7 + 6\,\zeta^6 + 2\,\zeta^5 + 4\,\zeta^4 + 10\,\zeta^3 + 11\,\zeta^2 + 5\,\zeta,$$

thus $\bar{f}(t^{-1})$ cannot be $(t-1)^{-1}\Delta(8_{13}, \chi_{12})$, and so $\Delta(8_{13}\#8_{13}, \chi_1 \oplus \chi_{12})$ does not factor as required. One can see by a similar analysis that $\Delta(8_{13}\#8_{13}, \chi_1 \oplus \chi_{17})$ also does not factor as required; thus the knot $8_{13}\#8_{13}$ is not slice, and 8_{13} is not order two in $\mathcal{C}_1$.

The knots 8_{13}, 9_{14}, 9_{19}, 9_{30}, 9_{33}, 9_{44}, 10_{10}, 10_{13}, 10_{26}, 10_{28}, 10_{34}, 10_{58}, 10_{60}, 10_{91}, 10_{102}, 10_{119}, 10_{135}, and 10_{165} are all shown to be not of order two by this same method. Their twisted Alexander polynomials are found in [12].

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