

Characteristic classes of 2-fold symmetric products of spheres.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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In this note we shall compute characteristic classes of $S^2 * S^2$, $S^4 * S^4$ and $S^8 * S^8$ in the sense of my previous paper [3]. The notation and terminology of this note are the same as in [3].

1. Cohomology groups of $S^n * S^n$.

Let $S^n * S^n = \{S^n * S^n, S^n \times S^n, S^n \times S^n, \varphi, G\}$ be the $2n$ -dimensional C^∞ - M -space of the symmetric product of S^n mentioned in [3, Example 1.3].

In case where n is even, non-trivial cohomology groups of $S^n * S^n$ are as follows (Liao [1]):

$$H^q(S^n * S^n; Z) \approx \begin{cases} Z & q = 0, n, 2n; \\ Z_2 & q = n + 2k + 1 \quad (0 < k \leq [(n-1)/2]). \end{cases}$$

Thus $S^n * S^n$ is not topological manifold unless $n = 2$, because the Poincaré duality relation does not hold for $S^n * S^n$. Let i' be the injection $S^n \rightarrow S^n \times S^n$ defined by

$$i'(x) = (x, x_0),$$

where x_0 is a point of S^n . Then a generator $\mu_n^{(n)}$ of $H^n(S^n * S^n; Z) \approx Z$ is given by

$$i'^* \varphi^* \mu_n^{(n)} = \{S^n\}, \quad (1.1)$$

where $\{S^n\}$ is the generator of $H^n(S^n; Z) \approx Z$ determined by the orientation of S^n . Furthermore we denote the generator of $H^{2n}(S^n * S^n; Z) \approx Z$ determined by the orientation of $S^n * S^n$ by $\mu_{2n}^{(n)}$.

Now let S_d^n be the diagonal sphere of $S^n \times S^n$ and let NS_d^n be a sufficiently small (closed) normal tube neighbourhood of S_d^n in $S^n \times S^n$. Let $\mathfrak{N}(S_d^n) = \{NS_d^n, p, S_d^n, \mathcal{L}^n, GL(n; R)\}$ be the normal bundle. ($\mathfrak{N}(S_d^n)$ is equivalent to the tangent bundle of S^n .) We consider fibre bundles $\mathfrak{N}^\circ(S_d^n) = \{\text{Int } NS_d^n, p, S_d^n, \text{Int } \mathcal{L}^n/G, \Gamma\}$ and their spectral sequences [3, Section 5].

Since Σ^n/G is contractible, we obtain

$$H_{\mathcal{K}}^n(\text{Int}(\Sigma^n/G); Z) \approx Z,$$

$$H_{\mathcal{K}}^1(\text{Int}(\Sigma^n/G); Z) = 0.$$

Therefore, we have ${}^{\varphi}d_s {}^{\varphi}\kappa_s {}^2\varphi e_2 = 0$ ($2 \leq s < \infty$) for any element ${}^{\varphi}e_2$ of $H^q(S_d^n; Z) \otimes H_{\mathcal{K}}^n(\text{Int}(\Sigma^n/G); Z)$. Moreover, as is easily verified, we have

$${}^{\varphi}J^{q+1, n-1} = 0 \quad (q = 0, n),$$

$$i_{\infty}^*({}^{\varphi}E_{\infty}^{q, n}) = H_{\mathcal{K}}^{q+n}(\varphi(\text{Int } NS_d^n); Z) \quad (q = 0, n).$$

Hence we obtain the following lemma:

LEMMA 1.1. $\mathbf{S}^n * \mathbf{S}^n$ satisfies the assumption (AVI) of [3, Section 5] for $r = n, 2n$.

The following diagram is commutative ($q = 0, n$):

$$\begin{array}{ccc} E_2^{q, n} = H^q(S_d^n; Z) \otimes H_{\mathcal{K}}^n(\text{Int } \Sigma^n; Z) & \xrightarrow{i_{\infty}^* \kappa_{\infty}^2} & H_{\mathcal{K}}^{q+n}(\text{Int } NS_d^n; Z) \\ \uparrow id. \otimes \varphi^* & & \uparrow \varphi^* \\ {}^{\varphi}E_2^{q, n} = H^q(S_d^n; Z) \otimes H_{\mathcal{K}}^n(\text{Int } (\Sigma^n/G); Z) & \xrightarrow{i_{\infty}^* {}^{\varphi}\kappa_{\infty}^2} & H_{\mathcal{K}}^{q+n}(\varphi(\text{Int } NS_d^n); Z). \end{array} \quad (1.2)$$

Denote by $\mu_d^{(0)}$ the generator of $H^0(S_d^n; Z) \approx Z$. Then we have

$$i'^* \tilde{\iota}_1^* i_{\infty}^* \kappa_{\infty}^2(\mu_d^{(0)} \otimes \{\text{Int } \Sigma^n\}) = \{S^n\},$$

where $\tilde{\iota}_1$ denotes the map $S^n \times S^n \rightarrow (S^n \times S^n, \partial NS_d^n)$. Hence we obtain by (1.1) and (1.2)

$$\iota_1^* i_{\infty}^* \kappa_{\infty}^2(\mu_d^{(0)} \otimes \{\text{Int}(\Sigma^n/G)\}) = 2\mu_n^{(n)}, \quad (1.3)$$

where ι_1 denotes the map $S^n * S^n \rightarrow (S^n * S^n, \varphi(\partial NS_d^n))$.

Furthermore the tangent D -bundle [3, Definition 3.3] of $\mathbf{S}^n * \mathbf{S}^n$ is given as the following collection (i), (ii):

(i) The tangent bundle $\mathfrak{T}(S^n \times S^n) = \{T(S^n \times S^n), p, S^n \times S^n, E^{2n}, SO(2n; R)\}$.

(ii) An isomorphism ${}^1\alpha$ of G into the group of bundle maps of $\mathfrak{T}(S^n \times S^n)$ defined by ${}^1\alpha(g) = dg$.

2. C-classes of $\mathbf{S}^2 * \mathbf{S}^2$.

Let us regard S^2 as the space of 2 homogeneous complex variables $[\alpha, \beta]$. Since $S^2 \times S^2$ has the complex analytic structure and G operates analytically on $S^2 \times S^2$, $\mathbf{S}^2 * \mathbf{S}^2 = \{S^2 * S^2, S^2 \times S^2, S^2 \times S^2, G, \varphi\}$ is a 2-dimensional complex analytic M -space.

Consider the complex projective plane CP^2 as the space of 3 homogeneous complex variables $[\alpha, \beta, \gamma]$. We define the analytic map $\tilde{h}_0$ of $S^2 \times S^2$ onto CP^2 by

$$\tilde{h}_0([\alpha, \beta], [\alpha', \beta']) = [\alpha\alpha', \alpha\beta' + \alpha'\beta, \beta\beta'].$$

Then $\tilde{h}_0$ gives the homeomorphic map h_0 of $S^2 * S^2$ onto CP^2 such that

$$\tilde{h}_0 = h_0 \circ \varphi.$$

Therefore $S^2 * S^2$ and CP^2 are homeomorphic.

Let $K = \{\sigma^0, \sigma^1, \sigma_1^2, \sigma_2^2\}$ be a cellular decomposition of S^2 such that

$$\sigma^0 = [1, 1], \quad \sigma^1 = \{[1, \beta]; |\beta| = 1\},$$

$$\sigma_1^2 = \{[1, \beta]; |\beta| \leq 1\}, \quad \sigma_2^2 = \{[\alpha, 1]; |\alpha| \leq 1\}.$$

Then $K \times K$ gives a cellular decomposition of $S^2 \times S^2$.

Since $S^2 - [0, 1] = \{[1, \beta]\}$ (resp. $S^2 - [1, 0] = \{[\alpha, 1]\}$) is diffeomorphic to the complex line (β) (resp. (α)), there exists a continuous orthonormal vector field $\{V_{[1, \beta]}\}$ on $S^2 - [0, 1]$ (resp. $\{V_{[\alpha, 1]}\}$ on $S^2 - [1, 0]$) such that each $V_{[1, \beta]}$ (resp. $V_{[\alpha, 1]}$) is the vector at $[1, \beta]$ (resp. $[\alpha, 1]$) parallel to the vector $\vec{01} = V_e$ in (β) (resp. (α)). Denote by $(\beta/|\beta|)V_{[1, \beta]}$ ($\beta \neq 0$) the vector at $[1, \beta]$ which is parallel to $(\beta/|\beta|)V_e$, where the multiplication of $\beta/|\beta|$ means the orthogonal transformation determined by $\beta/|\beta|$. It is easily shown that

$$(\beta/|\beta|)V_{[1, \beta]} = -(|\beta|/\beta)'V_{[1/\beta, 1]} \quad (\beta \neq 0). \quad (2.1)$$

Now let K_d be a cellular decomposition of S_d^2 consisting of $\{\sigma_d^0, \sigma_d^1, \sigma_{d1}^2, \sigma_{d2}^2\}$ such that

$$\sigma_d^0 = (\sigma^0, \sigma^0), \quad \sigma_d^1 = \{(x, x); x \in \sigma^1\}, \quad \sigma_{d1}^2 = \{(x, x); x \in \sigma_1^2\},$$

$$\sigma_{d2}^2 = \{(x, x); x \in \sigma_2^2\}.$$

Then $\{p^{-1}(\sigma_d^0), p^{-1}(\sigma_d^1), p^{-1}(\sigma_{d1}^2), p^{-1}(\sigma_{d2}^2)\}$ gives a cellular decomposition of $\text{Int } NS_d^2$. Denote by $(\sigma \times \sigma') \sim (\sigma, \sigma' \in K)$ the set $\sigma \times \sigma' - \text{Int } NS_d^2$. Then the collection of $p^{-1}(\sigma_d)$ ($\sigma_d \in K_d$) and suitable cellular subdivisions of $(\sigma \times \sigma') \sim (\sigma, \sigma' \in K)$ define an admissible cellular decomposition $\tilde{K}$ of $S^2 \times S^2$. We can assume without loss of generality that

$$|\tilde{K}^2| - \text{Int } NS_d^2 - (\sigma_2^2 \times \sigma^0) - (\sigma^0 \times \sigma_2^2) \subset (S^2 - [0, \beta]) \times (S^2 - [0, \beta]). \quad (2.2)$$

Denote the subcomplex of $\tilde{K}$ which gives a cellular decomposition of NS_d^2 (resp. $S^2 \times S^2 - \text{Int } NS_d^2$) by ${}_N\tilde{K}$ (resp. $\tilde{K}_C$).

In the following let us denote by V_N the normalized vector of non-zero vector V .

First we compute the first C-class of $S^2 * S^2$.

Let v_1 be the continuous field of normalized normal vectors with the outer direction defined on ${}_N\tilde{K} \cap \tilde{K}_C^2$. We can regard v_1 as an extension of the set of vectors

$$\{((\beta - \beta')V_{[1, \beta]}, (\beta' - \beta)V_{[1, \beta']})_N; ([1, \beta], [1, \beta']) \in \partial p^{-1}(S_d^2 - U([0, \beta]))\},$$

where $U([0, \beta])$ is a sufficiently small neighbourhood of $[0, \beta]$. Let v_2 be

the continuous field of normalized vectors defined on ${}_N\tilde{K} \cap \tilde{K}_C^2$ which is the restriction of the set of vectors

$$\{(V_{[1,\beta]}, V_{[1,\beta']})_N; ([1,\beta], [1,\beta']) \in (S^2 - U([0,\beta])) \times (S^2 - U([0,\beta]))\}.$$

v_1 and v_2 are two non-zero mutually orthogonal vector fields, and define a continuous field of orthonormal 2-frames $f^{(2)}$ on ${}_N\tilde{K} \cap \tilde{K}_C^2$. Obviously $f^{(2)}$ is invariant under dg , that is, $f^{(2)}$ is a G -cross section of $\mathfrak{T}^{(2)}(S^3 \times S^2)$ on ${}_N\tilde{K} \cap \tilde{K}_C^2$. We take $f^{(2)}$ as the standard cross section of $\mathfrak{T}^{(2)}(S^2 * S^2)$. The continuous field of normalized vectors defined on $(S^2 - U([0,\beta])) \times (S^2 - U([0,\beta])) - \text{Int } NS_d^2$

$$\bar{v}_1 = \{((\beta - \beta')V_{[1,\beta]}, (\beta' - \beta)V_{[1,\beta']})_N; ([1,\beta], [1,\beta']) \in (S^2 - U([0,\beta])) \times (S^2 - U([0,\beta])) - \text{Int } NS_d^2\}$$

is an extension of v_1 . The continuous field of normalized vectors defined on $(S^2 - U([0,\beta])) \times (S^2 - U([0,\beta]))$

$$\bar{v}_2 = \{(V_{[1,\beta]}, V_{[1,\beta']})_N; ([1,\beta], [1,\beta']) \in (S^2 - U([0,\beta])) \times (S^2 - U([0,\beta]))\}$$

is an extension of v_2 . The vectors of $\bar{v}_1$ and $\bar{v}_2$ are mutually orthogonal at each point of $(S^2 - U([0,\beta])) \times (S^2 - U([0,\beta])) - \text{Int } NS_d^2$. Therefore the restrictions of $\bar{v}_1$ and $\bar{v}_2$ on $\tilde{K}_C^1$ define a continuous field $\tilde{f}^{(2)}$ of orthonormal 2-frames which is invariant under dg .

Let us consider the obstruction $\tilde{c}(\tilde{f}^{(2)})$. By (2.1) the restrictions of $\bar{v}_1, \bar{v}_2$ on $(\sigma^1 \times \sigma^1)^\sim$ are

$$\begin{aligned} &\{((1/\alpha') - (1/\alpha))\alpha^2 V_{[\alpha,1]}, ((1/\alpha) - (1/\alpha'))\alpha'^2 V_{[\alpha',1]})_N; \\ &([\alpha, 1], [\alpha', 1]) \in (\sigma^1 \times \sigma^1)^\sim\}, \\ &\{(-\alpha^2 V_{[\alpha,1]}, -\alpha'^2 V_{[\alpha',1]})_N; ([\alpha, 1], [\alpha', 1]) \in (\sigma^1 \times \sigma^1)^\sim\}. \end{aligned}$$

Let f_0 be the map $(\sigma^1 \times \sigma^0)^\sim \rightarrow U(2)$ defined by

$$f(\alpha, \sigma^0) = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1 - (1/\alpha))/|\sqrt{2} - (\sqrt{2}/\alpha)| & 1/\sqrt{2} \\ ((1/\alpha) - 1)/|\sqrt{2} - (\sqrt{2}/\alpha)| & 1/\sqrt{2} \end{pmatrix}.$$

The map $(\sigma^1 \times \sigma^0)^\sim \rightarrow |f(\alpha, \sigma^0)| = \alpha^2(1 - (1/\alpha))/|1 - (1/\alpha)|$ is homotopic to a generator of $\pi_1(U(2))$, because the maps from $\partial(\sigma^1 \times \sigma^0)^\sim$ (Fig. 1) into S^1 defined by

$$\begin{aligned} (\alpha, 1) &\rightarrow \alpha^2, \\ (\alpha, 1) &\rightarrow (1 - (1/\alpha))/|1 - (1/\alpha)| \end{aligned}$$

have degree 2, -1 respectively. Thus we have

$$\tilde{c}(\tilde{f}^{(2)})(\sigma_2^2 \times \sigma^0)^\sim = \tilde{c}(\tilde{f}^{(2)})(\sigma^0 \times \sigma_2^2)^\sim = 1.$$

Moreover it is obvious by (2.2) that

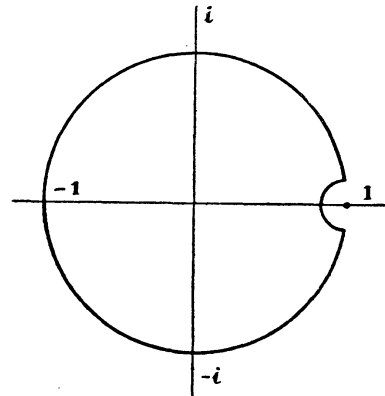


Fig. 1.

$$\tilde{c}(\tilde{f}^{(2)})(\tilde{\sigma}) = 0 \quad (\tilde{\sigma} \in \tilde{K}_c^2, \tilde{\sigma} \neq (\sigma_2^2 \times \sigma^0)^\sim, (\sigma^0 \times \sigma_2^2)^\sim)$$

Therefore, as is easily verified, the cohomology class $\{c'(f)\}$ ([3, Section 5]) satisfies

$$i'^* \tilde{\iota}_1^* \varphi^* (\{c'(f)\}) = \{S^2\}.$$

Using (1.1), we obtain

$$\iota_1^* (\{c'(f)\}) = \mu_2^{(2)}. \quad (2.3)$$

On the other hand, we have

$$\tilde{c}(\tilde{f}^{(2)})(p^{-1}(\sigma_a^0)) = 1.$$

Denote $\{c_j(f)\}$ in [3, Section 5] simply by $\{c_N(f)\}$. Then we have by (1.3)

$$\iota^* (\{c_N(\tilde{f}^{(2)})\}) = 2\mu_2^{(2)}.$$

Therefore $\tilde{c}(\mathfrak{T}^{(2)}(\mathbf{S}^2 * \mathbf{S}^2))$ is given by

$$\tilde{c}(\mathfrak{T}^{(2)}(\mathbf{S}^2 * \mathbf{S}^2)) = \{c(\tilde{f}^{(2)})\} = 3\mu_2^{(2)}.$$

Furthermore the second C-class of $\mathbf{S}^2 * \mathbf{S}^2$ is $3\mu_4^{(2)}$ by [3, Theorem 8.1].

Hence we obtain the following theorem:

THEOREM 2.1. *C-classes of the 2-dimensional complex analytic M-space $\mathbf{S}^2 * \mathbf{S}^2$ are as follows:*

$$C_1(\mathbf{S}^2 * \mathbf{S}^2) = 3\mu_2^{(2)},$$

$$C_2(\mathbf{S}^2 * \mathbf{S}^2) = 3\mu_4^{(2)}.$$

REMARK 2.1. In a forthcoming paper [4] this theorem will be reproved by considering the invariance of characteristic classes of M-spaces under the isomorphism in the wider sense.

3. P-classes of $\mathbf{S}^4 * \mathbf{S}^4$, $\mathbf{S}^8 * \mathbf{S}^8$.

First, we shall compute P-classes of $\mathbf{S}^4 * \mathbf{S}^4$.

Let o and o' be two poles of $\mathbf{S}^4$. We can regard the open set $\mathbf{S}^4 - o'$ (resp. $\mathbf{S}^4 - o$) of $\mathbf{S}^4$ as the set of quaternions $\{(q); \|q\| < 1\}$ (resp. $\{(q)'; \|q\| < 1\}$). By means of the identification (q) and $((1 - \|q\|)q / \|q\|)'$ ($q \neq 0$), we obtain the natural differentiable structure of $\mathbf{S}^4$.

Let $K = \{\sigma^0, \sigma^3, \sigma_1^4, \sigma_2^4\}$ be a cellular decomposition of $\mathbf{S}^4$ consisting of cells

$$\sigma^0 = \langle 1/2 \rangle, \quad \sigma^3 = \{(q); \|q\| = 1/2\},$$

$$\sigma_1^4 = \{(q); \|q\| \leq 1/2\}, \quad \sigma_2^4 = \{(q)'; \|q\| \leq 1/2\}.$$

The quaternion space (q) (resp. $(q)'$) has a continuous field of normalized vectors $\{V_1(q)\}$ (resp. $\{V_1'(q)\}$), where $V_1(q)$ (resp. $V_1'(q)$) is the parallel translation of the normalized vectors at the origin defined by real axis of (q) (resp. $(q)'$).

It is obvious that

$$V_1(q) = -V_1'((1 - \|q\|)q/\|q\|) \quad (q \neq 0). \quad (3.1)$$

Since the first Pontrjagin class of S^4 is zero, there exists a continuous field of unitary 4-frames $\{(V_1^e(x), V_2^e(x), V_3^e(x), V_4^e(x)); x \in S^4\}$, i. e., a cross section of the fibre bundle $\mathfrak{A}^{[4]}(S^4)$.

Now let $K_d = \{\sigma_d^0, \sigma_d^3, \sigma_{d_1}^4, \sigma_{d_2}^4\}$ be a cellular decomposition of S_d^4 such that

$$\begin{aligned} \sigma_d^0 &= (\sigma^0, \sigma^0), \quad \sigma_d^3 = \{(x, x); x \in \sigma^3\}, \\ \sigma_{d_1}^4 &= \{(x, x); x \in \sigma_1^4\}, \quad \sigma_{d_2}^4 = \{(x, x); x \in \sigma_2^4\}. \end{aligned}$$

Then $p^{-1}(K_d)$ gives a cellular decomposition of $\text{Int } NS_d^4$. Denote by $(\sigma \times \sigma')^\sim$ $(\sigma, \sigma' \in K)$ the set $\sigma \times \sigma' - \text{Int } S_d^4$. Then the collection of $p^{-1}(\sigma_d)$ $(\sigma_d \in K_d)$ and suitable cellular subdivisions of $(\sigma \times \sigma')^\sim$ $(\sigma, \sigma' \in K)$ define an admissible cellular decomposition $\tilde{K}$ of $S^4 \times S^4$. We can assume without loss of generality that

$$|\tilde{K}^4| - \text{Int } NS_d^4 - (\sigma_2^4 \times \sigma^0) - (\sigma^0 \times \sigma_2^4) \subset (S^4 - o') \times (S^4 - o'). \quad (3.2)$$

We denote the subcomplex of $\tilde{K}$ which gives a cellular decomposition of NS_d^2 (resp. $S^4 \times S^4 - \text{Int } NS_d^4$) by ${}_N\tilde{K}$ (resp. $\tilde{K}_C$).

$$\begin{aligned} \mathfrak{v}^e &= \{(V_1^e(x), V_1^e(x'))_N, (V_2^e(x), V_2^e(x'))_N, (V_3^e(x), V_3^e(x'))_N, (V_4^e(x), V_4^e(x'))_N; \\ &\quad (x, x') \in S^4 \times S^4\} \end{aligned}$$

is a continuous field of unitary 4-frames. Also

$$\begin{aligned} \mathfrak{v}_1^e &= \{((q - q')V_1^e((q)), (q' - q)V_1^e((q')))_N, ((q - q')V_2^e((q)), (q' - q)V_2^e((q')))_N, \\ &\quad ((q - q')V_3^e((q)), (q' - q)V_3^e((q')))_N; ((q), (q')) \in (S^4 - o') \times (S^4 - o') - S_d^4\} \end{aligned}$$

is a continuous field of unitary 3-frames defined on $(S^4 - o') \times (S^4 - o') - S_d^4$, where the left multiplication of $q - q'$ means the transformation determined by $q - q'$ in an obvious way. Both $\mathfrak{v}^e$ and $\mathfrak{v}_1^e$ are invariant under the operation of dg . Hence $\mathfrak{v}^e$ and $\mathfrak{v}_1^e$ define a G -cross section $f^{[7]}$ of $\mathfrak{A}^{[7]}(S^4 \times S^4)$ over $(S^4 - o') \times (S^4 - o') - S_d^4$. We take the restriction of $f^{[7]}$ on ${}_N\tilde{K} \cap \tilde{K}_C^4$ as the standard cross section of $\mathfrak{A}^{[7]}(S^4 * S^4)$.

Now we compute $\tilde{c}(f^{[7]})$. Obviously we have by (3.2)

$$\tilde{c}(f^{[7]})(\tilde{\sigma}) = 0 \quad (\tilde{\sigma} \in \tilde{K}^4, \tilde{\sigma} \neq (\sigma_2^4 \times \sigma^0)^\sim, (\sigma^0 \times \sigma_2^4)^\sim). \quad (3.3)$$

Furthermore $f^{[7]}$ defines the map $f_1^{[7]}: \partial((\sigma_2^4 \times \sigma^0)^\sim) \rightarrow U(7)$ such that

$$f_1^{[7]}((q)', (1/2)) = j_4(Q_1),$$

$$Q_1 \in SO(4), \quad Q_1(q') = (((1 - \|q\|)q/\|q\|) - (1/2))_N q',$$

where $j_4: SO(4) \rightarrow U(7)$ is the composition of inclusion maps $SO(4) \rightarrow U(4)$ and $U(4) \rightarrow U(7)$. Hence we obtain (Tamura [2])

$$\tilde{c}(f^{[7]})(\sigma_2^4 \times \sigma^0)^\sim = \tilde{c}(f^{[7]})(\sigma^0 \times \sigma_2^4)^\sim = -2. \quad (3.4)$$

On the other hand, since $f^{[7]}$ defines the map $f_2^{[7]}: \partial p^{-1}(\sigma_a^0) \rightarrow U(7)$ such that

$$f_2^{[7]}(q) = j_4(Q_2), \quad Q_2 \in SO(4), \quad Q_2(q') = qq',$$

we have

$$\tilde{c}(f^{[7]})(p^{-1}(\sigma_a^0)) = 2. \quad (3.5)$$

Therefore, by (1.3), (3.3), (3.4) and (3.5), we have

$$\tilde{c}(\mathfrak{T}^{[7]}(\mathbf{S}^4 * \mathbf{S}^4)) = \{\tilde{c}(f^{[7]})\} = 2\mu_4^{(4)}.$$

Next we consider the second P-class of $\mathbf{S}^4 * \mathbf{S}^4$. Let

$$\begin{aligned} \mathfrak{v}_1 = \{ & ((1 - \|q_1\|)(1 - \|q_2\|)(q_1 - q_2)V_1(q_1) + J(1 - \|q_1'\|)(1 - \|q_2'\|)(q_1' - q_2')V_1'(q_1'), \\ & (1 - \|q_1\|)(1 - \|q_2\|)(q_2 - q_1)V_1(q_2) + J(1 - \|q_1'\|)(1 - \|q_2'\|)(q_2' - q_1')V_1'(q_2'))_N; \\ & (q_1) = (q_1')', (q_2) = (q_2')' \} \end{aligned}$$

be a continuous field of complex vectors defined on $S^4 \times S^4 - (S_a^4 \cup (o, o') \cup (o', o))$. As is easily verified, $\mathfrak{v}^c$ and $\mathfrak{v}_1$ define a continuous field of unitary 5-frames, which is invariant under dg , i. e., a G -cross section $f^{[5]}$ of $\mathfrak{T}^{[5]}(S^4 \times S^4)$ over $S^4 \times S^4 - (S_a^4 \cup (o, o') \cup (o', o))$. We take the restriction of $f^{[5]}$ on ${}_N\tilde{K} \cap \tilde{K}_c$ as the standard cross section.

Let us compute $\{c(f^{[5]})\}$. Let $U(o, o')$ be a sufficiently small ball containing (o, o') whose orientation agrees with that of $S^4 \times S^4$. Let $f_1^{[5]}: \partial U(o, o') \rightarrow S^7$ be the map defined by

$$f((q_1, q_2')) = ((1 - \|q_1\|)(\|q_2'\|q_1 - (1 - \|q_2'\|)q_2'), (1 - \|q_2'\|)(1 - \|q_1\|)q_1 - \|q_1\|q_2'))_N.$$

Since, as is easily verified, $f_1^{[5]}$ has degree 1, we have

$$\tilde{c}(f^{[5]})(\sigma_1^4 \times \sigma_2^4) = -1. \quad (3.6)$$

By the way, we have

$$\begin{aligned} & \tilde{c}(f^{[5]})(\sigma_1^4 \times \sigma_2^4) + \tilde{c}(f^{[5]})(\sigma_2^4 \times \sigma_1^4) \\ & + \tilde{c}(f^{[5]})(p^{-1}(\sigma_{a1}^4)) + \tilde{c}(f^{[5]})(p^{-1}(\sigma_{a2}^4)) = 0, \end{aligned} \quad (3.7)$$

because the Pontrjagin classes of $S^4 \times S^4$ vanish.

(1.2), (3.6) and (3.7) enable us to compute $\{c(f^{[5]})\}$:

$$\begin{aligned} \{c(f^{[5]})\}[S^4 * S^4] &= \tilde{c}(f^{[5]})(\sigma_1^4 \times \sigma_2^4) + \tilde{c}(f^{[5]})(p^{-1}(\sigma_{a1}^4)) + \tilde{c}(f^{[5]})(p^{-1}(\sigma_{a2}^4)) \\ &= -\tilde{c}(f^{[5]})(\sigma_1^4 \times \sigma_2^4) = 1. \end{aligned}$$

Hence we obtain the following theorem:

THEOREM 3.1. *P-classes of the 8-dimensional C^∞ -M-space $\mathbf{S}^4 * \mathbf{S}^4$ are as follows:*

$$\begin{aligned} P_1(\mathbf{S}^4 * \mathbf{S}^4) &= 2\mu_4^{(4)}, \\ P_2(\mathbf{S}^4 * \mathbf{S}^4) &= \mu_8^{(4)}. \end{aligned}$$

In a similar way we obtain the following theorem, making use of the Cayley numbers (Tamura [2]):

THEOREM 3.2. *P-classes of the 16-dimensional C^∞ -M-space $S^8 * S^8$ are as follows:*

$$P_2(S^8 * S^8) = 6\mu_8^{(8)},$$

$$P_4(S^8 * S^8) = \mu_{16}^{(8)}.$$

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