Bordism theory and the Kervaire semi-characteristic

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Abstract By using the bordism group, this paper provides an alternative proof of Weiping Zhangs' theorem on counting Kervaire semi-characteristic.

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1 Introduction

Let M be a closed connected smooth manifold. The classical Hopf index theorem asserts that the vanishing of the Euler characteristic of the manifold M is the necessary and sufficient condition for the existence of a nowhere vanishing vector field on M. Let V be a vector field with isolated zeros on M, then the Hopf index theorem takes on the more precise form: the sum of the indices of the vector field V on M is equal to the Euler characteristic of M. It is natural to be concerned with the problems of existence of $r > 1$ linearly independent vector fields instead of a single vector field. However the situation is much more complicated. For background information see refs. [1—3], and especially ref. [4].

Let M be a closed connected oriented manifold of dimension $4q+1 (q \geqslant 1)$. The (real) Kervaire semi-characteristic $k(M)$ of M is a mod 2 integer defined by

$$k(M) = \left(\sum b_{2i}\right) \bmod 2,$$

where b_i denotes the i-th betti number of M. Using the mod 2 index of a real skew-adjoint elliptic operator, Atiyah[2] showed that the Kervaire semi-characteristic has an analytical interpretation.

We consider 2 vector fields V_1, V_2 on the closed oriented $(4q+1)$ -manifold M and we assume that they are linearly independent except at a finite set of points (the singularities). The index of such a 2-field is an element of the homotopy group $\pi_{4q}(V_{4q+1,2}) \cong Z_2$ of the Stiefel manifold $V_{4q+1,2}$ of orthogonal 2-frames in the Euclidean space R^{4q+1}. Atiyah (ref. [2], Theorem (5.1)) proved the following formula

$$\mathrm{Ind}(V_1, V_2) = k(M),$$

as mod 2 integers. It has led to an analogue of the Hopf index theorem mentioned before. However it is worth noticing that Atiyah's formula exists only when the $4q$-th Stiefel-Whitney characteristic class of M vanishes, since M admits a 2-field with finite singularities if and only if $w_{4q}(M) = 0$ (cf. [1]).

In the quite recent paper[5], Zhang adopted a different approach. His new formula for the Kervaire semi-characteristic is generic, without the assumption that $w_{4q}(M)=0$ which Atiyah[2] based on. Following ref. [5], let V be a smooth nowhere vanishing vector field on M, a closed oriented manifold of dimension $4q+1$. The existence of V is guaranteed by the Hopf index theorem. Choose a Riemannian metric g^{TM} on M whose associated Levi-Civita connection will be denoted by $\bigtriangledown^{TM}$. For each $e\in TM$, let $e^*\in T^*M$ correspond to e via the metric g^{TM} and let $c(e),\hat{c}(e)$ be the Clifford operators acting on the exterior algebra bundle $\wedge^*(T^*M)$ defined by

$$c(e)=e^*\wedge -i_e,\qquad \hat{c}(e)=e^*\wedge +i_e,$$

where $e^*\wedge$ and i_e are the standard notation for exterior and inner multiplications, respectively. Without loss of generality, we will assume that V is a unit vector field.

Denoting by 1_V the oriented line bundle spanned by V, we have an oriented codimension one sub-bundle E of TM. Without loss of generality, we may take E to be the orthogonal complement to 1_V in TM.

We next choose a transversal section X of E. Then the set of zeros of X, saying F, consists of a union of disjoint circles $F_1,\cdots,F_p$. Let $i:F\hookrightarrow M$ be the natural embedding. As explained in ref. [5], we may assume that $1_V|_F$ is tangent to F and that i^*E is the normal bundle to F in M.

For any $x\in F$, let $e_0=V,e_1,\cdots,e_{4q}$ be an oriented orthonormal basis near x, and let $y_0,\cdots,y_{4q}$ be the normal coordinate system near x associated to $e_0(x),\cdots,e_{4q}(x)$. Then near x, the map X can be expressed as

$$X=\sum_{i=1}^{4q}f_i(y)e_i.$$

By the transversality of X, it follows that the following endomorphism of E_x is invertible:

$$C(x)=\{c_{ij}(x)\}_{1\leqslant i,j\leqslant 4q}\ \text{with}\ c_{ij}(x)=\frac{\partial f_i}{\partial y_j}(0),$$

where the matrix is with respect to the basis $e_1(x),\cdots,e_{4q}(x)$.

Let $|C(x)|=\sqrt{C^*(x)C(x)}$, where $C^*(x)$ is the adjoint of $C(x)$ with respect to g^E, the induced metric on E from g^{TM}. We finally define an endomorphism $K(x)$ of $\wedge^*(E_x^*)$ by the formula

$$K(x)=Tr\,[|C(x)|]+\sum_{i,j=1}^{4q}c_{ij}(x)c(e_j(x))\hat{c}(e_i(x)).$$

It is easily seen that $K(x)$ is independent of the choice of the basis $e_1(x),\cdots,e_{4q}(x)$ (see refs. [5,6]). Thus it induces an endomorphism K of the exterior algebra bundle $\wedge^*(E^*)|_F$ over F.

Zhang[5] asserted that Ker K forms a real line bundle L over F, and the orientability of L is independent of the choice of the Riemannian metric on M.

For any connected component F_j of F, denote by L_j the restriction of L on F_j. The main result of ref. [5] is the following elegant formula.

Theorem (ref. [5], Theorem 1.3). The Kervaire semi-characteristic $k(M)$ is equal to

$$\#\{j|L_j \text{ is orientable over } F_j\} \bmod 2.$$

While the formula above is purely topological, Zhang's proof is analytic. He first constructed a real skew-adjoint elliptic operator whose mod 2 index provides an alternative analytic interpretation of $k(M)$[7], which is different from that of Atiyah[2]. Then he deformed this operator in a way similar to what Witten[8] used in the analytic proof of the Hopf index theorem. By applying the localization techniques of Bismut and Lebeau[9] to these deformed operators, he finally got his proof.

The main purpose of the present paper is to give a topological proof of Zhang's theorem by the normal framed bordism theory (see refs. [10, 3], for example).

It should be remarked that in fact, Zhang has gotten similar formulas for the manifolds of arbitrary dimensions (see res. [5] Theorem 3.3 for details). Fortunately, our methods still work in every case.

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2 Normal framed manifolds

We begin by recalling what Pontrjagin[10] used when he calculated π_1^s, the first stable homotopy group. By using the Pontrjagin-Thom construction, one associates each smooth map from an $(n+k)$-dimensional sphere into an n-dimensional sphere with a smooth normal framed submanifold N^k of the Euclidean space R^{n+k}. By a normal framed manifold N^k we mean that at every point x of N^k, there is a given system $U(x) = \{u_1(x), \cdots, u_n(x)\}$ of linearly independent vectors orthogonal to N^k, where $u_i(x)$ continuously depends on $x \in N^k$. The manifold N^k together with its frame U is called a normal framed manifold and is denoted by (N^k, U). One has also the concept of normal framed bordism (Pontrjagin called it homology) between two normal framed manifolds embedded in the same Euclidean space R^{n+k}. It turns out that every smooth normal framed manifold (N^k, U) corresponds to some map from S^{n+k} into S^n, moreover two maps from S^{n+k} into S^n are homotopic if and only if their corresponding smooth normal framed manifolds are normal framed bordant. Thus the problem of classification of the maps from a sphere into a sphere reduces to the problem of classification of smooth normal framed manifolds.

We want now to consider the special case when $k = 1$. Let (N^1, U) be a normal framed manifold in the Euclidean space $R^{n+1} (n \geqslant 3)$. Let $U(x) = \{u_1(x), \cdots, u_n(x)\}$ be an orthonormal frame of N^1, and let $u_0(x)$ be the unit vector tangent to N^1 at $x \in N^1$. The system $U'(x) = \{u_0(x), u_1(x), \cdots, u_n(x)\}$ is derived from a fixed orthonormal basis of R^{n+1} by means of a rotation $h(x)$. Thus, one gets a continuous map h from N^1 into the manifold $SO(n+1)$ of rotations of R^{n+1}. For a one-component curve N^1, the invariant δ is taken equal to zero if h is not homotopic to zero, and equal to unity otherwise (It is well known that $\pi_1 SO(n+1) \cong Z_2$ if $n > 1$). For a multicomponent curve, δ is defined to be the sum modulo 2 of the values of the invariants for the components. Thus one gets a normal framed bordism invariant $\delta(N^1, U)$ of a normal framed manifold in the Euclidean space. Pontrjagin established the following

Theorem (ref. [10] Theorem 21). For $n \geqslant 3$ the homomorphism δ from the group π_1^s

into the group of residues modulo two is an isomorphism.

We pass next to normal bordism groups[3,11].

Let Y be a topological space, φ be a virtual real vector bundle over Y, i.e. an ordered pair (φ_1, φ_2) of vector bundles written $\varphi = \varphi_1 - \varphi_2$. Now consider triples of the form $(S, g, \bar{g})$

(i) S is an r-manifold without boundary;

(ii) $g : S \to Y$ is a continuous map;

(iii) $\bar{g} : R^s \oplus TS \oplus g^*(\varphi_1) \to R^t \oplus g^*(\varphi_2)$ is a vector bundle isomorphism for suitable integers r and s. Here R^s and R^t stand for trivial bundles of dimensions s and t, respectively.

The set of bordism classes $[S, g, \bar{g}]$ of triples $(S, g, \bar{g})$, with the group structure given by disjoint union, is called the r-th normal bordism group of Y with coefficients in φ, and is denoted by $\Omega_r(Y, \varphi)$.

If Y is a point and φ is trivial, then $\Omega_r(Y, \varphi)$ is canonically isomorphic to the r-th stable homotopy group π_r^s. In particular Ω_1 (point, trivial) is isomorphic to Z_2, and the generator of $\Omega_1(\text{point}, \text{trivial})$ is represented by the invariant framed circle S^1. In fact, an element of $\Omega_1(\text{point}, \text{trivial})$ is presented by a circle S^1 equipped with an isomorphism

$$\bar{g} : TS^1 \oplus R^n \to R^{n+1}.$$

This will give rise to a map $h : S^1 \to SO(n+1)$. According to the classification theory of Pontrjagin, it follows that $[S^1, \bar{g}]$ generates $\Omega_1(\text{point}, \text{trivial})$ if and only if the homotopy class of h is zero.

Returning to our closed oriented $(4q+1)$-manifold M we recall that one can choose a nowhere vanishing vector field V on M by the Hopf index theorem. Then the tangent bundle TM of M is splitted into $TM = 1_V \oplus E$, where $E \to M$ is a $4q$-dimensional oriented vector bundle. As usual M is embedded into the total space E as zero section. Then the map $X : M \to E$ is transversal to the subset M in E by the assumption. Denote by F the set of zeros of X. It is well known that the normal bundle $V(F, M)$ of F in M is isomorphic to the restriction of E on F via the differential $\mathrm{d}X$. Hence we have a bundle isomorphism:

$$\bar{g} : TF \oplus E|_F \to TF \oplus V(F, M) \to TM|_F \to 1 \oplus E|_F.$$

These data give rise to the well defined invariant (cf. ref. [3], (12.5))

$$\chi''(M, V) = [F, \bar{g}] \in \Omega_1(\text{point}, \text{trivial}) \cong Z_2.$$

Note that the closed manifold F is of dimension one, thus

$$F = F_1 \cup \cdots \cup F_p,$$

where the union is disjoint and every $F_j (j = 1, 2, \cdots, p)$ is a circle S^1.

For any $x \in F$, let $e_0 = V, e_1, \cdots, e_{4q}$ be again the oriented orthonormal basis near x as before. By the transversality of X, for every point x in F, one has a matrix $C = (c_{ij})_{4q \times 4q}$ in $GL(4q; R)$ given by

$$\mathrm{d}X(e_1, \cdots, e_{4q}) = (e_1, \cdots, e_{4q})(c_{ij}).$$

Note that this cannot define a map from F to $GL(4q; R)$ since $C = (c_{ij})$ depends on the choice of the basis $e_0 = V, e_1, \cdots, e_{4q}$. However we can get a well-defined element of $[F, GL(4q; R)]$ which is the set of homotopy classes of the mappings. By a homotopy equivalence $GL(4q;R) \simeq O(4q)$ and a construction $\tilde{C} = \mathrm{diag}(\det(C), C)$ we get finally an element $\tilde{C}$ in $[F, SO(4q+1)]$. Put $\tilde{C}_j = \tilde{C}|_{F_j}$ and observe that $\pi_1 SO(4q+1) \cong Z_2$ since $q \geqslant 1$. The arguments above have established the following

Lemma 2.1. $\chi''(M,V) = \#\{j|\tilde{C}_j = 0 \text{ in } \pi_1 SO(4q+1)\} \bmod 2$.

On the other hand, it turns out that the invariant $\chi''(M,V)$ is independent of the choice of V and is equal to the Kervaire semi-characteristic of the manifold.

Lemma 2.2 (ref. [3], (15.16)). $\chi''(M,V) \equiv k(M) \bmod 2$ if $\dim M \equiv 1 \bmod 4$.

In order to complete a topological proof of Zhang's formula, it suffices to show the following criterion.

Lemma 2.3. $\tilde{C}_j$ is zero if and only if the line bundle L_j over S^1 is orientable or trivial.

Proof. The line bundle L_j over S^1 is constructed by means of the element $\tilde{C}_j$ in $[F_j, SO(4q+1)]$. Thus it remains to check a special example. Define $C: S^1 = \{e^{i\theta}\} \to SO(2)$ by

$$C(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Composing by an inclusion $SO(2) \to SO(4q) \to SO(4q+1)$, we have a map $S^1 \to SO(4q+1)$ which will be denoted by $\tilde{C}$. Clearly the homotopy class $[\tilde{C}]$ is a generator of the homotopy group $\pi_1 SO(4q+1) \cong Z_2$. It is straightforward to verify that the associated line bundle L over S^1 has no nowhere vanishing section, hence being the Hopf line bundle which is nonorientable. It completes the proof.

Now, the combination of Lemmas 2.1, 2.2, with 2.3 will provide a topological proof of Zhang's formula.

We conclude with one remark. It is interesting to note that the line bundle L_j constructed by Zhang[5] is in fact isomorphic to the pull back of the associated line bundle of the 2-fold covering $Spin(4q) \to SO(4q)$ by the map $C_j : S^1 \to O(4q)$, where a homeomorphism between two components of $O(4q)$ is used if necessary.

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