TANIYAMA, K. and YASUHARA, A.
KOBE J. MATH.,
11 (1994) 117—127

ON C-DISTANCE OF KNOTS

By Kouki TANIYAMA and Akira YASUHARA

(Received May 25, 1993)
(Revised September 22, 1993)

Abstract

In this paper we define a distance on knots in terms of surfaces in S^3. This distance is related to the genera of knots, unknotting numbers and surfaces in 4-space spanning the knots.

1. Introduction

Throughout this paper we work in the piecewise linear category. We consider oriented knots in S^3 up to ambient isotopy.

Let g be a non-negative integer. Two knots k_1 and k_2 are said to be *g-cobordant* if there is a genus g compact connected oriented surface F in S^3 such that ∂F has two components, one is ambient isotopic to k_1 and the other is ambient isotopic to $-k_2$ where $-k_2$ denotes the knot k_2 with reversed orientation. Then F is called a *g-cobordism* between k_1 and k_2. The *C-distance* of k_1 and k_2 is defined by

$$d_C(k_1, k_2) = \min\{g \mid k_1 \text{ is } g\text{-cobordant to } k_2\}.$$

In other words, $d_C(k_1, k_2)$ is the minimum among the genera of 2-component links whose components are ambient isotopic to k_1 and $-k_2$ respectively.

An $SH(3)$-move is a local change defined in [2] as illustrated in Figure 1.1.

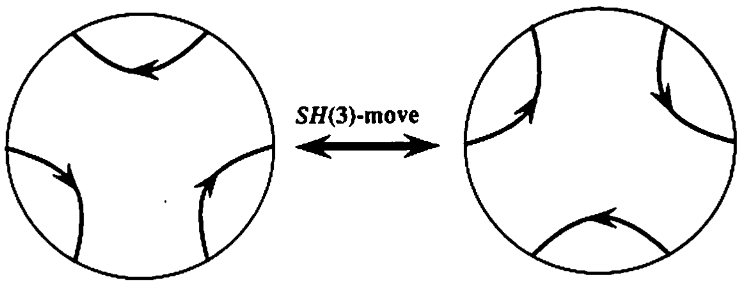

Fig. 1.1

It is known in [2] that all knots are transformed into each other by a finite sequence of $SH(3)$-moves. Let $d_3(k_1, k_2)$ be the minimum number of $SH(3)$-moves that is needed to transform k_1 into k_2.

A properly and locally flatly embedded compact connected oriented surface E in $S^3 \times [0,1]$ is called a *g-concordance* between k_1 and k_2 if the genus of E is g and

$$\partial E = k_1 \times \{0\} \cup (-k_2) \times \{1\}.$$

Then we say that k_1 is *g-concordant* to k_2. Let

$$d_4(k_1, k_2) = \min\{g \mid k_1 \text{ is } g\text{-concordant to } k_2\}.$$

We call $d_4(k_1, k_2)$ the *4-distance* of k_1 and k_2. A concordance E is called *regular* if the restriction of the natural projection

$$\pi : S^3 \times [0,1] \to [0,1]$$

is a Morse function on E. Then the restriction $\pi|_E$ has only finitely many maximal points, minimal points and saddle points. Let $c(E)$ be the number of these critical points. Then $c(E)$ is an even number. Let

$$c(k_1, k_2) = \frac{1}{2} \min\{c(E) \mid E \text{ is a regular concordance between } k_1 \text{ and } k_2\}.$$

The following result is central in this paper.

THEOREM 1.1. *For any knots k_1 and k_2,*

$$d_C(k_1, k_2) = d_3(k_1, k_2) = c(k_1, k_2).$$

Thus C-distance is in fact a distance on knots because d_3 clearly satisfies the axioms of the distance. The *Gordian distance* $d_G(k_1, k_2)$ is the minimum number of crossing changes that is needed to transform k_1 into k_2 [4]. Let $\sigma(k)$ be the signature of k. Let $e(k)$ be the minimum number of generators of $H_1(\widetilde{X})$ as a $Z[t, t^{-1}]$-module where $\widetilde{X}$ is the infinite cyclic covering space of $S^3 - k$. Then we have the following theorem.

THEOREM 1.2. *For any knots k_1 and k_2,*

1. $\dfrac{|\sigma(k_1) - \sigma(k_2)|}{2} \leq d_4(k_1, k_2) \leq d_C(k_1, k_2) \leq d_G(k_1, k_2).$
2. $\dfrac{|e(k_1) - e(k_2)|}{2} \leq d_C(k_1, k_2).$

We denote the trivial knot by o. The *C-genus* $g_C(k)$ is defined by $g_C(k) = d_C(k, o)$. Then it is clear that $g_C(k) \leq g(k)$ where $g(k)$ is the genus of k. In [2] an integer $sh(k)$ is defined for a knot k. It is not explicitly mentioned in [2] but follows by the arguments in [2] and this paper that $sh(k) = 2su_3(k) + 1$ where $su_3(k) = d_3(k, o) = d_C(k, o) = g_C(k)$ in our notation. Therefore C-genus gives a geometric interpretation of $sh(k)$.

We will observe the behavior of C-distance and C-genus under the connected sum and band sum in Section 4. An unoriented version of C-distance is defined and argued in Section 5. Various examples of knots are shown in each section.

2. Proofs of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1 it is sufficient to prove the following three assertions

(1) If $d_C(k_1, k_2) \leq n$, then $d_3(k_1, k_2) \leq n$.
(2) If $d_3(k_1, k_2) \leq n$, then $c(k_1, k_2) \leq n$.
(3) If $c(k_1, k_2) \leq n$, then $d_C(k_1, k_2) \leq n$.

Proof of (1). Let F be an n-cobordism between k_1 and k_2. Since F is a twice punctured genus n connected oriented surface, F is abstractly homeomorphic to a surface obtained from a disk by attaching $2n+1$ bands as illustrated in Figure 2.1. Therefore by an ambient isotopy of S^3, we can deform F as illustrated in Figure 2.2. By an $SH(3)$-move we can cut a pair of bands as illustrated in Figure 2.3. Thus after n-time $SH(3)$-moves k_1 is transformed into a knot that bounds an annulus with $-k_2$. Therefore the knot is ambient isotopic to k_2. □

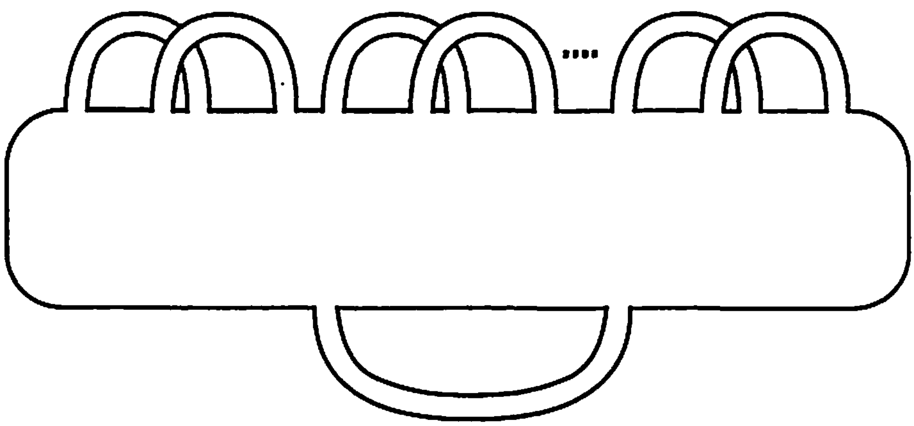

Fig. 2.1

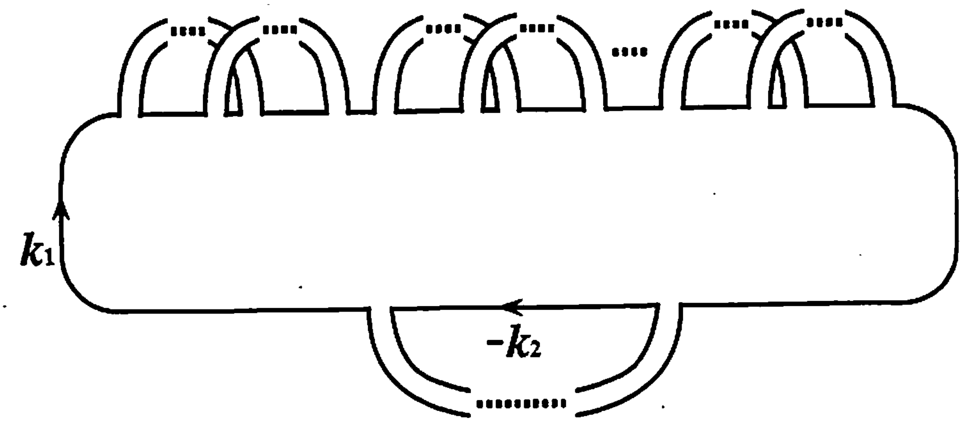

Fig. 2.2

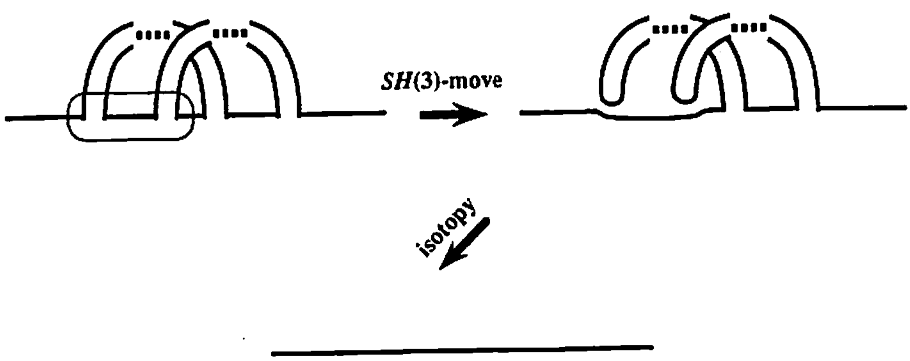

Fig. 2.3

PROOF OF (2). An $SH(3)$-move is realized by twice hyperbolic transformation as illustrated in Figure 2.4. This shows the conclusion. □

PROOF OF (3). Let E be a regular concordance between k_1 and k_2 with $c(E) = n$. By the surgery as illustrated in Figure 2.5, we may suppose that all critical points of E are saddle points. We regard each saddle point as a saddle band in the sence of [3]. We can deform E so that each saddle band lies in $S^3 \times \{1/2\}$ (cf. [3]). Then we have that k_2 is obtained from k_1 by adding $2n$ coherent bands. Let k_1' be a parallel of k_1 and A an annulus bounded by k_1 and $-k_1'$. If we attach the $2n$ bands to k_1' then the union of A and the $2n$ bands yields a required n-cobordism between k_1 and k_2. □

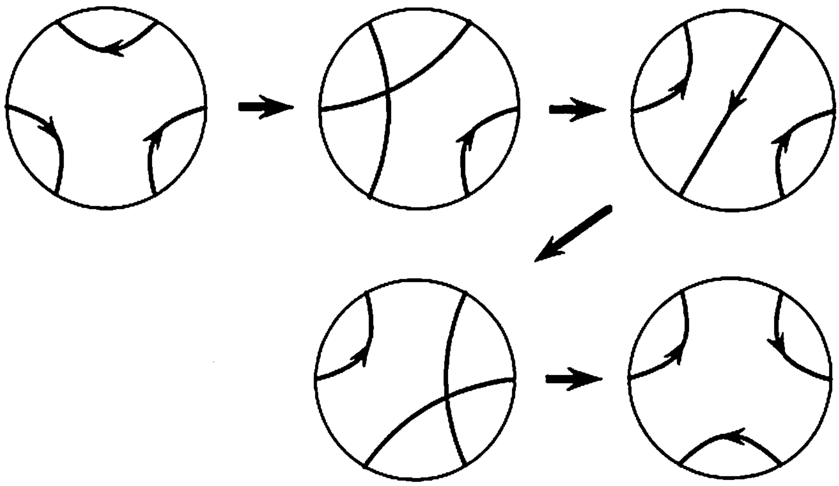

Fig. 2.4

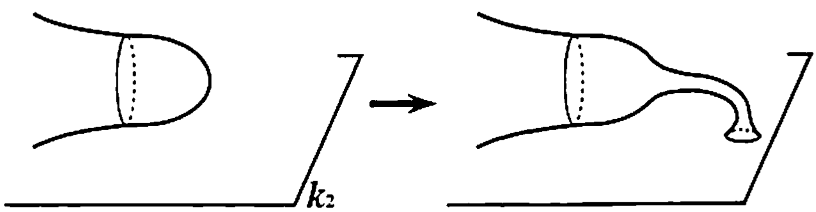

Fig. 2.5

PROPOSITION 2.1. *A local change as illustrated in* Figure 2.6 *is achieved by an* $SH(3)$*-move.*

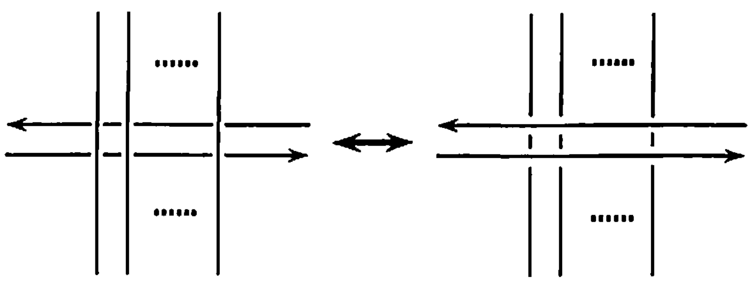

Fig. 2.6

PROOF. The proof is indicated in Figure 2.7. □

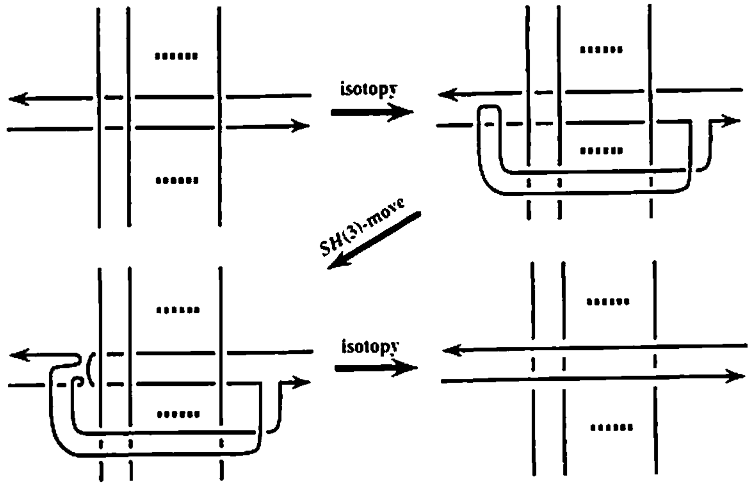

Fig. 2.7

PROOF of THEOREM 1.2. 1. The first inequality is known in [6]. By pushing the cobordism in S^3 into the concordance in $S^3 \times [0,1]$ we have the second inequality. The final inequality follows the fact that a crossing change is realized by an $SH(3)$-move [2]. See also Figure 2.8 and Proposition 2.1. □

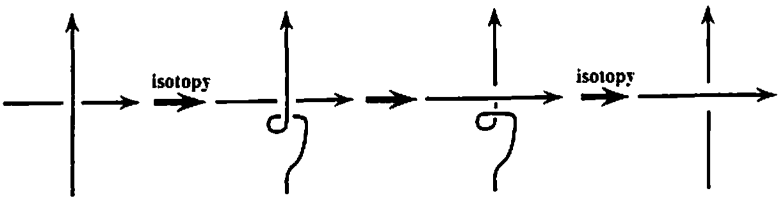

Fig. 2.8

2. We note that $|e(k_1)-e(k_2)|/2 \leq c(k_1,k_2)$ is known in [8]. Since $c(k_1,k_2) = d_C(k_1,k_2)$ we have the conclusion. □

3. C-genus

We recall that the C-genus $g_C(k)$ is defined by $d_C(k,o)$ where o is the trivial knot. Similarly let $g_4(k) = d_4(k,o)$ be the 4-genus of k. The unknotting number $u(k)$ is equal to $d_G(k,o)$. Then by Theorem 1.2 we have the following theorem.

THEOREM 3.1. *For any knot k.*

1. $\frac{|\sigma(k)|}{2} \leq g_4(k) \leq g_C(k) \leq g(k)$.
2. $\frac{e(k)}{2} \leq g_C(k) \leq u(k)$.

EXAMPLE 3.2. (1) Let k be a nontrivial ribbon knot with 1-fusion. Then

$$g_C(k) = c(k, o) = 1.$$

In particular if k is a 2-bridge knot with $e(k) = 1$, then

$$1 = \frac{e(k\#(-k!))}{2} = g_C(k\#(-k!)),$$

where $-k!$ is the reflected inverse of k and $\#$ means the connected sum of knots.

(2) Let k be the $(2, 2n+1)$-torus knot with $n \geq 1$. Then

$$\frac{|\sigma(k)|}{2} = g_4(k) = g_C(k) = g(k) = u(k) = n, \quad \text{and} \quad e(k) = 1.$$

(3) Let $k = \#_n(3_1\#(-3_1!))$ where 3_1 is the right handed trefoil knot. Then k is a slice knot, i.e., $g_4(k) = 0$. It is easy to see that

$$\frac{e(k)}{2} = g_C(k) = n.$$

Since the genus is additive under the connected sum we have $g(k) = 2n$. Since $e(k) \leq u(k)$ (see [7]) and $e(k) = 2n$ we have $u(k) = 2n$.

The above examples show that the inequalities in Theorem 3.1 are both best possible and possibly have arbitrarily large gaps.

4. Behavior of C-distance and C-genus under the connected sum and band sum

PROPOSITION 4.1. *For any knots k_1 k_2, k_3 and k_4,*

$$d_C(k_1\#k_2, k_3\#k_4) \leq d_C(k_1, k_3) + d_C(k_2, k_4).$$

PROOF. Since the C-distance is a distance on knots we have

$$d_C(k_1\#k_2, k_3\#k_4) \leq d_C(k_1\#k_2, k_3\#k_2) + d_C(k_3\#k_2, k_3\#k_4).$$

It is easy to see that

$$d_C(k_1\#k_2, k_3\#k_2) \leq d_C(k_1, k_3) \quad \text{and} \quad d_C(k_3\#k_2, k_3\#k_4) \leq d_C(k_2, k_4).$$

Therefore we have the conclusion. □

COROLLARY 4.2. *For any knots k_1 and k_2,*

$$g_C(k_1\#k_2) \leq g_C(k_1) + g_C(k_2).$$

The following example implies that the above inequality is best possible and has an arbitrarily large gap.

EXAMPLE 4.3. (1) Let k_1 and k_2 be the $(2, 2n+1)$-torus knot. Then

$$g_C(k_1\#k_2) = g_C(k_1) + g_C(k_2) = 2n.$$

(2) Let k_1 be the $(2, 2n+1)$-torus knot and $k_2 = -k_1!$. Then

$$1 = g_C(k_1\#k_2) < g_C(k_1) + g_C(k_2) = 2n.$$

Let $k_1\#_b k_2$ denote a band sum of k_1 and k_2.

PROPOSITION 4.4. *For any knots k_1, k_2, k_3 and k_4,*

$$d_C(k_1\#_b k_2, k_3\#_b k_4) \leq d_C(k_1, k_3) + d_C(k_2, k_4) + 1.$$

PROOF. We recall that

$$d_C(k_1\#_b k_2, k_3\#_b k_4) = c(k_1\#_b k_2, k_3\#_b k_4).$$

Let E_i $(i = 1, 2)$ be a regular concordance of k_i and k_{i+2} realizing $c(k_i, k_{i+2})$. Attach two bands b_1 and b_2 to E_1 and E_2 so that

$$\partial(E_1 \cup E_2 \cup b_1 \cup b_2) = (k_1\#_b k_2) \times \{0\} \cup (-(k_3\#_b k_4)) \times \{1\}.$$

Thus we have

$$c(k_1\#_b k_2, k_3\#_b k_4) \leq c(k_1, k_3) + c(k_2, k_4) + 1.$$

This completes the proof. □

COROLLARY 4.5. *For any knots k_1 and k_2,*

$$g_C(k_1\#_b k_2) \le g_C(k_1) + g_C(k_2) + 1.$$

5. Unoriented version

In this section we consider unoriented knots in S^3. Two knots k_1 and k_2 are said to be *g-bordant* if there is a compact connected (possibly nonorientable) surface F in S^3 with the first Betti number $\beta_1(F) = g+1$ such that ∂F has two components, one is ambient isotopic to k_1 and the other is ambient isotopic to k_2. Let

$$\tilde{d}_C(k_1, k_2) = \min\{g \mid k_1 \text{ is } g\text{-bordant to } k_2\}.$$

It follows the definitions that

$$\tilde{d}_C(k_1, k_2) \le 2d_C(k_1, k_2).$$

An $H(2)$-move is local change defined in [2] as illustrated in Figure 5.1.

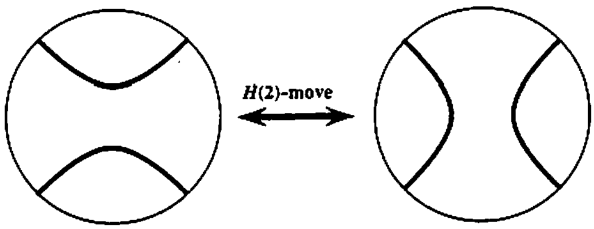

Fig. 5.1

Let $d_2(k_1, k_2)$ be the minimum number of $H(2)$-moves that is needed to transform k_1 into k_2 where we allow $H(2)$-moves to change the number of components. We remark here that only $H(2)$-moves that preserve the number of components were considered in [2]. Let $\tilde{c}(k_1, k_2)$ be the minimum number of critical points of a locally flat (possibly nonorientable) surface in $S^3 \times [0,1]$ bounded by $k_1 \times \{0\}$ and $k_2 \times \{1\}$. Then we have the following result.

THEOREM 5.1. *For any knots k_1 and k_2,*

$$\tilde{d}_C(k_1, k_2) = d_2(k_1, k_2) = \tilde{c}(k_1, k_2).$$

The proof is similar to that of Theorem 1.1 and we omit it.

Let $e_p(k)$ be the minimum number of generators of $H_1(X_p)$ where X_p is the p-fold cyclic branched covering space of (S^3, k).

PROPOSITION 5.2. *For any knots k_1 and k_2 and an integer $p \geq 2$.*

$$\frac{|e_p(k_1) - e_p(k_2)|}{p-1} \leq \tilde{d}_C(k_1, k_2).$$

PROOF. By the argument similar to [2, Proof of Theorem 4] we have

$$\frac{|e_p(k_1) - e_p(k_2)|}{p-1} \leq d_2(k_1, k_2).$$

Since $\tilde{d}_C(k_1, k_2) = d_2(k_1, k_2)$ we have the desired result. □

Let $\tilde{g}_C(k) = \tilde{d}_C(k, o)$. In [2] an integer $h(k)$ is defined for a knot k. As in the oriented case we can see easily that $h(k) = 2\tilde{g}_C(k)+1$, so $\tilde{g}_C(k)$ gives a geometric interpretation of $h(k)$. Let $\tilde{g}(k)$ be the *crosscap number* of k defined in [1], [5]. Note that $\tilde{g}_C(k) \leq \tilde{g}(k)$. Let $cr(k)$ be the minimum crossing number of k. Then we have

PROPOSITION 5.3. *For any knot k,*

$$\frac{e_p(k)}{p-1} \leq \tilde{g}_C(k) \leq \tilde{g}(k) \leq \left[\frac{cr(k)}{2}\right],$$

where $[x]$ is the maximum integer that is not greater than x.

PROOF. The first inequality is induced by Proposition 5.2. The final inequality is known in [5]. □

Acknowledgment. The authors would like to thank Professor Shin'ichi Suzuki for his encouragement.

Note added in proof. We note that for any two knots k_1 and k_2 with $k_1 \cap k_2 = \emptyset$, there is a properly embedded twice punctured torus F in a 4-disk such that one of the components of ∂F is k_1 and the other is $-k_2$.

References

[1] B.E. Clark: Crosscaps and knots, Intern. J. Math. & Math. Sci., 1 (1978), 113–123.
[2] J. Hoste, Y. Nakanishi and K. Taniyama: Unknotting operations involving trivial tangles, Osaka J. Math., 27 (1990), 555–566.
[3] A. Kawauchi, T. Shibuya and S. Suzuki: Descriptions on surfaces in four-space, I. Normal forms, Math. Sem. Notes, Kobe Univ., 10 (1982), 75–125.
[4] H. Murakami: Some metrics on classical knot, Math. Ann., 270 (1985), 35–45.
[5] H. Murakami and A. Yasuhara: Crosscap number of a knot, to appear in Pacific J. Math.

[6] K. Murasugi: On a certain numerical invariant of link types. Trans. Amer. Math. Soc., 117 (1965), 387–422.
[7] Y. Nakanishi: A note on unknotting number, Math. Sem. Notes, Kobe Univ., 9 (1981), 99–108.
[8] Y. Nakanishi: A remark on critical points of link cobordism. Kobe J. Math., 3 (1987), 209–212.

Kouki Taniyama
Department of Mathematics
College of Arts and Sciences
Tokyo Woman's Christian University
Zempukuji, Suginami-ku
Tokyo 167, Japan

Akira Yasuhara
Department of Mathematical Sciences
College of Science and Engineering
Tokyo Denki University
Hatoyamamachi, Hiki-gun
Saitama-ken 350-03, Japan