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Surgery on Paracompact Manifolds

by

Laurence R. Taylor

Abstract

In this thesis we solve the problem of surgery for an arbitrary, finite dimensional, paracompact manifold. The problem of surgery is to decide whether, given a proper map $f : M \to X$, a bundle (vector, PL-micro, or TOP-micro), ν, over X, and a stable bundle map $F : \nu_M \to \nu$ over f (ν_M is the normal bundle of M, so we must assume M is respectively a differentiable, a PL, or a topological, finite dimensional, paracompact manifold), we can find a cobordism W with $\partial W = M \cup N$, a proper map $g: W \to X$ with $g|M = f$, a stable bundle map $G: \nu_W \to \nu$ with $G|\nu_M = F$, such that $g|N$ is a proper homotopy equivalence.

If this problem can be solved, we show this forces conditions on X, ν, and f. In particular, X must be a Poincaré duality space (Chapter 2), ν must lift the Spivak normal fibration of X, and f must be degree 1.

If X, ν, and f satisfy these conditions, there is a well-defined obstruction to solving this problem if m, the dimension of M, is at least five (Theorem 3.2.1). This obstruction lies in a naturally defined group, $L_m(X,w)$, and every element of this group can be realized, in a specific fashion, as the obstruction to a surgery

problem, provided $m \geq 6$ (Theorem 3.2.4). $L_m(X,w)$ depends only on the system of fundamental groups of X (Theorem 3.2.3).

Finally, we have applications for paracompact manifolds along the same lines as the compact case. Perhaps the most interesting of these is the theoretical solution of the related questions of when a Poincaré duality space has the proper homotopy type of a paracompact manifold, and if a proper homotopy equivalence between paracompact manifolds can be properly deformed to a homeomorphism, diffeomorphism, or PL-equivalence (Theorem 3.2.4).

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INTRODUCTION

The object of this work is to give an adequate theory of surgery for paracompact manifolds and proper maps. By adequate we mean first that it should contain the theory of surgery for compact manifolds. Secondly, the theory should be general enough to permit extensions of the theoretical results of compact surgery.

These objectives are largely realized. We obtain surgery groups which characterize the problem in dimensions greater than or equal to five. These groups depend only on the proper 2-type of the problem. Using these groups one can classify all paracompact manifolds of a given proper, simple homotopy type (see [33] or [10] for a definition of simple homotopy type).

The first chapter constitutes the chief technical results of this work. In [33], Siebenmann gives a "geometric" characterization of proper homotopy equivalence (Proposition IV). This characterization was also discovered by Farrell-Wagoner [9] from whom I learned it.

In section 2 we develop an algebraic process to handle this characterization. In section 3 we apply this process to construct groups which are the analogue of the homotopy and homology groups. Thus we get actual groups measuring by how much a map fails to be a proper homotopy equivalence. These groups also satisfy a version of the Hurewicz and Namioka theorems, so one can often use the homology groups, which satisfy a version of excision, Mayer-Vietoris, etc.

In section 4 we construct a cohomology theory for our theory. We get various products for this theory. Section 5 is devoted to an analysis of simple homotopy type along the lines set out by Milnor in [23]. Section 6 is devoted to constructing locally compact CW complexes with a given chain complex (see Wall [38] for a treatment of the compact case of this problem).

Chapter 2 is devoted to an analysis of Poincare duality for paracompact manifolds and its generalization to arbitrary locally compact, finite dimensional CW complexes.

In Chapter 3 the actual surgeries are performed. It has been observed by several people (especially Quinn [29] and [30]) that all the surgery one needs to be able to do is the surgery for a pair $(X,\partial X)$ for which $\partial X \subseteq X$ is a proper 1-equivalence (in the compact case this means the inclusion induces isomorphisms on components and on π_1). We do this in the first section. In the second section, we sketch the general set up and applications of the theory of paracompact surgery.

A word or two is in order here about internal referencing. A reference reads from right to left, so that Corollary 3.4.1.5 is the fifth corollary to the first theorem of section 4 in chapter 3. If the reference is made from chapter 3 it would be Corollary 4.1.5, and if from section 4, Corollary 1.5. Theorem (Proposition, Lemma) 3.4.6 is the sixth theorem of section 4 of chapter 3.

Perhaps we should also remark that our use of the term n-ad agrees with the use of the term in Wall [14] (see especially Chapter 0). For an n-ad, K, $\partial_i K$ denotes the (n-1)-ad whose total space is the i^{th} face of K and with the (n-1)-ad structure induced by intersecting the other faces of K. $\delta_i K$ is the (n-1)-ad obtained by deleting the i^{th} face. $s_n K$ denotes the (n+1)-ad obtained by making K the $(n+1)$-st face (it can also be regarded as the (n+1)-ad, $K \times I$, where I has the usual pair structure).

Lastly, several acknowledgements are in order. This thesis was written under the direction of J. Wagoner, to whom I am indebted for many suggestions during the preparation of this work. I am greatly indebted to him and to T. Farrell for sharing their results and intuition on proper homotopy with me at the very beginning. Thanks are also due to G. Cooke for many helpful discussions. Many other friends likewise deserve thanks for their help. The National Science Foundation should also be thanked for its support during my graduate career.

Chapter I

The Proper Homotopy Category and Its Functors

Section 1. Introduction, elementary results, and homogamous spaces.

The purpose of this chapter is to recall for the reader some of the basic results we will need and to describe a "good" category in which to do proper homotopy theory.

The notion of a proper map is clearly essential. We define a map to be proper iff the inverse image of every closed compact set is contained in a closed compact set. We note that this definition is also found in Bredon [2], page 56.

With this definition of a proper map we immediately have the notions of proper homotopy, proper homotopy equivalence, etc., and we can define the category of all topological spaces and proper maps. Classically there are several functors which apply to this situation. As examples we have sheaf cohomology with compact supports and Borel-Moore homology with closed supports (see Bredon [2]).

We prefer to use singular theory whenever possible. Here too we have cohomology with compact supports and homology with locally finite chains. Most of the results concerning such groups are scattered (or non-existent) in the literature. As a partial remedy for this situation we will write out the definitions of these groups and at least indicate the results we need.

<u>Definition</u>: A collection of subsets of X is said to be locally finite if every closed, compact subset of X intersects only finitely many elements of this collection.

<u>Definition</u>: $S_q^{\ell.f.}(X;\Gamma)$, where Γ is a local system of R-modules on X (see Spanier [35] pages 58; 281-283), is defined to be the R-module which is the set of all formal sums $\Sigma \alpha_\sigma \sigma$, where σ is a singular q-simplex of X, and $\alpha_\sigma \in \Gamma(\sigma(V_0))$ is zero except for a set of σ whose images in X are locally finite.

$S^q(X:\Gamma)$ is the module of functions φ assigning to every singular q-simplex σ of X an element $\varphi(\sigma) \in \Gamma(\sigma(V_0))$.

For a family of supports ψ on X (see Bredon [2] page 15 for a definition) let $S_q^\psi(X;\Gamma)$ denote the submodule of $S_q^{\ell.f.}(X;\Gamma)$ such that the union of all the images of the σ occurring with non-zero coefficient in a chain lies in some element of ψ. $S_\psi^q(X;\Gamma)$ consists of the submodule of all function φ for which there exists an element $c \in \psi$ such that if Image $\sigma \cap c = \emptyset$, $\varphi(\sigma) = 0$.

These modules become chain complexes in the usual fashion. Note that for the family of compact supports, c,

$S_q^c(X;\Gamma)$ is just the ordinary singular chains with local coefficients.

For a proper subspace $A \subseteq X$ (inclusion is a proper map) we get relative chain groups $S_q^{\psi}(X,A;\Gamma)$ and $S_{\psi}^q(X,A;\Gamma)$. Actually proper subspace is sometimes stronger than we need; i.e. $S_q^c(X,A;\Gamma)$ and $S^q(X,A;\Gamma)$ are defined for any $A \subseteq X$. There is a similar definition for the chain groups of a (proper) n-ad.

The homology of $S_*^{\psi}(X,A;\Gamma)$ will be denoted $H_*^{\psi}(X,A;\Gamma)$ except when $\psi = c$ when we just write $H_*(X,A;\Gamma)$. The homology of $S_{\psi}^*(X,A;\Gamma)$ will be written $H_{\psi}^*(X,A;\Gamma)$.

Now $S_c^q(X,A;\Gamma) \subseteq S^q(X,A;\Gamma)$. The quotient complex will be denote $S_{end}^q(X,A;\Gamma)$ and its homology $H_{end}^q(X,A;\Gamma)$. We have similar definitions for proper n-ads and also for homology.

We will next set out the properties of these groups we will use. Some of the obvious properties such as naturality and long exact sequences will be omitted.

Cup products: There is a natural cup product

$$H_{\psi}^q(X:A_1,\ldots,A_n;\Gamma_1) \otimes H^k(X:A_{n+1},\ldots,A_m;\Gamma_2) \xrightarrow{\cup}$$

$H_{\psi}^{q+k}(X:A_1,\ldots,A_n,\ldots,A_m;\Gamma_1 \otimes \Gamma_2)$ for a proper $(m+1)$-ad $(X:A_1,\ldots,A_m)$. It is associative and commutative in the graded sense (i.e. $a \cdot b = (-1)^{\deg a \cdot \deg b}\, b \cdot a$).

Since $S_{\psi}^q \subseteq S^q$, all this follows easily from the ordinary cup product with local coefficients once one checks that if a cochain was supported in $c \in \psi$, then its product with any other cochain is supported in c if one uses the Alexander-Whitney diagonal approximation (Spanier [35] page 250).

Cross products: There are natural products

$$H^q_\psi(X:A_1,\ldots,A_n;\Gamma_1) \otimes H^k(Y:B_1,\ldots,B_m;\Gamma_2) \xrightarrow{\times}$$

$$H^{q+k}_{\pi_1^{-1}\psi}(X\times Y: X\times B_1,\ldots,X\times B_m,A_1\times Y,\ldots,A_n\times Y;\ \Gamma_1\times\Gamma_2) \text{ and}$$

$$H^\psi_q(X:A_1,\ldots,A_n;\Gamma_1) \otimes H^{\ell.f.}_k(Y:B_1,\ldots,B_m;\Gamma_2) \xrightarrow{\times}$$

$$H^{\psi\times Y}_{q+k}(X\times Y: X\times B_1,\ldots,X\times B_m,A_1\times Y,\ldots,A_n\times Y;\ \Gamma_1\times\Gamma_2)\ ,$$

where $\pi_1^{-1}(\psi) = \{K \subseteq X\times Y | \pi_1(K) \in \psi\}$ and $\psi\times Y = \{K\times Y \subseteq X\times Y | K \in \psi\}$. These satisfy the usual properties of the cross product.

We discuss this case in some detail. Let us first define $\tau : S_n^{\ell.f.}(X\times Y) \to \sum_{i+j=n} S_i^c(X) \hat{\otimes} S_j^c(Y)$, where $\hat{\otimes}$ is the completed tensor product, i.e. infinite sums are allowed. If $\sigma : \Delta^n \to X\times Y$, and if π_1 and π_2 are the projections, $\tau(\sigma) = \sum_{i+j=n} {}_i(\pi_1\sigma) \otimes (\pi_2\sigma)_j$, where ${}_i(\ )$ is the front i-face and $(\ )_j$ is the back j-face (see Spanier [35] page 250). This extends over all of $S_n^{\ell.f.}$ and is a natural chain map.

The cohomology cross product is then defined on the chain level by $(c\times d)(\sigma) = c({}_i(\pi_1\sigma)) \otimes d((\pi_2\sigma)_j)$, where c is an i-cochain, d a j-cochain, and σ an (i+j)-chain. One checks it has the usual properties.

We next define $\lambda : S_i^{\ell.f.}(X) \otimes S_j^{\ell.f.}(Y) \to S_{i+j}^{\ell.f.}(X\times Y)$

as follows. Let $h_{i,j} : \Delta^{i+j} \to \Delta^i \times \Delta^j$ be a homeomorphism such that ${}_i(h_{i,j}) : \Delta^i \to \Delta^i \times \Delta^j$ by $x \to (x,0)$ and such that $(h_{i,j})_j : \Delta^j \to \Delta^i \times \Delta^j$ by $y \to (0,y)$. Define $\lambda(\sigma_X \otimes \sigma_Y) = h_{ij} \circ (\sigma_X \times \sigma_Y)$ and extend "linearly"; i.e. $\lambda(\Sigma\, \alpha\, \sigma_\alpha \otimes \Sigma\, \beta\, \sigma_\beta) = \sum_{\alpha,\beta} \alpha \otimes \beta \cdot \lambda(\sigma_\alpha \otimes \sigma_\beta)$. λ then becomes a chain map, and the homology cross product is then defined on the chain level as above. It has the usual properties.

Slant product: There are natural products

$$H_c^q(Y:B_1,\ldots,B_m;\Gamma_1) \otimes H_{q+k}^{\ell.f.}(X \times Y: A_1 \times Y,\ldots,A_n \times Y,$$

$$X \times B_1,\ldots,X \times B_m;\Gamma_2 \times \Gamma_1) \xrightarrow{\;/\;} H_k^{\ell.f.}(X:A_1,\ldots,A_n;\Gamma_1 \otimes \Gamma_2)$$

and $H^q(Y:B_1,\ldots,B_m;\Gamma_1) \otimes H_{q+k}(X \times Y:A_1 \times Y,\ldots,A_n \times Y$,

$$X \times B_1,\ldots,X \times B_m;\ \Gamma_2 \times \Gamma_1) \xrightarrow{\;/\;} H_k(X:A_1,\ldots,A_n;\Gamma_1 \otimes \Gamma_2) .$$

The product is defined on the chain level by

$$c/\sigma = c/\Sigma\, \alpha\, \sigma_\alpha = \sum_\alpha \Big(\sum_{i+j=q+k} {}_i(\pi_1 \sigma_\alpha) \otimes c(\pi_2 \sigma)_j) \otimes \alpha \Big) ,$$

where c applied to a chain is zero if the dimensions do not agree. The slant product is natural on the chain level and has all the usual properties.

Cap product: There is a natural product

$$H_\psi^q(X:A_1,\ldots,A_n;\Gamma_1) \otimes H_{q+k}^\varphi(X:A_1,\ldots,A_n,B_1,\ldots,B_m;\Gamma_2) \xrightarrow{\;\cap\;}$$

$H_k^{\varphi\cap\psi}(X : B_1,\ldots,B_m;\Gamma_1 \otimes \Gamma_2)$. It is given by $u \cap z = u/d_* v$, where $d : X \to X \times X$ is the diagonal map. The cap product has all the usual properties. We get better support conditions for our cap product than we did for an arbitrary slant product because d_* of a chain in $X \times X$ is "locally finite" with respect to sets of the form $c \times X$ and $X \times c$ for any closed, compact $c \subseteq X$.

One of the most useful of the usual properties of the cap product is the

Browder Lemma: ([3], [4]). Let (X,A) be a proper pair (A is a proper subspace), and let $Z \in H_n^{\psi}(X,A;\Gamma_2)$. Then $\partial Z \in H_{n-1}^{\psi}(A;\Gamma_2|A)$ is defined. The following diagram commutes

$$\begin{array}{ccccccc}
H_{\varphi}^{*-1}(A;\Gamma_1|A) & \longrightarrow & H_{\varphi}^{*}(X,A;\Gamma_1) & \longrightarrow & H_{\varphi}^{*}(X;\Gamma_1) & \longrightarrow & H_{\varphi}^{*}(A;\Gamma_1|A) \\
\downarrow \cap(-1)^n \partial Z & & \downarrow \cap Z & & \downarrow \cap Z & & \downarrow \cap \partial Z \\
H_{n-*}^{\varphi\cap\psi}(A;(\Gamma_1 \otimes \Gamma_2)|A) & \to & H_{n-*}^{\varphi\cap\psi}(X;\Gamma_1 \otimes \Gamma_2) & \to & H_{n-*}^{\varphi\cap\psi}(X,A;\Gamma_1 \otimes \Gamma_2) & \to & H_{n-1-*}^{\varphi\cap\psi}(A;\Gamma_1|A).
\end{array}$$

In two cases, we also have a universal coefficient formula relating cohomology and homology. We first have the ordinary universal coefficient formula; namely
$0 \to \mathrm{Ext}(H_{*-1}(\ ,\Gamma),Z) \to H^*(\ ,\mathrm{Hom}(\Gamma,Z)) \to H_*(\ ,\Gamma),Z) \to 0$
is split exact (see Spanier [35], page 283).

We have a natural chain map
$\alpha : S_*^{\ell.f.}(\ ,\mathrm{Hom}(\Gamma,Z)) \longrightarrow \mathrm{Hom}(S_c^*(\ ,\Gamma), Z)$
given by $\alpha(c)(\varphi) = \varphi(c)$. If the space X is HCL Bredon [2], shows that α induces a homology isomorphism, so we get $0 \to \mathrm{Ext}(H_c^{*+1}(X,\Gamma),Z) \to H_*^{\ell.f.}(X,\mathrm{Hom}(\Gamma,Z))$ $\to \mathrm{Hom}(H_c^*(X,\Gamma),Z) \to 0$ is split exact.

Write $\bar{\Gamma}$ for $\mathrm{Hom}(\Gamma,Z)$. Then if $c \in H^k(\ ,\Gamma)$, the following diagram commutes

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{*-1}(\ ,Z),\ Z) & \longrightarrow & H^*(\ ,Z) & \longrightarrow & \mathrm{Hom}(H_*(\ ,Z),Z) & \to 0 \\ & \downarrow \mathrm{Ext}(\cap c) & & \downarrow c\cup & & \downarrow \mathrm{Hom}(\cap c) & \\ 0 \to & \mathrm{Ext}(H_{*+k-1}(\ ,\Gamma),Z) & \to & H^{*+k}(\ ,\bar{\Gamma}) & \to & \mathrm{Hom}(H_*(\ ,\Gamma),Z) & \to 0 \end{array}$$

If $c \in H_k^{\ell.f.}(\ ,\Gamma)$, and if the spaces in question are HCL, the following diagram commutes

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{k-*-1}(\ ,Z),Z) & \to & H^{k-*}(\ ,Z) & \to & \mathrm{Hom}(H_{k-*}(\ ,Z),Z) & \to 0 \\ & \downarrow \mathrm{Ext}(\cap c) & & \downarrow \cap c & & \downarrow \mathrm{Hom}(\cap c) & \\ 0 \to & \mathrm{Ext}(H_c^{*+1}(\ ,\Gamma),Z) & \to & H_*^{\ell.f.}(\ ,\Gamma) & \to & \mathrm{Hom}(H_c^*(\ ,\Gamma),Z) & \to 0\ . \end{array}$$

These formulas can actually be seen on the chain level by picking representatives and using the Alexander-Whitney diagonal approximation.

These homology and cohomology groups enjoy other pleasant properties. One which we shall exploit heavily throughout the remainder of this work is the existence of a transfer map for any arbitrary cover. For particulars, let $\pi : \tilde{X} \to X$ be a covering map. Then we have homomorphisms tr: $H_*^{\ell.f.}(X;\Gamma) \to H_*^{\ell.f.}(\tilde{X};\pi^*\Gamma)$ and

tr: $H_c^*(\tilde{X};\pi^*\Gamma) \to H_c^*(X;\Gamma)$. The first of these is given by defining $tr(\sigma)$ for a simplex σ and extending "linearly." $tr\,\sigma = \sum_{p\in\pi^{-1}(v_o)} \sigma_p$, where p runs over all the points in $\pi^{-1}(v_o)$ (v_o is a vertex of σ) and σ_p is σ lifted so that v_o goes to p . It is not hard to check tr is a chain map. For the cohomology trace, define $tr(c)$ as the cochain whose value on the simplex σ in X is $c(tr(\sigma))$; i.e. $(tr(c))(\sigma) = c(tr(\sigma))$. If $f : X \to Y$ is a proper map, and if $\pi : \tilde{Y} \to Y$ is a cover, then, for the cover $\tilde{X} \to X$ which is induced from π by f, $\tilde{f}_*(tr\ Z) = tr\ \tilde{f}_* Z$ and $tr(\tilde{f}^* c) = f^*(tr\ c)$. Warning: The trace tends to be highly unnatural except in this one situation.

As an easy exercise, one may check that if $c \in H^k_c(\tilde{X};\pi^*\Gamma_1)$ and if $Z \in H^{\ell.f.}_{q+k}(X;\Gamma_2)$, then, in $H_q(X;\Gamma_1 \otimes \Gamma_2)$, $\pi_*(c \cap tr\ Z) = tr\ c \cap Z$.

In the coming pages, we will want to study spherical fibrations and paracompact manifolds. For the former objects we have

<u>Thom Isomorphism Theorem</u>: Let ξ be a spherical fibration of dimension $(q-1)$ over B. Let $S(\xi)$ be its total space, and let $D(\xi)$ be the total space of the associated disc bundle. Then there is a class $U_\xi \in H^q(D(\xi),S(\xi);\ p^*_\xi(\Gamma_\xi))$ (where $p : D(\xi) \to B$ is the projection, and Γ_ξ is the local system on B given at $b \in B$ by $H^q(p^{-1}(b),\ p^{-1}(b) \cap S(\xi);Z))$ such that

$$\cup\, U_\xi\colon H^*_\varphi(B;\Gamma) \longrightarrow H^{*+q}_{p^{-1}(\varphi)}(D(\xi),S(\xi);\ p^*(\Gamma \otimes \Gamma_\xi))$$

is an isomorphism. One also has

$$U_\xi \cap\colon H_*(D(\xi),\ S(\xi);p^*(\Gamma)) \longrightarrow H_{*-q}(B;\ \Gamma_\xi \otimes \Gamma)$$

is an isomorphism.

Note that we have been (and will continue to be) a little sloppy. If $c \in H^*_\varphi(B;\Gamma)$, $c \cup U_\xi$ should actually be $p^*(c) \cup U_\xi$. A similar notational amalgamation has occurred when we write $U_\xi \cap$.

This theorem is proved by a spectral sequence argument (see [26]), so one need only check that we still have a Serre spectral sequence with the appropriate supports.

For a paracompact manifold (i.e. a locally Euclidean, paracompact, Hausdorff space), possibly with boundary, we have

<u>Lefschetz Duality</u>: ([20], [44]). If $M,\partial M)$ is a paracompact manifold pair of dimension n , there is a class $[M] \in H_n^{\ell.f.}(M,\partial M;\Gamma_M)$ (where Γ_M is the local system for the bundle ν , the normal bundle of M) such that the maps

$$\cap[M]\colon H^*_\psi(M,\partial M;\Gamma) \to H^\psi_{n-*}(M;\Gamma \otimes \Gamma_M)$$

and $\cap[M]\colon H^*_\psi(M;\Gamma) \to H^\psi_{n-*}(M,\partial M;\Gamma \otimes \Gamma_M)$ are isomorphisms.

This completes the first objective of this section, so we turn to the second. The functors above already give us much non-trivial information on the category of all spaces and proper maps, but they are insufficient even to determine if a map is a proper homotopy equivalence on the subcategory of locally compact, finite dimensional CW complexes, a category in which we are surely going to be interested. In fact, the next two sections will be concerned precisely with the problem of constructing functors which will determine whether a map is or is not a proper homotopy equivalence in this category.

If we restrict ourselves to finite complexes, the Whitehead Theorem ([43]) already provides the answer. Notice that to solve the problem, even for finite complexes, we are forced to consider homotopy, which means base points. In order to solve the problem for locally finite complexes, we are going to have to consider lots of base points simultaneously. The category of spaces we are about to define is about the largest in which we can place our points nicely. It is also closed under proper homotopy equivalence.

Definition: A set B of points of X is said to be a set of base points for X provided

a) every path component of X contains a point of B

b) given any closed, compact set $c \subseteq X$, there is a closed compact set D such that there is a point of B in every path component of $X - c$ which is not contained in D.

Definition: A set of base points, B, for a path connected space X is said to be irreducible if, for any set of base points C for X with $C \subseteq B$, the cardinality of C is equal to the cardinality of B.

A set of base points for any space X is said to be irreducible provided it is an irreducible set of base points for each path component of X .

Definition: Two locally finite sets of points are said to be equivalent (~) provided there is a 1-1 correspondence between the two sets which is given by a locally finite set of paths.

Definition: Consider the following two properties of a space X :

1) Every set of base points for X has an irreducible, locally finite subset.

2) Any two irreducible, locally finite sets of base points for X are equivalent.

A space X is said to be homogamous provided $X \times I$ satisfies 1) and 2).

Proposition 1: If X has the proper homotopy type of an homogamous space, then X has properties 1) and 2).

Proof: We first prove two lemmas.

Lemma 1: Let $f : X \to Y$ be a proper map which induces injections of $H^0(Y)$ into $H^0(X)$ and of $H^0_{end}(Y)$ into $H^0_{end}(X)$. Then, if $\{p\}$ is a set of base points for X , $\{f(p)\}$ is a set of base points for Y.

Proof: Since f induces an injection on H^0 , there is an f(p) in every path component of Y .

Now look at the path components of $Y - c$, where c is some closed, compact subset of Y . Let $\{W_\alpha\}$ be the set of path components of $Y-c$ such that $f^{-1}(W_\alpha)$ contains no point of $\{p\}$. Since $\{p\}$ is a set of base points for X, $\bigcup_\alpha f^{-1}(W_\alpha) \subseteq D$, where D is some closed, compact subset of X. Then $f(X-D) \cap W_\alpha = \emptyset$ for all α.

Define a cochain β by

$$\beta(q) = \begin{cases} 1 & q \in W_\alpha \\ 0 & q \notin W_\alpha \end{cases} .$$

Then $\delta\beta(\lambda) = \beta(\lambda(1)) - \beta(\lambda(0)) = 0$ if $\lambda \cap c = \emptyset$. Hence $\delta\beta = 0$ in $S^1_{end}(Y;Z)$. But since $f(X-D) \cap W_\alpha = \emptyset$, $f^*\beta = 0$ in $S^0_{end}(X;Z)$. Since f^* is an injection on H^0_{end}, $\beta = 0$ in $H^0_{end}(Y;Z)$. But this implies $\bigcup_\alpha W_\alpha$ is contained in some compact set. Q.E.D.

Lemma 2: Let f be a map properly homotopic to the identity. Let $\{p\}$ be a locally finite set of points. Then $\{f(p)\}$ is equivalent to a subset of $\{p\}$.

Proof: We have $F : X \times I \to X$ a proper map. The set $\{p \times I\}$ is clearly locally finite. Since F is proper, $\{F(p \times I)\}$ is easily seen to be locally finite. But $\{F(p \times I)\}$ provides an equivalence between $\{f(p)\}$ and some subset of $\{p\}$ (more than one p may go to a given $f(p)$). Q.E.D.

Now let X have the proper homotopy type of Y, an homogamous space. Hence we have proper maps $f\colon X \to Y$ and $g\colon Y \to X$ with the usual properties.

Let $\{p\}$ be a set of base points for X. Then by Lemma 1, $\{f(p)\}$ is a set for Y, and $\{f(p) \times 0\}$ is a set for $Y \times I$. Since Y is homogamous, there is an irreducible, locally finite subset $\{f(p') \times 0\}$. By Lemma 1, $\{g \circ f(p')\}$ is a locally finite set of base points for X. But by Lemma 2, there is a further refinement, $\{p''\}$, of $\{p\}$ such that $\{p''\} \sim \{g \circ f(p')\}$. But then $\{p''\}$ is easily seen to be a set of base points also. Now $\{p''\}$ is in 1-1 correspondence with $\{f(p'')\}$, and $\{f(p'') \times 0\}$ is a set of base points for $Y \times I$ by Lemma 1. $\{f(p'') \times 0\}$ is a subset of $\{f(p') \times 0\}$ and is thus irreducible. Hence $\{p''\}$ is easily seen to be irreducible, and therefore X satisfies 1).

Let $\{p\}$ be an irreducible, locally finite set of base points for X. We claim that there is an irreducible, locally finite set of base points $\{q\}$ for $Y \times I$ such that $\{p\} \sim \{g \circ \pi(q)\}$, where $\pi : Y \times I \to Y$ is projection.

By the argument in Lemma 2, we see that we have a locally finite set of paths $\{\lambda_p\}$ from $\{p\}$ to $\{g \circ f(p)\}$. However, $(g \circ f)^{-1}(g \circ f)(p)$ may contain more points of $\{p\}$ than just p. But since $\{\lambda_p\}$ is locally finite, there are only finitely many such points, say $p_1, \ldots, p_n$. Let $q = f(p) \times 0$ and define $q_i = f(p) \times 1/i$ for $1 \leq i \leq n$. The resulting set of points, $\{q\}$, is easily seen to be locally finite, and by several applications of Lemma 1, $\{q\}$ is an irreducible set of base points for $Y \times I$.

So suppose given $\{p\}$ and $\{p'\}$, irreducible, locally finite sets of base points for X. Pick $\{q\}$ and $\{q'\}$ as above to be irreducible, locally finite sets of base points for $Y \times I$. Since Y is homogamous, $\{q\} \sim \{q'\}$, so $\{g \circ \pi(q)\} \sim \{g \circ \pi(q')\}$. Thus $\{p\} \sim \{p'\}$, so X satisfies 2). $\square$

Corollary 1.1: A space which is the proper homotopy type of an homogamous space is homogamous.

Corollary 1.2: The mapping cylinder of a proper map whose range is homogamous is homogamous.

Proposition 2: Let $\{\mathcal{O}\}$ be a locally finite open cover of X. Further assume that each $\mathcal{O}$ is path connected and that each $\bar{\mathcal{O}}$ is compact. Then X is homogamous.

Corollary 2.1: A locally compact, locally path connected, paracompact space is homogamous.

Corollary 2.2: A locally compact CW complex is homogamous.

Corollary 2.3: A paracompact, topological manifold is homogamous.

Proof: If $\{\mathcal{O}\}$ is the collection for X, $\{\mathcal{O} \times I\}$ is a cover for $X \times I$ with the same properties, so, if we can show 1) and 2) hold for X, we are done.

Since each $\mathcal{O}$ is path connected, each path component of X is open. Also the complement of a path component is open, so each path component is both open and closed. Hence X is homogamous iff each path component is, so we assume X is path connected.

We claim X is σ-compact, i.e. the countable union of compact sets. In fact, we will show $\{\mathcal{O}\}$ is at most countable. As a first step, define a metric d on X as follows. If $p \neq q$, look at a path λ from p to q. λ is compact, so it is contained in a finite union of $\mathcal{O}$'s. Hence λ is contained in a closed, compact set so λ intersects only finitely many $\mathcal{O}$'s. Let $r(\lambda;p,q)$ = the number of $\mathcal{O}$'s that λ intersects (non-empty). Define $d(p,q) = \min_{\lambda} r(\lambda;p,q)$. This is a natural number, so there is actually some path, λ, such that $d(p,q) = r(\lambda;p,q)$. If $p = q$, set $d(p,q) = 0$. d is easily seen to be a metric.

Let us fix $p \in X$. Then to each $\mathcal{O}$ we can associate a number $m(\mathcal{O},p) = \min_{q \in \mathcal{O}} d(p,q)$. We claim that, for any $n, m(\mathcal{O},p) \leq n$ for only finitely many $\mathcal{O}$. For $n = 0$, this is an easy consequence of the fact that $\{\mathcal{O}\}$ is locally finite. Now induct on n. Let $\mathcal{O}_1,\ldots,\mathcal{O}_k$ be all the $\mathcal{O}$'s such that $m(\mathcal{O},p) \leq n-1$. Let

$$c = \bigcup_{i=1}^{k} \bar{\mathcal{O}}_i \,.$$

c is compact.

Suppose $\bar{\mathcal{O}} \cap c = \emptyset$. Then we claim $m(\mathcal{O},p) \geq n+1$. To see this, pick $q \in \mathcal{O}$, and any path λ from p to q. If we can show $r(\lambda;p,q) \geq n+1$, we are done. Let $[0,x]$ be the closed interval which is the first component of $\lambda^{-1}(c)$, where $\lambda : I \to X$ is the path. Since $c \cap \bar{\mathcal{O}} = \emptyset$, $\lambda^{-1}(\bar{\mathcal{O}}) \geq s$, where $s > x$. Pick $x < t < s$. Then $\lambda(t) \not\in c$, so the path from p to $\lambda(t)$ already intersects at least n of the $\mathcal{O}$'s , so, from p to q , it must intersect at least $n+1$.

Therefore, if $m(\mathcal{O},p) \leq n$, $\bar{\mathcal{O}} \cap c \neq \emptyset$. But since $\{\mathcal{O}\}$ is locally finite, there are only finitely many $\mathcal{O}$ for which this is true. This completes the induction.

Hence the cover $\{\mathcal{O}\}$ is at most countable. If $\{\mathcal{O}\}$ is finite, X is compact and hence is easily seen to satisfy 1) and 2). Hence we assume $\{\mathcal{O}\}$ is infinite.

Enumerate $\{\mathcal{O}\}$, and set $C_k = \bigcup_{i=o}^{k} \bar{\mathcal{O}}_i$. Since C_k is compact, there are but finitely many $\mathcal{O}$'s such that $\bar{\mathcal{O}} \cap C_k \neq \emptyset$. Let E be the union of c and these $\mathcal{O}$'s . Then E is compact, as is ∂E , the frontier of E in X . Let $\{W_\alpha\}$ be the path components of $X - C_k$ not contained entirely in E .

Look at $W_\alpha \cap \partial E$. It might be empty, in which case W_α is actually a component of X since ∂E separates the interior of E and $X - E$. But X is connected, so $W_\alpha \cap \partial E \neq \emptyset$. Now if $p \in \partial E$, $p \in \mathcal{O}$ with $\mathcal{O} \cap C_k = \emptyset$. Now $\mathcal{O}$ is a path connected set missing C_k with $\mathcal{O}$ not contained entirely in E , so $\mathcal{O} \subseteq W_\alpha$ for some α . Hence the W_α cover ∂E .

The W_α are disjoint, so, as ∂E is compact, there are only finitely many of them. Some $\bar{W}_\alpha$ may be compact

Set $D_k = E \cup (\text{compact } \bar{W}_\alpha)$. Then D_k is compact.

Since the C_k are cofinal in the collection of all compact subsets of X , we may assume, after refinement, that $C_o \subseteq D_o \subseteq C_1 \subseteq D_1 \subseteq \cdots \subseteq C_k \subseteq D_k \subseteq C_{k+1} \subseteq \cdots$

Now let $\{p\}$ be a set of base points for X . Let $\{W_{\alpha,k}\}$ be the set of unbounded path components of $X - C_k$, which we saw above was finite. Since $\{p\}$ is a set of base points, in each $W_{\alpha,k}$ there are infinitely many $p \in \{p\}$ for which there exists an $\mathfrak{I} \in \{\mathfrak{I}\}$ such that $p \in \mathfrak{I} \subseteq W_{\alpha,k}$. We get a locally finite subset $\{p'\} \subseteq \{p\}$ by picking one element of $\{p\} \cap \mathfrak{I}$ for each such non-empty intersection as $\mathfrak{I}$ runs over $\{\mathfrak{I}\}$. By the above remarks, this set is a set of base points. It is clearly locally finite, so X satisfies 1).

Now let $\{p_k\}$ and $\{q_k\}$ be locally finite irreducible sets of base points (they are of necessity both countable). Look at all the p_k's in D_o . Join them by paths to some q_ℓ not in D_o . Join the q_k's in D_o to some p_ℓ's not in D_o . Note that the number of paths intersecting $C_o \leq$ (number of p_k in D_o) + (number of q_k in D_o).

For the inductive step, assume we have joined all the p_k's in D_{n-1} to some q_k's and vice versa. Suppose moreover that the number of paths intersecting $C_{n-i} \leq$ (number of p_k in D_{n-i}) + (number of q_k in D_{n-i}) for $1 \leq i \leq n$.

Look at the p_k's in $D_n - D_{n-1}$ which have not already been joined to some q_ℓ in D_{n-1}. Each of these lies in some $W_{\alpha,n-1}$; i.e. in an unbounded component of $X - C_{n-1}$. Join the p_k in $W_{\alpha,n-1} \cap (D_n - D_{n-1})$ which have not already been fixed up to some q_ℓ in $W_{\alpha,n-1} - D_n$ by a path in $W_{\alpha,n-1}$; i.e. outside of C_{n-1}. (Recall there are an infinite number of p_k [and q_k] in each $W_{\alpha,\ell}$, so we can always do this.) Do the same for the q_k in $D_n - D_{n-1}$.

Now each of these new paths misses C_{n-1}, so the number of paths intersecting $C_{n-i} \leq$ (number of p_k in D_{n-i}) + (number of q_k in D_{n-i}) for $1 \leq i \leq n$. For $i = 0$, the number of paths intersecting $C_n \leq$ (number of p_k in D_n) + (number of q_k in D_n). This completes the induction and shows X satisfies 2). □

Local compactness and σ-compactness are easily seen to be proper homotopy invariants, so we redefine an homogamous space to be locally compact, σ-compact, in addition to homogamous. Note now that any (irriducible) set of base points for an homogamous space is countable.

Section 2. The $\varepsilon - \Delta$ construction.

In this section we describe our construction. It will enable us to produce a proper homotopy functor on any homogamous space from an ordinary homotopy functor (a homotopy functor is a functor from the category of based topological spaces and based homotopy classes of maps to some category).

Now our homotopy functor, say H , takes values in some category $\mathcal{A}$. Associated to any homogamous space, X , we have an irreducible set of locally finite base points, I . We also have a diagram scheme, $\mathcal{D}$, consisting of the closed, compact subsets of X (see the definition below for the definition of a diagram scheme). Our basic procedure is to associate an element in $\mathcal{A}$ to the collection H(X - C, p) , where C is a closed compact subset of X , and $p \in I$. In order to be able to do this, we must impose fairly strenuous conditions on our category $\mathcal{A}$, but we prefer to do this in two stages.

<u>Definition</u> (see [25] page 42): A diagram scheme is a triple $\mathcal{D} = (J,M,d)$, where J is a set whose elements are called vertices, M is a set whose elements are called arrows, and $d : M \to J \times J$ is a map. Given a diagram scheme $\mathcal{D}$ and a category $\mathcal{A}$, a diagram over $\mathcal{D}$ is a map from J to the objects of $\mathcal{A}$ $(j \to A_j)$ and a map from M to the morphisms of $\mathcal{A}$ such that, if $d(m) = (i,j)$ m goes to an element of $Hom(A_i, A_j)$.

Notation: $[\mathcal{D}, \mathcal{A}]$ denotes the category of all diagrams in $\mathcal{A}$ over $\mathcal{D}$. (A map between diagrams over $\mathcal{D}$ is a collection of morphisms $f_j : A_j \to B_j$ such that $f_j \circ m = \bar{m} \circ f_i$, where $m \in Hom(A_i,A_j)$, and $\bar{m} \in Hom(B_i, B_j)$ correspond to the same element in M). If I is an index set (i.e. a set) $\mathcal{A}^I$ denotes the category whose objects are sets of objects in $\mathcal{A}$ indexed by I . The morphisms are sets of morphisms in $\mathcal{A}$

indexed by I . Finally, if $\mathcal{A}$ and $\mathcal{B}$ are categories, $\{\mathcal{A},\mathcal{B}\}$ is the category of covariant functors from $\mathcal{A}$ to $\mathcal{B}$ (see [25] page 63).

Definition: A category $\mathcal{A}$ is weakly regular with respect to an index set I provided:

1) $\mathcal{A}$ has products and zero objects.

2) Let $\mathfrak{F}(I) = \{T \mid T \subseteq I \text{ and } T \text{ is finite}\}$. If $\{G_i\}$ is an object in $\mathcal{A}^I$, each $T \in \mathfrak{F}(I)$ induces an endomorphism of $\{G_i\}$ by

$$\begin{cases} G_i \to G_i \text{ is the identity if } i \notin T \\ G_i \to G_i \text{ is the zero map if } i \in T . \end{cases}$$

This induces a unique map $X_T : \underset{i\in I}{\times} G_i \to \underset{i\in I}{\times} G_i$. We require that there exist an object $\underset{i\in I}{\mu}(G_i)$ and a map $\underset{i\in I}{\times} G_i \to \underset{i\in I}{\mu}(G_i)$ which is the coequalizer of the family of morphisms X_T for all $T \in \mathfrak{F}(I)$.

We easily check

Lemma 1: $\mu: \mathcal{A}^I \to \mathcal{A}$ is a functor when $\mathcal{A}$ is a weakly regular category with respect to I . Q.E.D.

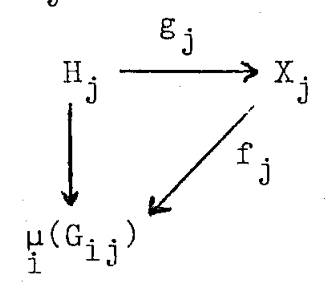
Examples: The categories of groups, abelian groups, rings, and pointed sets are all weakly regular with respect to any index set I . μ in each case is given as follows. We define an equivalence relation R on $\underset{i\in I}{\times} G_i$ by $x R y$ iff (the i^{th} component of x) = (the i^{th} component of y) for all but finitely many $i \in I$. Then $\underset{i\in I}{\mu}(G_i) = \underset{i\in I}{\times} G_i/R$.

Lemma 2: If $\mathcal{D}$ is a diagram scheme, and if $\mathcal{A}$ is a weakly regular category with respect to I , then $[\mathcal{D},\mathcal{A}]$ is also weakly regular with respect to I .

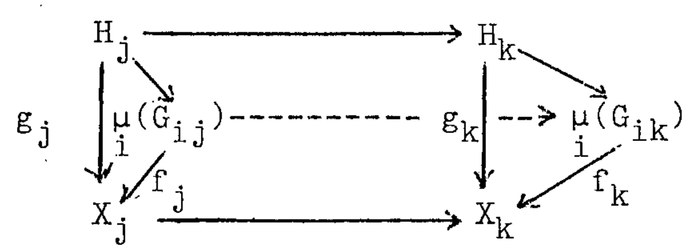
Proof: $[\mathcal{D},\mathcal{A}]$ is easily seen to have a zero object. $[\mathcal{D},\mathcal{A}]$ has products, for to each object in $[\mathcal{D},\mathcal{A}]^I$, $(\{G_{ij}\},\{m_i\})$, we associate the diagram $(\bigtimes_{i\in I} G_{ij}$, $\bigtimes_{i\in I} m_i)$. It is not hard to check that this diagram has the requisite universal properties.

To see condition 2), to $\{G_{ij}\}$ associate $\mu_{i\in I}(G_{ij})$. Then $\bigtimes_{i\in I} m_i$ induces $\mu_{i\in I}(m_i)$, so we do get a diagram.

To show it is a coequalizer, let X_j be the objects of a diagram. Set $H_j = \bigtimes_{i\in I} G_{ij}$. We are given $g_j\colon H_j \to X_j$ which commute with the diagram maps. If $T_1,T_2 \in \mathfrak{J}(I)$ we also have $g_j \circ X_{T_1} = g_j \circ X_{T_2}$. Hence by the universality for μ for $\mathcal{A}$, we get unique maps $f_j : \mu_i (G_{ij}) \to X_j$ such that



commutes. If we have a map in $\mathcal{D}$ from j to k , we get



with the front and back squares and both end triangles commutative. By the uniqueness of the map $\underset{i}{\mu}(G_{ij}) \to X_k$, the bottom square also commutes and we are done. Q.E.D.

Suppose given a functor $F \in \{\mathcal{A},B\}$. If F does not preserve products, it seems unreasonable to expect it to behave well with respect to μ , so assume F preserves products. Then we get a natural map $\mu \circ F^I \to F \circ \mu$ (F^I is the obvious element in $\{\mathcal{A}^I,B^I\}$). F preserves μ iff this map is an isomorphism.

Now suppose $\mathcal{A}$ is complete with respect to a diagram scheme $\mathcal{D}$. Then we have a functor $\lim_{\mathcal{D}} : [\mathcal{D},\mathcal{A}] \to \mathcal{A}$, the limit functor (see [25], page 44).

<u>Definition</u>: If $\mathcal{D}$ is a diagram scheme, and if $\mathcal{A}$ is a $\mathcal{D}$-complete, weakly regular category with respect to I , then we define $\varepsilon : [\mathcal{D},\mathcal{A}]^I \to \mathcal{A}$ to be the composite $\lim_{\mathcal{D}} \circ \mu$.

<u>Proposition 1</u>: Let $\mathcal{D}$ be a diagram scheme, and let $\mathcal{A}$ and B be two $\mathcal{D}$-complete, weakly regular categories with respect to I . Let $F \in \{\mathcal{A},B\}$. Then $\varepsilon : [\mathcal{D},\mathcal{A}]^I \to \mathcal{A}$ is a functor. If F preserves products and $\lim_{\mathcal{D}}$, there is a natural map $\varepsilon \circ F_{\#}^I \to F \circ \varepsilon$ (where $F_{\#} : [\mathcal{D},\mathcal{A}] \to [\mathcal{D},B]$ is the induced functor). If F preserves μ , this map is an isomorphism.

<u>Proof</u>: Trivial. $\square$

Unfortunately, the limit we are taking is an inverse limit, which is notorious for causing problems. In some cases however (and in all the cases in which we shall be interested) it is possible to give a discription of ε as a direct limit. In fact, we will describe the Δ-construction as a direct limit and then investigate the relationship between the two.

Definition: A lattice scheme $\mathfrak{D}$ is a diagram scheme (J,M,d) such that J is a partially ordered set with least-upper and greatest lower bounds for any finite subset of J. We also require that $d:M \to J \times J$ be a monomorphism and that Image $d = \{(j,k) \in J \times J | j > k)$.

To the lattice scheme $\mathfrak{D}$ and the index set I, we associate a diagram scheme $\mathfrak{D}_I$ as follows ($\mathfrak{D}_I$ is the diagram of "cofinal subsequences of $\mathfrak{D}$"). If $\alpha \in \bigtimes_{i \in I} J$, define $J_\alpha = \{j \in J | j = p_i(\alpha)$ for some $i \in I\}$. p_i is just the i^{th} projection, so J_α is just the subset of J we used in making up α. Define $\rho_\alpha: I \to J$ by $\rho_\alpha(i) = p_i(\alpha)$. Set $J_I = \{\alpha \in \bigtimes_{i \in I} J | J_\alpha$ is cofinal in J and $\bigcup_{j \leq k} \rho_\alpha^{-1}(j)$ is finite for all $k \in J\}$. (A subset of J is cofinal iff given any $j \in J$, there is an element k of our subset so that $k \geq j$. J_I may be thought of as the set of "locally finite, cofinal subsets of J).

We say $\alpha \geq \beta$ iff $p_i(\alpha) \geq p_i(\beta)$ in J for all $i \in I$. Set $M_I = \{(\alpha,\beta) \in J_I \times J_I | \alpha \geq \beta\}$ and let d_I

be the inclusion. Given $\alpha, \beta \in J_I$, define $\gamma \in J_I$ by $p_i(\gamma)$ = least-upper bound of $p_i(\alpha)$ and $p_i(\beta)$. (It is not hard to see $\gamma \in J_I$) . Greatest-lower bounds can be constructed similarly. Hence $\mathscr{D}_I$ is also a lattice scheme.

Now if J does not have any cofinal subsets of cardinality $\leq$ card (I) , $J_I = \emptyset$. Since J has upper bounds for finite sets, if J has finite cofinal subsets, then J has cofinal subsets of cardinality $\geq$ card(I) - N , where N is some natural number, then the condition that $\bigcup_{j \leq k} \rho_\alpha^{-1}(j)$ be finite forces $J_I = \emptyset$. Empty diagrams are a nuisance, so we define an I-lattice scheme as a lattice scheme with cofinal subsets of cardinality = card(I) .

We can now define $\delta\colon [\mathscr{D},\mathscr{A}]^I \to [\mathscr{D}_I,\mathscr{A}]$ as follows. If $\{d_i\} \in [\mathscr{D},\mathscr{A}]^I$, $\delta(d)$ has for objects $\delta_\alpha = \bigtimes_{i \in I} G_{ip_i(\alpha)}$, where G_{ij} is the j^{th} object in the diagram for d_i ($\alpha \in J_I$, $j \in J$, $i \in I$). If $\alpha \geq \beta$, we define $\delta_\alpha \to \delta_\beta$ by the maps $G_{ip_i(\alpha)} \to G_{ip_i(\beta)}$ which come from the diagram d_i .

We can also define maps $\delta_\alpha \to \mu_j$ as follows. Map $G_{ip_i(\alpha)} \to G_{ij}$ by the unique map in d_i if $p_i(\alpha) \geq j$, and by the zero map if $j > p_i(\alpha)$. (Notice that there are at most finitely many i such that $p_i(\alpha) < j$ by the second defining condition on J_I) . These maps induce a unique map $\delta_\alpha \to \bigtimes_{i \in I} G_{ij}$. Composing with the projection, we get a unique map

$\delta_\alpha \to \underset{i\in I}{\mu}(G_{ij}) = \underset{j}{\mu}$.

If $k \geq j$, $\begin{array}{ccc} \delta_\alpha & \longrightarrow & \mu_j \\ & \searrow & \downarrow \\ & & \mu_k \end{array}$ commutes as one easily checks. If $\alpha \geq \beta$, $\begin{array}{ccc} \delta_\alpha & & \\ \downarrow & \searrow & \\ & & \mu_j \\ & \nearrow & \\ \delta_\beta & & \end{array}$ also commutes.

Lemma 3: $\delta\colon [\mathcal{D},\mathcal{A}]^I \to [\mathcal{D}_I,\mathcal{A}]$ is a functor.

Proof: The proof is easy and can be safely left to the reader. Q.E.D.

Now suppose $\mathcal{A}$ is $\mathcal{D}_I$-cocomplete. Then we have a colimit functor $\underset{\mathcal{D}_I}{\operatorname{colim}}$.

Definition: If $\mathcal{D}$ is an I-lattice scheme, and if $\mathcal{A}$ is a $\mathcal{D}_I$-cocomplete, weakly regular category with respect to I , then we define $\Delta : [\mathcal{D},\mathcal{A}]^I \to \mathcal{A}$ to be the composition $\underset{\mathcal{D}_I}{\operatorname{colim}} \circ \delta$.

Proposition 2: Let $\mathcal{D}$ be a diagram scheme and let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{D}_I$-cocomplete, weakly regular categories with respect to I . Let $F \in \{\mathcal{A},\mathcal{B}\}$. Then $\Delta\colon [\mathcal{D},\mathcal{A}]^I \to \mathcal{A}$ is a functor. There is always a natural map $\Delta \circ F^I_{\#} \to F \circ \Delta$. If F preserves products and $\underset{\mathcal{D}_I}{\operatorname{colim}}$, this map is an isomorphism.

Proof: Trivial. □

The maps we constructed from δ_α to μ_j combine to give us a natural transformation from Δ to ε whenever both are defined. We would like to study this natural transformation in order to get information about both Δ and ε. A $(\mathcal{D},I)$-regular category is about the most general category in which we can do this successfully, and it includes all the examples we have in mind.

Definition: A category $\mathcal{A}$ is said to be $(\mathcal{D},I)$-regular provided

1) $\mathcal{A}$ is weakly regular with respect to I

2) $\mathcal{A}$ has images and inverse images

3) There is a covariant functor F from $\mathcal{A}$ to the category of pointed sets and maps such that

a) F preserves kernels, images, products, limits over $\mathcal{D}$, increasing unions, and μ.

b) F reflects kernels, images, and isomorphisms

4) $\mathcal{A}$ is $\mathcal{D}$-complete and $\mathcal{D}_I$-cocomplete

5) I is countable.

Examples: The categories of groups, abelian groups, rings, and pointed sets are all $(\mathcal{D},I)$-regular for any I-lattice scheme. The functor F is just the forgetful functor.

Lemma 4: Let $\mathcal{A}$ be a $(\mathcal{D},I)$-regular category. Then $\mathop{\times}_{i \in I}$ and μ preserve kernels and images.

<u>Proof</u>: $\underset{i\in I}{\times}$ is known to preserve kernels (Mitchell [25], page 67, Corollary 12.3).

Since F preserves images, if Im(f) is the image of $A \xrightarrow{f} B$, then F(A) is onto F (Im(f)) and F(Im(f)) injects into F(B). Let K_i be the image of $A_i \xrightarrow{f_i} B_i$. Then, since F preserves products, $\underset{i\in I}{\times} F(A_i) \longrightarrow \underset{i\in I}{\times} F(K_i)$ is onto. $\underset{i\in I}{\times} K_i \longrightarrow \underset{i\in I}{\times} B_i$ is a monomorphism since $\underset{i\in I}{\times}$ is a monofunctor. Since F preserves monomorphisms, $\underset{i\in I}{\times} F(K_i) \longrightarrow \underset{i\in I}{\times} F(B_i)$ is seen to be a monomorphism as F also preserves products. Since F reflects images, $\underset{i\in I}{\times} K_i$ is the image of $\underset{i\in I}{\times} A_i \longrightarrow \underset{i\in I}{\times} B_i$.

Let $K_i \to A_i \to B_i$ be kernels. Then

$$(*) \qquad \begin{array}{ccccc} X\,K_i & \xrightarrow{h} & X\,A_i & \xrightarrow{g} & X\,B_i \\ \downarrow & & \downarrow & & \downarrow \\ \mu(K_i) & \longrightarrow & \mu(A_i) & \xrightarrow{f} & \mu(B_i) \end{array}$$

commutes. Since F reflects kernels, we need only show that $F(\mu(K_i))$ injects into $F(\mu(A_i))$ and is onto $F(f)^{-1}(0)$. Since F preserves μ, we may equally consider $\mu(F(K_i))$, etc. Since F preserves products, we may as well assume the diagram (*) is in the category of pointed sets.

We show $\mu(K_i)$ is onto $f^{-1}(0)$. Let $x \in f^{-1}(0) \subseteq \mu(A_i)$.

Lift x to $y \in \times A_i$, which we may do since $\underset{i\in I}{\times}$ is onto $\underset{i\in I}{\mu}$ in the category of pointed sets. Now $g(y) \in \times B_i$ can have only finitely many non-zero components since it goes to 0 in $\mu(B_i)$.

Define $\bar{y}$ by

$$p_i(\bar{y}) = \begin{cases} p_i(y) & \text{if } p_i(g(y)) = 0 \\ 0 & \text{if } p_i(g(y)) \neq 0 \end{cases} .$$

Then $\bar{y}$ also lifts x and $g(\bar{y}) = 0$. There is a $z \in \times K_i$ such that $h(Z) = \bar{y}$, so $\mu(K_i)$ maps onto $f^{-1}(0)$. A similar argument shows $\mu(K_i)$ injects into $\mu(A_i)$. Hence μ preserves kernels.

Now let K_i be the image of $A_i \to B_i$. Then

$$\begin{array}{ccccc} \times A_i & \longrightarrow & \times K_i & \longrightarrow & \times B_i \\ \downarrow & & \downarrow & & \downarrow \\ \mu(A_i) & \longrightarrow & \mu(K_i) & \longrightarrow & \mu(B_i) \end{array} .$$

commutes. By general nonsense, it suffices to prove the result assuming we are working in the category of pointed sets.

Since $\underset{i\in I}{\times}$ preserves images, $\times A_i \to \times K_i$ is onto, so $\mu(A_i) \to \mu(K_i)$ is easily seen to be onto. Since μ preserves kernels, $\mu(K_i)$ injects into $\mu(B_i)$, so $\mu(K_i)$ is the image of $\mu(A_i) \to \mu(B_i)$. Q.E.D.

<u>Theorem 1</u>: Let $\mathcal{A}$ be a $(\mathfrak{D},I)$-regular category. Then ε preserves kernels and images.

<u>Proof</u>: By Mitchell [25] (page 67, Corollary 12.2) $\lim\limits_{\mathcal{D}}$ preserves kernels, so $\varepsilon = \lim\limits_{\mathcal{D}} \circ \mu$ also does using Lemma 4 and general nonsense.

Now let K_{ij} be the image of $A_{ij} \to B_{ij}$. We claim that, if $x \in \varepsilon(K_{ij})$, then there exists $\alpha \in J_I$ such that x is in the image of $\delta_\alpha(K_{ij})$. Assuming this for now we proceed as follows.

Since $\delta_\alpha(K_{ij}) = \underset{i\in I}{\times} K_{ip_i(\alpha)}$, δ_α preserves kernels and images by Lemma 4. Hence

$$\begin{array}{ccccc} \delta_\alpha(A_{ij}) & \longrightarrow & \delta_\alpha(K_{ij}) & \longrightarrow & \delta_\alpha(B_{ij}) \\ \downarrow & & \downarrow & & \downarrow \\ \varepsilon(A_{ij}) & \longrightarrow & \varepsilon(K_{ij}) & \longrightarrow & \varepsilon(B_{ij}) \end{array}$$

commutes and $\delta_\alpha(K_{ij})$ is the image of $\delta_\alpha(A_{ij}) \to \delta_\alpha(B_{ij})$. By the usual abstract nonsense, we may as well assume we are in the category of pointed sets (note F preserves ε by Proposition 1).

Now using our claim we can easily get $\varepsilon(A_{ij}) \to \varepsilon(K_{ij})$ is onto. $\varepsilon(K_{ij}) \to \varepsilon(B_{ij})$ injects since ε preserves kernels. Hence $\varepsilon(K_{ij})$ is the image of $\varepsilon(A_{ij}) \to \varepsilon(B_{ij})$. $\square$

We prove a stronger version of our claim than we have yet used.

<u>Lemma 5</u>: Let $\mathcal{A}$ be the category of pointed sets. Let $\{G_{ij}\}$ be an object in $[\mathcal{D},\mathcal{A}]^I$. Then if $x \in \varepsilon(G_{ij})$, there is an $\alpha \in J_I$ such that $\delta_\alpha(G_{ij})$ contains x in its image. If $y,z \in \delta_\alpha(G_{ij})$ both hit x, then there

is a $\beta \leq \alpha$ such that, in $\delta_\beta(G_{ij})$, the images of y and z differ in only finitely many coordinates. In fact, if there is a $j \epsilon J$ such that $j \leq p_i(\alpha)$ for all $i \epsilon I$ and such that y and z agree in $\bigtimes_{i\epsilon I} G_{ij}$, then β can be chosen so that $y = z$ in $\delta_\beta(G_{ij})$.

<u>Proof</u>: If $x \epsilon \varepsilon(G_{ij})$, there exist unique $a_j \epsilon \mu_{i\epsilon I}(G_{ij})$ such that x hits a_j. Since $\bigtimes_{i\epsilon I}$ is onto $\mu_{i\epsilon I}$, we may lift a_j to $b_j \epsilon \bigtimes G_{ij}$. Since J has countable cofinal subsets, let the natural numbers $j = 1,2,...$ be one such. Since I is countable (and infinite or our result is easy) we also assume it to be the natural numbers.

Now look at b_2 and b_1. Since they agree in $\mu(G_{i1})$, b_2 projected into $\bigtimes G_{i1}$ differs from B_1 in only finitely many coordinates. Let $I_1 \subseteq I$ be the finite subset which indexes these unequal coordinates, together with the element $1 \epsilon I$.

Next look at the pairs (b_3, b_2) and (b_3, b_1). As before, projected into $\bigtimes G_{i2}$, b_3 and b_2 agree in all but finitely many coordinates. In $\bigtimes G_{i1}$, b_3 and b_1 differ in only finitelylmany coordinates. Set $I_2 \subseteq I$ to be the finite subset of I which indexes the unequal coordinates of (b_3,b_2) or (b_3,b_1) which lie in $I - I_1$, together with the smallest integer in $I - I_1$.

Define I_k to be the finite subset of I which indexes the unequal coordinates of $(b_k,b_{k-1}),...,(b_k,b_2)$, $(b_k,1)$ which lie in $I - (I_{k-1} \cup ... \cup I_2 \cup I_1)$, together with the smallest integer in $I - (I_{k-1} \cup \ldots \cup I_2 \cup I_1)$.

Then $I = \bigcup_{k=1}^{\infty} I_k$ as a disjoint union. Define α by $p_i(\alpha) = k$, where $i \in I_k$. Since I is countable, but not finite, and since each I_k is finite, $\alpha \in J_I$. Define $y \in \delta_\alpha(G_{ij})$ by $p_i(y) = p_i(b_{p_i(\alpha)})$. A chase through definitions shows y hits each a_j through the map $\delta_\alpha(G_{ij}) \to \mu_{i \in I}(G_{ij})$. Thus y hits x in $\varepsilon(G_{ij})$.

Now suppose $y, z \in \delta_\alpha(G_{ij})$ both map to x. Then they map to the same element in each $\mu_{i \in I}(G_{ij})$. Let a_j be the image of y in $\bigtimes_{i \in I} G_{ij}$ under the map $\delta_\alpha(G_{ij}) \to \bigtimes_{i \in I} G_{ij}$ which we defined just before Lemma 3. Set b_j to be the image of z in $\bigtimes_{i \in I} G_{ij}$. Then a_j and b_j differ in only finitely many coordinates.

Let I_1 be the finite subset of I which indexes the unequal coordinates of a_1 and b_1. If there is a $j \le p_i(\alpha)$ for all $i \in I$ such that y and z agree in $\bigtimes_{i \in I} G_{ij}$, we may assume $j = 1$, so $a_1 = b_1$, and $I_1 = \emptyset$.

Define I_k as the finite subset of I which indexes the unequal coordinates of (a_k, b_k) which lie in $I - (I_{k-1} \cup \ldots \cup I_1)$. Define β by

$$p_i(\beta) = \begin{cases} k-1 & \text{if } i \in I_k \text{ for some } k \ge 2 \\ p_i(\alpha) & \text{if } i \notin I_k \text{ for any } k \ge 2 \end{cases} .$$

Note $p_i(\beta) \leq p_i(\alpha)$, since $i \in I_k$, this says $p_i(a_k) \neq p_i(b_k)$. But if $k > p_i(\alpha)$, $p_i(a_k) = 0 = p_i(b_k)$ by the definition of our map from δ_α to $\underset{i\in I}{\times}$. Hence $k \leq p_i(\alpha)$, so $p_i(\beta) \leq p_i(\alpha)$.

Let $\bar{y}$ be the projection of y into $\delta_\beta(G_{ij})$, and let $\bar{z}$ be the projection of z into $\delta_\beta(G_{ij})$. $p_i(\bar{y}) = p_i(a_{p_i(\beta)})$ and $p_i(\bar{z}) = p_i(b_{p_i(\beta)})$. If $p_i(a_{p_i(\beta)}) \neq p_i(b_{p_i(\beta)})$, then $i \notin I_k$ for any $k \geq 2$, since $i \in I_k$ for $k \geq 2$ says that $p_i(a_k) \neq p_i(b_k)$ but $p_i(a_{k-1}) = p_i(b_{k-1})$. If $i \notin I_k$ for any k, it says that $p_i(y) = p_i(z)$. Thus $p_i(\bar{y}) = p_i(\bar{z})$ if $i \notin I_1$. Hence they agree for all but finitely many $i \in I$. In fact, if $I_1 = \emptyset$, $\bar{y} = \bar{z}$. Q.E.D.

We can now describe $\varepsilon(G_{ij})$ as a colimit (direct limit). Let $\underset{\alpha}{\mu}(G_{ij})$ be the μ functor applied to $\{G_{ip_i(\alpha)}\}$. Then the map $\delta_\alpha(G_{ij}) \to \varepsilon(G_{ij})$ factors through $\underset{\alpha}{\mu}(G_{ij})$.

<u>Theorem 2</u>: Let $\mathcal{A}$ be a $(\mathcal{D},I)$-regular category. Then the natural map $\underset{\mathcal{D}_I}{\text{colim}}\, \mu_\alpha \to \varepsilon$ is an isomorphism.

Hence ε is a cokernel, kernel preserving functor.

<u>Proof</u>: Let us first show F preserves $\underset{\mathcal{D}_I}{\text{colim}}$; i.e. we must show that the natural map $\text{colim}\, F(A_\alpha) \xrightarrow{f} F(\underset{\mathcal{D}_I}{\text{colim}}\, A_\alpha)$ is an isomorphism. To do this, we first

compute Im(f). If $Im(f_\alpha)$ is the image of $F(A_\alpha) \to \text{colim } F(A_\alpha) \to F(\text{colim } A_\alpha)$, then by Mitchell [25] (Proposition 2.8, page 46), $Im(f) = \bigcup_\alpha Im(f_\alpha)$. Let $Im(g_\alpha)$ be the image of $A_\alpha \to \text{colim } A_\alpha$. Then, since F preserves images, $F(Im(g_\alpha)) = Im(f_\alpha)$, so $\bigcup_\alpha Im(f_\alpha) = \bigcup_\alpha F(Im(g_\alpha))$.

Now $\{\alpha\}$ has a cofinal subsequence (which is countable and, if I is finite, it is also finite) $\{\alpha_i\}$ such that $\alpha_o < \alpha_1 < \cdots < \alpha_n < \cdots$. Therefore $\bigcup_\alpha Im(f_\alpha) = \bigcup_{i=o}^{\infty} Im(f_{\alpha_i}) = \bigcup_{i=o}^{\infty} F(Im(g_{\alpha_i}))$ since $\{\alpha_i\}$ is cofinal.

Again by Mitchell [25] (Proposition 2.8, page 46), $\text{colim } A_\alpha = \bigcup_\alpha Im(g_\alpha) = \bigcup_{i=o}^{\infty} Im(g_{\alpha_i})$. Thus $F(\text{colim } A_\alpha) = F(\bigcup_{i=o}^{\infty} Im(g_{\alpha_i}))$. Since $\mathcal{A}$ is a $(\mathcal{D},I)$-regular category, the natural map $\bigcup_{i=o}^{\infty} F(Im(g_{\alpha_i})) \subseteq F(\bigcup_{i=o}^{\infty} Im(g_{\alpha_i}))$ is an isomorphism. Thus the map $\bigcup_\alpha Im(f_\alpha) \subseteq F(\bigcup_\alpha Im(g_\alpha))$ is an isomorphism. But this map is just the natural map $\underset{\mathcal{D}_I}{\text{colim}}\, F(A_\alpha) \to F(\underset{\mathcal{D}_I}{\text{colim}}\, A_\alpha)$.

The natural map $\text{colim } \mu_\alpha \to \varepsilon$ is the map which comes from the maps $\mu_\alpha \to \varepsilon$. To show it is an isomorphism, it is enough to show it is for pointed sets by the result above and the fact that F reflects isomorphisms. But this is exactly what Lemma 5 says.

Now ε preserves kernels by Theorem 1, and it preserves cokernels since colimits preserve cokernels by

Mitchell [25] (page 67, Corollary 12.2 dualized). □

Corollary 2.1: Let $\{G^n_{ij}\} \in [\mathcal{D},\mathcal{A}]^I$ be a collection of exact sequences in a $(\mathcal{D},I)$-regular category $\mathcal{A}$ (i.e. there are maps $f^n_{ij}: G^n_{ij} \to G^{n-1}_{ij}$ which are maps of diagrams such that $\mathrm{Im}(f^n_{ij}) = \ker(f^{n-1}_{ij}))$. Then the sequence $\cdots \to \varepsilon(G^n_{ij}) \xrightarrow{\varepsilon(f^n_{ij})} \varepsilon(G^{n-1}_{ij}) \to \cdots$ is also exact.

Corollary 2.2: Let $\mathcal{A}$ be a $(\mathcal{D},I)$-regular abelian category. Let $\{G^*_{ij}, f^*_{ij}\}$ be a collection of chain complexes in $[\mathcal{D},\mathcal{A}]^I$. Then $\{\varepsilon(G^*_{ij}), \varepsilon(f^*_{ij})\}$ is a chain complex, and $H_*(\varepsilon(G^*_{ij})) = \varepsilon(H_*(G^*_{ij}))$, where H_* is the homology functor (see Mitchell [25], page 152).

Proofs: The first corollary is easily seen to be true. (It is, in fact, a corollary of Theorem 1.)

The second corollary is almost as easy. If $\{Z^n_{ij}\}$ are the n-cycles, and if $\{B^{n+1}_{ij}\}$ are the (n+1)-boundaries, $0 \to B^{n+1}_{ij} \to Z^n_{ij} \to H_n(G^*_{ij}) \to 0$ is exact. Applying ε , we get $0 \to \varepsilon(B^{n+1}_{ij}) \to \varepsilon(Z^n_{ij}) \to \varepsilon(H_n(G^*_{ij})) \to 0$ is exact. But as ε preserves kernels and images, $\varepsilon(Z^n_{ij})$ is the collection of n-cycles for $\varepsilon(G^*_{ij})$ and $\varepsilon(B^{n+1}_{ij})$ is the collection of (n+1)-boundaries. Hence $H_n(\varepsilon(G^*_{ij})) \to \varepsilon(H_n(G^*_{ij}))$ is an isomorphism. □

Now suppose J has a unique minimal element j_o .

Then we get a square

$$\begin{array}{ccc} \Delta(G_{ij}) & \longrightarrow & \mathop{\times}\limits_{i\in I} G_{ij_o} \\ \big\downarrow & & \big\downarrow \\ \varepsilon(G_{ij}) & \longrightarrow & \mathop{\mu}\limits_{i\in I}(G_{ij_o}) \end{array}$$

<u>Theorem 3</u>: In a $(\mathcal{D},I)$-regular category, the above diagram is a pullback in the category of pointed sets, so if F reflects pullbacks the above square is a pullback.

<u>Proof</u>: As we showed in the proof of Theorem 2 that F and $\operatorname{colim}_{\mathcal{D}_I}$ commute, we have $F(\Delta(G_{ij})) = \Delta(F(G_{ij}))$, so we may work in the category of pointed sets.

The omnipresent Lemma 5 can be used to show the above square is a pullback. The pullback is the subset of $\varepsilon(G_{ij}) \times (\mathop{\times}\limits_{i\in I} G_{ij_o})$ consisting of pairs which project to the same element in $\mathop{\mu}\limits_{i\in I}(G_{ij_o})$. Given any element, x, in $\varepsilon(G_{ij})$ we can find $\alpha \in J_I$ such that the element is in the image of $\delta_\alpha(G_{ij})$. Lift the image of x in $\mu(G_{ij_o})$ to $y \in \mathop{\times}\limits_{i\in I} G_{ij_o}$. Let $z \in \delta_\alpha(G_{ij_o})$ be an element which hits x. Then y pushed into $\times\, G_{ij_o}$ and z agree, except in finitely many places. It is then easy to find $\beta \in J_I$ with $\beta \leq \alpha$ and an element $q \in \delta_\alpha(G_{ij})$ such that q hits x and y. This says precisely that our square is a pullback. $\square$

Remarks: In all our examples, F reflects pullbacks. The analogues of Corollaries 2.1 and 2.2 may be stated and proved by the reader for the Δ functor.

Theorem 4: In a $(\mathfrak{d}, I)$-regular category, $\varepsilon(G_{ij}) = 0$ iff given any $j \in J$ there exists a $k \geq j$ such that $G_{ik} \to G_{ij}$ is the zero map for all but finitely many i.

Proof: Suppose given j we can find such a k. Then we can produce a cofinal set of j's, $j_0 \leq j_1 \leq \cdots$, such that the map $\mu(G_{ij_k}) \to \mu(G_{ij_{k-1}})$ is the zero map. Hence $\varepsilon = 0$.

Conversely, suppose for some j_0 that no such k exists. This means that for every $k \geq j_0$ there are infinitely many i for which $G_{ik} \to G_{ij_0}$ is not the zero map.

As usual, it suffices to prove the result for pointed sets, so assume we have $Z_{ik} \in G_{ik}$ which goes non-zero into G_{ij_0}. Pick $j_0 \leq j_1 \leq j_2 \leq \cdots$ a countable cofinal subsequence of J. We define an element α of $\mathfrak{d}_I$ as follows. Well order I. Then $\alpha(i) = j_0$ until we hit the first element of I for which a Z_{ij_k} is defined. Set $\alpha(i) = j_k$ for this i and continue defining $\alpha(i) = j_k$ until we hit the next element of I for which a $Z_{ij_{k_1}}$ is defined with $k_1 \geq k$. Set $\alpha(i) = j_{k_1}$ until we hit the next $Z_{ij_{k_2}}$ with $k_2 \geq k_1$. Continuing in this fashion is seen to give an element of $\mathfrak{d}_I$. Define Z_α by $Z_{i\alpha(i)} = 0$

unless i is one of the distinguished elements of I, in which case set $Z_{i\alpha(i)} = Z_{ij_k}$, where $j_k = \alpha(i)$.

Then $Z_\alpha \in \Delta_\alpha(G_{ij})$. It is non-zero in $\underset{i\in I}{\mu}(G_{ij_o})$ by construction, so $\varepsilon(G_{ij}) \neq 0$. □

Section 3. Proper homotopy functors and their relations.

We begin by clarifying the concept of an ordinary homotopy functor. A homotopy functor is a functor, h, from the category of pointed topological spaces to some category, $\mathcal{A}$. Given a space X and two base points p_1 and p_2, and a path λ from p_1 to p_2, there is a natural transformation $\alpha_\lambda : h(X,p_1) \to h(X,p_2)$ which is an isomorphism and depends only on the homotopy class of λ rel end points. Furthermore, $h(X,p) \to h(X \times I, p \times t)$ given by $x \to (x,t)$ is an isomorphism for $t = 0$ and 1.

For any homotopy functor we are going to associate a proper homotopy functor defined on the category of homogamous spaces and countable sets of locally finite irreducible base points.

To be able to do this is the generality we need, we shall have to digress momentarily to discuss the concept of a covering functor.

Let X be a homogamous space, and let $\{x_i\}$ be a countable, locally finite, irreducible set of base points for X. (From now on we write just "set of base points" for "countable, locally finite, irreducible set of base points.") Let $\mathcal{D}_X$ be some naturally defined collection of subsets of X (by naturally defined we mean that if $f : X \to Y$ is a proper map, $f^{-1}\mathcal{D}_Y \subseteq \mathcal{D}_X$). $\mathcal{D}_X$ is a diagram with arrows being inclusion maps. Assume $\mathcal{D}_X$ is an $\{x_i\}$-lattice, and assume $\emptyset \in \mathcal{D}_X$.

<u>Definition</u>: A covering functor for $\mathcal{D}_X$ is a function, S, which assigns to each $\pi_1(X-C,x_i)$ a subgroup $S\pi_1(X-C,x_i)$ subject to

$$\begin{array}{ccc} S\pi_1(X-C,x_i) & \subseteq & \pi_1(X-C,x_i) \\ \downarrow & & \downarrow \\ S\pi_1(X-D,x_i) & \subseteq & \pi_1(X-D,x_i) \end{array}$$

commutes whenever $D \subseteq C$, $x_i \notin C$, and where the vertical maps are induced by inclusion (C and D are any elements of $\mathcal{D}_X$).

Remarks: We have two examples for $\mathcal{D}_X$ in mind. In this section we can use the set of all closed, compact subsets of X for $\mathcal{D}_X$. For cohomology however, we will have to use the set of open subsets of X with compact closure for $\mathcal{D}_X$.

Examples: There are three useful examples we shall define.

1) no covering functor (the subgroup is the whole group)

2) the universal covering functor (the subgroup is the zero group)

3) the universal cover of X but no more covering functor (the subgroup is the kernel of $\pi_1(X-C,x_i) \to \pi_1(X,x_i)$) .

Definition: A compatible covering functor for $\mathcal{D}_X$ is a covering functor S such that, for any $C \in \mathcal{D}_X$, the cover of the component of $X - C$ contain x_i corresponding to $S\pi_1(X-C,x_i)$ exists. We write $(X,\sim)$ for a compatible covering functor for $\mathcal{D}_X$ (which is inferred from context) to denote a collection of pointed spaces $(\widetilde{(X-C)}^i,\hat{x}_i)$, where $\widetilde{(X-C)}^i$ is the covering space of the component of X-C containing x_i, and $\hat{x}_i$ is a lift of x_i to this cover such that $\pi_1(\widetilde{X-C},\hat{x}_i) = S\pi_1(X-C,x_i)$. Notice this notation is mildly ambiguous since if we change the $\hat{x}_i$ we get a different object. As the two objects are homeomorphic this tends to cause no problems so we use the more compact notation.

We say $(X,\sim) \leq (X,---)$ provided the subgroup of $\pi_1(X-C,x_i)$ corresponding to --- contains the one corresponding to $\sim$. Hence any $(X,\sim) \leq$ (X, no cover), and, if the universal covering functor is compatible with $\mathcal{D}_X$, (X , universal cover) $\leq (X,\sim)$.

Now the no covering functor is compatible with any $\mathcal{D}_X$. If X is semi-locally 1-connected, the universal cover of X but no more covering functor is compatible with any $\mathcal{D}_X$. If $\mathcal{D}_X$ is the collection of closed, compact subsets of X , and if X is locally 1-connected, the universal covering functor is compatible with $\mathcal{S}_X$, as in any other covering functor. Hence a CW complex is compatible with any covering functor (see Lundell and Weingram [21] page 67, Theorem 6.6) for $\mathcal{S}_X$.

We can now describe our construction. Let $(X,\sim)$ be a covering functor for X. Assume from now on that our homotopy functor takes values in a $(\mathcal{S}_X,\{x_i\})$-regular category for all homogamous X with base points $\{x_i\}$. We apply the ε and Δ constructions to the collection

$$G_{iC} = \begin{cases} h((\widetilde{X-C})^i,\hat{x}_i) & \text{if } x_i \in X-C \\ 0 & \text{if } x_i \notin X-C \end{cases} .$$

If $D \subseteq C$, there is a unique map $((\widetilde{X-D})^i,\hat{x}_i) \to ((\widetilde{X-C})^i,\hat{x}_i)$ if $x_i \in X-D$ by taking the lift of the inclusion which takes $\hat{x}_i$ in $(\widetilde{X-D})^i$ to $\hat{x}_i$ in $(\widetilde{X-C})^i$. Hence we get a map $G_{iD} \to G_{iC}$. We denote these groups by $\varepsilon(X{:}h,\{\hat{x}_i\},\sim)$ and $\Delta(X:h,\{\hat{x}_i\},\sim)$.

<u>Theorem 1</u>: Let $\{x_i\}$ and $\{y_i\}$ be two sets of base points for X, X homogamous with countable base points. Let Λ be a locally finite collection of paths giving an equivalence between $\{x_i\}$ and $\{y_i\}$. Then there are natural transformations $\alpha_\Lambda : \varepsilon(X{:}h, \{\hat{x}_i\},\sim) \to \varepsilon(X:h, \{\hat{y}_i\}, -)$ and $\alpha_\Lambda : \Delta(X{:}h, \{\hat{x}_i\},\sim) \to \Delta(X:h, \{\hat{y}_i\}, -)$ which are isomorphisms and depend only on the proper homotopy class of Λ rel end points. ($-$ is the covering functor induced by the set of paths Λ.)

<u>Proof</u>: Define α_Λ as follows. By relabeling if necessary we may assume x_i goes to y_i by a path

in Λ . Map $h(\widetilde{(X-C)}^i,\hat{x}_i)$ to $h(\overline{(X-C)}^i,\hat{y}_i)$ by the zero map if the path from x_i to y_i hits C . If the path misses C , map $h(\widetilde{(X-C)}^i,\hat{x}_i)$ to $h(\overline{(X-C)}^i,\hat{y}_i)$ by lifting the path from x_i to y_i into $\widetilde{(X-C)}^i$ beginning at $\hat{x}_i$ (say it now ends at z , and then map $(\widetilde{(X-C)}^i,z)$ to $(\overline{(X-C)}^i,\hat{y}_i)$ by the unique homeomorphism covering the identity which takes z to $\hat{y}_i$. This defines a homomorphism α_Λ on ε and Δ .

If by Λ^{-1} we mean the collection of paths from y_i to x_i given by the inverse of the path from x_i to y_i , we can also define $\alpha_{\Lambda^{-1}}$.

$\alpha_\Lambda \circ \alpha_{\Lambda^{-1}}$ takes $h(\overline{(X-C)}^i, \hat{y}_i)$ to itself by the zero map if the path hits C and by the identity otherwise. Since all but finitely many paths miss C , this induces the identity on ε . Since the empty set is the minimal element of $\mathcal{S}_X$, $\alpha_\Lambda \circ \alpha_{\Lambda^{-1}}$ is the identity on $\times_{i\in I} h(X,y_i)$ and $\mu(h(X,\hat{y}_i))$. Hence it is also the identity on Δ . A similar argument shows $\alpha_{\Lambda^{-1}} \circ \alpha_\Lambda$ is the identity, so they are both isomorphisms.

The same sort of argument shows α_Λ depends only on the proper homotopy type of Λ . It can be safely left to the reader. □

If h is actually a homotopy functor on the category of pairs (n-ads) we can define $\gamma(X,A:h,\{\hat{x}_i\},\sim)$ for the pair (X,A) (where γ denotes ε or Δ) using

$G_{iC} = h((X-C)^i, (\tilde{A} \cap (\widetilde{X-C})^i) \cup (\hat{x}_i), \hat{x}_i)$, where $\tilde{A} \cap (\widetilde{X-C})^i = \pi^{-1}(A \cap (X-C)^i)$, $\pi : (\widetilde{X-C})^i \to (X-C)^i$, if $x_i \not\in C$ and is 0 otherwise (for n-ads use $h((\widetilde{X-C})^i , (\tilde{A}_1 \cup (\widetilde{X-C})^i) \cup \{\hat{x}_1\}, \ldots, (\tilde{A}_{n-1} \cup (\widetilde{X-C})^i) \cup \hat{x}_i, \hat{x}_i)$ or 0) .

Now suppose we have a connected sequence of homotopy functors h_* ; i.e. each h_n is a functor on some category of pairs and we get long exact sequences. By applying our construction to (X,A) , one would hope to get a similar long exact sequence for the ε or Δ theories.

Several problems arise with this naive expectation. To begin, we can certainly define groups which fit into a long exact sequence. Define $\gamma(A;X:h_*,\{x_i\},\sim)$ where $\gamma = \varepsilon$ or Δ from $G_{ic} = h_*((\tilde{A} \cap (\widetilde{X-C})^i) \cup (\hat{x}_i), \hat{x}_i)$ if $x_i \not\in C$ and 0 if $x_i \in C$. Then Corollary 2.1, or its unstated analogue 3.1, shows we get a long exact sequence $\cdots \to \gamma(A;X : h_n,\{x_i\},\sim) \to \gamma(X: h_n, \{x_i\},\sim) \to \gamma(X,A:h_n,\{\hat{x}_i\},\sim) \to \gamma(A;X:h_{n-1},\{\hat{x}_i\},\sim) \to \cdots$ The problem of course is to describe $\gamma(A;X:\text{etc.})$ in terms of A .

We clearly have little hope unless A is homogamous, and for convenience we insist $A \subseteq X$ be a proper map. Such a pair is said to be homogamous, and for such a pair we can begin to describe $\gamma(A;X : \text{etc.})$.

Pick a set of base points for A , and then add enough new points to get a set of base points for X .

Such a collection is a set of base points for (X,A). Two such are equivalent provided the points in $X-A$ can be made to correspond via a locally finite collection of paths in X all of which lie in $X-A$. A set of base points for (X,A) is irreducible provided any subset which is also a set of base points for $(X-A)$ has the same cardinality. (Note an irreducible set of base points for (X,A) is not always an irreducible set of base points for X. (S^1,S^0) is an example.) We can construct ε and Δ groups for X based on an irreducible set of base points for (X,A), and whenever we have a pair, we assume the base points are an irreducible set of base points for the pair. If X has no compact component, then any irreducible set of base points for (X,A) is one for X. Over the compact components of X, the Δ group is just the direct product of $h(\tilde{X},p)$ for one p in each component of A. As in the absolute case, we drop irreducible and write "set of base points" for "irreducible set of base points".

With a set of base points for (X,A), there is a natural map $\gamma(A;h\ ,\{\hat{x}_i\},\sim_F) \to \gamma(A;X\colon h\ ,\{\hat{x}_i\},\sim)$, where $\sim_F$ is the covering functor over A induced as follows. Let $\mathcal{D}(X)$ denote the following category. The objects are closed compact subsets $C \subseteq X$. The morphisms are the inclusions. Given $A \subseteq X$ a closed subset, there is a n natural map $\mathcal{D}(X) \to \mathcal{D}(A)$ given by $C \to C \cap A$. A lift functor $F : \mathcal{D}(A) \to \mathcal{D}(X)$ is a functor such that

$\wp(A) \xrightarrow{F} \wp(X) \to \wp(A)$ is the identity and such that the image of F is cofinal in $\wp(X)$. $\sim_F$ is the covering functor whose subgroups are the pullbacks of

$$\begin{array}{ccc} & & S\pi_1((X-F(C))^i,x_i) \\ & & \cap\downarrow \\ \pi_1((A-C)^i\ ,x_i) & \longrightarrow & \pi_1((X-F(C))^i,x_i) \end{array}$$

for $x_i \in A-C$, $C \in \wp(A)$. The existence of our natural map $\gamma(A: \cdots \sim_F) \longrightarrow \gamma(A;X: \cdots \sim)$ presupposes $\sim_F$ is compatible with A , but this is always the case since the appropriate cover of $(A-C)^i$ is sitting in $\widetilde{(X-F(C))}^i$. We denote this natural map by $\tau(A,X)$.

Notice first that $\tau(A,X)$ is a monomorphism since each map is. Moreover, $\tau(A,X)$ is naturally split. The splitting map is induced as follows. We need only define it on some cofinal subset of $\wp(X)$, so we define it on $\{F(C)|C \in \wp(A)\}$. $h_*((\tilde{A} \cap \widetilde{(X-F(C))}^i) \cup (\hat{x}_i),\hat{x}_i)$ goes to 0 if $x_i \notin A$, and it goes to $h_*(\widetilde{(A-C)}^i,\hat{x}_i)$ if $x_i \in A$, where $\sim$ in this last case is the cover given by the covering functor $\sim_F$. $\tilde{A} \cap \widetilde{(X-F(C))}^i$ is just several disjoint copies of $\widetilde{(A-C)}^i$ union covers of other components of $A-C$. The map collapses each of these covers of other components of $A-C$ to $\hat{x}_i$ and on the copies of $\widetilde{(A-C)}^i$ it is just the covering projection. At this point, this is all we can say about $\tau(A,X)$. This map however has many more properties and we shall return to it again.

Now let $f: X \to Y$ be a proper map between homogamous spaces. We have the mapping cylinder M_f. (M_f, X) is an homogamous pair (Corollary 1.1.2). Let $\{x_i\}$ be a set of base points for (M_f, X). We also have the homogamous pair (M_f, Y). By Lemma 1.2 a set of base points for Y is also a set of base points for (M_f, Y). If $\{y_i\}$ is such a set, $\tau(Y, M_f)$ is an isomorphism. This is seen by showing the splitting map is a monomorphism. But if we use the lift functor $F(C) = I \times f^{-1}(C) \cup C \subseteq M_f$ this is not hard to see.

Given a covering functor on M_f, it induces covering functors on X and Y, and these are the covering functors we shall use. Given a covering functor on Y, we can get a covering functor on M_f as follows. The subgroups to assign to $\pi_1(M_f - f^{-1}(C) \times I \cup C)$ are the subgroups for $\pi_1(Y - C)$. One can then assign subgroups to all other required sets in such a way as to get a covering functor. If we use the obvious lift functor for Y, the induced cover is the original.

By taking the cofinal collection $F(C)$, it is also not hard to see $\gamma(M_f, Y\colon h_n, \{\hat{y}_i\}, \sim) = 0$. We define $f: \gamma(X\colon h_n, \{\hat{x}_i\}, \sim) \to \gamma(Y\colon h_n, \{\hat{y}_i\}, \sim)$ if no component of Y is compact by $\gamma(X\text{:etc.}) \xrightarrow{\tau(X, M_f)} \gamma(X; M_f\text{:etc.})$

$$\longrightarrow \gamma(M_f\colon h_n, \{\hat{x}_i\}, \sim) \xrightarrow{\alpha_\Lambda} \gamma(M_f\colon h_n, \{\hat{y}_i\}, \sim) \xleftarrow{\cong}$$

$\gamma(Y\text{:etc.})$. Notice that this map may depend on the paths used to join $\{\hat{x}_i\}$ to $\{\hat{y}_i\}$. If f is properly 1/2-connected, (i.e. f induces isomorphisms on H^0 and H^0_{end} : compare this definition and the one in [11]) there is a natural choice of paths.

This choice is obtained as follows. Take a set of base points $\{x_i\}$ for X. By Lemma 1.1, $\{f(x_i)\}$ is a set of base points for Y. Let $\{x_i'\} \subseteq \{x_i\}$ be any subset obtained by picking precisely one element of $\{x_i\}$ in each $f^{-1}f(x_i)$. By Lemma 1 below, $\{x_i'\}$ is a set of base points for X. Thus we can always find a set of base points for X on which f is 1-1 and whose image under f is a set of base points for Y. Take such a set of points as a set of base points for (M_f, X). Take their image in Y as a set of base points for (M_f, Y). The paths joining these two sets are just the paths

$$\lambda_{x_i}(t) = \begin{cases} x_i \times t & 0 \leq t \leq 1 \\ f(x_i) & t = 1 \end{cases} .$$

Given a properly 1/2-connected map f, we can get another definition of the induced map. Pick a set of base points $\{x_i\}$ as in the last paragraph. Then we have $f_* : \gamma(X{:}h, \{\hat{x}_i\}, \sim) \longrightarrow \gamma(Y{:}h, \widehat{\{f(x_i)\}}, \sim)$ defined by taking $h(\widetilde{(X - C)}, \hat{x}_i) \longrightarrow h(\widetilde{Y - F(C)}, \widehat{f(x_i)})$ by f, where F is a lift functor which splits $\mathfrak{D}(Y) \longrightarrow \mathfrak{D}(X)$ and F is the lift functor used to get the covering functor for X from the one over Y. One sees easily the two definitions of f_* agree.

Now suppose we consider $i : A \subseteq X$ for an homogamous pair. Then we can define i_* as above. It is not hard to see

$$\begin{array}{ccc} \gamma(A;X:\text{etc.}) & \longrightarrow & \gamma(X:\text{etc.}) \\ \uparrow \tau(A,X) & \nearrow i_* & \\ \gamma(A:\text{etc.}) & & \end{array}$$

commutes, where the paths we use in defining i_* are $\lambda_{x_i}(t) = x_i \times t$ in $A \times I \cup X \times 1 = M_i$.

<u>Lemma 1</u>: If $f : X \to Y$ is a proper map which induces epimorphisms on H^0 and H^0_{end} , then, if $\{f(p)\}$ is a set of base points for Y , $\{p\}$ is a set of base points for X .

<u>Proof</u>: Since f is an epimorphism on H^0 , each path component of X has a point of $\{p\}$ in it.

Now define a cochain in $S^0(X)$ for some closed compact set $D \subseteq X$, φ_D , as follows. $\varphi_D(q) = 1$ if q is in a path component of $X - D$ with no point of $\{p\}$ in it and is 0 otherwise. $\delta\varphi_D = 0$ in S^1_{end} .

Since f is an epimorphism on H^0_{end} , there must be a chain in $S^0(Y)$, ψ , such that $f^*\psi = \varphi$ in $S^0_{end}(X)$. But this means there is some closed compact set $C \subseteq X$ such that $f^*\psi$ and φ agree for any point in $X - C$. Hence there is a closed, compact set $E \subseteq Y$ such that $f^{-1}(E) \supseteq C \cup D$. There is also a closed, compact $F \subseteq Y$ such that there is an $f(p)$ in each path component of $Y - E$ which is not contained in F. ψ restricted to $Y - E$ must be 0 since some component of $X - D$ which is not contained in $f^{-1}(E)$ has a point of $\{p\}$ has a point of $\{p\}$ in it. Hence φ restricted to $X - f^{-1}(E)$ is 0, so we are done. Q.E.D.

Definition: An homogamous pair (X,A) is properly 0-connected if the inclusion induces monomorphisms on H^0 and H^0_{end}. We have already defined properly 1/2-connected. If (X,A) is properly 0-connected we can choose a set of base points for the pair to be a set of base points for A. We say (X,A) is properly n-connected, $n \geq 1$ provided it is properly 1/2-connected, and, with base points chosen as above, $\Delta(X,A : \pi_k, \{x_i\}, \text{no cover}) = 0$, $1 \leq k \leq n$. It is said to be properly n-connected at ∞ provided it is properly 1/2-connected and $\varepsilon(X,A : \pi_k, \{x_i\}, \text{no cover}) = 0$, $1 \leq k \leq n$.

Proposition 1: If (X,A) is properly 1/2-connected, and if $i_* : \Delta(A: \pi_1, \{x_i\}, \text{no cover}) \to \Delta(X: \pi_1, \{x_i\}, \text{no cover})$ is onto, (X,A) is properly 1-connected and conversely.

Proof: If (X,A) is properly 1/2-connected, $\Delta(A: \pi_0, \{x_i\}, \text{no cover}) \to \Delta(X: \pi_0, \{x_i\}, \text{no cover})$ is seen to be an isomorphism by applying Theorem 2.4 to the kernel and cokernel of this map, together with the definition of a set of base points.

Hence $\Delta(A;X: \pi_1) \to \Delta(X:\pi_1) \to \Delta(X,A: \pi_1) \to 0$

is exact.

$$\begin{array}{ccc} \Delta(A;X:\pi_1) & \longrightarrow & \Delta(X:\pi_1) \\ \uparrow{\scriptstyle \tau(A,X)} & \nearrow{\scriptstyle i_*} & \\ \Delta(A:\pi_1) & & \end{array}$$

commutes, and i_* is an epimorphism. Hence $\Delta(X,A:\pi_1) = 0$, so (X,A) is properly 1-connected.

The converse follows trivially from Proposition 2 and the definitions. □

<u>Proposition 2</u>: Let (X,A) be a properly 1-connected pair. Then $\tau(A,X)$ is an isomorphism if the base points for the pair are a set of base points for A. We may use any lift functor to induce the covering functor.

<u>Proof</u>: If τ is an isomorphism on the ε objects, we need only show $h(\tilde{A}, \hat{x}_i) = h(\tilde{A} \cap \tilde{X}, \hat{x}_i)$. But $\tilde{A} = \tilde{A} \cap \tilde{X}$ if $\pi_1(A) \to \pi_1(X)$ is onto, so if we can show the result for the ε objects we are done.

We need only show τ is onto. By Theorem 2.4 applied to the cokernels of the maps inducing τ, we need only show that for each $C \in \mathcal{D}(X)$, there is a $D \supseteq C$ in $\mathcal{D}(X)$ such that

$$\begin{array}{ccc} h(\widetilde{(A-D)}^i, \hat{x}_i) & \longrightarrow & h((\tilde{A} \cap \widetilde{(X-F(D))}^i) \cup \hat{x}_i, \hat{x}_i) \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ h(\widetilde{(A-C)}^i, \hat{x}_i) & \xrightarrow{\tau_*} & h((\tilde{A} \cap \widetilde{(X-F(C))}^i) \cup \hat{x}_i, \hat{x}_i) \end{array}$$

satisfies Image $i_* \subseteq$ Image τ_* for all $x_i \not\subset D$.

We saw $\tilde{A} \cap \widetilde{(X-F(C))}^i$ was just some copies of $\widetilde{(A-C)}^i$, together with covers of components of $A - C \subseteq (X-F(C))^i$. Since (X,A) is properly 1/2-connected, we can find D so that Image $i_* \subseteq h$(copies of $\widetilde{(A-C)}^i$); i.e. we can find D so that $(X - F(D))^i \cap (A-C) = (X-F(D))^i \cap (A-C)^i$.

Since (X,A) is properly 1-connected, we can find $D_1 \supseteq D$ so that

$$\begin{array}{c} \pi_1(X-F(D_1),\ A-D_1,\ x_i) \\ \big\downarrow \\ \pi_1(X-F(C),\ A-C,\ x_i) \end{array}$$

is zero for all $x_i \not\subset D_1$. But this says all the copies of $\widetilde{(A-D_1)}^i$ in $\tilde{A} \cap \widetilde{(X-F(D_1))}^i$ go to the same copy of $\widetilde{(A-C)}^i$ in $\tilde{A} \cap \widetilde{(X-F(C))}^i$, namely the one containing $\hat{x}_i$. □

Theorem 2.4 can also be used to get

The subspace principle: Let (X,A) be an arbitrary homogamous pair. Then $\gamma(A;X : h, \{x_i\},\sim) = 0$ iff $\gamma(A: h, \{\hat{x}_i\},\sim) = 0$ provided, for the if part,

1) if A_α is a collection of disjoint subsets of A , $h(\bigcup_\alpha \tilde{A}_\alpha \cup p,p) \cong \bigoplus_\alpha h(\tilde{A}_\alpha \cup p,p)$

2) if $E \subseteq B$ are subsets of A , and if there is a

$q \in \tilde{E}$ such that $h(\tilde{E},q) \longrightarrow h(\tilde{B},q)$ is the zero map, then $h(\tilde{E} \cup p,p) \longrightarrow h(\tilde{B} \cup p,p)$ is the zero map for any p. h need only be natural on subsets of A.

<u>Proof</u>: Only if is clear as $\tau(A,X)$ is naturally split, so we concentrate on the if part.

$\gamma(A:h, \{\hat{x}_i\},\sim) = 0$ implies by Theorem 2.4 that we can find a cofinal sequence $C_0 \subseteq C_1 \subseteq \cdots$ of closed, compact subsets of A such that $h((\widetilde{A - C_j})^i,\hat{x}_i) \longrightarrow h((\widetilde{A - C_{j-1}})^i,\hat{x}_i)$ is the zero map for all $x_i \not\in C_j$. If $\gamma = \Delta$, $h(A)$ is also zero.

We then claim $h((\tilde{A} \cap (\widetilde{X - F(C_j)})^i \cup \hat{x}_i,\hat{x}_i) \longrightarrow h((\tilde{A} \cap (\widetilde{X - F(C_{j-1})})^i \cup \hat{x}_i,\hat{x}_i)$ is the zero map, and, if $\gamma = \Delta$, $h(\tilde{A} \cap \tilde{X}, \hat{x}_i) = 0$.

This last is easy since $\tilde{A} \cap \tilde{X}$ is the disjoint union of copies of $\tilde{A}$. Now $\tilde{A} \cap (\widetilde{X - F(C_j)})^i = \bigcup_\beta \bigcup_{\alpha_\beta} Z_{\alpha_\beta}$ where β runs over the path components of $A - C_j$ in $(X - F(C_j))^i$, and α_β runs over the path components of $\pi^{-1}((A - C_j)^\beta)$ where $\pi : (\widetilde{X - F(C_j)})^i \longrightarrow (X - F(C_j))^i$ is the covering projection and $(A - C_j)^\beta$ is the component of $A - C_j$ corresponding to β. Z_{α_β} is the α_β-th component of $\pi^{-1}((A - C_j)^\beta)$.

Similarly $\tilde{A} \cap (\widetilde{X - F(C_{j-1})})^i = \bigcup_\alpha \bigcup_{\alpha_b} Z_{\alpha_b}$. The map we are looking at is just the map induced on the direct sum by the maps $h(Z_{\alpha_\beta} \cup \hat{x}_i, \hat{x}_i) \to h(Z_{a_b} \cup \hat{x}_i, \hat{x}_i)$ for the unique a_b such that Z_{a_b} is mapped into by $Z_{\alpha\beta}$. $Z_{\alpha_\beta} = (A - C_j)^\beta$, so if $\hat{x}_i \in Z_{\alpha_\beta}$, the map is the zero map since it is then a map of the form $h(\widetilde{(A - C_j)}^i, \hat{x}_i) \to h(\widetilde{(A - C_{j-1})}^i, \hat{x}_i)$, which we know to be zero.

If $\hat{x}_i \not\in Z_{\alpha_\beta}$, the map is now a map of the form $h(\widetilde{(A - C_j)}^\beta \cup \hat{x}_i, \hat{x}_i) \to h(\widetilde{(A - C_{j-1})}^\beta \cup \hat{x}_i, \hat{x}_i)$, which is still zero by the properties of h. □

We now investigate the invariance of our construction.

<u>Theorem 2</u>: Let $f, g : X \to Y$ be properly homotopic maps between homogamous spaces. Then there is a set of paths Λ such that

$$\begin{array}{ccc} \gamma(X : h, \{x_i\}, \sim) & \xrightarrow{f_*} & \gamma(Y : h, \{\hat{y}_i\}, \sim) \\ & \searrow^{g_*} & \downarrow \alpha_\Lambda \\ & & \gamma(Y : h, \{\hat{y}_i\}, \sim) \end{array}$$

commutes.

<u>Proof</u>: Let $F : X \times I \to Y$ be the homotopy, and let M_F be its mapping cylinder. Then it is possible to pick paths so that

$$\begin{array}{ccc} \gamma(M_F : h, \{\hat{x}_i\} \times 0, \sim) & \longrightarrow & \gamma(M_F : h, \{\hat{y}_i\}, \sim) \\ \downarrow & & \downarrow \alpha_\Lambda \\ \gamma(M_F : h, \{\hat{x}_i\} \times 1, \sim) & \longrightarrow & \gamma(M_F : h, \{\hat{y}_i\}, \sim) \end{array}$$

commutes, where the horizontal maps are the maps induced by the paths joining $\{\hat{x}_i\} \times t$ to $\{\hat{y}_i\}$ $(t = 0, 1)$ and the left hand vertical map is the map induced by the canonical path $\hat{x}_i \times 0$ to $\hat{x}_i \times 1$ in $X \times I \subseteq M_F$.

It is now a chase of definitions to show the desired diagram commutes. []

<u>Corollary 2.1</u>: Let $f : X \to Y$ be a proper homotopy equivalence between two homogamous spaces. Then f_* is an isomorphism.

<u>Proof</u>: There is a standard derivation of the corollary from the theorem. []

<u>Corollary 2.2</u>: A proper homotopy equivalence between homogamous spaces is proper n-connected for all n (i.e. its mapping cylinder modulo its domain is a properly n-connected pair).

<u>Proof</u>: (M_f, X) is clearly properly 1/2-connected. $i_* : \Delta(X: \pi_1) \to \Delta(Y: \pi_1)$ is onto, so it is easy to show (M_f, X) is properly 1-connected. Then

$\gamma(X; M_f: \pi_k) \cong \gamma(X: \pi_k) \cong \gamma(M_f: \pi_k)$, so

$\gamma(M_f, X: \pi_k) = 0$. []

<u>Corollary 2.3</u>: If $f : X \to Y$ is a proper homotopy equivalence, $\gamma(M_f, X: h, \{\hat{x}_i\}, \sim) = 0$.

<u>Proof</u>: Since f is properly 1-connected, $\gamma(X; M_f : h, \text{etc.}) \cong \gamma(X : h, \text{etc.})$ by Proposition 2. $\gamma(X:h, \text{etc.}) \cong \gamma(M_f : h, \text{etc.})$ by Corollary 2.1. Hence $\gamma(M_f, X : h, \text{etc.}) = 0$. □

In the other direction we have

<u>Theorem 3</u>: (Proper Whitehead) Let $f : X \to Y$ be properly n-connected. Then for a locally finite CW complex, K , of dimension $\leq n$, $f_\#: [K,X] \to [K,Y]$ is an epimorphism. If f is properly (n+1)-connected, $f_\#$ is a bijection.

Remarks: $[K,X]$ denotes the proper homotopy classes of proper maps of K to X . For a proof of this result, see [11] Theorem 3.4 and note the proof is valid for X and Y homogamous.

<u>Definition</u>: An homogamous space Z is said to satisfy Dn provided the statement of Theorem 3 holds for Z in place of K and for each proper map f between homogamous spaces.

<u>Proposition 3</u>: Let Z be properly dominated by a space satisfying Dn . Then Z satisfies Dn .

<u>Proof</u>: We leave it to the reader to modify the proof of Proposition 1.1 to show Z is homogamous iff it is properly dominated by an homogamous space. Let K be a space satisfying Dn and properly dominating Z .

Then $[Z,X]$ is a natural summand of $[K,X]$ for any homogamous X, so the result follows. □

We finish this section by proving a proper Hurewicz and a proper Namioka theorem.

Definition: A (as opposed to the) universal covering functor for X is a covering functor ~ such that $\varepsilon(X:\pi_1,\sim) = \Delta(X:\pi_1,\sim) = 0$. Note that if the universal covering functor is compatible with X, then it is a universal covering functor for X. There are other examples however.

We start towards a proof of the Hurewicz theorem. The proof mimics Spanier [35] pages 391-393. We first prove

Lemma 2: Suppose $\mathcal{G} = \{G_{ij}\}$ is a system of singular chain complexes on spaces X_{ij}. Suppose the projection maps $G_{ij} \to G_{ij-1}$ are induced by continuous maps of the spaces $X_{ij} \to X_{ij-1}$. Assume $i \geq 0$, $j \geq 0$.

Assume we are given a system $C = \{C_{ij}\}$, where each C_{ij} is a subcomplex of G_{ij} which is generated by the singular simplices of G_{ij} which occur in C_{ij}. Also assume that the projection $G_{ij} \to G_{ij-1}$ takes $C_{ij} \to C_{ij-1}$.

Lastly assume that to every singular simplex $\sigma : \Delta^q \to X_{ij}$ for $j \geq n$ (n is given at the start and held fixed throughout) there is assigned a map $P_{ij}(\sigma): \Delta^q \times I \to X_{ij-n}$ which satisfies

a) $P_{ij}(\sigma)(z,0) = \bar{\sigma}(z)$, where $\bar{\sigma} : \Delta^q \xrightarrow{\sigma}$ $X_{ij} \xrightarrow{\text{projection}} X_{ij-n}$

b) Define $\sigma_1 : \Delta^q \longrightarrow X_{ij-n}$ by $\sigma_1(z) = P_{ij}(\sigma)(z,1)$. Then we require that $\sigma_1 \in C_{ij-n}$, and, if $\sigma \in C_{ij}$, then $\sigma_1 = \bar{\sigma}$.

c) If $e_q^k : \Delta^{q-1} \to \Delta^q$ omits the k^{th} vertex, then $P_{ij}(\sigma) \circ (e_q^k \times 1) = P_{ij}(\sigma^{(k)})$.

Then $\varepsilon(C) \subseteq \varepsilon(\mathcal{S})$ is an homology equivalence. (Compare Spanier [35], page 392, Lemma 7).

Proof: Let $\alpha(i,k) : C_{ik} \subseteq G_{ik}$ be the inclusion, and let $\tau(i,k) : G_{ik} \to C_{ik-n}$ be defined by $\tau(i,k)(\sigma) = \bar{\sigma}_1$ and extend linearly. (Here we must assume $k \geq n$). Define $\rho_r : G_{ik} \to G_{ik-r}$ to be projection.

One easily checks that condition c) makes $\tau(i,k)$ into a chain map. $\tau(i,k) \circ \alpha(i,k) : C_{ik} \to C_{ik-n}$ is just the map induced on the C_{ij} by ρ_n on the G_{ij} . This follows from condition b).

We claim $\alpha(i,k-n) \circ \tau(i,k) : G_{ik} \to G_{ik-n}$ is chain homotopic to ρ_n . To show this, let $D_g : S(\Delta^q) \to S(\Delta^q \times I)$ be a natural chain homotopy between $\Delta(h_1)$ and $\Delta(h_o)$, where $h_o, h_1 : \Delta^q \to \Delta^q \times I$ are the obvious maps (S is the singular chain functor).

Define a chain homotopy $D_{ik} : S(X_{ik}) \to S(X_{ik-n})$ by $D_{ik}(\sigma) = S(P_{ik}(\sigma))\,(D_q(\xi_q))$ (where $\xi_q : \Delta^q \subseteq \Delta^q$ is the identity) where σ is a q-simplex. One checks, using

c) and the naturality of D_q , that $\partial D_{ik} + D_{ik}\partial = \rho_n - \alpha(i,k-n) \circ \tau(i,k)$.

By definition, $\varepsilon(\mathcal{G}) = \varprojlim_k \mu(G_{ik})$ and $\varepsilon(C) = \varprojlim_k \mu(C_{ik})$. Since

$$\begin{array}{ccc} C_{ik} & \xrightarrow{\alpha(i,k)} & G_{ik} \\ \downarrow & & \downarrow \\ C_{ik-1} & \xrightarrow{\alpha(i,k-1)} & G_{ik-1} \end{array}$$

commutes, we get a chain map $\alpha : \varepsilon(C) \to \varepsilon(\mathcal{G})$, which is just the inclusion.

Since $\tau(i,k) \circ \alpha(i,k) = \rho_n$,

$$\begin{array}{ccccc} C_{ik} & \xrightarrow{\alpha(i,k)} & C_{ik} & \xrightarrow{\tau(i,k)} & C_{ik-n} \\ \downarrow{\rho_1} & & \downarrow{\rho_1} & & \downarrow{\rho_1} \\ C_{ik-1} & \xrightarrow{\alpha(i,k-1)} & G_{ik-1} & \xrightarrow{\tau(i,k-1)} & C_{i,k-1-n} \end{array}$$

commutes along the outside square. Unfortunately the right-hand square may not commute as we have made no stipulation as to the behavior of P_{ij} with respect to ρ_1 . Similarly

$$\begin{array}{ccccc} G_{ik} & \xrightarrow{\tau(i,k)} & C_{ik-n} & \xrightarrow{\alpha(i,k-n)} & G_{ik-n} \\ \downarrow{\rho_1} & & & & \downarrow{\rho_1} \\ G_{ik-1} & \xrightarrow{\tau(i,k-1)} & C_{ik-1-n} & \xrightarrow{\alpha(i,k-1-n)} & G_{ik-1-n} \end{array}$$

may not commute. However, since $\alpha(i,k-n) \circ \tau(i,k)$ is chain homotopic to ρ_n ,

$$
\begin{array}{ccccc}
H_*(G_{ik}) & \xrightarrow{H(\tau)} & H_*(C_{ik-n}) & \xrightarrow{H(\alpha)} & H_*(G_{ik-n}) \\
\downarrow{\scriptstyle H(\rho_1)} & & & & \downarrow{\scriptstyle H(\rho_1)} \\
H_*(G_{ik-1}) & \xrightarrow{H(\tau)} & H_*(C_{ik-1-n}) & \xrightarrow{H(\alpha)} & H_*(G_{ik-1-n}
\end{array}
$$

does commute.

Define $\beta(i,k)\colon G_{ik} \to C_{ik-2n}$ for $k \geq 2n$ by $\beta(i,k) = \tau(i,k-n) \circ \alpha(i,k-n) \circ \tau(i,k)$. We claim

$$
\begin{array}{ccc}
H_*(G_{ik}) & \xrightarrow{H(\beta)} & H_*(C_{ik-2n}) \\
\downarrow{\scriptstyle H(\rho_1)} & & \downarrow{\scriptstyle H(\rho_1)} \\
H_*(G_{ik-1}) & \xrightarrow{H(\beta)} & H_*(C_{ik-1-2n}
\end{array}
$$

commutes. To see this, look at

$$
\begin{array}{ccccccc}
H_*(G_{ik}) & \xrightarrow{H(\tau)} & H_*(C_{ik-n}) & \xrightarrow{H(\alpha)} & H_*(G_{ik-n}) & \xrightarrow{H(\tau)} & H_*(G_{ik-2n}) \\
\downarrow{\scriptstyle H(\rho_1)} & \text{I} & \downarrow{\scriptstyle H(\rho_1)} & \text{II} & \downarrow{\scriptstyle H(\rho_1)} & \text{III} & \downarrow{\scriptstyle H(\rho_1)} \\
H_*(G_{ik-1}) & \xrightarrow{H(\tau)} & H_*(C_{ik-1-n}) & \xrightarrow{H(\alpha)} & H_*(G_{ik-1-n}) & \xrightarrow{H(\tau)} & H_*(G_{ik-1-2n})
\end{array}
$$

The square II commutes since it already does on the chain level. Similarly the square II + III commutes. The square I + II commutes on the homology level. The desired commutativity is now a diagram chase.

Now define $\tau : \varepsilon(H_*(\mathcal{G})) \to \varepsilon(H_*(C))$ using the

$H(\beta)$'s . We also have $H(\alpha) : H_*(\varepsilon(C)) \to H_*(\varepsilon(\mathscr{S}))$. By Corollary 1.2.2 we have $H(\alpha): \varepsilon(H_*(C)) \to \varepsilon(H_*(\mathscr{S}))$. $\tau \circ H(\alpha)$ and $H(\alpha) \circ \tau$ are both induced from the maps $H(\rho_{2n})$, and hence are the identities on the inverse limits. Q.E.D.

Lemma 3: Let X be an homogamous space. Then we can find a countable, cofinal collection of closed, compact sets $C_j \subseteq X$, with $C_j \subseteq C_{j+1}$. Let $G_{ij} = S((\widetilde{X - C_j})^i, \hat{x}_i)$, the singular chain groups on $(\widetilde{X - C_j})^i$. Let $C_{ij} = S((\widetilde{X - C_j})^i , \tilde{A} \cap (\widetilde{X - C_j})^i, \hat{x}_i)^n$ (see Spanier [35], page 391 for a definition).

Suppose (X,A) is properly 1-connected and properly n-connected at ∞ for $n \geq 0$. Then the inclusion map $\varepsilon(C) \subseteq \varepsilon(\mathscr{S})$ is an homology equivalence. (Notice that if we pick a set of base points x_i for A , they are a set for the pair, and $\varepsilon(\mathscr{S}) = \varepsilon(X;H_*,\{\hat{x}_i\},\sim)$.)

Proof: Let $r = \min(q,n)$. Then we produce for every $\sigma \in G_{ij}$ a map $P_{ij}(\sigma) : \Delta^q \times I \to ((\widetilde{X - C_{j-r}})^i, \hat{x}_i)$ which satisfies

a) $P_{ij}(\sigma)(z,0) = \bar{\sigma}: p^q \xrightarrow{\sigma} ((\widetilde{X - C_j})^i) \xrightarrow{\text{projection}} ((\widetilde{X - C_{j-r}})^i)$.

b) If $\sigma_1(z) = P_{ij}(\sigma)(z,1)$, $\sigma_1 \in C_{ij-r}$, and if $\sigma \in C_{ij}$, $P_{ij}(\sigma) : \Delta^q \times I \xrightarrow{\text{proj}} \Delta^q \xrightarrow{\sigma} ((\widetilde{X - C_j})^i) \xrightarrow{\text{projection}} ((\widetilde{X - C_{j-r}})^i)$

c) $P_{ij}(\sigma) \circ (C_q^k \times 1) =$

$$\begin{cases} (\text{projection 1 step}) \circ P_{ij}(\sigma^{(k)}) & q \leq n \\ P_{ij}(\sigma^{(k)}) & q > n . \end{cases}$$

From such a P, it is easy to see how to get a P as required by our first lemma. We remark that C_{ij} and G_{ij} satisfy all the other requirements to apply the lemma. Hence Lemma 2 will then give us the desired conclusion.

We define P_{ij} by induction on q. Let $q = 0$. Then $\sigma \in G_{ij}$ is a map $\sigma : \Delta^0 \to ((\widetilde{X - C_j})^i)$. Since the point $\sigma(\Delta^0)$ lies in the same path component of $(\widetilde{X - C})^i$ as $\hat{x}_i$, there is a path joining them. Let $P_{ij}(\sigma)$ be such a path. If $\sigma(\Delta^0) = \hat{x}_i$, $P_{ij}(\sigma)$ should be the constant path. This defines P_{ij} for $q = 0$, and P is easily seen to satisfy a) - c).

Now suppose P_{ij} is defined for all σ of degree $< q$, $0 < q \leq n$ so that it has properties a) - c).

If $\sigma \in C_{ij}$, b) defines $P(\sigma)$, and P then satisfies a) and c). So suppose $\sigma \notin C_{ij}$. a) and c) define P_{ij} on $\Delta^q \times 0 \cup \dot{\Delta}^q \times I$; i.e. we get a map $f : \Delta^q \times 0 \cup \dot{\Delta}^q \times I \longrightarrow (\widetilde{X - C_{j-q+1}})^i$. There is a homeomorphism $h : E^q \times I \approx \Delta^q \times I$ such that $h(E^q \times 0) = \Delta^q \times 0 \cup \dot{\Delta}^q \times I$; $h(S^{q-1} \times 0) = \dot{\Delta}^q \times 1$; and $h(S^{q-1} \times I \cup E^q \times 1) = \Delta^q \times 1$.

Let $g : (E^q, S^{q-1}) \to ((\widetilde{X - C_{j-q+1}})^i, \tilde{A} \cap (\widetilde{X - C_{j-q+1}})^i)$ be defined by $g = f \circ h$.

Because $q \leq n$ and (X,A) is properly n-connected at ∞, we could have chosen (and did) the C_j so that

$$\pi_q(X - C_k, A \cap (X - C_k),*) \longrightarrow \pi_q(X - C_{k-1}, A \cap (X - C_{k-1}),*)$$

is the zero map for $q \leq n$. Thus we get a homotopy

$$H : (E^q, S^{q-1}) \times I \longrightarrow ((\widetilde{X - C_{j-q}})^i , \tilde{A} \cap (\widetilde{X - C_{j-q}})^i)$$

between $\rho_1 \circ g$ and an element of C_{ij-q}.

Define $P_{ij}(\sigma)$ to be the composite

$\Delta^q \times I \xrightarrow{h^{-1} \times id} E^q \times I \xrightarrow{H} (\widetilde{X - C_{j-q}})^i$. P_{ij} clearly satisfies a) and b). Since h was chosen carefully, c) is also satisfied.

In this way P is defined for all simplicies of degree $\leq n$. Note that a singular simplex of degree $> n$ is in C_{ij} iff every proper face is in C_{ij}.

Suppose that P has been defined for all degrees $< q$, where $q > n$. If $\sigma \in C_{ij}$, we define $P_{ij}(\sigma)$ by b) as usual. It satisfies a) and c). So suppose $\sigma \notin C_{ij}$. Then a) and c) define a map $f : \Delta^q \times 0 \cup \dot{\Delta}^q \times I \longrightarrow (\widetilde{X - C_{j-n}})^i$. By the homotopy extension property we can extend f to some map $P(\sigma) : \Delta^q \times I \longrightarrow (\widetilde{X - C_{j-n}})^i$. It clearly satisfies a) and c). It also satisfies b) since every proper face of σ_1 is in C_{ij-n}. Hence we have defined our P.

Q.E.D.

Now define $\varepsilon^{(n)}(X,A\colon H_q,\sim)$ to be

$$\varepsilon(H_q(S(\widetilde{(X-C_j)}^i,\ \tilde{A}\cap\widetilde{(X-C_j)}^i,\hat{x}_i)^n \big/ S(\widetilde{(X-C_j)}^i\ ,\ \tilde{A}\cap\widetilde{(X-C_j)}^i,\ \hat{x}_i)^n \cap S(\tilde{A}\cap\widetilde{(X-C_j)}^i,\ \hat{x}_i)))\ .$$

Then there are natural maps

$$\varepsilon^{(n)}(X,A : H_q,\sim) \to \varepsilon^{(n-1)}(X,A:H_q,\sim) \to \cdots$$

$$\to \varepsilon(X,A : H_q,\ \{\hat{x}_i\},\sim)\ .$$

Lemma 4: Assume (X,A) is a properly 1-connected pair which is properly n-connected at ∞ for some $n \geq 0$. Then the natural map $\varepsilon^{(n)}(X,A\colon H_q,\sim) \to \varepsilon(X,A\colon H_q,\{\hat{x}_i\},\sim)$ is an isomorphism for all q.

Proof: We have the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & S(\widetilde{(X-C_j)}^i,\text{etc.})^n \cap S(\tilde{A}\cap\widetilde{(X-C_j)}^i) & \longrightarrow & S(\widetilde{(X-C_j)}^i,\text{etc.}) \\
 & \downarrow \alpha & & \downarrow \beta \\
0 \longrightarrow & S(A \cap \widetilde{(X-C_j)}^i) & \longrightarrow & S(\widetilde{(X-C_j)}^i)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & (\text{the quotient complex}) & \longrightarrow 0 \\
 & \downarrow \gamma & \\
\longrightarrow & S(\widetilde{(X-C_j)}^i\ ,\ \tilde{A}\cap\widetilde{(X-C_j)}^i) & \longrightarrow 0
\end{array}$$

where (the quotient complex) was used in defining $\varepsilon^{(n)}(X,A)$.

Now, since (X,A) is properly 1-connected, the subspace groups $\varepsilon^{(n)}(A;X : \text{etc.})$ and $\varepsilon(A;X : \text{etc.})$ are the absolute groups. Since (A,A) is properly 1-connected and properly n-connected at infinitely for all n, Lemma 3 says $\varepsilon(\alpha)$ is an isomorphism on homology. Similarly Lemma 3 says $\varepsilon(\beta)$ is an isomorphism on homology. Thus $\varepsilon(\gamma)$ is an isomorphism on homology as asserted. Q.E.D.

<u>Theorem 4</u>: Suppose (X,A) is properly 1-connected and properly (n-1)-connected at ∞ for some $n \geq 2$. Then the Hurewicz map $\varepsilon(X,A\colon \pi_n', \{\hat{x}_i\},\sim) \longrightarrow \varepsilon(X,A : H_n,\{\hat{x}_i\},\sim)$ is an isomorphism, where $\pi_n'((\widetilde{X - C_j})^i, \tilde{A} \cap (\widetilde{X - C_j})^i \cup \hat{x}_i, \hat{x}_i)$ is π_n quotiented out by the action of $\pi_1(\tilde{A} \cap (\widetilde{X - C_j})^i \cup \hat{x}_i, \hat{x}_i)$.

<u>Proof</u>: The usual Hurewicz theorem contains the fact that $\pi_n'((\widetilde{X - C_j})^i, \tilde{A} \cap (\widetilde{X - C_j})^i, \hat{x}_i) \longrightarrow H_n^{(n-1)}((\widetilde{X - C_j})^i, \tilde{A} \cap (\widetilde{X - C_j})^i, \hat{x}_i)$ is an isomorphism. Thus $\varepsilon(X,A : \pi_n', \{\hat{x}_i\},\sim) \to \varepsilon^{(n-1)}(X,A : H_n,\sim)$ is an isomorphism. But Lemma 4 says $\varepsilon^{(n-1)}(X,A\colon H_n,\sim) \longrightarrow \varepsilon(X,A\colon H_n, \{\hat{x}_i\},\sim)$ is an isomorphism. $\square$

<u>Theorem 5</u>: Suppose that $\varepsilon(A\colon\pi_1, \{\hat{x}_i\},\sim_F) = 0$ where $\sim_F$ is the cover over A induced by the lift functor F from a cover $\sim$ over X. Then the natural surjection $\varepsilon(X,A\colon \pi_n,\{\hat{x}_i\},\sim) \longrightarrow \varepsilon(X,A : \pi_n', \{\hat{x}_i\},\sim)$ is an isomorphism.

Proof: Set $G_{ij} = \pi_n(\widetilde{(X-C_j)}^i, \tilde{A} \cap \widetilde{(X-C_j)}^i \cup \hat{x}_i, \hat{x}_i)$ and $H_{ij} = \pi_n'(\widetilde{(X-C_j)}^i, \tilde{A} \cap \widetilde{(X-C_j)}^i \cup \hat{x}_i, \hat{x}_i)$. Define K_{ij} to be the kernel of $G_{ij} \to H_{ij} \to 0$. K_{ij} is generated by elements of the form $x - \alpha x$, where $x \in \pi_n(\widetilde{(X-C_j)}^i, \tilde{A} \cap \widetilde{(X-C_j)}^i \cup \hat{x}_i, \hat{x}_i)$ and $\alpha \in \pi_1(\tilde{A} \cap \widetilde{(X-C_j)}^i \cup \hat{x}_i, \hat{x}_i)$.

Since $\varepsilon(A : \pi_1, \{\hat{x}_i\}, \sim_F) = 0$, the subspace principle says that we can assume the map

$$\pi_1(\tilde{A} \cap \widetilde{(X-C_j)}^i \cup \hat{x}_i, \hat{x}_i) \longrightarrow \pi_1(\tilde{A} \cap \widetilde{(X-C_{j-1})}^i \cup \hat{x}_i, \hat{x}_i)$$

is the zero map. Then $K_{ij} \to K_{ij-1}$ takes $x - \alpha x$ to $i_*(x) - i_*(\alpha x) = i_*(x) - i_\#(\alpha)$. $i_*(x) = i_*(x) - i_*(x) = 0$, so this map is the zero map. □

Theorem 6: Let (X,A) be a properly 1-connected pair. Then, for any covering functor $\sim$ on X, the natural map $\varepsilon(X,A : \pi_n, \{\hat{x}_i\}, \sim) \longrightarrow \varepsilon(X,A : \pi_n, \{\hat{x}_i\},$ no cover) is an isomorphiam.

Proof: We have

$$\begin{array}{ccccc} \cdots \to \varepsilon(A:\pi_k,\{x_i\},\sim_F) & \longrightarrow & \varepsilon(X: \pi_k,\ldots,\sim) & \longrightarrow & \varepsilon(X,A: \text{etc.}) \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \to \varepsilon(A:\pi_k,\{\hat{x}_i\},\text{no cover}) & \longrightarrow & \varepsilon(X:\text{etc.}) & \longrightarrow & \varepsilon(X,A: \text{etc.}) \end{array}$$

commutes. The first two maps are clearly isomorphisms for $k \geq 2$, so the third is for $k \geq 3$. Moreover

$$\begin{array}{ccc} \varepsilon(A : \pi_1, \sim_F) & \longrightarrow & \varepsilon(X, \pi_1, \sim) \\ \downarrow & & \downarrow \\ \varepsilon(A : \pi_1, \text{no cover}) & \longrightarrow & \varepsilon(X : \text{etc.}) \end{array}$$

is a pullback since it is obtained as the ε construction applied to pullbacks. Hence the theorem remains true for $k = 2$. □

<u>Corollary 6.1</u>: Suppose (X,A) is a properly 1-connected pair which is $(n-1)$-connected at ∞ for some $n \geq 2$. If $n = 2$, assume $\varepsilon(A:\pi_1$, no cover $\rightarrow$ $\varepsilon(X:\pi_1$, no cover) is an isomorphism. Then the Hurewicz map $\varepsilon(X,A: \pi_n$, no cover) $\longrightarrow \varepsilon(X,A: H_n, \sim)$ is an isomorphism, where $\sim$ is any universal covering functor for X.

<u>Theorem 7</u>: Theorems 4, 5, and 6 are true (after appropriate changes) with Δ instead of ε. They are also true for the absolute groups.

<u>Proof</u>: Easy. □

Now suppose (X,A) is a locally compact CW pair. Then we might hope to improve our Hurewicz theorems by getting information about the second non-zero map (see [42]). We do this following Hilton [13].

<u>Definition</u>: Two proper maps $f,g: X \to Y$ are said to be properly n-homotopic if for every proper map $\emptyset : K \to X$, where K is a locally compact CW complex of dimension $\leq n$, $f\emptyset$ is properly homotopic to $g\emptyset$.

X and Y are of the same proper n-homotopy type provided there exist proper maps $f : X \to Y$ and $g : Y \to X$ with $f \circ g$ and $g \circ f$ properly n-homotopic to the identity. Two locally compact CW complexes, K and L, are said to be of the same proper n-type iff K^n and L^n have the same proper (n-1)-type. A proper cellular map $f : K \to L$ is said to be a proper n-equivalence provided there is a proper map $g: L^{n+1} \to K^{n+1}$ with $f|K^{n+1} \circ g$ and $g \circ f|K^{n+1}$ properly n-homotopic to the identity.

A proper J_m-pair, (X,A), is a properly 1-connected, locally compact CW pair such that the maps

$$\Delta(X^{n-1} \cup A,\ A:\ \pi_n,\ \text{no cover}) \longrightarrow \Delta(X^n \cup A,\ A:\ \pi_n,\ \text{no cover})$$

are zero for $2 \leq n \leq m$. A proper J_m-pair at ∞ is the obvious thing.

<u>Lemma 5</u>: The property of being a J_m-pair is an invariant of m-type.

<u>Proof</u>: See Hilton [13]. Q.E.D.

<u>Theorem 8</u>: Let (X,A) be a proper J_m-pair at ∞. Then the Hurewicz map $h_n : \varepsilon(X,A: \infty_n, \{\hat{x}_i\}, \text{no cover}) \to \varepsilon(X,A: H_n,\{\hat{x}_i\},\sim)$, where $\sim$ is a universal covering functor for X, satisfies h_n is an isomorphism for $n \leq m$, and h_{m+1} is an epimorphism.

<u>Proof</u>: See Hilton [13]. □

<u>Corollary 8.1</u>: The same conclusions hold for a proper J_m-pair with the Δ groups.

<u>Corollary 8.2</u>: Let (X,A) be a properly $(n-1)$-connected, locally compact CW pair, for $n \geq 2$. If $n = 2$, let $\Delta(A: \pi_1, \text{no cover}) \to \Delta(X: \pi_1, \text{no cover})$ be an isomorphism. Then the Hurewicz map $h_n : \Delta(X,A: \pi_n, \text{no cover}) \to \Delta(X,A: H_n, \sim)$ is an isomorphism, where $\sim$ is a universal covering functor for X. h_{n+1} is an epimorphism.

<u>Proof</u>: In section 5 we will see there is a locally finite 1-complex $T \subseteq A$ such that (A,T) is a proper 1/2-equivalence and $\Delta(T: \pi_k, \text{no cover}) = 0$ for $k \geq 1$. Then (T, T) is certainly a proper J_n-complex. $(T, T) \subseteq (X,A)$ is a proper $(n-1)$-equivalence, so (X, A) is a J_n-complex by Lemma 5. □

<u>Theorem 9</u>: (Namioka [28]) Let $\emptyset : (X, A) \to (Y, B)$ be a map of pairs of locally compact CW complexes. Let $\emptyset|X$ and $\emptyset|A$ be properly n-connected, $n \geq 1$ ($\emptyset|X$ and $\emptyset|A$ should induce isomorphisms on $\Delta(\ : \pi_1, \text{no cover})$ if $n = 1$). Then the Hurewicz map $h_{n+1}: \Delta((M_\emptyset : M_{\emptyset|A}, X) : \pi_{n+1}, \text{no cover}) \longrightarrow \Delta((M_\emptyset : M_{\emptyset|A}, X): H_{n+1}, \sim)$, where $\sim$ is a universal covering functor of $M_\emptyset$, is an epimorphism.

<u>Proof</u>: $(M_\emptyset : M_{\emptyset|A}, X)$ is a triad, and the groups in question are the proper triad groups. The reader should have no trouble defining these groups. We can pick a set of base points for (X,A), and it will also be a set for our triad.

The triad groups fit into a long exact sequence $\Delta((M_{\emptyset|A},A);\ M_\emptyset) \to \Delta((M_\emptyset,X)) \to \Delta((M_\emptyset\colon M_{\emptyset|A},X)) \to \cdots$, where again we get subspace groups. Since $\emptyset|A$ and $\emptyset|X$ are properly n-connected. h_m for $(M_\emptyset,X)$ is an isomorphism $m \le n$ and an epimorphism for $m = n+1$. By the subgroup principle, h_m for $((M_{\emptyset|A},A);\ M_\emptyset)$ is an isomorphism for $m \le n$ and an epimorphism for $m = n+1$. The strong version of the 5-lemma now shows the triad h_n an isomorphism and the triad h_{n+1} an epimorphism. □

Notation: $\Delta_*(X,A\colon \sim)$ will hereafter denote $\Delta(X,A\colon H_*,\{\hat{x}_i\},\sim)$ for some set of base points for the pair (X,A) . Similar notation will be employed for homology n-ad groups, subspace groups, etc.

We conclude this section with some definitions and computations.

<u>Definition</u>: An homogamous space X is said to have monomorphic ends, provided $\Delta(X\colon\pi_1,\text{no cover}) \to \underset{i\in I}{\times}\ \pi_1(X,x_i)$ is a monomorphism (equivalently $\varepsilon \to \mu_i$ is a monomorphism). A space has epimorphic ends provided the above map is onto, and isomorphic ends if the map is an isomorphism.

As examples, if X is an homogamous space which is not compact, $X \times R$ has one, isolated end (see [32]) which is epimorphic. $X \times R^2$ has isomorphic ends. These results use Mayer-Vietoris to compute the number of ends of $X \times R$ and van-Kampen to yield the π_1 information, using the following pushout

$$\begin{array}{ccc} (X - C) \times (Y - D) & \longrightarrow & X \times (Y - D) \\ \downarrow & & \downarrow \\ (X - C) \times Y & \longrightarrow & X \times Y - C \times D \end{array}$$

In fact, this diagram shows that if X and Y are not compact, $X \times Y$ has one end, which is seen to be epimorphic since $\pi_1(X \times Y - C \times D) \to \pi_1(X \times Y)$ is easily seen to be onto. If X has epimorphic ends, $\pi_1(X - C, p) \to \pi_1(X, p)$ must always be onto, so if X and Y have epimorphic ends, $X \times Y$ has one isomorphic end.

Monomorphic ends are nice for then the third example of covering functor that we gave (the universal cover of X but no more) becomes a universal covering functor. Farrell and Wagoner ([9] or [11]) then showed that a proper map $f : X \to Y$, X, Y locally compact CW, with X having monomorphic ends is a proper homotopy equivalence provided it is a properly 1-connected map; a homotopy equivalence; and $f^*: H_c^*(\tilde{Y}) \to H_c^*(\tilde{X})$ is an isomorphism where ~ denotes the universal cover (coefficients are the integers).

Section 4. Proper cohomology, coefficients and products

In attempting to understand ordinary homotopy theory, cohomology theory is an indispensable tool. In ordinary compact surgery, the relationship between homology and cohomology in Poincaré duality spaces forms the basis of many of the results. To extend surgery to paracompact objects, we are going to need a cohomology theory.

If one grants that the homology theory that we constructed in section 3 is the right one, then the correct cohomology theory is not hard to intuit. To be loose momentarily, in homology we associate to each compact set C the group $H_*(\widetilde{M-C})$. If $M-C$ is a manifold with boundary, Lefschetz duality tells us this is dual to $H_c^*(\widetilde{\overline{M-C}}, \widetilde{\partial C})$, where $\overline{M-C}$ is the closure of $M-C$. If $C \subseteq D$, we have a map $H_*(\widetilde{M-D}) \longrightarrow H_*(\widetilde{M-C})$, so we need a map $H_c^*(\widetilde{\overline{M-D}}, \partial D) \to H_c^*(\widetilde{\overline{M-C}}, \widetilde{\partial C})$. A candidate for this map is $H_c^*(\widetilde{\overline{M-D}}, \widetilde{\partial D}) \xrightarrow{tr}$ $H_c^*(\overline{\overline{M-D}}, \overline{\overline{\partial D}}) \xleftarrow{ex} H_c^*(\widetilde{\overline{M-C}}, \widetilde{D-C}) \xrightarrow{inc} H_c^*(\widetilde{M-C}, \widetilde{\partial C})$, where $\overline{\overline{M-D}} = \pi^{-1}(M-D)$ ($\pi: \widetilde{M-C} \to M-C$). inc is the map induced by inclusion, tr is the trace, and ex is an excision map.

The first problem that arises is that ex need not be an isomorphism. This problem is easily overcome. We define $\mathcal{O}(X)$ to be the category whose objects are open subsets of X whose closure (in X) is compact. If $U, V \in \mathcal{O}(X)$, there is a morphism $U \to V$ iff $\bar{U} \subseteq V$ or $U = V$. $\mathcal{O}(X)$ will be our diagram scheme. Note we have a functor $\mathcal{O}(X) \to D(X)$ which sends $U \to \bar{U}$. Since X is locally compact, this functor has a cofinal image (X is homogamous, hence locally compact).

The second problem which arises concerns covering functors. Since $X - U$, $U \in \mathcal{O}(X)$ is closed, it is hard to get conditions on X so that $X - U$ has arbitrary covers. There are two solutions to this problem. We can restrict $\mathcal{O}(X)$ (e.g. if X is an homogamous CW complex, and if we pick sets U so that $X - U$ is a subcomplex, then we always have covers), or we can ignore the problem. We choose the latter alternative, and when we write $\sim$ is a covering functor for X, we mean $\sim$ is compatible with $X - U$ for each $U \in \mathcal{O}(X)$. It is not hard to see that if X is locally 1-connected, then universal covering functors exist despite the fact that the universal covering functor need not.

Now we could have defined homology and homotopy groups using $\mathcal{O}(X)$ instead of $\mathcal{B}(X)$. Given a covering functor for $\mathcal{O}(X)$ there is an obvious one for $\mathcal{B}(X)$. It is not hard to show that the homology and homotopy groups for X are the same whether one uses $\mathcal{O}(X)$ or $\mathcal{B}(X)$.

<u>Definition</u>: $\Delta_*(X;A_1,\ldots,A_n\colon \sim,\Gamma)$, where Γ is a local system on X, denotes the Δ-construction applied to $G_{iU} = H_*((\widetilde{X-U})^i;\ \widetilde{A}_1 \cap (\widetilde{X-U})^i,\ldots,\widetilde{A}_n \cap (\widetilde{X-U})^i\colon i^*\Gamma)$, where the homology group is the ordinary (singular) n-ad homology group with coefficients $i^*\Gamma$, where $i^*\Gamma$ is the local system induced from Γ by the composite $(\widetilde{X-U})^i \xrightarrow{\pi} X - U \subseteq X$.

<u>Definition</u>: $\Delta^*(X\colon\sim,\Gamma)$ is the Δ-construction applied to $G_{iU} = H_c^*((\widetilde{X-U})^i,\ \widetilde{\partial U} \cap (\widetilde{X-U})^i;\ i^*\Gamma)$.

(∂U = frontier of U in X.)

$\Delta^*(X,A: \sim,\Gamma)$ is the Δ-construction applied to $G_{iU} = H_c^*((\widetilde{X-U})^i;\ \widetilde{\partial U} \cap (\widetilde{X-U})^i,\ \widetilde{A} \cap (\widetilde{X-U})^i\ ;\ i^*\Gamma)$.

Caution: (X,A) must be a proper pair (i.e. $A \subseteq X$ is proper) before $H_c^*(X,A)$ makes sense. A similar remark applies for n-ads.

$\Delta^*(X;A_1,\ldots,A_n:\sim,\Gamma)$ is defined similarly.

In our definition we have not defined our maps $G_{iV} \to G_{iU}$ if $\bar{U} \subseteq V$. If $\overline{X-V} = \pi^{-1}(X-V)$, where $\pi : (\widetilde{X-U})^i \to X-U$, then the map is the composite $H_c^*((\widetilde{X-V})^i,\ \widetilde{\partial V} \cap (\widetilde{X-V})^i; i^*\Gamma) \xrightarrow{tr} H_c^*(\overline{X-V},\overline{\partial V};\ \Gamma_1) \xleftarrow[\cong]{ex} H_c^*((\widetilde{X-U})^i,\ (\widetilde{V-U}) \cap (\widetilde{X-U})^i;\Gamma_2) \xrightarrow{inc} H_c^*((\widetilde{X-U})^i,\ \widetilde{\partial U} \cap (\widetilde{X-U})^i;\ i^*\Gamma)$ where Γ_1 and Γ_2 are the obvious local systems. A similar definition gives the map in the pair and n-ad cases.

Once again we get long exact sequences modulo the usual subspace difficulties. We let $\Delta^*(A;X:\sim,\Gamma)$ denote the subspace group with a similar notation for sub-n-ad groups. Again we get a subspace principle. Lastly the cohomology groups are "independent" of base points (compare Theorem 3.2) and are invariant under proper homotopy equivalence. The proofs of these results should be easy after section 3, and hence they are omitted.

One reason for the great power of cohomology is that we have various products. The first product we investigate is the cup product.

<u>Theorem 1</u>: There is a natural bilinear pairing, the cup product

$$H^m(X,A;\Gamma_1) \times \Delta^n(X,B:\sim,\Gamma_2) \to \Delta^{m+n}(X;A,B:\sim,\Gamma_1 \otimes \Gamma_2) .$$

If $\{A,B\}$ is a properly-excisive pair, the natural map $\Delta^*(X, A \cup B: \sim,\Gamma_1 \otimes \Gamma_2) \to \Delta^*(X;A,B:\sim,\Gamma_1 \otimes \Gamma_2)$ is an isomorphism, so we get the "usual" cup product.

<u>Proof</u>: Given $\varphi \in H^m(X,A;\Gamma_1)$, define, for any $U \in \mathcal{O}(X)$, $\varphi_U \in H^m(\widetilde{X-U})^i, \widetilde{A} \cap (\widetilde{X-U})^i; i^*\Gamma_1)$ via $H^m(X,A;\Gamma_1) \to H^m((X-U), (\widetilde{A-U});\Gamma_1) \xrightarrow{\pi^*} H^m((\widetilde{X-U})^i, \tilde{A} \cap (\widetilde{X-U})^i, i^*\Gamma)$. One then checks that if $G_{iU} = H^n_c((\widetilde{X-U})^i , \widetilde{B} \cap (\widetilde{X-U})^i, \partial\widetilde{U} \cap (\widetilde{X-U})^i, i^*\Gamma_2)$ and if H_{iU} = the group for $\Delta^*(X;A,B:\sim,\Gamma_2 \otimes \Gamma_2)$, then

$$\begin{array}{ccc} G_{iU} & \xrightarrow{\cup\varphi_U} & H_{iU} \\ \downarrow & & \downarrow \\ G_{iV} & \xrightarrow{\cup\varphi_V} & H_{iV} \end{array}$$

commutes. Hence the maps $\cup\, \varphi_U$ give us a map $\Delta^n(X,B:\sim,\Gamma_2) \to \Delta^{m+n}(X;A,B:\sim,\Gamma_1 \otimes \Gamma_2)$. One easily checks this map gives us a natural bilinear pairing.

Now we have a natural map $\Delta^*(X,A \cup B) \to \Delta^*(X;A,B)$. We get a commutative diagram

$$\begin{array}{ccccccc} \cdots \to \Delta^*(X,A \cup B) & \longrightarrow & \Delta^*(X, A) & \longrightarrow & \Delta^*(A \cup B, A ;X) & \to \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ \cdots \to \Delta^*(X;A,B) & \longrightarrow & \Delta^*(X, A) & \longrightarrow & \Delta^*(B,A \cap B; X) & \end{array}$$

where the rows are exact. $\{A,B\}$ a properly-excisive pair implies $\Delta^*(A \cup B,A) \to \Delta^*(B,A \cap B)$ is an isomorphism for a set of base points in $A \cap B$ which is a set for A, B, and $A \cup B$. The subspace principle now shows the right hand map is an isomorphism. The middle map is the identity, so the left hand map is an isomorphism. This establishes the last part of our claim. □

For completeness we give the definition of a properly-excisive pair.

<u>Definition</u>: A pair $\{A,B\}$ of homogamous spaces is said to be properly excisive with respect to some covering functor $\sim$, provided

$$\Delta^*(A \cup B;A,B:\sim,\Gamma) = \Delta_*(A \cup B;A,B:\sim,\Gamma) = H^0_{end}(A \cup B;\tilde{A},\tilde{B};\Gamma) = 0$$

for any local system Γ.

The pair is properly-excisive if it is properly excisive with respect to all covering functors compatible with $A \cup B$.

The other product of great importance is the cap product. We get two versions of this (Theorems 2 and 3).

<u>Theorem 2</u>: There is a natural bilinear pairing, the cap product

$$\Delta^m(X,A:\sim,\Gamma_1) \times H^{\ell.f.}_{n+m}(X;A,B;\Gamma_2) \to \Delta_n(X,B:\sim,\Gamma_1 \otimes \Gamma_2)$$

If $\{A,B\}$ is a properly-excisive pair, we can define the "usual" cap product.

<u>Proof</u>: Let $C \in H^{\ell.f.}_{n+m}(X;A,B;\Gamma_2)$. Define

$$C_U \in H^{\ell.f.}_{n+m}((\widetilde{X-U})^i; \tilde{A} \cap (\widetilde{X-U})^i, \tilde{B} \cap (\widetilde{X-U})^i, \widetilde{\partial U} \cap (\widetilde{X-U})^i; i^*\Gamma_2)$$

from $H^{\ell.f.}_*(X;A,B;\Gamma_2) \to H^{\ell.f.}_*(X;A,B,U;\Gamma_2) \xleftarrow[\cong]{ex}$

$$H^{\ell.f.}_*(X-U;\ A-U,\ B-U,\ \partial U;\ \Gamma_2)$$
$$\downarrow tr$$
$$H^{\ell.f.}_*((\widetilde{X-U})^i;\ \hat{A} \cap (\widetilde{X-U})^i,\ \hat{B} \cap (\widetilde{X-U})^i,\ \widehat{\partial U} \cap (\widetilde{X-U})^i; i^*\Gamma_2).$$

One can check that $\cap\, C_U$ satisfies the necessary commutativity relations to define a map $\Delta^m(X,A:\sim,\Gamma_1) \to \Delta_n(X,B:\sim,\Gamma_1 \otimes \Gamma_2)$.

If $\{X,A\}$ is properly excisive, $H^*_c(A \cup B;A,B) = 0$ from $\Delta^* = 0$. Universal coefficients shows $H^{\ell.f.}_*(A \cup B; A,B) = 0$, so the standard exact sequence argument shows $H^{\ell.f.}_*(X;A,B;\Gamma_2) \cong H^{\ell.f.}_*(X,A \cup B;\ \Gamma_2)$. □

Theorem 3: There is a natural bilinear pairing, the cap product

$$H^m(X,A;\Gamma_2) \times \Delta_{n+m}(X;A,B:\sim\Gamma_2) \longrightarrow \Delta_n(X,B:\sim,\Gamma_1 \otimes \Gamma_2) .$$

If $\{A,B\}$ is a properly-excisive pair, we can define the "usual" cup product.

Proof: Given $\varphi \in H^m(X,A;\Gamma_1)$, define $\varphi_U \in H^m((\widetilde{X-U})^i,\ \tilde{A} \cap (\widetilde{X-U})^i,\ i^*\Gamma_1)$ by $H^m(X,A;\Gamma_1) \to H^m(X-U,\ A-U;\Gamma_1) \xrightarrow{\pi^*} H^m((\widetilde{X-U})^i, \tilde{A} \cap (\widetilde{X-U})^i; i^*\Gamma_1)$. One checks again that the necessary diagrams commute. The statement about $\{A,B\}$ follows from the 5-lemma and the subspace principle. □

We will also need a version of the slant product for our theory. To get this we need to define a group for the product of two - ads. As usual we apply the Δ construction to a particular situation. Pick a set of base points for X and a set for Y. Our indexing set is the cartesian product of these two sets. Our diagram is $\mathcal{O}(X) \times \mathcal{O}(Y) = \{U \times V \subseteq X \times Y \mid U \in \mathcal{O}(X),\ V \in \mathcal{O}(Y)\}$.

$$G^{i \times j}_{U \times V} = H_*((\widetilde{X-U})^i \times (\overline{Y-V})^j; (\widetilde{A}_1 \cap (\widetilde{X-U})^i) \times (\overline{Y-V})^j, \ldots, \widetilde{A}'_n \cap (\widetilde{X-U})^i) \times (\overline{Y-V})^j,\ (\widetilde{X-U})^i \times (\widetilde{B}_1 \cap (\overline{Y-V})^j), \ldots, (\widetilde{X-U})^i \times (\widetilde{B}_m \times (\overline{Y-V})^j\ ;\ i^*\Gamma)\ .$$

The resulting group will be denoted $\Delta_*((X;A_1,\ldots,A_n) \times (Y;B_1,\ldots,B_m)\colon \sim,\ ---,\ \Gamma)$ (Γ is some local system on $X \times Y$).

Theorem 4: There is a natural bilinear pairing, the slant product

$$H^m(Y;B_1,\ldots,B_m;\Gamma_1) \times \Delta_{m+n}((X;A_1,\ldots,A_n) \times (Y;B_1,\ldots,B_m);\ \Gamma_2 \times \Gamma_1) \to \Delta_n(X;A_1,\ldots,A_n;\ \Gamma_1 \otimes \Gamma_2)\ .$$

Proof: For $\varphi \in H^m(Y;B_1,\ldots,B_m;\Gamma_1)$, define φ_V as in Theorem 1. These give us the necessary maps. □

Corollary 4.1: If $d : X \to X \times X$ is the diagonal, and if $C \in \Delta_{n+m}(X;A,B\colon \sim, \Gamma_1)$, and if $\varphi \in H^m(X,A;\ \Gamma_2)$, then

$$\varphi \cap C = \varphi / d_*C\ . \quad \square$$

Using our slant product, we can define the cap product of Theorem 3 "on the chain level". There are two basic chain groups we would like to use. For an homogamous CW complex we would like to use the cellular chains, and when X is a paracompact manifold with a locally finite handlebody decomposition, we want to use the chains based on the handles. We do the former case and leave the reader to check the theory still holds in the latter.

If X is an homogamous CW complex, we define $P_*(X;A,B:\sim,\Gamma) = \Delta_*(X^*;X^{*-1},A^*,B^*:\sim,\Gamma)$ (where $A^* = A \cap X^*$) for $* \geq 2$. If $* = 0$ or 1, we must use subspace groups $\Delta_*((X^*;X^{*-1},A^*,B^*);\ X : \sim,\Gamma)$. A and B are subcomplexes. Similarly define

$$P^*(X;A,B:\sim,\Gamma) = \Delta^*(X^*;X^*-1,A^*,B^*:\sim,\Gamma)$$

$$C_*^{\ell.f.}(X:A,B) = H_*^{\ell.f.}(X^*;X^{*-1},A^*,B^*)$$

$$C^*(X : A, B) = H^*(X^*;X^{*-1},A^*, B^*) \quad .$$

The triple (X^*,X^{*-1},X^{*-2}) gives us a boundary map $P_* \to P_{*-1}$, $P^* \to P^{*+1}$, etc. This boundary map makes the above objects into chain complexes ($\partial\partial = 0$), and by Corollary 2.2.2, the homology of these complexes is just what one expects.

A diagonal approximation $h_*\colon P_*(X;A,B:\sim,\Gamma) \longrightarrow \Delta_*(((X,A) \times (X,B))^*: \sim,\Gamma \times \Gamma)$ is a cellular approximation to $d\colon X \to X \times X$ with a homotopy $H\colon X \times I \to X \times X$ such that

$\pi_1 \circ H$ and $\pi_2 \circ H$ are proper. $(((X,A) \times (X,B))^*$ is just $\bigcup_k (X,A)^k \times (X,B)^{*-k})$. Any two such diagonal approximations are cellularly homotopic so that the homotopy composed with projection is proper.

Theorem 5: Given any diagonal approximation h , there is a bilinear pairing.

$$B_h : C^m(X,A;\Gamma_1) \times P_{n+m}(X;A,B:\sim,\Gamma_2) \longrightarrow P_n(X,B:\sim,\Gamma_1 \otimes \Gamma_2).$$

If $f \in C^m(X,A,\Gamma_1)$ and $c \in P_{n+m}(X;A,B:\sim,\Gamma_2)$, then $\partial B_h(f,c) = (-1)^{n+m} B_h(\delta f,c) + B_h(f,\partial c)$. Hence we get an induced pairing on the homology level. Any two $B_h(f,\ )$ are chain homotopic, so the pairing on homology does not depend on the diagonal approximation. This pairing is the cap product of Theorem 3.

Proof: $h_*(c) \in \Delta_{n+m}(((X,A) \times (X,B))^{n+m}$, etc.). $\Delta_{n+m}((X^n;X^{n-1},A^n) \times (X^m;X^{m-1},B^m) : \sim,\Gamma_2 \times \Gamma_2)$ lies as a natural summand of this first group. Let p^n_m be the projection. Then $B_h(f,c) = f|p^n_m(h_*(c))$. The rest of the proof involves checking this definition has all the asserted properties. □

We also want to define the cap product of Theorem 2 on the chain level. Unfortunately, there is no slant product of the needed type, so we must use brute force.

Theorem 6: Given any diagonal approximation h , there is a bilinear pairing

$$B_h : P^n(X,A:\sim,\Gamma_1) \times C^{\ell.f}_{n+m}(X;A,B;\Gamma_2) \longrightarrow P_n(X,B:\sim,\Gamma_1 \otimes \Gamma_2).$$

If $f \in P^m(X,A:\sim,\Gamma_1)$ and $c \in C^{\ell.f.}_{n+m}(X;A,B;\Gamma_2)$, then

$$\partial B_h(f,c) = (-1)^{n+m} B_h(\delta f,c) + B_h(f,\partial c) .$$

Hence we get an induced pairing (independent of h) on the homology level. This pairing is the cap product of Theorem 2.

<u>Proof</u>: Let $c \in C^{\ell.f.}_{n+m}(X;A,B;\Gamma_2)$. Define $C_U \in C^{\ell.f.}_{n+m}((\widetilde{X-U})^i;\ \tilde{A} \cap (\widetilde{X-U})^i,\ \tilde{B} \cap (\widetilde{X-U})^i, \partial\tilde{U} \cap (\widetilde{X-U})^i;\ i^*\Gamma_2)$ by excision and trace as in Theorem 2. We define $B_h(\ ,c)$ from the maps $H^m_c(\widetilde{(X-U)^i \cap X^m};\ \widetilde{(X-U)^i \cap X^{m-1}}, \text{etc.}) \xrightarrow{\ |\, b_U\ } H_n(\widetilde{(X-U)^i \cap X^n};\ \widetilde{(X-U)^i \cap X^{n-1}}, \text{etc.})$ where $|$ is the slant product and b_U is the homology class given via $H^{\ell.f.}_{n+m}(\widetilde{(X-U)^i \cap X^m}; \text{etc.}) \xrightarrow{h_*}$ $H^{\ell.f.}_{n+m}(\widetilde{((X-U)^i \cap (X-U)^i) \cap (X \sim X)^{n+m}}; \text{etc.}) \xrightarrow{p^n_m}$ $H^{\ell.f.}_{n+m}(\widetilde{((X-U)^i \cap X^m)} \times \widetilde{((X-U)^i \cap X^n)}; \text{etc.})$ (superscript i denotes a component containing $\hat{x}_i$, and superscripts n, m and n+m denote skeletons.

Note in passing that $h_*(\text{tr } b_U) \neq \text{tr}(h_* b_U)$, which is why we are unable to define a general slant product like Theorem 4 to cover this case.

The rest of the proof incolves verifying diagrams commute and verifying our equation. □

Lastly we prove the Browder lemma, which will be essential in our study of Poincare duality.

Theorem 7: Let (X,A) be a proper pair, and let $c \in H_n^{\ell.f.}(X,A;\ \Gamma_2)$. Then

$$\begin{array}{ccccc}
\Delta^{*-1}(A;X:\sim,\ \Gamma_1) & \longrightarrow & \Delta^{*}(X,A:\sim,\Gamma_1) & \longrightarrow \\
\Big\downarrow{\scriptstyle (-1)^n \cap \partial c} & & \Big\downarrow{\scriptstyle \cap c} & \\
\Delta_{n-*}(A;X:\sim,\Gamma_1 \otimes \Gamma_2) & \longrightarrow & \Delta_{n-*}(X:\sim,\Gamma_1 \otimes \Gamma_2 & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\Delta^{*}(X:\ \sim,\ \Gamma_1) & \longrightarrow & \Delta^{*}(A;X:\ \sim,\ \Gamma_1) \\
\Big\downarrow{\scriptstyle \cap c} & & \Big\downarrow{\scriptstyle \cap\ \partial c} \\
\Delta_{n-*}(X,A:\sim,\Gamma_1 \otimes \Gamma_2) & \longrightarrow & \Delta_{N-1-*}(A;X:\sim,\Gamma_1 \otimes \Gamma_2)
\end{array}$$

commutes.

Proof: The usual Browder lemma (see section 1) says that the corresponding diagram commutes for ordinary homology and cohomology with compact supports. Commutativity is then trivial for the above diagram. (While we have not defined a cap product for subspace groups, the reader should have no difficulty writing down the necessary maps.) □

Section 5. Chain complexes and simple homotopy type.

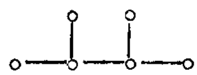
In our Δ-construction as applied to the homology or homotopy functors, we still have some structure that we have not utilized.

As an example of this extra structure, let us consider $\varepsilon(X: \pi_1)$. This is an inverse limit $\varinjlim \mu(\pi_1(X - C, x_i))$. Now many of the $\pi_1(X - C, x_i)$ are isomorphic. (Unfortunately this isomorphism is not natural but depends on a path joining x_i to x_j.) Our ε-construction makes no use of this fact. In order to be able to make effective use of this extra structure, we need a way to choose the above isomorphisms.

We will do this through the concept of a tree. A tree for an homogamous space X will be a 1-dimensional, locally finite, simplicial complex, T , such that

1) $\Delta(T : \pi_k) = 0$ for $k > 0$

2) If $T' \subseteq T$ is a subcomplex of T , T' has the proper homotopy type of T iff $T = T'$.

(This last condition is to insure that  ... is not a tree for R^2 , but rather ... is.) We also require a map $f : T \to X$ which is properly 1/2-connected.

Two trees (T,f) and (S,g) are equivalent provided there is a proper homotopy equivalence $h: T \to S$ with $h \circ g$ properly homotopic to f .

A space X is said to have a tree provided X is homogamous and there is a tree for X . Any locally path connected homogamous space has a tree. To see this, let {p} be a set of base points for our space X .

We claim $H_1(T;Z) = 0$, and in fact, if H_1 is computed from the simplicial chains then there are no 1-cycles. This is fairly clear, so it will be left to the reader. Now any locally finite 1-complex with $H_1(T) = 0$ satisfies $\Delta(T: \pi_k) = 0$ for $k > 0$. One shows f is properly 1/2-connected by showing that $Z^o_{end}(X) \to Z^o_{end}(T)$ is an isomorphism (Z^o_{end} are the o-cycles in S^o_{end}) . But this follows from our construction. Lastly suppose $T' \subseteq T$ is a connected subcomplex, and suppose $p \in T - T'$. Now by definition p is in an essential component of $X - C_i$ for all $i \leq n$ for some n . Since each essential component of $X - C_i$ has infinitely many base points in it, let $\{q\}$ be the set of base points in the component of $X - C_n$ containing p . Then $\{q\} \subseteq T - T'$, as is easily seen. Hence $H^o_{end}(T) \to H^o_{end}(T')$ has a kernel, and so $T' \subseteq T$ is not a proper homotopy equivalence. Hence X has a tree.

From now on in this section we restrict ourselves to the category of homogamous CW complexes. We will denote this hCW complex.

Given X , an hCW complex, we have the category $\mathcal{C}(X)$ whose objects are all sets $A \subseteq X$ such that

1) A is a subcomplex
2) A is connected
3) There exists an element of $\mathfrak{I}(X)$, U , such that A is an essential component of $X - U$.

The morphisms are inclusions.

Let $\{C_i\}$ be a cofinal collection of compact subsets of X. We can assume X is path connected since we can do each path component separately. We may assume $\{p\} \cap C_o \neq \emptyset$. Pick a point $p_o \in \{p\} \cap C_o$. Look at the components of $X - C_o$ with a point of p in them. As we showed in the proof of Proposition 1.2, there are only finitely many components of $X - C_o$. The components whose closure is not compact are called essential components. We may assume $\{p\} \cap$ (each essential component of $X - C_o) \cap C_1 \neq \emptyset$ since this is true for some compact set. Let $p_1^{\alpha_1}, p_1^{\alpha_2}, \ldots, p_1^{\alpha_n}$ be a subset of $\{p\} \cap C_1$, one for each essential component of $X - C_o$. Join $p_1^{\alpha_i}$ to p_o by a path $\lambda_{1,i}$. Now look at the essential components of $X - C_1$. Pick $p_2^{\alpha_1}, \ldots, p_2^{\alpha_m}$ (which we may assume are in C_2), one for each essential component of $X - C_1$. Each $p_2^{\alpha_i}$ lies in an essential component of $X - C_o$, so pick paths $\lambda_{2,i}$ which join $p_2^{\alpha_i}$ to the appropriate element in $\{p_1^{\alpha}\}$. These paths should lie in $X - C_o$. Continue in this fashion to get $\{p_j^{\alpha}\}$, one for each essential component of $X - C_{j-1}$. $\{p_j^{\alpha}\}$ may be assumed to lie in C_j. We can also get paths λ_{j,α_i} which join $p_j^{\alpha_i}$ to the appropriate p_{j-1}^{α} and which lie in $X - C_{j-1}$.

Now T has $\{p_j^{\alpha}\}$ for vertices and $(p_j^{\alpha_i}, p_k^{\alpha_\ell})$ is a 1-simplex iff $k = j-1$ and λ_{j,α_i} joins $p_j^{\alpha_i}$ to $p_k^{\alpha_\ell}$. The map $f : T \to X$ is the obvious one.

Now given a tree (T,f) for X, we get a functor $\mathcal{C}(X) \xrightarrow{\mathcal{C}(f)} \mathcal{C}(T)$ (f is always assumed to be cellular.

Definition: A lift of $\mathcal{C}(f)$ is a covariant functor $F : \mathcal{C}(T) \to \mathcal{C}(X)$ such that $\mathcal{C}(f) \circ F$ is the identity and such that the image of F is cofinal. The set of all lifts is a diagram scheme by defining $F \leq G$ iff $F(A) \subseteq G(A)$ for all $A \in \mathcal{C}(T)$. We denote this diagram scheme by $\mathcal{L}(f)$.

Definition: A tree of rings is a covariant functor $R : \mathcal{C}(T) \to \mathcal{R}$, where $\mathcal{R}$ is the category of all rings (rings have units and all ring homomorphisms preserve units). A tree of modules over R is a collection of modules M_A, $A \in \mathcal{C}(T)$, where M_A is a unitary R_A-module. A tree of right (left) R-modules requires each M_A to be a right (left) R_A-module. If $A \subseteq B$ in $\mathcal{C}(T)$, there is a unique map $p_{AB}: M_A \to M_B$, which is an $R(A \subseteq B)$-linear map; i.e. if $f : R_A \to R_B$ is the ring homomorphism associated to $A \subseteq B$ by R,

$$p_{AB}(a\alpha + b\beta) = p_{AB}(a)f(\alpha) + p_{AB}(b)f(\beta)$$

for $\alpha, \beta \in R_A$; $a, b \in M_A$.

An R-module homomorphicm $f : M \to M'$ is a set of maps $f_A: M_A \to M'_A$ for each $A \in \mathcal{C}(T)$ such that

1) f_A is an R_A-module homomorphism

2) For $A \subset B$,

$$\begin{array}{ccc} M_A & \xrightarrow{f_A} & M'_A \\ \downarrow{\scriptstyle p_{AB}} & & \downarrow{\scriptstyle p'_{AB}} \\ M_A & \xrightarrow{f_B} & M'_B \end{array}$$

commutes, where the vertical maps come from the tree structure on M and M' .

Example: Given an hCW complex X with a tree (T,f) and given $F \in \mathcal{L}(f)$, we get a tree of rings from $R_A = Z\pi_1(F(A),f(p))$ where if $A \neq T$, p is the vertex ∂A , the set theoretic frontier of A . If $A = T$, pick a vertex for a base point and use it. This will be the tree of rings we will consider for our geometry, and we will denote it by $Z\pi_1$.

The tree of $Z\pi_1$-modules we will consider will be various chain modules. The basic idea is given by $M_A = H_i(\widetilde{F(A)}^i, \widetilde{F(A)}^{i-1}, f(p))$, where ~ denotes the universal cover of $F(A)$, and $\widetilde{F(A)}^i$ is π^{-1} of the i-skeleton of $F(A)$ in $\widetilde{F(A)}(\pi : \widetilde{F(A)} \to F(A))$.

Now given an R-module M , we can form $\Delta(M)$ by applying the Δ-construction with index set the vertices of T , and with diagram scheme $\mathcal{O}(X)$. Given $U \in \mathcal{O}(X)$, there are finitely many $A \in \mathcal{C}(T)$ for which $A \cap \bar{U} =$ a vertex. Set

$$G_{pU} = \begin{cases} M_A & \text{if } p \in A \\ 0 & \text{otherwise} \end{cases}$$

for some A such that $A \cap \bar{U}$ = a vertex. An R-module homomorphism $f : M \to M'$ clearly induces a map $\Delta(f) : \Delta(M) \to \Delta(M')$. An R-module homomorphism, f, which induces an isomorphism $\Delta(f)$ is said to be a strong equivalence and the two modules are said to be strongly equivalent. Note that this relation on R-modules seems neither symmetric nor transitive. Nevertheless we can define two R-modules M and M' to be equivalent iff there is a (finite) sequence of R-modules $M = M_o, M_1, \ldots, M_n = M'$ such that either M_i is strongly equivalent to M_{i+1} or M_{i+1} is strongly equivalent to M_i.

We tend only to be really interested in the equivalence class of M (indeed, we are often interested merely in $\Delta(M)$). The relation of equivalence is not however very nice. We would like M equivalent to M' iff there were "maps" $f : M \to M'$ and $g : M' \to M$ whose composites were the identities. To do this properly we need a short digression.

<u>Definition</u>: A functor F which assigns to each $A \in \mathcal{C}(T)$ a cofinite subcomplex of A, $F(A)$, such that $F(A) \subset F(B)$ whenever $B \subseteq A$ and such that $F(T) = T$ will be called a shift functor. $\mathcal{S}(T)$ will denote the set of all shift functors on T. $\mathcal{S}(T)$ is partially ordered via $F \geq G$ iff $F(A) \subseteq G(A)$ for for all $A \in \mathcal{C}(T)$. $(F \cap G)(A) = F(A) \cap G(A)$, and one checks it is a shift functor. $F \cap G \geq F$ and $F \cap G \geq G$.

Given a tree of R-modules and a shift functor F, we get a tree of R-modules, M_F, in a natural way; i.e. F is going to induce a functor from the category of R-modules to itself. M_F is defines as follows. Let $A \in \mathcal{C}(T)$. Then $F(A) = \bigcup_{i=1}^{n} A_i$, with $A_i \in \mathcal{C}(T)$. $(M_F)_A = \bigoplus_{i=1}^{n} M_{A_i} \otimes R_A$, where the tensor product is formed using the homomorphisms $R_{A_i} \to R_A$. Note that there is an R_A-module map

$(M_F)_A \to M_A$. $(p_F)_{AB} : \bigoplus_{i=1}^{n} M_{A_i} \otimes R_A \to \bigoplus_{i=1}^{m} M_{B_i} \otimes R_B$

is defined as follows. Since $A \subseteq B$, $F(A) \subseteq F(B)$, so each A_i is contained in a unique B_j. Let p_{ij} be $p_{A_iB_j}$ if $A_i \subseteq B_j$ and 0 otherwise. f_{ij} is the map $R_{A_i} \to R_{B_j}$ if $A_i \subseteq B_j$ and 0 otherwise. g is the map $R_A \to R_B$. Then $(p_F)_{AB} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} p_{ij} \otimes f_{ij} \otimes g$.

Notice that

$$\begin{array}{ccc} (M_F)_A & \longrightarrow & M_A \\ \downarrow (p_F)_{AB} & & \downarrow p_{AB} \\ (M_F)_B & \longrightarrow & M_B \end{array}$$

commutes.

If $f : M \to M'$ is a map, $(f_F)_A = \bigoplus_{i=1}^{n} f_{A_i} \otimes g_{A_iA}$, where $g_{A_iA} : R_{A_i} \to R_A$, defines a map

$f_F : M_F \to M'_F$ so that

$$\begin{array}{ccc} M_F & \xrightarrow{f_F} & M'_F \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

commutes.

For the natural map of M_F into M we write $M_F \subseteq M$.

If $G \geq F$ there is a natural map $M_G \to M_F$ induced by the inclusion of each component of $G(A)$ in $F(A)$.

<u>Lemma 1</u>: $M_F \subseteq M$ is a strong equivalence.

<u>Proof</u>: We must show $\Delta(M_F) \to \Delta(M)$ is an isomorphism. Suppose $B \in \mathcal{C}(T)$ and $B \subseteq F(A)$. Then

$$\begin{array}{ccc} (M_F)_B & \longrightarrow & M_B \\ \downarrow & & \downarrow \\ (M_F)_A & \longrightarrow & M_A \end{array}$$

commutes and there is a map $h : M_B \to (M_F)_A$ so that the resulting triangles commute. But then clearly $\Delta(M_F) \cong \Delta(M)$. Q.E.D.

As motivation for our next definition we prove

<u>Lemma 2</u>: Let $f : M \to N$ be a strong equivalence. Then there is a shift functor F and a map $N_F \to M$ such that

$$\begin{array}{ccc} & \nearrow & M \\ & & \downarrow \\ N_F & \subseteq & N \end{array}$$

commutes.

<u>Proof</u>: By Theorem 2.4 applied to kernel and cokernel, f is a strong equivalence iff for any $A \in \mathcal{C}(T)$ there is a $U \in \mathcal{O}(T)$ such that for any $B \in \mathcal{C}(T)$ with $B \subseteq A - U$

$$\begin{array}{ccc} M_B & \xrightarrow{f_B} & N_B \\ \downarrow p^M_{AB} & & \downarrow p^N_{AB} \\ M_A & \xrightarrow{f_A} & N_A \end{array}$$

satisfies

1) $\ker f_B \subseteq \ker p^M_{AB}$ and

2) Image $p^N_{AB} \subseteq$ Image f_A .

For each $A \in \mathcal{C}(T)$, pick such an element in $\mathcal{O}(T)$, U_A. Now let $F(A) = A - \bigcup_{A \subseteq D} U_D$ F is easily seen to be a shift functor, and for any $B \in \mathcal{C}(T)$ with $B \subseteq F(A)$, 1) and 2) hold.

Now look at

$$\begin{array}{ccc} M_{A_2} & \longrightarrow & N_{A_2} \\ \downarrow & & \downarrow p \\ M_{A_1} & \xrightarrow{f_{A_1}} & N_{A_1} \\ \downarrow q & & \downarrow \\ M_A & \longrightarrow & N_A \end{array}$$

where $A_1 \subseteq F(A)$ $A_2 \subseteq F(A_1)$. Then there exists a map $h: N_{A_2} \to M_A$ defined by $h(x) = q(f_{A_1})^{-1}\, p(x)$ for all $x \in N_{A_2}$. By properties 1) and 2), h is well-defined, and if $g: R_{A_2} \to R_A$ is the homomorphism given by the tree, h is easily seen to be g-linear.

Define a shift functor $F \circ G$ by $F \circ G(A) = \bigcup_{i=1}^{n} F(A_i)$, where $G(A) = \bigcup_{i=1}^{n} A_i$. Then one checks that the h defined above yields a map $N_{F \circ F} \to M$. Q.E.D.

<u>Definition</u>: A T-map $f: M \to N$ is a map $M_F \to N$, where $F \in \mathcal{S}(T)$. $M_F \to N$ induces a natural map $M_G \to N$ for all $G \geq F$. We say f is defined on M_G for all $G \geq F$. Two T-maps $f, g: M \to N$ are equal provided that, for some $F \in \mathcal{S}(T)$ such that f and g are defined

on M_F , the two maps $M_F \to N$ are equal.

Remarks: If f is defined on M_F , and if g is defined on M_G , f and g are both defined on $M_{F\cap G}$. With this remark it is easy to see equality of T-maps is an equivalence relation. It is also easy to see how to add or subtract two T-maps, and it is easy to check that if $f_1 = f_2$ and $g_1 = g_2$, then $f_1 \pm g_1 = f_2 \pm g_2$.

Hence, if $\mathrm{Hom}_T(M,N)$ is the set of equivalence classes of T-maps from M to N , $\mathrm{Hom}_T(M,N)$ has the structure of an abelian group. An equivalence class of T-maps is called a map-germ.

We can compose two T-maps $f: M \to N$ and $g: N \to P$ as follows. g is defined on N_F and f is defined on M_F . Hence $f: M_F \to N$ is an actual map, and we define the T-map $g \circ f$ to be the map $g \circ f_G: (M_F)_G \longrightarrow N_G \to P$. Note $(M_F)_G = M_{G\circ F}$. One can check that the map-germ $g \circ f$ is well-defined.

Hence Lemma 2 becomes

<u>Lemma 3</u>: M and N are equivalent iff they are T-equivalent.

<u>Proof</u>: If M and N are equivalent, Lemma 2 shows how to get T-maps $M \to N$ and $N \to M$ using the sequence of strong equivalences.

If M and N are T-equivalent, we have T-maps

$f\colon M \to N$ and $g\colon N \to M$ such that $f \circ g = id_N$ and $g \circ f = id_M$. Now a T-map $f\colon M \to N$ induces a unique map $\Delta(f)\colon \Delta(M) \to \Delta(N)$ via $\Delta(f) = \Delta(f) \circ \Delta(inc)^{-1}$ where f is defined on M_F and $inc\colon M_F \subseteq M$. It is clear that $\Delta(f)$ depends only on the map-germ of f. Hence in our case, g induces an equivalence of M and N by $N \supseteq N_G \xrightarrow{g} M$. Q.E.D.

Also useful is

Lemma 4: Let f and g be T-maps. Then $f = g$ iff $\Delta(f) = \Delta(g)$.

Proof: $f = g$ iff $f - g = 0$. $\Delta(f-g) = \Delta(f) - \Delta(g)$. Thus we need only show $h = 0$ iff $\Delta(h) = 0$. Since $\Delta(h)$ depends only on the map-germ, and since $\Delta(0) = 0$, one way is easy.

So assume we are given a T-map $h\colon M \to N$ with $\Delta(h) = 0$. We may as well assume h is an actual map, since otherwise set $M = M_H$ and proceed. We have a submodule $\ker h \subseteq M$ defined in the obvious way. Since $\ker h \subseteq M$ is a strong equivalence, Lemma 2 says we can find F such that $M_F \to \ker h \subseteq M$. But then $M_F \to N$ is the zero map. Q.E.D.

Definition: If R is a tree of rings, let $\mathcal{M}_R$ be the category of trees of R-modules and germs of maps. Let $\mathcal{M}_{\Delta(R)}$ be the category of $\Delta(R)$-modules.

Proposition 1: $\mathcal{M}_R$ is an abelian category. The natural functor $\Delta: \mathcal{M}_R \to \mathcal{M}_{\Delta(R)}$ is an exact, additive, faithful functor.

Proof: The functor just takes M to $\Delta(M)$ and $[f]$ to $\Delta(f)$ ($[f]$ denotes the map-germ of f). Δ is additive moreorless by definition, and faithful by Lemma 4.

Δ preserves kernels: Let $M \xrightarrow{[g]} N$ be a map-germ in $\mathcal{M}_R$. We can find G such that $M_G \xrightarrow{g} N$ is a representative. Clearly any kernel for $[g]$ is equivalent to $\ker g \subseteq M_G$, where $\ker g$ is the obvious submodule. But $\Delta(\ker g)$ is clearly a kernel for $\Delta(g)$.

An entirely similar argument shows Δ preserves cokernels, so Δ is exact.

To see $\mathcal{M}_R$ is normal and conormal, take representatives for the germs and construct the quotient or the kernel module.

$\mathcal{M}_R$ has pullback and pushouts again by finding representatives and then constructing the desired modules. Now by [25], Theorem 20.1 (c), page 33, $\mathcal{M}_R$ is abelian. $\square$

We want to do stable algebra, and for this we need an analogue of finitely-generated projective. Projective is easy, we just insist that a projective R-module is projective in the category $\mathcal{M}_R$ (see [25], page 69-71 for definitions and elementary properties).

For the analogue of finitely-generated, we first produce the analogue of a finitely-generated, free module.

Definition: Let T be a tree and let S be a set. A partition of S is a functor $F: \mathcal{C}(T) \to 2^S$ (where 2^S is the category of subsets of S and inclusion maps) satisfying:

1) $\pi(T) = S$.

2) If $A \cap B = \emptyset$, $\pi(A) \cap \pi(B) = \emptyset$ $(A,B \in \mathcal{C}(T))$.

3) Let $A_i \in \mathcal{C}(T)$, $i = 1,\ldots,n$. If $T - \bigcup_{i=1}^{n} A_i$ is compact, $\pi(T) - \bigcup_{i=1}^{n} \pi(A_i)$ is finite.

4) Let $s \in S$. Then there exist $A_i \in \mathcal{C}(T)$, $i = 1,\ldots,n$ such that $T - \bigcup_{i=1}^{n} A_i$ is compact and $s \notin \pi(A_i)$ for any $i = 1,\ldots,n$.

Definition: Let R be a tree of rings over T . Let π be a partition of S . The free R-module based on π , F_π , is the tree of R-modules defined by $(F_\pi)_A$ is the free R_A-module based on $\pi(A)$, and if $A \subseteq B$, $p_{AB}: (F_\pi)_A \longrightarrow (F_\pi)_B$ is induced by the inclusion $\pi(A) \subseteq \pi(B)$.

Definition: A tree of R-modules, M , is said to be locally-finitely generated iff there is a set of generators, S , and a partition, π , of S , such that there is an epimorphism $F_\pi \to M$.

Let us briefly discuss partitions. If π and ρ are two partitions of a set S , we say $\pi \subseteq \rho$ iff

$\pi(A) \subseteq \pi(\rho)$ for all $A \in \mathcal{C}(T)$. (Hence we could talk about the category of partitions, but we shall largely refrain.) Two partitions are equivalent iff there exists a finite sequence $\pi = \pi_0, \pi_1, \ldots, \pi_n = \rho$ of partitions with $\pi_i \subseteq \pi_{i+1}$, or $\pi_{i+1} \subseteq \pi_i$. (This is clearly an equivalence relation.) Given two sets X and Y, and partitions π and ρ, $\pi \cup \rho$ is the partition of $X \cup Y$ given by $(\pi \cup \rho)(A) = \pi(A) \cup \rho(A)$.

<u>Lemma 5</u>: Let R be a tree of rings over T, and let X and Y be sets. Then if π and π' are equivalent partitions of X, F_π is isomorphic to $F_{\pi'}$ in $\mathcal{M}_R$. If ρ is a partition of Y, $F_{\pi \cup \rho} = F_\rho \oplus F_\pi$ (X and Y are disjoint).

<u>Proof</u>: To show the first statement we need only show it for $\pi \subseteq \pi'$. In this case there is a natural map $f: F_\pi \to F_{\pi'}$. For each $A \in \mathcal{C}(T)$, $(F_\pi)_A \to (F_{\pi'})_A$ is injective, so f is a monomorphism. If $\pi \subseteq \pi'$, then $\pi'(A) - \pi(A)$ has only finitely many elements. To see this observe we can find $A_i \in \mathcal{C}(T)$, $i = 1, \ldots, n$ such that $A \cap A_i = \emptyset$, and $T - \bigcup_{i=1}^{n} A_i - A$ is compact. Then by 2) $\pi'(A) \subseteq \pi'(T) - \bigcup_{i=1}^{n} \pi'(A_i)$, so $\pi'(A) - \pi(A) \subseteq \pi'(T) - \bigcup_{i=1}^{n} \pi'(A_i) - \pi(A) \subseteq \pi(T) - \bigcup_{i=1}^{n} \pi(A_i) - \pi(A)$, which is finite. Since $\pi'(A) - \pi(A)$ is finite, $f_A (F_\pi)_A \to (F_{\pi'})_A$ has finitely generated

cokernel, so when the Δ construction is applied to it, 4) guarantees that $\Delta(f)$ is onto, so f is an equivalence. The second statement is the definition of $\pi \cup \rho$ and $F_\pi \oplus F_\rho$. Q.E.D.

It is not hard to see that if we have a partition of S for the tree T , then S has at most countably many elements if T is infinite, and at most finitely many if T is a point. In the case S is infinite, we have a very handy countably infinite set lying around, namely the vertices of T , There is an obvious partition, π , where $\pi(A) = \{p \mid p$ is a vertex of $A\}$. Denote F_π by $F^{(1)}$. If T = point, let $F^{(1)}$ denote the free module on one generator; i.e. still F_π for the above partition π . $F^{(n)} = F^{(n-1)} \oplus F^{(1)}$ for $n \geq 2$.

<u>Lemma 6</u>: Let π be any partition of a set S for the tree T , and let R be a tree of rings. Then $F_\pi \oplus F^{(1)}$ is equivalent to $F^{(n)}$ for some $n \geq 1$. If T is infinite, n can be chosen to be 1 .

<u>Proof</u>: If T = point, this is obvious, so assume T is infinite. $F_\pi \oplus F^{(1)}$ is just $F_{\pi \cup \rho}$, where ρ is the standard partition on V , the vertices of T . Since $V \cup S$ is infinite, there is a 1-1 correspondence $\alpha : V \cup S \to V$. Any such α induces an equivalence of categories $\alpha : 2^{V \cup S} \longrightarrow 2^V$. We show that we can pick α so that $\alpha \circ (\pi \cup \rho)$ is equivalent to ρ . (We will show in Lemma 7 that $\alpha \circ (\pi \cup \rho)$ is a partition for any α.)

Our α is defined by picking a strictly increasing sequence of finite subcomplexes, $C_o \subseteq C_1 \subseteq \dots$, so that $\bigcup_{i=o}^{\infty} C_i = T$. Let $A_k(i)$ be the essential components of $T - C_i$. Set $A_1(-1) = T$, and let $K_{ki} = (\pi \cup \rho)(A_k(i)) - \bigcup_{\ell} (\pi \cup \rho) A_{\ell}(i+1)$. Note $K_{ki} \cap K_{k'i} = \emptyset$ and $K_{ki} \cap K_{k'i+1} = \emptyset$ by 2), so $K_{ki} \cap K_{\ell j} \neq \emptyset$ iff $k = \ell$ and $i = j$.

Now K_{ki} is finite. We define α on K_{ki} by induction on i. Let $L_{ki} = \rho(A_k(i)) - \bigcup_{\ell} \rho(A_{\ell}(i+1))$, and note that the cardinality of K_{ki} is greater than or equal to the cardinality of L_{ki}. Define α on K_{1-1} by mapping some subset of it to L_{1-1} and mapping any left over elements to any elements of V (α should be injective).

Suppose α defined on K_{ki-1} so that $\alpha(K_{kj}) \subseteq \rho(A_k(j))$ for $j \leq i-1$. We need only define α on K_{ki} so that $\alpha(K_{ki}) \subseteq \rho(A_k(i))$ to be done. Look at $M = L_{ki} - \bigcup_{\substack{\text{all } \ell \\ j \leq i-1}} \text{Image } \alpha(K_{\ell j})$. Map some subset of K_{ki} to M. Map the rest of K_{ki} to any elements of $\rho(A_k(i))$ at all.

By 4), $V \cup S = \bigcup_{\substack{\text{all } k \\ \text{all } i}} K_{ki}$, and $S = \bigcup_{\substack{\text{all } k \\ \text{all } i}} L_{ki}$ (as disjoint unions as we saw). Since α is onto each L_{ki}, and since it injects when restricted to each K_{ki}, α is 1-1. Furthermore, $\tau = \alpha \circ (\pi \cup \rho)$ satisfies

$\tau(A_k(i)) \subseteq \rho(A_k(i))$ by construction.

Set $\lambda(A) = \tau(A) \cap \rho(A)$. We claim λ is a partition. Clearly λ is a functor $\mathcal{C}(T) \to 2^V$. 1) and 2) are trivial and 4) is not much harder (1), 2) and 4) hold for the intersection of any two partitions, it is only 3) which might fail). To show 3), note $\lambda(A_k(i)) = \tau(A_k(i))$. If $T - \bigcup_{j=1}^{n} B_j$ is compact, there is a minimal i such that B_j contains $A_k(i)$ for some k (perhaps several, say $k = 1,\ldots,m$). Then $\bigcup_{k=1}^{m} \lambda(A_k(i)) \subseteq \lambda(B_j)$. $\lambda(T) - \bigcup_{j=1}^{n} \lambda(B_j) \subseteq \lambda(T) - \bigcup \lambda(A_k(i))$ $= \tau(T) - \bigcup \tau(A_k(i))$. The last two unions are over all $A_k(i) \subseteq B_j$ for $j = 1,\ldots,n$. The last set is finite, so 3) holds. Hence λ is a partition and thus τ is equivalent to ρ .

The map from $F_{\pi\cup\rho} \to F_\tau$ induced by α is the obvious map: $(F_{\pi\cup\rho})_A \to (F_\tau)_A$ is the isomorphism induced by the equivalence of bases $\alpha : (\pi \cup \rho)(A) \longleftrightarrow \tau(A)$. Lemma 5 completes the proof modulo the proof of Lemma 7.

Lemma 7: Let X and Y be two (disjoint) sets, and let π be a partition of X for the tree T . Any 1-1 correspondence $\alpha : X \to Y$ induces a partition $\alpha \circ \pi$ of Y for the tree T .

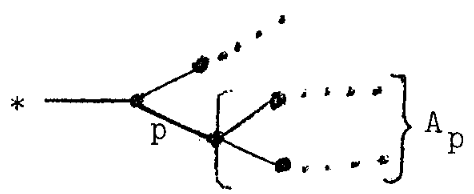
Proof: The easy proof is omitted.

Lemma 8: F_π is projective.

Proof: By Lemma 6 and standard nonsense, it is enough to prove the result for $F^{(1)}$. By Mitchell [25] Proposition 14.2, page 70, we need only show $M \xrightarrow{[f]} F^{(1)}$ splits whenever $[f]$ is an epimorphism (note $\mathcal{M}_R$ is abelian by Proposition 1 so we may apply Mitchell).

By taking a representative for $[f]$, we may as well assume that we have a map $f: M \to F = F^{(1)}$ which is an epimorphism. Now there is a partition π with $\pi \subseteq \rho$ (ρ the standard partition for $F^{(1)}$), such that the inclusion of $(F_\pi)_A$ in F_A lies in the image of M_A under f_A ; i.e. define $\pi(A) = \{x \in \rho(A) | x \in \text{Image } f_A\}$. Since f is an epimorphism, one can easily check $\rho(A) - \pi(A)$ is finite, and from this result one easily deduces π is a partition.

Now pick a base point $* \in T$. This choice immediately orders all the vertices of T by saying $p \geq q$ provided the minimal path from p to $*$ hits q . $A_p \in \mathcal{C}(T)$ for each p a vertex of T , $p \neq *$, is defined as the unique $A \in \mathcal{C}(T)$ such that $q \in A$ implies $q \geq p$.



Given a partition π , define a new partition τ by $\tau(A) = \bigcup_{A_p \subseteq A} \pi(A_p)$ (again, $\pi(A) - \tau(A)$ is finite, $\tau(A) \subseteq \pi(A)$, so one can check τ is a partition).

Since $\tau \subseteq \pi$, $(F_\tau)_A \subseteq F_A$ lies in Image (f_A).

Now given any vertex v of T, there is a unique p such that $v \in \tau(A_p)$ and $v \in \tau(A)$ iff $A_p \subseteq A$, unless $v \notin \tau(A_p)$ for any A_p (there are only finitely many of the latter). To see this, set $A = \bigcap_{v \in \tau(A_p)} A_p$.

Now $A_p \cap A_q \neq \emptyset$ implies $A_p \subseteq A_q$ (or $A_q \subseteq A_p$). By 4) the intersection runs over finitely many objects, so $A = A_p$ for some p. This A_p has the properties we claimed.

Define $x_v \in M_{A_p}$ to be any element such that $f_{A_p}(x_v)$ hits the image of the generator in $(F_\pi)_{A_p}$ corresponding to v. Define $h : F_\tau \to M$ by $h_A : (F_\tau)_A \to M_A$ takes the generator corresponding to v to $p_{A_pA}(x_v)$. We extend linearly. Notice that if the generator corresponding to v lies in $(F_\tau)_A$, $A_p \subseteq A$, so p_{A_pA} makes sense.

It is not hard to check the h_A induce a map $h : F_\tau \to M$, and $f \circ h : F_\tau \to F$ is just the inclusion. Q.E.D.

If $\mathcal{P}_R$ is the category of locally-finitely generated trees of projective R-modules, we have

<u>Lemma 9</u>: Let $0 \to M \to N \to Q \to 0$ be a short exact sequence of R-modules. Then if $N, Q \in \mathcal{P}_R$, $M \in \mathcal{P}_R$. If $M, Q \in \mathcal{P}_R$, $N \in \mathcal{P}_R$. Lastly, any $P \in \mathcal{P}_R$ is a summand of a locally-finitely generated free module.

Proof: The proof is easy.

Remarks: $\mathcal{P}_R$ is a suitable category in which to do stable algebra (see Bass [1]). $\mathcal{P}_R$ has a product, the direct sum. $\mathcal{P}_R$ is also a full subcategory of $\mathcal{M}_R$, which is abelian by Proposition 1. Hence we may use either of Bass's definitions of the K-groups. Note $\mathcal{P}_R$ is semi-simple (Bass [1]) so the two definitions agree.

Notation: $K_o(R) = K_o(\mathcal{P}_R)$ and $K_1(R) = K_1(\mathcal{P}_R)$ for R a tree of rings.

Given a map of trees of rings $R \to S$ ($R_A \to S_A$ takes units to units) we can define $M \otimes_R S$ for M a right R-module by taking $(M \otimes_R S)_A = M_A \otimes_{R_A} S_A$. $\otimes$ induces a functor $\mathcal{M}_R \to \mathcal{M}_S$. The only non-trivial part of this is to show $\otimes$ is well-defined on map-germs. But since

$$\begin{array}{ccc} \mathcal{M}_R & \xrightarrow{\otimes_R S} & \mathcal{M}_S \\ \downarrow & & \downarrow \\ \mathcal{M}_{\Delta(R)} & \xrightarrow{\otimes_{\Delta(R)} \Delta(S)} & \mathcal{M}_{\Delta(S)} \end{array}$$

commutes, this is easy. $\otimes$ is, as usual, an additive, right exact functor.

Now given a partition π, $F_\pi^R \otimes_R S = F_\pi^S$, where F_π^R is the free R-module based on π (F_π^S similarly). Hence it is easy to see $\otimes$ takes $\mathcal{P}_R$ to $\mathcal{P}_S$. $\otimes$ is cofinal in the sense of Bass [1], so we get a relative group $K_o(f)$, where $f : R \to S$ is the map of trees of rings. There is an exact sequence

$$K_1(R) \to K_1(S) \to K_o(f) \to K_o(R) \to K_o(S) .$$

We denote by $K_i(T)$, $i = 0,1$, the result of applying the K-groups to $\mathcal{P}_T$, where $\mathcal{P}_T$ is the category of locally-finitely generated projective modules over the tree of rings "T" , where $("T")_A = Z$ for all A , and $p_{AB} = id$. There is always a functor $\mathcal{P}_T \to \mathcal{P}_R$ induced by the unit map "T" $\to R$. The relative K_o of this map will be called the reduced K_1 of R , written $\overline{K}_1(R)$.

Remarks: If the tree of rings is a point the functor $\mathcal{M}_R \to \mathcal{M}_{\Delta(R)}$ induces a functor $\mathcal{P}_R \to \mathcal{P}_{\Delta(R)}$, where $\mathcal{P}_{\Delta(R)}$ is the category of finitely-generated projective $\Delta(R)$-modules. This functor induces an isomorphism on K_o and K_1 . For the compact case $(T = pt.)$, torsions lie in quotients of $K_1(\mathcal{P}_{\Delta(R)})$. This, together with Proposition 2 below is supposed to motivate our choice of $\mathcal{P}_R$ as the category in which to do stable algebra.

<u>Definition</u>: Let W be an hCW complex of finite dimension. Let X and Y be subcomplexes. Let (T,f) be a tree for W . Lastly let $F \in \mathcal{L}(f)$. Then $Z\pi_1(W,F,f)$ is the tree of rings we have earlier as an example. Pick a locally finite set of paths, Λ , from the cells of W to the vertices of $f(T)$ (the paths all begin at the baracenter of each cell).

$C_*(W;X,Y:\Lambda,F)$ is the tree of $Z\pi_1(W,F,f)$-modules given at A by $H_*(\widetilde{F(A)^*}; \widetilde{F(A)^{*-1}}, \widetilde{F(A)^* \cap X}, \widetilde{F(A)^* \cap Y})$, where ~ is the universal cover of $F(A)$, so, for example, $\widetilde{F(A)^* \cap Y}$ is the part of the universal cover of $F(A)$ lying over $Y \cap$ (the $*$-skeleton of $F(A)$). In each $\widetilde{F(A)}$ pick a base point covering the vertex ∂A. These choices give us maps $\widetilde{F(A)} \to \widetilde{F(B)}$ whenever $A \subseteq B$.

$C^*(W;X,Y:\Lambda,F)$ is defined from the cohomology groups $H_c^*(\widetilde{F(A)^*}; \widetilde{F(A)^{*-1}}, \partial\widetilde{F(A)^*}, \widetilde{F(A)^* \cap X}, \widetilde{F(A)^* \cap Y})$. The maps are the ones we defined in section 4.

<u>Proposition 2</u>: $C_*(W;X,Y:\Lambda,F)$ $(C^*(W;X,Y:\Lambda,F))$ is a locally-finitely generated, free, right (left) $Z\pi_1(W,F,f)$-module. If $G \in \mathcal{L}(f)$ satisfies $G \geq F$, there is an induced map $Z\pi_1(W,F,f) \to Z\pi_1(W,G,f)$. $C_*(W;X,Y:\Lambda,F) \otimes Z\pi_1(W,G,f)$ is equivalent to $C_*(W;X,Y:\Lambda,G)$. $Z\pi_1(W,G,f) \otimes C^*(W;X,Y:\Lambda,F)$ is equivalent to $C^*(W;X,Y:\Lambda,G)$. The Δ-functor applied to $C_*(W;X,Y:\Lambda,F)$ is $P_*(W;X,Y:\sim)$; $\Delta(C^*(W;X,Y:\Lambda,F)) = P^*(W;X,Y:\sim)$ (the P were defined in section 4)).

<u>Proof</u>: The assertions are all fairly obvious. Note in passing that the set S for C_* (C^*) is the set of all $*$-cells in $W-(X \cup Y)$. □

<u>Proposition 3</u>: The choice of path Λ determines a basis for C_* (C^*).

<u>Proof</u>: Let S be the set of all $*$-cells in $W-(X \cup Y)$. Partition S by $\pi(A)$ = the set of all

$*$-cells in $W(X \cup Y)$ such that the cell and its associated path both lie in $F(A)$. π is seen to be a partition, and F_π is equivalent to C_*. The path also determines a lift of the cell into $F(A)$, so each $(F_\pi)_A$ is based. □

Apparently our tree of rings and modules is going to depend on the lift functor we choose. This is not the case, and we proceed to prove this. Given a shift functor F and a tree of rings R, R_F is the tree of rings given by $(R_F)_A = \bigoplus_{i=1}^{n} R_{A_i}$ where the A_i are the essential components of $F(A)$. p_{AB} is just $\oplus\, p_{ij}$, where p_{ij} is the projection $p_{A_i B_j}$ where $A_i \subseteq B_j$.

We now redefine M_F. M_F is going to be an R_F-module. $(M_F)_A = \bigoplus_{i=1}^{n} M_{A_i}$ with the obvious R_F-module structure. Note $M_F \otimes_{R_F} R$ is just our old M_F.

Now a T-map of rings is just a map $R_F \to S$. As in the case of modules, we can define a map-germ between two rings, and the category of trees of rings and map-germs is an additive category.

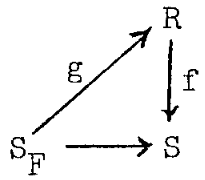
<u>Lemma 10</u>: The maps $K_i(R_F) \to K_i(R)$, $i = 0,1$, are isomorphisms.

<u>Proof</u>: $M \to M_F$, $f \to f_F$ defines a functor $\mathcal{P}_R \to \mathcal{P}_{R_F}$. Using this functor, one checks $\mathcal{P}_{R_F} \to \mathcal{P}_R$ is an equivalence of categories. The result is now easy. Q.E.D.

Hence given a map-germ $f: R \to S$, we get well-defined induced maps $K_i(R) \to K_i(S)$, $i = 0,1$, and

$\overline{K}_1(R) \to \overline{K}_1(S)$.

<u>Lemma 11</u>: Let $f : R \to S$ be a map such that $\Delta(f)$ is an isomorphism. Then there is a shift functor F and a map $g: S_F \to R$ such that



commutes.

<u>Proof</u>: The proof is just like that of Lemma 2. Q.E.D.

<u>Lemma 12</u>: Let $[f]: R \to S$ be a map germ such that $\Delta(f)$ is an isomorphism. Then the maps $K_o(R) \to K_o(S)$; $K_1(R) \to K_1(S)$; and $\overline{K}_1(R) \to \overline{K}_1(S)$ are isomorphisms.

<u>Proof</u>: This proof is easy and will be left to the reader. Q.E.D.

Remarks: By Lemma 12, the K-groups we get will not depend on which lift functor we use. Let $K_i(X:f) = \varinjlim_{F \in \mathcal{L}(f)} K_i(Z\pi_1(X,F,f))$. Since all the maps in our direct limit are isomorphisms, $K_i(X:f)$ is computable in terms of $K_i(Z\pi_1(X,F,f))$ for any F . $\overline{K}_1(X:f)$ is defined similarly.

<u>Definition</u>: A stably free (s-free) tree of R-modules is an element, P, of $\mathcal{P}_R$ such that $[P]$ is in the image of $K_o(T)$. Let P be an s-free R-module.

An s-basis for P is an element $F \in \mathcal{P}_T$ and an isomorphism $b : F \otimes_T R \to P \oplus F_1 \otimes_T R$, where $F_1 \in \mathcal{P}_T$.

Two s-bases $b : F \otimes_T R \to P \oplus F_1 \otimes_T R$ and $c : F_2 \otimes_T R \to P \oplus F_3 \otimes_T R$ are equivalent $(b \sim c)$ iff $0 = (F \oplus F_3, (b \oplus id_{F_3}) \circ tw \circ (c \oplus id_{F_1})^{-1}, F_2 \oplus F_1)$ in $\overline{K}_1(R)$, where $tw: (P \oplus F_1 \otimes_T R) \oplus F_3 \otimes_T R \longrightarrow (P \oplus F_3 \otimes_T R) \oplus F_1 \otimes_T R$ is the obvious map.

We can now give an exposition of torsion following Milnor [23]. Given a short exact sequence $0 \to E \xrightarrow{i} F \xrightarrow{p} G \to 0$ and s-bases b for E and c for G, define an s-basis bc for F by picking a splitting $r : G \to F$ for p and then taking the composition $F_1 \oplus F_2 \xrightarrow{(b,c)} (E \oplus F_3) \oplus (G \oplus F_4) \xrightarrow{h} F \oplus (F_3 \oplus F_4)$, where $h(e,x,g,z)$ goes to $(i(e) + r(g),x,z)$. It is not hard to check that this s-basis does not depend on the choice of splitting map.

We use Milnor's formulation. Let $F_o \subseteq F_1 \subseteq \dots \subseteq F_k$ and suppose each F_i/F_{i-1} has an s-basis b_i. Then $b_1 b_2 \dots b_k$ is seen to be well-defined; i.e. our construction is associative.

Let E and F be submodules of G. Then $E + F$ is the submodule of G generated by E and F. $E \cap F$ is the pullback of

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ F & \longrightarrow & G \end{array} .$$

Lemma 13: (Noether) The natural map $E/E \cap F \longrightarrow E+F/F$ is an isomorphism.

Proof: Apply the ordinary Noether isomorphism to each term. Q.E.D.

Now let $E/E \cap F$ have an s-basis b, and let $F/E \cap F$ have an s-basis c. Base $E+F/F$ by b composed with the Noether map (we will continue to denote it by b). Similarly base $E+F/E$ by c. Then $bc \sim cb$ as s-bases for $E+F/E \cap F$.

Definition: Let b and c be two s-bases for P. Then $[b/c] \in \overline{K}_1(R)$ is defined as follows: if $F \xrightarrow{b} P \oplus F_1$; $G \xrightarrow{c} P \oplus F_2$, then $[b/c] = (F \oplus F_2, h, G \oplus F_1)$, where $h : F \oplus F_2 \xrightarrow{b \oplus id} (P \oplus F_1) \oplus F_2) \longrightarrow (P \oplus F_2) \oplus F_1 \xrightarrow{c^{-1} \oplus id} G \oplus F_1$. Two s-basis are equivalent iff $[b/c] = 0$. The formulas $[b/c] + [c/d] = [b/d]$ and $[b/c] + [d/e] = [bd/ce]$ are easy to derive from the relations in the relative K_o.

We next define a torsion for chain complexes. A free chain complex is a set of s-free modules, P_n, together with map-germs $\partial_n : P_n \rightarrow P_{n-1}$ such that $\partial_n \partial_{n-1} = 0$. A finite free chain complex is one with only finitely many non-zero P_n. A positive free chain complex has $P_n = 0$ for $n < 0$.

Definition: Let $\{P_n, \partial_n\}$ be a finite free chain complex. Let P_n be s-based by c_n, and suppose each homology group H_i is s-free and s-based by h_i .

The sequences $0 \to B_{n+1} \to Z_n \to H_n \to 0$ and $0 \to Z_n \to P_n \to B_n \to 0$, where $B_n = \text{Image } (P_n \to P_{n-1})$ and $Z_n = \text{kernel } (\partial_n)$, are short exact. Let b_n be an s-basis for B_n , which exist by an inductive argument.

$$\tau(P_*) = \sum_n (-1)^n [b_n h_n b_{n-1}/c_n] \in \overline{K}_1 \ (R) .$$

It is easy to show $\tau(P_*)$ does not depend on the choice of b_n . Let $0 \to P'_* \to P_* \to P''_* \to 0$ be a short exact sequence of finite chain complexes. There is a long sequence

$$\begin{array}{ccc} H_*(P') & \longrightarrow & H_*(P) \\ & {}_{\partial}\nwarrow \quad \swarrow & \\ & H_*(P'') & \end{array} .$$

Suppose each homology module is s-based. Then we have a torsion associated to $\mathcal{H}$, where

$$\mathcal{H}_{3n} = H_n(P''), \ \mathcal{H}_{3n+1} = H_n(P) \ , \ \mathcal{H}_{3n+2} = H_n(P'') \ ,$$

since $\mathcal{H}$ is acyclic.

<u>Theorem 1</u>: $\tau(P_*) = \tau(P'_*) + \tau(P''_*) + \tau(\mathcal{H})$.

<u>Proof</u>: See Milnor [23], Theorems 3.1 and 3.2. □

We next describe the algebraic Subdivision Theorem of Milnor [23] (Theorem 5.2). Given a chain complex C_* , suppose it is filtered by $C_*^{(0)} \subseteq C_*^{(1)} \subseteq \ldots \subseteq C_*^{(n)} = C_*$ such that the homology group $H_i(C^{(\lambda)}/C^{(\lambda-1)}) = 0$ for $i \neq \lambda$. $(C_*^{(-1)} = 0)$.

Then we have a chain complex $(\overline{C}_*, \overline{\partial}_*)$ given by $\overline{C}_\lambda = H_\lambda(C^{(\lambda)}/C^{(\lambda-1)})$ and $\overline{\partial}$ is given by the boundary in the homology exact sequence of the triple $(C^{(\lambda)}, C^{(\lambda-1)}, C^{(\lambda-2)})$. There is a well-known canonical isomorphism $H_i(\overline{C}) \xrightarrow{\cong} H_i(C)$ (see Milnor, Lemma 5.1).

Now suppose each $C_i^{(\lambda)}/C_i^{(\lambda-1)}$ has an s-basis c_i^λ : each $\overline{C}_\lambda$ has an s-basis $\overline{c}_\lambda$: each $H_i(\overline{C})$ has an s-basis h_i. Assume C_* is a finite complex. Then so is $\overline{C}_*$.

Each $C^{(\lambda)}/C^{(\lambda-1)}$ has a torsion. If C_i is s-based by $c_i^o\, c_i^1 \ldots c_i^n$, and $H_i(C)$ is based by h_i composed with the canonical isomorphism, then the torsion of C is defined. Lastly the torsion of $\overline{C}$ is also defined.

<u>Theorem 2</u>: (Algebraic Subdivision Theorem)

$$\tau(C) = \tau(C) + \sum_{\lambda=o}^{n} \tau(C^{(\lambda)}/C^{(\lambda-1)}) .$$

<u>Proof</u>: The proof is the same as Milnor's [23], Theorem 5.2. One does the same induction, but one just shows $\tau(C^{(k)}) = \tau(\overline{C}^{(k)}) + \sum_{\lambda=o}^{k} \tau(C^{(\lambda)}/C^{(\lambda-1)})$ (notation is the same as Milnor's). ▯

Now let (K,L) be a pair of finite dimensional hCW complexes with L a proper deformation retract of K. We have the modules $C_*(K,L\colon \Lambda, F)$. The exact sequence of a triple makes C_* into a chain complex, whose homology is zero since L is a proper deformation retract of K. The paths Λ gives us a basis for C_* up to sign; i.e. we must orient each cell, which we can do arbitrarily. $\tau(K,L:\Lambda,F) \in \overline{K}_1(Z\pi_1(K,F,f))$ is the torsion of this complex with the basis given by Λ. We proceed to show it does not depend on the choice of signs.

Let τ' be the torsion with a different choice of signs. Then, by Lemma 14 below, $\tau'-\tau = \sum_* (-1)^*[c_*/c_*^1]$, where c_* and c_*^1 are maps $F_\pi \to C_*$, one with the signs for τ and the other with the signs for τ'. But $c_*^{-1} \; c_*^1 : F_\pi \to F_\pi$ lies in the image of $\mathcal{P}_T \to \mathcal{P}_R$, and so $[c_*/c_*^1] = 0$ in $\overline{K}_1(R)$.

<u>Lemma 14</u>: Let C_* be a chain complex. Let c_* and $c_*^1 : F_\pi \to C_*$ be two free bases for C_*. Suppose $H_*(C)$ is s-based. Let τ and τ' be the torsions from the bases c_* and c_*^1 respectively. Then $\tau' - \tau = \sum_* (-1)^*[c_*/c_*^1]$.

<u>Proof</u>: This is a fairly dull computation. Q.E.D.

Now suppose G is a different lift functor with $F \leq G$. Then by Proposition 2, the basis $c_*: F_\pi \to C_*(F)$ goes to $c_* : F_\pi \to C_*(G)$ under $\otimes_{Z\pi_1(F)} Z\pi_1(G)$. Let $c_*^1 : F_\rho \to C_*(G)$ be the usual basis. Then $\pi \subseteq \rho$, and $F_\pi \to F_\rho \xrightarrow{c_*^1} C_*$ is c_*. The inclusion $F_\pi \to F_\rho$ lies in the image of $\mathcal{P}_T$ in $\mathcal{P}_R$, so $[c_*/c_*^1] = 0 \in \overline{K}_1(Z\pi_1(K,G,f))$. Hence $i_* \tau(K,L:F,\Lambda) - \tau(K,L:G,\Lambda) = 0$ where $i_* : \overline{K}_1(Z\pi_1(K,F,f)) \to \overline{K}_1(Z\pi_1(K,G,f))$. Therefore we can define $\tau(K,L:\Lambda) \in \overline{K}_1(K:f)$.

$\tau(K,L:\Lambda)$ depends strongly on Λ . We would like this not to be the case, so we pass to a quotient of $\overline{K}_1$.

<u>Definition</u>: Let G be a tree of groups with associated tree of rings ZG . The Whitehead group of G , $Wh(G) = \overline{K}_1(ZG)/(\Delta(G))$, where $(\Delta(G))$ is the subgroup generated by all objects of the form $(F^{(1)}, [g], F^{(1)})$ where $[g]$ is the map-germ of $F^{(1)}$ to itself induced by any element $g \in \Delta(G)$ as follows: g can be represented by a collection $\{g_p\}$, where $g_p \in G_{A(p)}$, with $p \in A(p)$ and $\{A(p)\}$ cofinal and locally finite. Define a partition, π , of the vertices of T by $\pi(A) = \{p \in T | A(p) \subseteq A\}$. π is seen to be a partition and $\pi \subseteq \rho$, the standard partition. Define a map $g : F_\pi \to F_\pi$ by $g_A : (F_\pi)_A \to (F_\pi)_A$ takes e_p to $e_p \cdot f_{A_pA}(g_p)$ where $f_{AB}: (ZG)_A \to (ZG)_B$. It is not hard to show this is a well-defined map-germ. What we have actually done is construct a homomorphism $\Delta(G) \to \overline{K}_1(ZG)$ defined by $g \to (F^{(1)},[g],F^{(1)})$. By definition, $\Delta(G) \to \overline{K}_1(ZG) \to Wh(G) \to 0$ is exact.

Given a homomorphism $f : G \to H$ between two trees of groups, we clearly get a commutative square

$$\begin{array}{ccc} \Delta(G) & \longrightarrow & \overline{K}_1(ZG) \\ \downarrow & & \downarrow \\ \Delta(H) & \longrightarrow & \overline{K}_1(ZH) \end{array} ,$$

so we get a homomorphism $Wh(G) \to Wh(H)$.

Lemma 15: Let $f : G \to H$ be a map between two trees of groups for which $\Delta(f)$ is an isomorphism. Then $Wh(G) \to Wh(H)$ is an isomorphism.

Proof: $\Delta(f) : \Delta(ZG) \to \Delta(ZH)$ is also an isomorphism, so apply Lemma 12 and the 5-lemmas. Q.E.D.

We can now define $Wh(X:f)$ as $\varinjlim_{F \in \mathcal{L}(f)} Wh(Z\pi_1(X,F,f))$.

Proposition 4: Let (K,L) be a pair of finite dimensional hCW complexes with L a proper deformation retract of K. Then if Λ and Λ' are two choices of paths, then $\tau(K,L:\Lambda) = \tau(K,L:\Lambda')$ in $Wh(X:f)$. Hence we can define $\tau(K,L) \in Wh(X:f)$.

Proof: We can pick any lift functor we like, say F. $C_*(K,L:\Lambda,F) = C_*(K,L:\Lambda',F)$, and each is naturally based. Let π_* be the partition asoociated to Λ (see Proposition 3) and let π_*' be the partition associated to Λ'. Let ρ_* be the partition $\rho(A) = \{e \mid e$ is a $*$-cell in $F(A)$ and the path for e in Λ lies in $F(A)$ and the path for e in Λ' also lies in $F(A)\}$.
$\rho_* = \pi_* \cap \pi_*'$.

The basis $F_{\rho_*} \to C_*$ is equivalent to the basis $F_{\pi_*} \to C_*$. Similarly $F_{\rho_*} \to C_*'$ is equivalent to the basis $F_{\pi_*'} \to C_*'$. ($C_* = C_*(\ldots,\Lambda)$; $C_*' = C_*(\ldots,\Lambda')$.)

$\tau' - \tau = \tau(K,L:\Lambda') - \tau(K,L:\Lambda) = \sum_* (-1)^* [\pi_*/\pi_*']$, by Lemma 14. If we can show $[\pi_*/\pi_*']$ is in the image of $\Delta(Z\pi_1)$ we are done. But this is not hard to see ($Wh(\ )$ was defined by factoring out by such things). □

Having defined a torsion, we prove it invariant under subdivision. We follow Milnor [23].

<u>Theorem 3</u>: The torsion $\tau(K,L)$ is invariant under subdivision of the pair (K,L); (K,L) a finite dimensional hCW pair.

<u>Proof</u>: Following Milnor [23] we prove two lemmas.

<u>Lemma 16</u>: Suppose that each component of $K-L$ has compact closure and is simply connected. If L is a proper deformation retract of K, then $\tau(K,L) = 0$.

<u>Proof</u>: (Compare Milnor [23] Lemma 7.2). Let $f : T \to K$ be the tree. We wish to find a set of paths Λ so that the boundary map in $C_*(K,L:F,\Lambda)$ comes from $\mathcal{P}_T$.

Let $\{M_i\}$ be the components of $K-L$. Pick a point $q_i \in M_i$, and join $\{q_i\}$ to T by a locally finite set of paths λ_i. Now join each cell in M_i to q_i by a path lying in M_i. Let Λ be the set of paths gotten by following the path from the cell to a q_i and then following the path λ_i. Clearly Λ is a locally finite set of paths joining the cells of $K-L$ to T.

Let e be a cell of $K-L$. Then if f is a cell of ∂e, to compute the coefficient of f in ∂e we join the baracenter of f to the baracenter of e by a path in e and look at the resulting loop. The path from e and the path from f hit the same q_i, and since $\pi_1(M_i,q_i) = 0$, the coefficient is ± 1, so the boundary map comes from $\mathcal{P}_T$. Q.E.D.

Lemma 17: Suppose that $H_*(C_*(K,L:\Lambda))$ is not zero, but is a free $Z\pi_1(K)$-module with a preferred basis. Suppose each basis element can be represented by a cycle lying over a single component of $K - L$. Assume as before that each component of $K - L$ is compact and simply connected. Then $\tau(K,L) = 0$.

Proof: Pick a set of paths as in Lemma 16 so that the boundary maps come from $\mathcal{P}_T$. Look at a cycle z, representing a basis element of H_*. What this means is the following. Let $C_0 \subseteq C_1 \subseteq \cdots$ be an increasing sequence of compact subcomplexes with $\bigcup C_i = K$ and $M_i \subseteq C_i$. Then $z \in H_*(\widetilde{K - C_i}, \widetilde{L - C_i})$ for a maximal C_i. Then z is represented by a cycle lying in some component of $\pi^{-1}(M_{i+1})$, where $\pi : \widetilde{K - C_i} \to K - C_i$. All the lifted cells of M_{i+1} lie in a single component of $\pi^{-1}(M_{i+1})$, so let $g \in \pi_1(K - C_i)$ be such that gx also lies in this distinguished component.

Then the torsion computed with this altered basis is zero since it again comes from $Wh(T) = 0$. But the new basis for H_* is clearly equivalent to the old one in $Wh(K)$. Q.E.D.

The proof of Theorem 3 now follows Milnor's proof of Theorem 7.1 word for word except for a renumbering of the requisite lemmas. □

Lemma 18: If $M \subseteq L \subseteq K$, where both L and M are proper deformation retracts of K, then

$\tau(K,L) = \tau(K,M) + i_*\tau(L,M)$, where $i_*: Wh(L:f) \to Wh(K : i \circ f)$ is the map induced by $i: L \subseteq K$. (Note the tree must be in L.)

<u>Proof</u>: This is a simple application of Theorem 1. Q.E.D.

Let $f: X \to Y$ be a proper, cellular map between two finite dimensional hCW complexes. Let M_f be the mapping cylinder. Y is a proper deformation retract of M_f and we have

<u>Lemma 19</u>: $\tau(M_f, Y) = 0$ in $Wh(M_f,t)$, where $t : T \to Y$ is a tree for $Y \subseteq M_f$.

<u>Proof</u>: Word for word Milnor [23] Lemma 7.5. Q.E.D.

<u>Definition</u>: For any cellular proper homotopy equivalence $f : X \to Y$, X and Y as above, there is a torsion, $\tau(f)$, defined as follows. Let $t : T \to Y$ be a tree for Y . Then, as in Lemma 19, t is also a tree for M_f under $T \to Y \subseteq M_f$. $\tau(f) = i_*\tau(M_f,X)$, where $r_* : Wh(M_f:t) \to Wh(Y:t)$, where r is the retraction.

Just as in Milnor we have

<u>Lemma 20</u>: If $i : L \to K$ is an inclusion map $\tau(i) = \tau(K,L)$ if either is defined.

<u>Lemma 21</u>: If f_o and f_1 are properly homotopic, $\tau(f_o) = \tau(f_1)$.

<u>Lemma 22</u>: If $f : X \to Y$ and $g : Y \to Z$ are cellular proper homotopy equivalences, then $\tau(g \circ f) = \tau(g) + g_*\tau(f)$, where $t : T \to Y$ is a tree for Y and $g_* : Wh(Y:t) \xrightarrow{\cong} Wh(Z:g \circ t)$.

Remarks: It follows from Lemma 21 that we may define the torsion of any proper homotopy equivalence between finite dimensional hCW complexes, since we have a proper cellular approximation theorem [11].

Now in [33], Siebenmann defined the notion of simple homotopy type geometrically. In particular, he got groups $\zeta(X)$ associated to any locally compact CW complex. If X is finite dimensional, we can define a map $\tau : \zeta(X) \to Wh(X:f)$ by choosing a tree $f: T \to X$. If $g: X \to Y$ is an element of $\zeta(X)$, g goes to $\tau(M_g{-1},Y)$, where $g^{-1}: Y \to X$ is a proper homotopy inverse for g .

τ is additive by Lemma 22 and depends only on the proper homotopy class of g by Lemma 21. That τ is well-defined reduces therefore to showing that g a simple homotopy equivalence implies $\tau(g) = 0$. We defer for the proof to Farrell-Wagoner [10], where it is also proved τ is an isomorphism. The inverse for τ is easy to describe. Let $\alpha \in Wh(X:f)$ be an automorphism of $F^{(n)}$ for some n . Wedge n 2-spheres to each vertex of the tree. Attach 3-cells by α to get an hCW complex Y with $Y - X$ 3-dimensional. Then

$i : X \subseteq Y$ is an element of $\varsigma(X)$ and $\tau(i) = \alpha$. Again we defer to [10] for the proof that this map is well-defined.

In [33] Siebenmann also constructs an exact sequence $0 \to Wh'\pi_1(X) \to \varsigma(X) \to K_o\ \pi_1 E(X) \to K_o\ \pi_1(X)$. We have

$$\begin{array}{ccccccc} & & \varsigma(X) & & & & \\ & \nearrow & \uparrow \tau^{-1} & \searrow & & & \\ 0 \to Wh'\pi_1(X) & & & & K_o\ \pi_1 E(X) & \to & K_o\ \pi_1(X) \\ & \underset{\alpha}{\searrow} & & \nearrow & & & \\ & & Wh(X:f) & & & & \end{array}$$

commutes. Farrell and Wagoner describe α and β and prove this diagram commutes. They they show the bottom row is exact, so τ^{-1} is an isomorphism.

Note now that if $g: T \to X$ is another tree for X, we have natural maps $Wh(X:f) \rightleftarrows Wh(X:g)$ which take $\tau(X,Y)$ computed with f to $\tau(X,Y)$ computed with g and vice-versa. This shows $Wh(X:f)$ does not really depend on the choice of tree. We content ourselves with remarking that the map $Wh(X:f) \to Wh(X:g)$ is not easy to describe algebraically.

In [33] Siebenmann derives some useful formulas which we name:

1) Sum formula

2) Product formula

3) Transfer formula.

Note if $\pi : Y \to Y$ is a cover, π induces $\pi^*: \varsigma(Y) \to \varsigma(Y)$. We are unable to say much about this map algebraically. The product formula is algebraically describable however.

Lemma 23: Let C_* be an s-based, finite chain complex over the tree of rings R. Let D_* be an s-based, finite chain complex on the ring S (the tree of rings over a point). Then $(C \otimes D)_*$ is defined. If C_* is acyclic with torsion τ, $(C \otimes D)_*$ is acyclic with torsion $\chi(D) \circ i_* \tau(C) \in Wh(R \times S)$, where $R \times S$ is the tree of rings $(R \times S)_A = R_A \times S$, and $i_* : Wh(R) \to Wh(R \times S)$ is the obvious split monomorphism. If D_* is acyclic, then so is $(C \otimes D)_*$, and if $\tau(D) = 0$, then $\tau(C \otimes D) = 0$.

Proof: The first formula is Siebenmann's product formula and is proved by inducting on the number of cells in D_*. The second formula is new, but it is fairly easy. It basically requires the analysis of maps $Wh(S) \to Wh(R \times S)$ of the form $D_* \to P \otimes D_*$ for P an s-based R-module. These maps are homomorphisms, and so, if $\tau(D_*) = 0$, $\tau(P \otimes D_*) = 0$. But $\tau((C \otimes D)_*) = \sum_k (-1)^k \tau(C_k \otimes D_*)$. (There is evidence for conjecturing that the map $Wh(S) \to Wh(R \times S)$ is always 0). Q.E.D.

We conclude this section by discussing the notion of duality. In particular, we would like a functor $* : \mathcal{M}_R \to \mathcal{M}_R^{\ell}$ which generalizes the usual duality $P \to Hom_R(P,R)$ in the compact case. Up until now $\mathcal{M}_R$ has denoted without prejudice either the category of right or left R-modules. We now fix it to be the category of right R-modules. $\mathcal{M}_R^{\ell}$ then denotes the category of left R-modules.

Actually, we are really only interested in $* : \mathcal{P}_R \to \mathcal{P}_R^{\ell}$. Hence we begin by discussing a functor $* : \mathfrak{F}_R \to \mathfrak{F}_R^{\ell}$, where $\mathfrak{F}_R$ is the category of locally-finitely generated free modules. $*$ will satisfy

1) $*$ is a contravariant, additive, full faithful functor

2) $**$ is naturally equivalent to the identity.

By this last statement we mean the following. Given $* : \mathfrak{F}_R \to \mathfrak{F}_R^{\ell}$ there will be another obvious duality $* : \mathfrak{F}_R^{\ell} \to \mathfrak{F}$. The composite of these two is naturally equivalent to the identity.

We proceed to define $*$. If F_A is a free right R_A-module based on the set A, there is also a free left R_A-module based on the same set, F_A^*. F_A^* can be described as $\mathrm{Hom}^c_{R_A}(F_A, R_A)$, where Hom^c_R is the set of all R - linear homomorphisms which vanish on all but finitely many generators. $\mathrm{Hom}^c_{R_A}(F_A, R_A)$ is easily seen to have the structure of a left R_A-module.

Let $A \subseteq B$, and let $f : R \to S$ be a ring homomorphism. Then we have

$$\mathrm{Hom}^c_{R_A}(F_A, R_A) \longrightarrow \mathrm{Hom}^c_{R_B}(F_A \otimes R_B, R_B) \xleftarrow{\text{ex}}$$

$$\mathrm{Hom}^c_{R_B}(F_B/F_{B-A}, R_B) \longrightarrow \mathrm{Hom}^c_{R_B}(F_B, R_B) .$$

The map ex is an isomorphism since $0 \to F_A \otimes R_B \to F_B \to F_{B-A} \to 0$ is split exact. Thus we get a well-defined homomorphism

$$\mathrm{Hom}^c_{R_A}(F_A, R_A) \longrightarrow \mathrm{Hom}^c_{R_B}(F_B, R_B) .$$

Now given F_π , let F_π be the tree of left modules over the tree of rings R defined by $(F^*_\pi)_A = \mathrm{Hom}^c_{R_A}(F_{\pi(A)}, R_A)$, and use the map discussed above to define p_{AB} .

Given a map $f : F_\pi \to F_\rho$, define $f^* : F^*_\rho \to F^*_\pi$ by $(f^*)_A = \mathrm{Hom}(f_A) : \mathrm{Hom}^c_{R_A}(F_{\rho(A)}, R_A) \to \mathrm{Hom}^c_{R_A}(F_{\pi(A)}, R_A)$. We must check that $(f^*)_A$ is defined and that the requisite diagrams commute. This last is trivial, so we concentrate on the first objective. To this end, let $\alpha \in \mathrm{Hom}^c_{R_A}(F_{\rho(A)}, R_A)$. We must show $\mathrm{Hom}(f_A)(\alpha)$ lies in $\mathrm{Hom}^c_{R_A}(F_{\pi(A)}, R_A) \subseteq \mathrm{Hom}_{R_A}(F_{\pi(A)}, R_A)$. Since α has compact support, α vanishes on the generators corresponding to a subset $S \subseteq \rho(A)$ with $\rho(A) - S$ finite. Hence there is a $B \in \mathcal{C}(T)$ so that $\rho(B) \subseteq S$; i.e. α vanishes on generators corresponding to $\rho(B)$. Let $\overline{F}_{\pi(B)} = F_{\pi(B)} \otimes_{R_B} R_A$; let $\overline{F}_{\rho(B)} = F_{\rho(B)} \otimes_{R_B} R_A$; and let $\overline{f}_B = f_B \otimes \mathrm{id}$. Then

$$\begin{array}{ccc}
\mathrm{Hom}(F_{\rho(A)}, R_A) & \xrightarrow{\mathrm{Hom}(f_A)} & \mathrm{Hom}(F_{\pi(A)}, R_A) \\
\Big\downarrow i & & \Big\downarrow j \\
\mathrm{Hom}(\overline{F}_{\rho(B)}, R_A) & \xrightarrow{\mathrm{Hom}(\overline{f}_B)} & \mathrm{Hom}(\overline{F}_{\pi(B)}, R_A)
\end{array}$$

α is in the kernel of i, so $\mathrm{Hom}(f_A)(\alpha) \in \ker j$. But this means $\mathrm{Hom}(f_A)(\alpha)$ has compact support.

There is a natural map $F \to F^{**}$ induced by the natural inclusion of a module in its double dual. This map is an isomorphism and

$$\begin{array}{ccc} F & \longrightarrow & F^{**} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{**}} \\ G & \longrightarrow & G^{**} \end{array}$$

commutes.

$*$ is clearly contravariant and a functor. If $\pi \subseteq \rho$, one sees $F_\rho^* \to F_\pi^*$ is an equivalence. Hence we can define $*$ for map-germs. $(f+g)^* = f^* + g^*$ is easy to see, so $*$ is additive. Since $**$ is naturally equivalent to the identity, $*$ must be both faithful and full, so 1) is satisfied.

We next define the subcategory on which we wish to define $*$. Let $\overline{\mathfrak{M}}_R$ be the full subcategory of $\mathfrak{M}_R$ such that $M \in \mathfrak{M}_R$ iff there exists $f \colon F_\rho \to F_\pi$ with coker $f \cong M$. Note $\mathcal{P}_R \subseteq \overline{\mathfrak{M}}_R$. We define $* : \mathfrak{M}_R \to \mathfrak{M}_R^\ell$ by $M^* = \ker(f^*)$.

Given $M, N \in \overline{\mathfrak{M}}_R$, a map $g \colon M \to N$, and resolutions $F_\rho \to F_\pi \to M \to 0$ and $F_\alpha \to F_\beta \to N \to 0$, note that we can compare resolutions. That is, we can find h and f so that

$$\text{A)}\qquad \begin{array}{ccc} F_\rho & \xrightarrow{\;f\;} & F_\alpha \\ \downarrow & & \downarrow \\ F_\pi & \xrightarrow{\;h\;} & F_\beta \\ \downarrow & & \downarrow \\ M & \xrightarrow{\;g\;} & N \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

commutes.

Define $g^*: N^* \to M^*$ by

$$\text{B)} \qquad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ N^* & \xrightarrow{g^*} & M^* \\ \downarrow & & \downarrow \\ F_\beta^* & \xrightarrow{h^*} & F_\pi^* \\ \downarrow & & \downarrow \\ F_\alpha^* & \xrightarrow{f^*} & F_\rho^* \end{array} .$$

We first note that the definition of g^* does not depend on h and f, for if we pick h_1 and f_1 such that A) commutes, there is a commutative triangle

$$\begin{array}{ccc} & & F_\alpha \\ & \overset{p}{\nearrow} & \downarrow \\ F_\pi & \xrightarrow{h - h_1} & F_\beta \end{array} .$$

Dualizing we get

$$\begin{array}{ccc} F_\beta^* & \xrightarrow{h^* - h_1^*} & F_\pi^* \\ \downarrow & \overset{p^*}{\nearrow} & \\ F_\alpha^* & & \end{array} .$$

Now this triangle shows that the map we get from f_1, h_1 is the same as we got from f, h.

To show M^* does not depend on the resolution is now done by comparing two resolutions and noting $(\mathrm{id})^* = \mathrm{id}$.

Unfortunately $(M^*)^*$ may not even be defined, so we have little hope of proving a result like 2). One useful result that we can get however is

<u>Lemma 24</u>: Let $f : P \to M$ be an epimorphism with $M \in \overline{\mathcal{M}}_R$ and $P \in \mathcal{P}_R$. Then $f^*: M^* \to P^*$ is a monomorphism.

<u>Proof</u>: The proof is easy. Q.E.D.

If we restrict ourselves to $\mathcal{P}_R$, we can get 1) and 2) to hold. It is easy to see $P^* \in \mathcal{P}^\ell_R$ for $P \in \mathcal{P}_R$. Now the equation $(P \oplus Q)^* = P^* \oplus Q^*$ is easily seen since direct sum preserves kernels. Thus $(P \oplus Q)^{**} = P^{**} \oplus Q^{**}$, so it is not hard to see $P \to P^{**}$ must be an isomorphism since if P is free the result is known. Lastly, $*$ is natural, i.e. if $f : R \to S$ is a map,

$$\begin{array}{ccc} \overline{\mathcal{M}}_R & \longrightarrow & \overline{\mathcal{M}}_S \\ \downarrow{*} & & \downarrow{*} \\ \mathcal{M}^\ell_R & \longrightarrow & \mathcal{M}^\ell_S \end{array}$$

commutes. That $\overline{\mathcal{M}}_R$ hits $\overline{\mathcal{M}}_S$ follows since $\otimes$ is right exact.

<u>Definition</u>: Let $\{M_i, \partial_i\}$ be a chain complex with $M_i \in \overline{\mathcal{M}}_R$. Then $\{\mathcal{M}_i^*, \partial_i^*\}$ is also a chain complex. The cohomology of $\{M_i, \partial_i\}$ is defined as the homology of $\{M_i^*, \partial_i^*\}$.

<u>Proposition 5</u>: Let (X,Y) be an hCW pair; let F be a lift functor; and let Λ be a set of paths. Then $\{C_*(X,Y\colon F,\Lambda), \partial_*\}$ is a chain complex as we saw. Its dual is $\{C^*(X,Y\colon F,\Lambda), \delta^*)$. Hence the cohomology of a pair is the same as the cohomology of its chain complex.

<u>Proof</u>: Easy. □

Notice that our geometric chain complexes lie in $\mathcal{P}_R$. For such complexes we can prove

<u>Theorem 4</u>: Let $\{P_r, \partial_r\}$ be a finite chain complex in $\mathcal{P}_R$. $H_k(P) = 0$ for $k < n$ iff there exist maps $D_r : P_r \to P_{r+1}$ for $r \leq n$ with $D_{r-1}\partial_r + \partial_{r+1}D_r = id_{P_r}$.

<u>Proof</u>: Standard. □

<u>Corollary 4.1</u>: (Universal coefficients). With $\{P_r, \partial_r\}$ as above, $H_k(P) = 0$ for $k \leq n$ implies $H^k(P) = 0$ $k \leq n$. $H^k(P) = 0$ for $k \geq n$ implies $H_k(P) = 0$ for $k \geq n$.

<u>Proof</u>: Standard. □

Now suppose $\{P_r, \partial_r\}$ is a chain complex in $\mathcal{P}_R$. Then $\operatorname{coker} \partial_r \in \overline{\mathcal{M}}_R$. By Lemma 24, $\ker \delta_{r+1} = (\operatorname{coker} \partial_r)^*$. Now

$$\begin{array}{ccc} & \overset{\partial_r}{\nearrow} & P_{r-1} \\ & & \uparrow \\ P_r & \longrightarrow & \operatorname{coker} \partial_r \\ & & \uparrow \\ & & H_r(P) \\ & & \uparrow \\ & & 0 \end{array}$$

commutes and is exact. If $H_r(P) \in \overline{\mathcal{M}}_R$, applying duality to this diagram yields

$$\begin{array}{ccccc} P^*_{r-1} & \xrightarrow{\alpha} & \ker(\delta_r) & \xrightarrow{\beta} & (H_r(P))^* \\ & \underset{\delta_{n-1}}{\searrow} & \downarrow & & \\ & & P^*_r & & \end{array}$$

By definition, $\operatorname{coker} \alpha = H^r(P)$. $\beta \circ \alpha = 0$, so there is a unique, natural map $H^r(P) \to (H_r(P))^*$.

Corollary 4.2: With $\{P_r, \partial_r\}$ as above, if $H_k(P) = 0$ for $k < n$, $H_n(P) \in \overline{\mathcal{M}}_R$. If $H_n(P) \in \mathcal{P}_R$, the natural map $H^n(P) \to (H_n(P))^*$ is an isomorphism.

Proof: By induction one shows $Z_n \in \mathcal{P}_R$, and since $P_{n+1} \xrightarrow{\partial_{n+1}} Z_n \longrightarrow H_n(P) \longrightarrow 0$ is exact, it is not hard to see $H_n(P) \in \overline{\mathcal{M}}_R$. If $H_n(P) \in \mathcal{P}_R$, $0 \longrightarrow (H_n(P))^* \longrightarrow Z_n^* \longrightarrow P_{n+1}^*$ is exact, so $H^n(P) \cong (H_n(P))^*$. □

Theorem 5: With $\{P_r, \partial_r\}$ as above, suppose $H_k(P) = 0$ for $k < n$ and $H^k(P) = 0$ for $k > n$. Then $H_n(P) \in \mathcal{P}_R$ and the natural map $H^n(P) \to (H_n(P))^*$ is an isomorphism. In $K_o(R)$, $[H_n(P)] = (-1)^n \chi(P)$, where $\chi(P) \in K_o(R)$ is $\sum_r (-1)^r [P_r]$.

Proof: Since $H_k(P) = 0$ for $k < n$, the sequence $\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots$ splits up as

$$\cdots \longrightarrow P_{n+1} \longrightarrow Z_n \longrightarrow 0$$

$$0 \longrightarrow Z_n \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots$$

The second sequence is exact, and

$\cdots \longrightarrow P_{n+1} \longrightarrow Z_n \longrightarrow H_n \longrightarrow 0$ is exact since $H_k(P) = 0$ for $k > n$ by Corollary 4.1.

By Corollary 4.2, $H_n \in \overline{\mathcal{M}}_R$. Dualizing, we get $\cdots \longleftarrow P_{n+1}^* \longleftarrow Z_n^* \longleftarrow (H_n)^* \longleftarrow 0$ is exact by Corollary 4.1 and Lemma 24. As in the proof of Theorem 4, we get a chain retraction up to $D : P^*_{n+1} \longrightarrow Z^*_n$. This shows $(H_n)^* \in \mathcal{P}_R$. But $\cdots \longleftarrow P^*_{n+1} \longleftarrow Z^*_n \longleftarrow (H_n)^* \longleftarrow 0$ $0 \longleftarrow Z^*_n \longleftarrow P^*_n \longleftarrow P^*_{n-1} \longleftarrow \cdots$ splice together to give the cochain complex. $H^n \longrightarrow (H_n)^*$ is now easily seen to be an isomorphism.

Now $\sum_{r \geq n+1} (-1)^r[P_r] + (-1)^n[Z_n] + (-1)^{n-1}[H_n] = 0$ and $\sum_{r \leq n} (-1)^r[P_r] + (-1)^{n+1}[Z_n] = 0$ in $K_o(R)$ by Bass [1], Proposition 4.1, Chapter VIII. Summing these two equations shows $\chi(P) + (-1)^{n-1}[H_n] = 0$. □

Now let us return and discuss the products we defined in section 4. We defined two versions of the cap product on the chain level (see Theorems 4.5 and 4.6). Notice that the maps we defined on $P_*(X;A,B)$ and $P^*(X;A,B)$ actually come from maps on the tree modules $C_*(X;A,B:\Lambda,F)$ and $C^*(X;A,B:\Lambda,F)$. Thus if f is a cocycle in $C^m(X,A:\Gamma)$, and if h is a diagonal approximation, Theorem 4.5 yields a chain map $C_{*+m}(X;A,B) \xrightarrow{\cap_h f} C_*(X,B)$. Note that in order for this to land in the asserted place, Γ pulled up to the universal cover of X must just be ordinary integer coefficients.

$f \cap_h$ dualizes to $f \cup_h : C^*(X,B) \longrightarrow C^{m+*}(X;A.B)$. Since we did not define cup products on the chain level, we may take this as a definition. Nevertheless we assert that on homology, $f \cup_h$ induces the cup product of

Theorem 4.1. This follows from the duality relations we wrote down between ordinary cohomology and homology (see the discussion around the universal coefficient theorems in section 1).

Now one easily sees $*$ induces a map $Wh(*)$: $Wh(R) \to Wh^{\ell}(R)$, where $Wh^{\ell}(R)$ is the group formed from left modules. If $f \cap$ (or $\cup f$) is a chain equivalence, we can compare $\tau(\cup f)$. We get $Wh(*)(\tau(f \cap)) = (-1)^m \tau(f \cup)$ by definition.

Next we study the cap product of Theorem 4.6. A cycle $c \in C_m^{\ell.f.}(X;A,B;\Gamma)$ yields maps $C^*(X,A) \to C_{m-*}(X,B)$. C^* is a left module, while C_* is a right module, so $\cap_h c$ is not a map of tree modules. If Γ has all its groups isomorphic to Z, which it must to yield the asserted product, we get a homomorphism $w : \Gamma_1(X) \to Z_2 = \mathrm{Aut}(Z)$ given by the local system. We can make C^* into a right module (or C_* into a left module) by defining $M_A \cdot a = \bar{a} \cdot M_A$, where $m_A \in (C^*)_A$, $a \in (Z\pi_1)_A$ and $\bar{\ }$ is the involution on $(Z\pi_1)_A$ induced by $g \in (\pi_1)_A$ goes to $w(g)g^{-1}$, where $w(g) \in Z_2 = \{1,-1\}$ is the image of g under the composition $(\pi_1)_A \to \pi_1(X) \xrightarrow{w} Z_2$. Let C_w^* be C^* with this right module structure.

Then $\cap_h c$: $C_w^*(X,A) \to C_{m-*}(X,B)$ is a chain map. If we dualize, we get a map $(\cap_h c)^*: C^{m-*}(X,B) \longrightarrow (C_w^*(X,A))^{\text{dualized}}$. C_w^* dualized is just C_*^w, and $(\cap_h c)^* = \cap_h c$.

The involution is seen to induce an isomorphism $Wh^{\ell}(G) \to Wh(G)$, and the composition $Wh(G) \xrightarrow{wh(*)} Wh^{\ell}(G) \longrightarrow Wh(G)$ is the map induced by $ZG \to ZG$ via $-$ (it is not hard to see this map induces a map on the Whitehead group level.) We will denote the map on $Wh(G)$ also by $-$.

If $\cap_h c$ is a chain isomorphism, either from $C_w^*(X,A) \to C_{m-*}(X,B)$ or $C_w^{m-*}(X,B) \longrightarrow C_*(X,A)$, we can compare the two torsions. We get the confusing equation $\tau(\cap_h c) = (-1)^m \overline{\tau(\cap_h c)}$ where despite their similar appearance, the two $\cap_h c$'s are not the same (which is which is irrelevant).

We conclude by recording a notational convention. We will sometimes have a map on homology such as $\cap c : \Delta^*(M) \to \Delta_*(M)$. If this map is an homology isomorphism we will often speak of the torsion of $\cap c$ (or $\cup f$, etc.). By this we mean that there is a chain map (which is clear from the context) and these maps on the chain level are equivalences. Note that by the usual nonsense, the torsions of these product maps do not depend on a choice of cycle (cocycle) within the homology (cohomology) class. Nor do they depend on lift functor or choice of paths. They are dependent on the tree at this stage of our discussion, but this too is largely fictitious. A better proof of independence is given at the end of section 6. Especially relevant for this last discussion are Theorem 2.1.2 and the discussion of the Thom isomorphism in the appendix to Chapter 2.

Section 6: The realization of chain complexes.

In [37] and [38], Wall discussed the problem of constructing a CW complex whose chain complex corresponds to a given chain complex. We discuss this same problem for locally compact CW complexes. Throughout this section, complex will mean a finite dimensional, locally compact CW complex.

If we have a chain complex A_*, there are many conditions it must satisfy if it is to be the chain complex of a complex. Like Wall [38] we are unable to find an algebraic description of these conditions in low dimensions. We escape the dilemma in much the same way.

Definition: A geometric chain complex is a positive, finite, chain complex A_* together with a 2-complex K, a tree $f : T \to K$, and a lift functor $f \in \mathcal{L}(f)$ such that 1) each A_k is a locally-finitely generated free $Z\pi_1(K,F,f)$-module

2) each $\partial_k: A_k \to A_{k-1}$ is a map (not a map-germ)

3) in dimensions ≤ 2, $C_*(K:F) = A_*$.

For 3) to make sense, we must define equality for two free tree modules. If A is free and based on (S,π) and if B is free and based on (R,ρ), $A = B$ iff there exists a 1-1 map $\alpha : S \leftrightarrow R$ such that $\alpha \circ \pi$ is equivalent to ρ . One easily checks this is an equivalence relation.

Notice that if A_* is going to be the chain complex of some complex, then all the above conditions are necessary.

Given two geometric chain complexes A_* and B_* , a map $f_*: A_* \to B_*$ is a map (not a germ) on each A_k and $\partial_k f_k = f_{k-1}\partial_k$ as maps.

Definition: A map $f_* : A_* \to B_*$ between two geometric chain complexes is admissible provided

1) if L is the 2-complex for B_* , $L = K$ wedged with some 2-spheres in a locally finite fashion

2) f_o and f_1 are the identity

3) f_2 is the identity on the 2-cells of K and takes any 2-sphere to its wedge point. (The tree for L is just the tree for K . The lift functor for L is just g^{-1}(lift functor for K), where $g : L \to K$ is the collapse map).

Remarks: It seems unlikely that we really need such strong conditions on a map before we could handle it, but in our own constructions we usually get this, and these assumptions save us much trouble.

The chief geometric construction is the following.

Theorem 1: Let X be a connected complex. Let A_* be a geometric chain complex with an admissible map $f_* : A_* \to C_*(X)$ which is an equivalence. Then we can construct a complex Z and a proper, cellular map $g : Z \to X$ so that $C_*(Z) = A_*$ and

$$\begin{array}{ccc} A_* & \xrightarrow{f_*} & C_*(X) \\ \| & \nearrow_{g_*} & \\ C_*(Z) & & \end{array}$$

commutes. g is a proper homotopy equivalence.

Proof: We construct Z skeleton by skeleton. Since f_* is admissible, $Z^2 = X^2$ wedge 2 spheres. $g_2 : Z^2 \to X$ is just the collapse map onto X^2. To induct, assume we have Z^r and $g_r : Z^r \to X$ so that $C_*(Z^r) = A_*$ in dimensions $\leq r$ and $(g_r)_* = f_*$ in these dimensions. If we can show how to get Z^{r+1}, g_{r+1} we are done since A_* is finite.

Now A_{r+1} is free, so pick generators $\{e_i\}$. We have a map $\partial : A_{r+1} \to A_r$ and $C_r(Z^r) = A_r$. Hence each ∂e_i is an r-chain in Z^r. We will show that these r-chains are locally finite and spherical (i.e. there is a locally finite collection of r-spheres $\cup S_i^r \subseteq Z^r$ such that ∂e_i is homologous to S_i^r, and, if h_i is an (r+1)-chain giving the homology, the $\{h_i\}$ may be picked to be locally finite). We will then attach cells by these spheres and extend the map.

Let us now proceed more carefully. For each A_i, $\partial e_i \in A_r$ and $\partial e_i \in (A_r)_{W_i}$ for some $W_i \in \mathcal{C}(T)$ with $\{W_i\}$ cofinal in the subcategory of $\mathcal{C}(T)$ consisting of all A such that $e_i \in (A_r)_A$. Since $C_r(Z^r) = A_r$,

$\partial e_i = c_i \in (C_r(Z^r))_{B_i}$ for some $B_i \in \mathcal{C}(T)$ with $B_i \leq W_i$ (we write $B_i \leq w_i$ provided $B_i \subseteq W_i$ and $\{B_i\}$ is cofinal in the subcategory of all $A \in \mathcal{C}(T)$ for which $e_i \in (A_r)_A$. c_i is now a real geometric chain. $\partial c_i = 0$ since ∂ is actually a map. Let $[c_i]$ be the homology class of c_i in $H_r(\widetilde{F_r(B_i)})$, where F_r is the lift functor for Z^r. Now $g_*[c_i] = 0$ in $H_r(\widetilde{F(U_i)})$, where F is the lift functor for X, and $U_i \leq B_i$.

Hence there is an $f_i \in H_{r+1}(\tilde{g}_r\colon \widetilde{F_r(U_i)} \to \widetilde{F(U_i)})$ with $f_i \to [c_i]$. But $g_r\colon Z^r \to X$ is properly r-connected (it induces an isomorphism of $\Delta(\ :\pi_1)$'s and H^0_{end}'s by assumption, so it is always 1-1/2-connected. Hence the universal covering functor for X is a universal covering functor for Z^r, so $\Delta(\ :\pi_k) = 0$ iff $\Delta(M_{g_r}, X\colon H_k, \sim) = 0$ for $k \leq r$ by the Hurewicz theorem. But $\Delta(M_{g_r}, X\colon H_k, \sim) = 0$ for $k \leq r$ iff $\Delta(H_k(g_r))$ is an isomorphism for $k < r$ and an epimorphism for $k = r$. But this is true if $\Delta(H_k(f)$ is an isomorphism for $k \leq r$, which it is.) Hence the Hurewicz Theorem gives us elements $s_i \in \pi_{r+1}(\tilde{g}_r\colon F_r(V_i) \to F(V_i))$ where $V_i \leq U_i$ and s_i hits the image of f_i in $H_{r+1}(\tilde{g}_r\colon \widetilde{F_r(V_i)} \to \widetilde{F(V_i)})$ under the Hurewicz map.

Let $Z^{r+1} = Z^r \cup$ a collection of $(r+1)$-cells, $\{e_i\}$ attached by s_i. $g_{r+1}\colon Z^{r+1} \to X$ is g_r on Z^r. Since $g_r \circ s_i : S^r \to Z^r \to X$ are properly null homotopic, choose a locally finite collection $\{Q_i\}$ of null homotopies of

$g_r \circ s_i$ to zero in $F(V_i)$. $g_{r+1}: Z^{r+1} \to X$ is defined by Q_i on each e_i . g_{r+1} is obviously still proper. $C_*(Z^r) \to C_*(Z^{r+1})$ induces an isomorphism for $* \leq r$. $C_{r+1}(Z^{r+1}) = A_{r+1}$ by taking the cell e_i to the generator e_i . $F_{r+1}(B) = F_r(B) \cup$ (all cells e_i for which the generator e_i lies in B less those for which $g_{r+1}(e_i) \not\subseteq F(B))$.

Then $g_{r+1}^{-1} F(B)) \supseteq F_{r+1}(B)$. Notice that if a cell e does not attach totally in $F_r(B)$, $g_{r+1}(e) \not\subseteq F(B)$, so $F_{r+1}(B)$ is a subcomplex. $F_{r+1}(B)$ is cofinal in B, so F_{r+1} is a lift functor.

Look at the chain map $(g_{r+1})_* : C_{r+1}(Z^{r+1}) \to C_{r+1}(X)$. e_i as a cell goes under $(g_{r+1})_*$ to the same element in $(C_{r+1}(X))_B$ as the generator e_i does under f_* for all $B \in \mathcal{O}(T)$ such that e_i is a cell in $F_{r+1}(B)$. Hence

$$\begin{array}{ccc} A_{r+1} & \xrightarrow{f_{r+1}} & C_{r+1}(X) \\ \| & \nearrow_{g_{r+1}} & \\ C_{r+1}(Z^{r+1}) & & \end{array}$$

commutes. $\square$

<u>Definition</u>: A relative geometric chain complex is a triple (A_*,K,L) consisting of a finite, positive chain complex A_* and a pair of complexes (K,L) . Understood is a tree and a lift functor. Then each A_k is a locally-finitely generated free $Z\pi_1(K)$ tree module; each ∂_k is a map; and in dimensions ≤ 2, $A_* = C_*(K,L)$.

An admissible map is a chain map, not a germ $f_* : A_* \to B_*$ and $K = K'$ wedge a locally finite collection of 2-spheres. f_0 and f_1 are the identity, and f_2 is the map induced by the collapse $K \to K'$.

Corollary 1.1: Let (X,Y) be a pair of complexes, X connected. Let A_* be a relative geometric chain complex with an admissible map $f_* : A_* \to C_*(X,Y)$ which is an equivalence. Then we can construct a complex Z with Y as a subcomplex and a proper cellular map $g: Z \to X$ which is the identity on Y such that $C_*(Z,Y) = A_*$ and

$$\begin{array}{ccc} A_* & \xrightarrow{f_*} & C_*(X,Y) \\ \| & \nearrow_{g_*} & \\ C_*(Z,Y) & & \end{array}$$

commutes.

g is a proper homotopy equivalence of pairs.

Proof: The proof parallels the proof of Theorem 1, except we must now use Namioka to show our elements are spherical. $\square$

Now let $f_*: A_* \to C_*(X)$ be an arbitrary chain equivalence. As in Wall [38], we would like to replace A_* by an admissible complex with f_* admissible while changing A_* as little as possible. Look at

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & A_3 & \xrightarrow{\partial} & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & A_{-1} & \longrightarrow & \cdots & \longrightarrow & 0 \\ & & \downarrow{f_3} & & \downarrow{f_2} & & \downarrow & & \downarrow & & & & & & \\ \cdots & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & 0 & & & & \end{array}$$

One might like to try the complex

$$\begin{array}{ccccccccc}
\cdots \longrightarrow A_3 & \xrightarrow{f_2 \circ \partial} & C_2 & \longrightarrow & C_1 & \longrightarrow & C_o & \longrightarrow & 0 \\
\downarrow f_3 & & \downarrow id & & \downarrow id & & \downarrow id & & \\
\cdots \longrightarrow C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_o & \longrightarrow & 0 \;.
\end{array}$$

The top complex is clearly admissible, but unfortunately the map is no longer an equivalence. The cycles in A_3 are now bigger with no new boundaries, and the boundaries in C_2 are smaller with no fewer cycles.

Note first that X is not of great importance. If we replace X by something in its proper homotopy class, we will not be greatly concerned. Let X' be X with 2-spheres wedged on to give a basis for A_2 and 3-cells attached to kill them. Then X' has the same simple homotopy type as X, $C_k(X') = C_k(X)$ except for $k = 2,3$, and $C_k(X') = C_k(X) \oplus A_2$ for $k = 2,3$. Let $f'_k = f_k$, $k \neq 2,3$, and let $f'_3 = (f_3,\partial)$ and $f'_2 = (f_2,id)$. Then $A_* \xrightarrow{f'_*} C_*(X')$ is still an equivalence and now f'_2 is a monomorphism. Let A' be the complex

$$\cdots \longrightarrow A_3 \xrightarrow{f'_2 \circ \partial} C'_2 = C_2(X') \longrightarrow C'_1 \longrightarrow C'_o \longrightarrow 0 \;.$$

Then $h_*: A'_* \to C'_*$ has homology in only one dimension: $0 \to H_2(h) \to H_2(A') \to H_2(C') \to 0$. Since $A_* \to A'_*$ and since the composition $A_* \to A'_* \to C'_*$ is an equivalence, $H_2(A') = H_2(h) \oplus H_2(C')$.

Now by Theorem 5.5, $H_2(h)$ is s-free, provided we can show $H^k(h) = 0$ for $k \geq 3$. But since we have a chain equivalence $A_* \to C'_*$ we have a chain homotopy inverse in each dimension. We then clearly get a chain homotopy inverse for $A'_k \to C'_k$, $k \geq 4$, and $h_3 \circ g'_3$ chain homotopic to $id_{C'_3}$. But this implies $H^k(h) = 0$, $k \geq 3$.

Since $H_2(h)$ is projective, we get a map $\rho = \partial \circ \rho'$, where $\rho' : H_2(h) \to C'_3$ is given as follows. Both A_3 and C'_3 map into $C_2 = C'_2$, and $0 \to \text{Image } A_3 \to \text{Image } C_3 \to H_2(h) \to 0$ is exact. Split this map by $\sigma: H_2(h) \to \text{Image } C'_3$ and note $\text{Image } A_3 \cap \text{Image } \sigma = \{1\}$. Now $C'_3 \to \text{Image } C'_3 \to 0$ is exact, so we can lift σ to $\rho': H_2(h) \to C'_3$. Since σ is a monomorphism, note $\text{Image } \rho' \cap \text{Image } f_3 = \{1\}$.

Form A''_* and h'_* by

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A_4 & \longrightarrow & A_3 \oplus H_2(h) & \xrightarrow{\partial + \rho} & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0 & \longrightarrow & 0 \\
 & & \downarrow f_4 & & \downarrow f_3 * \rho' & & \downarrow id & & \downarrow id & & \downarrow id & & \\
\cdots & \longrightarrow & C'_4 & \longrightarrow & C'_3 & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0 & \longrightarrow & 0
\end{array}$$

Note $\ker(\partial + \rho) = (\ker \partial, 0)$ since ρ is a monomorphism and if $\rho(x) \in \text{Image } \partial$, $\rho(x) = \{1\}$ as $\text{Image } A_3 \cap \text{Image } \sigma = \{1\}$. Likewise note $\text{Image}(\partial + \rho) = \text{Image } C'_3$ since $\text{Image } A_3 \oplus \text{Image } \rho = \text{Image } C'_3$. Hence h'_* is an equivalence.

Note $\rho : H_2(h) \to C'_2$ is a direct summand. We split ρ as follows.

$$\begin{array}{ccccccc} \cdots \longrightarrow & A_4 & \longrightarrow & A_3 \oplus H_2(h) & \xrightarrow{\partial \circ \rho} & C_2' & \longrightarrow \cdots \\ & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle f_2} & \\ \cdots \longrightarrow & A_4 & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow \cdots \end{array}$$

, where α is inclusion on the first factor commutes. These maps must define a chain equivalence, so the dual situation is also an equivalence.

$$\begin{array}{ccccccc} \cdots \longleftarrow & A_4^* & \xleftarrow{\delta_3} & A_3^* \oplus (H_2(h))^* & \xleftarrow{\partial^* + \rho} & (C_2')^* & \longleftarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle f_2^*} & \\ \cdots \longleftarrow & A_4^* & \xleftarrow{\delta_3^A} & A_3^* & \xleftarrow{\delta_2^A} & A_2^* & \longleftarrow \cdots \end{array}$$

$\ker \delta_3 = \ker \delta_3^A \oplus (H_2(h))^*$, and $\text{Image}\,(\partial^* + \rho^*) =$ $\text{Image}\,\partial^* \oplus \text{Image}\,\rho^*$. $H_3(\text{Top complex}) =$ $\ker \delta_3/\text{Image}(\partial^* + \rho^*) = (\ker \delta_3^A/\text{Image}\,\partial^*) \oplus ((H_2(j))^*/\text{Image}\,\rho^*)$. $H_3(\text{Bottom complex}) = \ker \delta_3^A/\text{Image}\,\delta_2^A$. $H_3(\alpha)\colon H_3(\text{Top}) \to H_3(\text{Bottom})$ is

$$\begin{array}{ccc} \ker \delta_3^A/\text{Image}\,\partial^* & & \ker \delta_3^A/\text{Image}\,\delta_2^A \\ \oplus & \longrightarrow & \oplus \\ (H_2(h))^*/\text{Image}\,\rho^* & & 0 \end{array}$$

Hence, if $H_3(\alpha)$ is an isomorphism, $\text{Image}\,\partial^* = \text{Image}\,\delta_2^A$, and $\rho^*\colon (C_2')^* \longrightarrow (H_2(h))^*$ is onto. $(H_2(h))^*$ is projective so split ρ^* . Dualizing splits $\rho\colon H_2(h) \to C_2'$.

$H_2(h)$ may not be free (it is only s-free). $A_3 \oplus H_2(h)$ is often free, but we prefer to keep A_3 . Hence form A_*^s and f_*^s by

$$\begin{array}{ccccccc} \cdots \longrightarrow A_4 & \longrightarrow & A_3 \oplus (H_2(h)_S \oplus F^{(n)}) & \longrightarrow & C_2' \oplus F^{(n)} & \longrightarrow \cdots \\ \downarrow f_4 & & \downarrow f_3 + (\rho_S' + 0) & & \downarrow \mathrm{id} + 0 & \\ \cdots \longrightarrow C_4' & \longrightarrow & C_3' & \longrightarrow & C_2' & \longrightarrow \cdots \end{array}$$

where S is a shift functor so that the map germs ρ and ρ' are actual maps.

By wedging on n 2-spheres at each vertex of the tree, we see A_*^S , f_*^S are admissible. Notice that exactly the same procedure makes a map $f_*: A_* \to C_*(X,Y)$ admissible.

In section 3, Proposition 3 we defined what it meant by X satisfies Dn. We briefly digress to prove

<u>Theorem 2</u>: The following are equivalent for $n \geq 2$, X a complex.

1) X satisfies Dn

2) X is properly dominated by an n-complex.

3) Δ^k (X: universal covering functor) = 0 for $k > n$.

<u>Proof</u>: 1) implies 2) as $X^n \subseteq X$ is properly n-connected and hence dominates X if X satisfies Dn . 2) implies 3) by computing Δ^k from the cellular chain complex of the dominating complex.

3) implies 2): Since $\Delta^k(X: \;) = 0$ for $k > n$, by Theorem 5.4 dualized to cohomology, we get chain retracts

$$0 \longrightarrow C_r \underset{D}{\overset{\partial}{\rightleftarrows}} C_{r-1} \underset{D}{\rightleftarrows} \cdots \longrightarrow C_{n+1} \underset{D}{\overset{\partial_{n+1}}{\rightleftarrows}}$$

$C_n \xrightarrow{\partial_n} \cdots$ where $r = \dim X < \infty$. By an induction argument, Image ∂_{n+1} is s-free, and $C_n = \text{Image } \partial_{n+1} \oplus A_n$ (dualize everything to get these results in the cochain complex and then dualize back). A_n is s-free and

$$\begin{array}{ccccccccccccc} & & 0 & \to & A_n & \to & C_{n-1} & \to & \cdots & \to & C_o & \to & 0 \\ & & & & \downarrow r & & \downarrow id & & & & \downarrow id & & \\ \cdots & \to & C_{n+1} & \to & C_n & \to & C_{n-1} & \to & \cdots & \to & C_o & \to & 0 \end{array}$$

gives us an n-complex and a chain equivalence. A_n is only s-free, so form $0 \to A_n \oplus F^{(m)} \to C_{n-1} \oplus F^{(m)} \to \cdots$ which is now a free complex. If $n \geq 3$, the complex and the map are clearly admissible, so by Theorem 1 we get an n-complex Y and a proper homotopy equivalence $g: Y \to X$, so X satisfies 2).

In $n = 2$, X has the proper homotopy type of a 3-complex by the above, so we assume X is a 3-complex. Its chain complex is then $0 \to C_3 \to C_2 \to C_1 \to C_o \to 0$ with $H^2(C) = 0$. Wedge 2-spheres to X at the vertices of the tree to get a chain complex $0 \to C_3 \to C_2 \oplus C_3 \to C_1 \to C_o \to 0$. Since $H^2(C) = 0$, $C_2 = C_3 \oplus \text{kernel } \partial_2$. Let $j : C_3 \to C_2$ be the inclusion. Then we have

$$\begin{array}{lccccccccc} A: & 0 \to C_3 & \xrightarrow{j} & C_2 \oplus C_3 & \to & C_1 & \to & C_o & \to & 0 \\ & \downarrow & & \downarrow r & & \downarrow id & & \downarrow id & & \\ B: & 0 & \to & C_2 & \to & C_1 & \to & C_o & \to & 0 \end{array}$$

where $r : (C_3 \oplus \ker \partial_2) \to (C_3 \oplus \ker \partial_2) \oplus C_3$ by $r(x,y) = (0,y,x)$. This is a chain equivalence between B and A.

Both A and B are the complexes for a space. The chain map is easily realized on the 1-skeleton as a map, and we show we can find a map $g\colon X^2 \to X \vee_j S_j^2$ realizing the whole chain map.

Let $\{e_i\}$ be the two cells of X^2. Their attaching maps determine an element in $\Delta(X^1\colon \pi_1, \sim)$, where this group denotes the Δ-construction applied to the groups $\pi_1(p^{-1}(X^1-c), \hat{x}_i)$, where $p\colon \tilde{X} \to X$ is the projection for the universal cover of X. (i.e. $\sim$ denotes the covering functor over X^1 induced in the above manner from the universal covering functor on X.) Let $g_1\colon X^1 \to X \vee S^2$ be the natural inclusion. As in the proof of Theorem 1, the $\{e_i\}$ determine an element of $\Delta(g_1\colon H_2, \sim)$. The following diagram commutes, and the rows are exact

$$\begin{array}{ccccccccc}
 & & \Delta(X^2\colon\pi_2,\sim) & \to & \Delta(g_1\colon\pi_2,\sim) & \to & \Delta(X^1\colon\pi_1,\sim) & \to & \Delta(X^2\colon\pi_1,\sim) = \\
 & & \downarrow h & & \downarrow & & \downarrow & & \\
0=\Delta(X^1\colon H_2,\sim) & \to & \Delta(X^2\colon H_2,\sim) & \to & \Delta(g_1\colon H_2,\sim) & \to & \Delta(X^1\colon H_1,\sim) & &
\end{array}$$

where $\Delta(X^1\colon H_1, \sim)$ and $\Delta(X^1\colon H_2, \sim)$ are defined similarly to $\Delta(X^1\colon \pi_1, \sim)$. $X^1 \subseteq X^2$ is properly 1-connected, the subspace groups are the groups asserted. h is an isomorphism by the Hurewicz theorem, so a diagram chase yields a unique element in $\Delta(g_1\colon \pi_2, \sim)$ which hits our element in both $\Delta(X^1\colon \pi_1, \sim)$ and $\Delta(g_1\colon H_2, \sim)$. Use this element to extend the map to $g_2\colon X^2 \to X \vee_j S_j^2$. By our choices g_2 induces an isomorphism of Δ_{π_1}'s. Hence g_2 is a proper homotopy equivalence. Thus 3) implies 2) for $n \geq 2$.

2) implies 1) is trivial. □

Corollary 2.1: If X satisfies Dn for $n \geq 3$, X has the proper homotopy type of an n-complex. □

Combining our admissibility construction with Theorem 1 gives

Theorem 3: Let $f_* : A_* \to C_*(X)$ be a chain map for a complex X. Then there exists a complex Y_o satisfying D2; a complex $Y \supseteq Y_o$ such that $C_*(Y,Y_o) = A_*$ in dimensions greater than or equal to 3; and a proper, cellular homotopy equivalence $g: Y \to X$ such that $g_* = f_*$ in dimensions greater than or equal to 4. The torsion of g may have any preassigned value.

Proof: Make A_*, f_* admissible. The new complex is $\cdots \longrightarrow A_4 \longrightarrow A_3 \oplus (?) \longrightarrow C_2 \oplus (?) \longrightarrow C_1 \longrightarrow \cdots$ Construct a Y from this complex as in Theorem 1. When we pick a basis for $A_3 \oplus (?)$, pick a basis for A_3 and one for (?) and use their union. Then there is a subcomplex $Y_o \subset Y$ whose chain complex is $0 \longrightarrow (?) \longrightarrow C_2 \oplus (?) \longrightarrow C_1 \longrightarrow C_o \longrightarrow 0$. The first (?) is $H_2(h) \oplus F^{(m)}$. It is not hard to show $\Delta^3(Y_o : \sim) = 0$, so Y^o satisfies D2. The remainder of the theorem is trivial except the remark about torsion. But for some $m \geq 0$, we can realize a given torsion by an automorphism

$\alpha : F^{(m)} \to F^{(m)}$. Hence by altering the basis in $F^{(m)}$, we can cause our map to have any desired torsion (we may have to take m bigger, although in the infinite case $m = 1$ will realize all torsions). □

Theorem 4: Let $f_* : A_* \to C_*(X,Z)$ be a chain map for a pair (X,Z) . Then there exists a complex Y_o such that $Y \supseteq Y_o \supseteq Z$; $C_*(Y,Y_o) = A_*$ in dimensions greater than or equal to 3; a proper cellular homotopy equivalence $g: Y \to X$ which is the identity on Z such that $g_* = f_*$ in dimensions greater than or equal to 4. The torsion of g may have any preassigned value.

Proof: Use Corollary 1.1. □

We conclude this chapter by returning briefly to the question of the invariance of torsion for chain maps under a change of tree. The natural map $Wh(X:f) \to Wh(X:g)$ is a homomorphism, so the property of being a simple chain equivalence is independent of the tree. But now use Theorem 3 to get a proper homotopy equivalence $X \to X_\tau$ with torsion τ . Suppose given a chain map, say for example, $\cap c : \Delta^*(X) \to \Delta_{m-*}(X)$ with torsion τ . Then the composition $\Delta^*(X) \xrightarrow{\cap c} \Delta_{m-*}(X) \longrightarrow \Delta_{m-*}(X_{(-1)^m\tau})$ is simple, so a change of trees leaves it simple. But the second map is a proper homotopy equivalence of spaces, and hence is independent of the tree. Hence so must be $\cap c$. (Note here we are using our convention of writing chain equivalences on the homology level.)

CHAPTER II

Poincaré Duality Spaces

Section 1. Introduction, definitions, and elementary properties

In this chapter we discuss the analogue of manifold in the proper homotopy category. We seek objects, to be called Poincaré duality spaces, which have the proper homotopy attributes of paracompact manifolds. To this end, we begin by discussing these attributes.

There is a well known Lefschetz duality between H_* and H_c^* or between H^* and $H_*^{\ell.f.}$ which is valid for any paracompact manifold with boundary. (see for instance Wilder [44].) This duality is given via the cap product with a generator of $H_N^{\ell.f.}$, perhaps with twisted coefficients. This generator is called the fundamental class.

Given any paracompact handlebody M, M can be covered by an increasing sequence of compact submanifolds with boundary. Let $\{C_i\}$ be such a collection. If $[M] \in H^{\ell.f.}_{\dim M}(M;Z^t)$ is the fundamental class, its image in $H^{\ell.f.}_{\dim M}(\overline{M - C_i},\partial C_i;Z^t)$ via inclusion and excision is the fundamental class for the pair $(\overline{M - C_i},\partial C_i)$. A word about notation: Z^t occurring as a coefficient group will always denote coefficients twisted by the first Stiefel-Whitney class of the manifold.

<u>Theorem 1</u>: The fundamental class [M] in $H_N^{\ell.f.}(M;Z^t)$ induces via cap product an isomorphism

$$\cap[M]: \Delta^{n-*}(M:\sim) \longrightarrow \Delta_*(M:\sim)$$

where ~ is any covering functor.

If M has a boundary, we get a fundamental class $[M] \in H_N^{\ell.f.}(M,\partial M;Z^t)$ and isomorphisms

$$\cap[M] : \Delta^{N-*}(M,\partial M:\sim) \longrightarrow \Delta_*(M:\sim)$$

$$\cap[M] : \Delta^{N-*}(M:\sim) \longrightarrow \Delta_*(M,\partial M:\sim)$$

A similar result holds for a manifold n-ad.

<u>Proof</u>: The proof is easy. On the cofinal subset of compact submanifolds with boundary of M , [M] induces, via inclusion and excision, the fundamental class for the pair ((n+1)-ad in general) $(\overline{M-C_i},\partial C_i)$, where C_i is a compact submanifold with boundary of M , and $\overline{M-C_i}$ is the closure of $M-C_i$ in M . ∂C_i is equally the boundary of C_i as a manifold or the frontier of C_i as a set. By the definition of $\cap[M]$, it induces an isomorphism for each base point and set C_i . Hence it must on the inverse limit. □

If one computes the homology and cohomology from chain complexes based on a PL triangulation, on a handle-body decomposition, or on a triangulation of the normal disc bundle, $\cap[M]$ induces a chain isomorphism. We can ask for the torsion of this map. We have

Theorem 2: If $(M,\partial M)$ is a manifold with (possibly empty) boundary, and if $[M] \in H_N^{\ell.f.}(M,\partial M;Z^t)$ is the fundamental class, $\cap[M] : \Delta^{N-*}(M,\partial M:\sim) \longrightarrow \Delta_*(M:\sim)$ and $\cap[M] : \Delta^{N-*}(M:\sim) \longrightarrow \Delta_*(M,\partial M:\sim)$ are simple equivalences as chain equivalences, where $\sim$ is the universal covering functor.

Proof: Given a handlebody decomposition, the proof is easy. The cap product with the fundamental class takes the cochain which is 1 on a given handle and zero on all the other handles to the dual of the given handle. Hence $\cap[M]$ takes generators in cohomology to generators in homology (up to translation by the fundamental group). The fact that the simple homotopy type as defined by a PL triangulation or by a triangulation of the normal disc bundle is the same as that defined by a handlebody has been shown by Siebenmann [34]. □

We are still left with manifolds which have no handlebody decomposition. Let $N = CP^4 \# S^3 \times S^5 \# S^3 \times S^5$. Then $\chi(N) = 1$. $N \times M$ has $[N] \times [M]$ as a fundamental class. For M we use the simple homotopy type defined by a triangulation of the normal disc bundle. Then $\cap[N] \times [M]$ is a simple equivalence iff $\cap[M]$ is by Lemma 1.5.23, since $\cap[N]$ is known to induce a simple equivalence. But $\cap[N] \times [M]$ is a simple equivalence since $N \times M$ has a handlebody structure (Kirby-Siebenmann [18]). Note Theorems 1 and 2 now hold for arbitrary paracompact manifolds.

With these two theorems in mind, we make the following definition.

Definition: A locally finite, finite dimensional CW pair $(X,\partial X)$ with orientation class $w_1 \in H^1(X;Z_2)$ is said to satisfy Poincaré duality with respect to $[X]$ and the covering functor ~ provided there is a class $[X] \in H_N^{\ell.f.}(X,\partial X;Z^{w_1})$ such that the maps

$$\cap[X] : \Delta^{N-*}(X:\sim) \longrightarrow \Delta_*(X,\partial X:\sim) \quad \text{and}$$

$$\cap[X] : \Delta^{N-*}(X,\partial X:\sim) \longrightarrow \Delta_*(X:\sim)$$

are isomorphisms. Z^{w_1} denotes coefficients twisted by the class w_1.

If X is an n-ad we require that all the duality products be isomorphisms.

Remarks: The two maps above are dual to one another, so if one is an isomorphism the other is also.

Suppose ~ is a regular covering functor for X, and suppose --- is another regular covering functor with --- $\gg$ ~ . Then the chain and cochain groups have the structure of $Z\pi_1'(X:F,f:\sim)$ modules, when $f: T \longrightarrow X$ is a tree and $F \in \mathcal{L}(f)$. The tree of groups $\pi_1'(X:F,f:\sim)$ is the tree given by $(\pi_1')_A = \pi_1(F(A),p)/\pi_1(\overline{F(A)},p)$ where p is the minimal vertex for A. There is a map of rings $Z\pi_1'(X:F,f:\sim) \longrightarrow Z\pi_1'(X:F,f:---)$, and the tensor product takes $\Delta(X:\sim)$ to $\Delta(X:---)$. Since $\cap[X]$ is an isomorphism for ~ , we can get chain homotopy inverses, so under tensor product, $\cap[X]$ still induces isomorphisms for ---.

As we have the Browder Lémma (Theorem 1.4.7), with patience we can prove a variety of cutting and gluing theorems. The following are typical.

<u>Theorem 3</u>: Let $(X\colon \partial_0 X, \partial_1 X)$ be a triad. Then the following are equivalent.

1) $(X\colon \partial_0 X, \partial_1 X)$ satisfies Poincaré duality with respect to $V \in H_N^{\ell.f.}(X{:}\partial_0 X, \partial_1 X{:}Z^{w_1})$ and $\sim$. (w_1 is the orientation class.)

2) $(\partial_0 X, \partial_{\{0,1\}} X)$ satisfies Poincaré duality with respect to $\partial V \in H_{N-1}^{\ell.f.}(\partial_0 X, \partial_{\{0,1\}} X{:}Z^{w_1})$ and $\sim$ where $\sim$ is induced from $\sim$ over X and w_1 is the orientation class induced from w_1 over X. Moreover, one of the maps

$$\cap V : \Delta^*(X, \partial_1 X{:}\sim) \longrightarrow \Delta_{N-*}(X, \partial_0 X{:}\sim)$$

$$\cap V : \Delta^*(X, \partial_0 X{:}\sim) \longrightarrow \Delta_{N-*}(X, \partial_1 X{:}\sim)$$

is an isomorphism. (Hence they are both isomorphisms.)

3) The same conditions as 2^{o}) but considering $(\partial_1 X, \partial_{\{0,1\}} X)$.

<u>Proof</u>: The proof is fairly standard. We look at one of the sequences associated to a triple, say

$$\begin{array}{ccccc}
\Delta^*(X{:}\partial_0 X, \partial_1 X{:}\sim) & \longrightarrow & \Delta^*(X, \partial_1 X{:}\sim) & \longrightarrow & \Delta^*(\partial_0 X, \partial_{\{0,1\}} X{:}X{:}\sim) \\
\downarrow \cap V & & \downarrow \cap V & & \downarrow \cap \partial V \\
\Delta_{N-*}(X{:}\sim) & \longrightarrow & \Delta_{N-*}(X, \partial_0 X{:}\sim) & \longrightarrow & \Delta_{N-1-*}(\partial_0 X{:}X{:}\sim)
\end{array}$$

1) implies both $\cap V$'s are isomorphisms, so the 5-lemma shows $\cap\partial V$ is an isomorphism. 2) implies one of the $\cap V$'s is an isomorphism and that

$$\cap\partial V : \Delta\,(\partial_0 X, \partial_{\{0,1\}} X:\sim) \longrightarrow \Delta_{N-1-*}(\partial_0 X:\sim)$$

is an isomorphism. Hence we must investigate how the subspace groups depend on the absolute groups. Make sure the set of base points for X contains a set for $\partial_0 X$. Then we have a diagram

$$\begin{array}{ccc} \Delta^*(\partial_0 X, \partial_{\{0,1\}} X:\sim) & \longrightarrow & \Delta^*(\partial_0 X, \partial_{\{0,1\}} X:X:\sim\} \\ \downarrow{\scriptstyle \cap\partial V} & & \downarrow{\scriptstyle \cap\partial V} \\ \Delta_{N-1-*}(\partial_0 X:\sim) & \longrightarrow & \Delta_{N-1-*}(\partial_0 X:\sim) \end{array}$$

which commutes. The horizontal maps are naturally split, so if $\cap\partial V$ on the subspace groups is an isomorphism, then it is also an isomorphism on the absolute groups. Hence 1) implies 2) and 3).

Now if $\cap\partial V$ on the absolute groups is an isomorphism, then it is also an isomorphism on the subspace groups by Theorem 1.3. Hence 2) or 3) implies 1). □

<u>Theorem 4</u>: Let $Z = Y \cup Y'$ and set $X = Y \cap Y'$. Then any two of the following imply the third.

1) Z satisfies Poincaré duality with respect to $[Z]$ and $\sim$.

2). (Y,X) satisfies Poincaré duality with respect to $\partial[Z]$ and $\sim$.

3) (Y',X) satisfies Poincaré duality with respect to $\partial[Z]$ and $\sim$

where $\sim$ is a covering functor over Z, which then induces $\sim$ over Y and Y'. An orientation class over Z which induces one over Y and Y' has been assumed in our statements.

<u>Proof</u>: The reader should have no trouble proving this. □

A map $\varphi : M \to X$, where M and X are locally compact CW n-ads which satisfy Poincaré duality with respect to $[M]$ and $\sim$, and $[X]$ and --- respectively, is said to be degree 1 provided it is a map of n-ads and

1) $\varphi^*(---) = \sim$, where $\varphi^*(---)$ is the covering functor over M induced by φ from --- over X.

2) If $w_1 \in H^1(X;Z_2)$ is the orientation class for X, $\varphi^* w_1$ is the orientation class for M.

3) $\varphi_*[M] = [X]$.

<u>Theorem 5</u>: Let $\varphi : M \to X$ be a map of degree 1. Then the diagram

$$\begin{array}{ccc} \Delta^r(M:\sim) & \xrightarrow{\varphi^*} & \Delta^r(X:\sim) \\ \downarrow \cap[M] & & \downarrow \cap[X] \\ \Delta_{n-r}(M:\sim) & \xrightarrow{\varphi^*} & \Delta_{n-r}(X:\sim) \end{array}$$

commutes. ($\sim$ over M is the covering functor induced from $\sim$ over X.) Hence $\cap[M]$ induces an isomorphism of the cokernel of φ^*, $K^r(M:\sim)$ onto the kernel of φ, $K_{n-r}(M:\sim)$. Thus φ is k-connected, φ_* and φ^* are isomorphisms for $r < k$ and $r > n - k$.

Similarly let $\varphi : (N,M) \to (Y,X)$ be a degree 1 map of pairs. Then φ_* gives split surjections of homology groups with kernels K_*, and split injections of cohomology with cokernels K^*. The duality map $\cap[M]$ induces isomorphisms $K^*(N:\sim) \to K_{n-*}(N,M:\sim)$ and $K^*(N,M:\sim) \to K_{n-*}(N:\sim)$.

Analogous results hold for n-ads.

Proof: The results follow easily from the definitions and the naturality of the cap product. □

Section 2. The Spivak normal fibration

One important attribute of paracompact manifolds is the existence of normal bundles. In [35] Spivak constructed an analogue for these bundles in the homotopy category. Although he was interested in compact spaces, he was often forced to consider paracompact ones. It is then not too surprising that his definition is perfectly adequate for our problem. This is an example of a general principle in the theory of paracompact surgery, namely that all bundle problems encountered are exactly the same as in the compact case. One does not need a "proper" normal bundle or a "proper" Spivak fibration.

Definition: Let $(X,\partial X)$ be a locally compact, finite dimensional CW pair. Embed $(X,\partial X)$ in (H^n,R^{n-1}), where H^n is the upper half plane and $R^{n-1} = \partial H^n$. Let $(N;N_1,N_2)$ be a regular neighborhood of X as a subcomplex of H^n; i.e. $X \subseteq N$, $\partial X \subseteq N_2$ and $N(N_2)$ collapses to $X(\partial X)$. Let $\mathcal{P}(N_1,N,X)$ be the space of paths starting in N_1, lying in N, and ending in X endowed with the CO topology. (If A,B,C are spaces with $A,C \subseteq B$, a similar definition holds for $\mathcal{P}(A,B,C)$.) There is the endpoint map $w: \mathcal{P}(N_1,N,X) \to X$. w is a fibration and is called the Spivak normal fibration. Its fibre is called the Spivak normal fibre.

Spivak showed that a necessary and sufficient condition for a finite complex to satisfy Poincare duality with respect to the universal covering functor was that the Spivak normal fibre of the complex should have the homotopy type of a sphere. He also showed that if one started with a compact manifold then the normal sphere bundle had the same fibre homotopy type as the Spivak normal fibration, at least stably. Before we can do this for paracompact manifolds, we will need to do some work.

In practice, the fact that the Spivak normal fibration is constructed from a regular neighborhood is inconvenient. More convenient for our purposes is a semi-regular neighborhood.

<u>Definition</u>: Let $(X,\partial X)$ be a pair of finite dimensional, locally compact CW complexes. A semi-regular neighborhood (s-r neighborhood) is a manifold triad $(M: M_1, M_2)$ and proper maps $i: X \to M$ and $j: \partial X \to M_2$ such that

$$\begin{array}{ccc} X & \xrightarrow{i} & M \\ \cup| & & \cup| \\ \partial X & \xrightarrow{j} & M_2 \end{array}$$

commutes, and such that i and j are simple homotopy equivalences. Lastly we require that M be parallelizable (equivalent to being stably parallelizable.)

Theorem 1: The fibration $w : \mathcal{P}(M_1, M, X) \to X$ is stably fibre homotopy equivalent to the Spivak normal fibration.

Proof: The proof needs

Lemma 1: If $((M, M_1, M_2), i, j)$ is an s-r neighborhood of X, then so is $((M, M_1, M_2) \times (D^n, S^{n-1}), i \times c, j \times c)$, when c denotes the constant map. The triad structure on the product is $(M \times D^n; M_1 \times D^n \cup M \times S^{n-1}, M_2 \times D^n)$. Let ξ be $\mathcal{P}(M_1, M, X) \longrightarrow X$ and let η be $\mathcal{P}(M_1 \times D^n \cup M \times S^{n-1}, M \times D^n, X) \longrightarrow X$. Then $\xi * (n)$ is fibre homotopy equivalent to η, where (n) is the trivial spherical fibration of dimension $n-1$ and $*$ denotes the fibre join.

Proof: The first statement is trivial and the second is Spivak [35], Lemma 4.3. Q.E.D.

Now if $(M: M_1, M_2)$ is an s-r neighborhood of X, then for some n, $(M: M_1, M_2) \times (D^n, S^{n-1})$ is homeomorphic to a regular neighborhood of X in R^{n+m}, where $m = \dim M$. If we can show this, then the lemma easily implies that ξ is stably equivalent to the Spivak normal fibration formed from this regular neighborhood.

By crossing with D^n if necessary, we may assume $\dim M \geq 2 \dim X + 1$, so we may assume i and j are embeddings. Since M is parallelizable, $(M, \partial M)$ immerses in (H^m, R^{m-1}) . If $m \geq 2 \dim X + 1$ we can subject i and j to a homotopy so that $i : X \to M \subset H^m$ and $j : \partial X \to M_2 \subset R^{m-1})$ become embeddings on open neighborhoods U and U_2 , where U is a neighborhood of X and $U_2 = M_2 \cap U$ so it is a neighborhood of ∂X . In U sits a regular neighborhood of X , $(N: N_1, N_2)$. Hence $(N: N_1, N_2) \subseteq (M: M_1, M_2)$ and excision gives a simple homotopy equivalence $\partial N \subseteq \overline{M - N}$ and $\partial N_2 \subseteq \overline{M - N_2})$ (this uses the fact that i and j are simple equivalences.) By the s-cobordism theorem (see [33] or [10]) these are products (assume $m \geq 6$) so $(M: M_1, M_2)$ is homeomorphic to a regular neighborhood of X in R^m. □

<u>Corollary 1.1</u>. The Spivak normal fibration is stably well defined.

Remarks: By definition we have a Spivak normal fibration for any regular neighborhood, so we can not properly speak of "the" Spivak normal fibration. By the corollary however they are all stably equivalent, so we will continue to speak of the Spivak normal fibration when we really mean any fibration in this stable class.

Now, for finite complexes we know that the complex satisfies Poincaré duality iff the Spivak normal fibre has the homology of a sphere. Unfortunately, this is not true for our case. In fact, Spivak has already shown what is needed to get the normal fibre a sphere. This information is contained in Theorems 2 and 3.

<u>Definition</u>: A locally compact, finite dimensional CW complex is a Spivak space provided the normal fibre of any Spivak normal fibration has the homology of a sphere. A Spivak pair is a pair, $(X,\partial X)$, of locally compact, finite dimensional CW complexes such that the normal fibre of any Spivak normal fibration has the homology of a sphere, and such that the Spivak normal fibration for X restricted to ∂X is the Spivak normal fibration for ∂X. A Spivak n-ad is defined analogously.

<u>Theorem 2</u>: The following are equivalent.

1) X is a Spivak space

2) There is a class $[X] \in H_N^{\ell.f.}(X:Z)$, when $\tilde{X}$ is the universal cover of X, such that

$\cap[\tilde{X}]: H_c^*(\tilde{X}) \to H_{N-*}(\tilde{X})$ is an isomorphism

3) $\cap[\tilde{X}]: H^*(\tilde{X}) \to H_{N-*}^{\ell.f.}(\tilde{X})$ is an isomorphism.

<u>Proof</u>: 2) implies 3) thanks to the following commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_c^{*+1}(\tilde{X}),Z) & \longrightarrow & H_*^{\ell.f.}(\tilde{X}) & \longrightarrow & \mathrm{Hom}(H_c^*(\tilde{X}),Z) & \longrightarrow 0 \\ & \uparrow{\scriptstyle \mathrm{Ext}(\cap[\tilde{X}])} & & \uparrow{\scriptstyle \cap[\tilde{X}]} & & \uparrow{\scriptstyle \mathrm{Hom}(\cap[\tilde{X}])} & \\ 0 \to & \mathrm{Ext}(H_{N-*-1}(\tilde{X}),Z) & \longrightarrow & H^{N-*}(\tilde{X}) & \longrightarrow & \mathrm{Hom}(H_{N-*}(\tilde{X}),Z) & \longrightarrow 0 \end{array}$$

3) implies 1) thanks to Spivak, Proposition 4.4, and the observation that the Spivak normal fibration for X pulled back over $\tilde{X}$ is the Spivak normal fibration for $\tilde{X}$. This observation is an easy consequence of Theorem 1, the definition of s-r neighborhood, and the fact that the

transfer map $\varsigma(X) \to \varsigma(\tilde{X})$ is a homomorphism.

1) implies 2) as follows. Look at

$$\begin{array}{ccc}
H_{N-*}(D(\tilde{X})) & \longrightarrow & H_{N-*}(\tilde{X}) \\
\uparrow \cap U & & \\
H_{N+k-*}(D(\tilde{X}),S(\tilde{X})) & \longleftarrow & H_{N+k-*}(N,\partial N) \\
 & & \uparrow \cap [N] \\
H_c^*(\tilde{X}) & \longleftarrow & H_c^*(N)
\end{array}$$

where U is the Thom class for the normal disc fibration $D(\tilde{X})$ with spherical fibration $S(\tilde{X})$. $(N,\partial N)$ is an s-r neighborhood for $\tilde{X}$. The horizontal maps are induced by the inclusion $X \subseteq N$ and the homotopy equivalence $(N,\partial N) \to (D(\tilde{X}),S(\tilde{X}))$. All horizontal maps are isomorphisms. The composite map $H_c^*(\tilde{X})$ to $H_{N-*}(\tilde{X})$ is essentially the cap product with $U \cap [N]$, where $U \cap [N]$ should actually be written $i_*(U \cap j_*[N]$, where $i_* : H_*^{\ell.f.}(D(\tilde{X})) \longrightarrow H_*^{\ell.f.}(\tilde{X})$ and $j_*: H_*^{\ell.f.}(N,\partial N) \longrightarrow H_*^{\ell.f.}(D(\tilde{X}),S(\tilde{X}))$. $H_*^{\ell.f.}(D(\tilde{X}))$ is the homology group of the infinite singular chains on $D(\tilde{X})$ which project to give locally finite chains on $\tilde{X}$. ($H_*^{\ell.f.}(D(\tilde{X}),S(\tilde{X}))$ is similar.) Now $[\tilde{X}] = U \cap [N]$ shows 2) is satisfied. [1) was used to get the Thom class U].□

Remarks: If w_1 is the first Stiefel-Whitney class of the Spivak normal fibration, $[\tilde{X}]$ comes via transfer from a class in $H_N^{\ell.f.}$ of X with coefficients twisted by w_1. This class, denoted $[X]$, is called the fundamental class of X. These same remarks are valid for pairs (and indeed for n-ads) after our next theorem.

Theorem 3: If ∂X is a Spivak space with fundamental class $[\partial X]$, then the following are equivalent.

1) $(X,\partial X)$ is a Spivak pair

2) There is a class $[\tilde{X}] \in H_N^{\ell.f.}(\tilde{X},\overline{\partial X})$ when $\tilde{X}$ is the universal cover of X ($\overline{\partial X}$ is the induced cover over ∂X) such that $\partial[\tilde{X}] = [\overline{\partial X}] \in H_{N-1}^{\ell.f.}(\overline{\partial X})$ and such that any one of the following four maps is an isomorphism

A) $\cap[\tilde{X}] : H_c^*(\tilde{X},\overline{\partial X}) \longrightarrow H_{N-*}(\tilde{X})$

B) $\cap[\tilde{X}] : H^*(\tilde{X},\overline{\partial X}) \longrightarrow H_{N-*}^{\ell.f.}(\tilde{X})$

C) $\cap[\tilde{X}] : H_c^*(\tilde{X}) \longrightarrow H_{N-*}(\tilde{X},\overline{\partial X})$

D) $\cap[\tilde{X}] : H^*(\tilde{X}) \longrightarrow H_{N-*}^{\ell.f.}(\tilde{X},\overline{\partial X})$

The equation $\partial[\tilde{X}] = [\overline{\partial X}]$ means the following. In $\tilde{X}$, $\overline{\partial X}$ consists of (many) copies of some cover of ∂X.

$$\partial : H_N^{\ell.f.}(\tilde{X},\overline{\partial X}) \rightarrow H_{N-1}^{\ell.f.}(\overline{\partial X})$$

takes the class $[\tilde{X}]$ to a class in each component of $\overline{\partial X}$. These classes are all equal. There is a transfer map from $H_{N-1}^{\ell.f.}(\partial X;Z^{w_1})$ to $H_{N-1}^{\ell.f.}(\overline{\partial X}$; some coefficients). What $\partial[\tilde{X}] = [\overline{\partial X}]$ means is that under this transfer map, $[\partial X]$ should go to $\partial[\tilde{X}]$. Note this requires that the coefficients Z^{w_1} untwist in $\overline{\partial X}$.

Proof: 2) implies the normal fibre is a sphere as follows. By the Browder lemma, the 5-lemma, and a diagram like the one in the proof of Theorem 2, A), B), or C) implies D) . D) implies that the Spivak normal fibre is a sphere by Spivak, Proposition 4.4.

The Spivak normal fibre is a sphere implies A) - D) as follows. The Thom isomorphism and Lefschetz duality imply A) - D) hold for $[\tilde{X}] = U \cap [N]$, where U is the Thom class of the fibration and [N] is the fundamental class for the regular neighborhood of X .

If 1) holds, then A) - D) hold for $[\tilde{X}] = U \cap [N]$. $[\overline{\partial X}] = U_2 \cap [N_2]$ where U_2 is the Thom class for the normal fibration for $\overline{\partial X}$ and $[N_2]$ is the fundamental class of the regular neighborhood for $\overline{\partial X}$. Since the fibration over $\tilde{X}$ restricts to the one over $\overline{\partial X}$, $i^*U = U_2$, where i^*U is defined as follows. The total space of the normal fibration for $\tilde{X}$ is $\mathcal{P}(N_1,N,X)$. The total space of this fibration restricted to $\overline{\partial X}$ is $\mathcal{P}(N_1,N,\overline{\partial X})$. The total space of the normal fibration for $\overline{\partial X}$ is $\mathcal{P}(N_1 \cap N_2,N_2,\overline{\partial X})$. The inclusions $\mathcal{P}(N_1 \cap N_2,N_2,\overline{\partial X}) \subseteq \mathcal{P}(N_1,N,\overline{\partial X}) \subseteq \mathcal{P}(N_1,N,\tilde{X})$ are all fibre maps. Let $E_1 \subseteq E_2 \subseteq E_3$ denote the corresponding disc fibrations. The Thom class U lives in $H^k(E_3,\dot{E}_3;p^*\Gamma_3)$, where p is the projection and Γ_3 is a system of local coefficients on X . (see Spanier [35], page 283). Let Γ_1 be the local coefficients for E_1 . Then $\Gamma_1 \to \Gamma_3$ by the fibre map. Hence U pulls back to $H^k(E_1, \mathring{E}_1; p^*\Gamma_3)$. U_2 lives in $H^k(E_1, \mathring{E}_1; p^*\Gamma_1)$, and hence goes over to a class in $H^k(E_1, \mathring{E}_1; p^*\Gamma_3)$ where it must be $\pm$ (U pulled back). To write $i^*U = U_2$ means the coefficient systems Γ_1 and Γ_3 are the same. Now $\partial[N] = [N_2]$, so one sees $\partial[\tilde{X}] = [\overline{\partial X}]$. Hence 1) implies 2).

Given 2) we know the normal fibre is a sphere and we must just check that the fibration restricts properly. Note first that the choice of $[\tilde{X}]$ is unique up to sign, so if 2) holds, the class $[\tilde{X}] = U \cap [N]$. $\partial[X] = i^*U \cap [N_2]$ always. Now $[\partial X] = U_2 \cap [N_2]$, so if $\partial[X] = [\partial X]$, $U_2 = i^*U$, as is not hard to see. (By the remark at the end of 2), the local coefficients for U and for U_2 have to be the same.) Hence the inclusion $(N_1 \cap N_2, N_2, \overline{\partial X}) \subseteq (N_1, N, \overline{\partial X})$ is a fibre homotopy equivalence. This shows 2) implies 1). □

Now suppose ξ is an arbitrary spherical fibration over a locally compact, finite dimensional CW complex X. The total space is not, in general, such a complex. Our techniques apply best to such spaces however, and we want to study these total spaces. Hence we wish to replace any such total space by a space with the proper homotopy type of a locally compact, finite dimensional CW complex.

<u>Definition</u>: Let $S(\xi)$ be the total space of a spherical fibration ξ over a locally compact, finite dimensional CW n-ad X . A cwation (CW-ation) of ξ

is an n-ad Y and a proper map $f : Y \to X$ such that the following conditions are satisfied. Y has the proper homotopy type of a locally compact, finite dimensional CW n-ad. There are maps $S(\xi) \underset{h}{\overset{g}{\rightleftarrows}} Y$ such that $h \circ g$ is a fibre map, fibre homotopic to the identity and such that $g \circ h$ is properly homotopic to the identity.

Lastly
$$\begin{array}{ccc} S(\xi) & \underset{h}{\overset{g}{\rightleftarrows}} & Y \\ & \searrow \quad \swarrow_{f} & \\ & X & \end{array}$$
should commute. The pair

$(M(f),Y)$ is seen to satisfy the Thom isomorphism for the Δ^* and Δ_* theories. (see the appendix for a discussion of the Thom isomorphism in these theories.) The simple homotopy type of Y is defined by any locally compact, finite dimensional complex having the same proper homotopy type of Y and for which the Thom isomorphisms are simple homotopy equivalences. For a bundle ξ, $(D(\xi), C(\xi))$ will denote the pair $(M(f), Y)$ with this simple homotopy type. Such a pair is said to be a simple cwation.

Remarks: Any spherical fibration of dimension two or more has a cwation. The proof of this fact is long and is the appendix to this chapter.

<u>Theorem 4</u>: Let ξ be any spherical fibration of dimension > 1 over a Spivak space X. Then $(D(\xi),C(\xi))$ is a Spivak space. If $[\xi]$ is the fundamental class, and if U_ξ is the Thom class, we have the formula $U_\xi \cap [\xi] = [X]$.

If $(X,\partial X)$ is a Spivak pair, then $(D(\xi):C(\xi|\partial X))$ is a Spivak triad. In general a cwation of a Spivak n-ad has an (n+1)-ad structure. We still have the formula $U_\xi \cap [\xi] = [X]$.

<u>Proof</u>: The n-ad case follows by induction from the pairs case, so we concentrate on the latter. To show $(D(\xi),C(\xi))$ is a Spivak space look at

$$\begin{array}{ccc} H^*(\widetilde{D(\xi)}) & \overset{\psi}{-\!-\!-\!-\!-\!-\!-\!-\!-} & H^{\ell.f.}_{N+k-*}(\widetilde{D(\xi)}, \overline{C(\xi) \cup D(\xi|\partial X)}) \\ & \searrow^{\cap[X]} & \Big\downarrow \cong\ U_\xi\cap \\ & & H^{\ell.f.}_{N-*}(\widetilde{D(\xi)},\overline{D(\xi|\partial X)}) \end{array}$$

U_ξ when pushed into $H^*(\widetilde{D(\xi)},\overline{C(\xi)})$ has integer coefficients. $[X]$ denotes the image if the fundamental class of X in $H^{\ell.f.}_N(\widetilde{D(\xi)}, \overline{C(\xi) \cup D(\xi|\partial X)})$, which again has integer coefficients as it factors through $H^{\ell.f.}_N(\tilde{X},\overline{\partial X})$. Let ψ be the isomorphism given by $(U_\xi \cup)^{-1} \circ (\cap[X])$. We claim $\psi(x) = x \cap \psi(1)$, where $1 \in H^0(\widetilde{D(\xi)})$ is the generator. But $U_\xi \cap \psi(1)) = (U_\xi \cup x) \cap \psi(1) = x \cap (U_\xi \cap \psi(1))$ $= x \cap [X] = U_\xi \cap \psi(X)$, so we are done. Let $A = \psi(1)$.

Since ξ has dimension > 1, $\overline{D(\xi|\partial X) \cup C(\xi)}$ is simply connected. $\cap A : H^*(\widetilde{D(\xi)}, \overline{C(\xi) \cup D(\xi|\partial X)}) \longrightarrow$ $H_{N+k-*}(\widetilde{D(\xi)})$ is also an isomorphism, so $\partial A: H^*(\overline{C(\xi) \cup D(\xi|\partial X)}) \longrightarrow H_{N+k-*-1}(\overline{C(\xi) \cup D(\xi|\partial X)})$ is also an isomorphism. Hence if $\partial X = \emptyset$, $(D(\xi), C(\xi))$ is a Spivak pair by Theorem 1. Then $(D(\xi): C(\xi),D(\xi|\partial X))$ can be shown to be a Spivak triad.

We must still verify our equation. If we can show the local coefficients behave correctly, we will get a diagram like the one we used to define ψ, only this time with local coefficients. Let Γ_ξ be the local coefficients in X for the Thom class. Let Γ_{w_1} be the local coefficients in X for $\cap[X]$. Let Γ be the local coefficients in $D(\xi)$ for $[\xi]$. Then if we could show $\Gamma \otimes p^*(\Gamma_\xi) = p^*(\Gamma_{w_1})$, where $p : D(\xi) \to X$, we would be done as one can easily check. To check that two local systems are equivalent it is enough to check that they agree for any $g \in \pi_1(D(\xi),x)$. But

$$\Gamma(g) = \begin{cases} +1 & \text{if } g_* \psi(1) = \psi(1) \\ -1 & \text{if } g_* \psi(1) = -\psi(1) \end{cases}$$

where $g_* : H^{\circ f \circ}_{N+k}(D(\xi), C(\xi) \cup D(\xi|\partial X))$ is the map induced by the covering transformation g.

$$p^*(\Gamma_\xi)(g) = \begin{cases} +1 & \text{if } g^* U_\xi = U_\xi \\ -1 & \text{if } g^* U_\xi = -U_\xi \end{cases} .$$

Hence

$$(\Gamma \otimes p^*(\Gamma_\xi))(g) = \begin{cases} +1 & \text{if } g_*[X] = [X] \\ -1 & \text{if } g_*[X] = -[X] \end{cases} ,$$

which is just $p^*(\Gamma_{w_1})(g)$. $\square$

<u>Corollary 4.1</u>: Let X be a locally compact, finite dimensional CW n-ad. Let ξ be a spherical fibration of dimension ≥ 2 over X. Then if $D(\xi)$ is a Spivak

(n+1)-ad, X is a Spivak n-ad.

Proof: Let $[X] = U_\xi \cap [\xi]$. Then $\cap[X]$ induces isomorphisms $H_c^*(\widetilde{D(\xi)};Z) \longrightarrow H_{N-*}(\widetilde{D(\xi)},\overline{D(\xi|\partial X)};Z)$, or equivalently $H_c^*(\tilde{X}) \longrightarrow H_{N-*}(\tilde{X},\overline{\partial X};Z)$. Inducting over the n-ad structure of X and applying Theorems 1 and 2, we get X is a Spivak n-ad. □

Theorem 5: X is a Spivak n-ad iff X is for any cover of X . If X is an n-ad and Y is an m-ad, $X \times Y$ is a Spivak (n+m-1)-ad iff X is a Spivak n-ad and Y is a Spivak m-ad.

Proof: By induction, if we can prove the result for pairs we are done modulo the easy result that if (Y,X) and Y',X) are Spivak ads, then so is $Y \cup Y'$. This is also shown by induction using cutting and gluing arguments to deduce the cap product isomorphisms of Theorems 2 and 3. A careful proof is left to the reader.

Our first statement is immediate from Theorem 1, since if N is an s-r neighborhood for X , $\tilde{N}$ is one for $\tilde{X}$.

For the second statement we prove

Lemma 2: If ν_Z is the Spivak normal fibration for any finite dimensional n-ad Z , and if X and Y are such complexes, $\nu_X * \nu_Y = \nu_{X\times Y}$.

Proof: Let D_Z be the disc fibration of ν_Z . Then $\nu_X * \nu_Y = \nu_X \times D_Y \cup D_X \times \nu_Y \subseteq D_X \times D_Y$. Let $(N : N_1,N_2)$ be the s-r neighborhood for X from which ν_X was formed.

$(M: M_1, M_2)$ is the corresponding object for Y. Then $\nu_{X\times Y}$ has $\mathcal{P}(N\times M_1 \cup N_1 \times M, N\times M, X\times Y)$ for total space. $\nu_X \times D_Y$ consists of triples $e \in \nu_X$, $f \in \nu_Y$, and $t \in [0,1]$ with $(f,0) = (g,0)$ if $f(1) = g(1) \in Y$. Then $\nu_X \times D_Y \to \nu_{X\times Y}$ given by (e,t,f) goes to the path $(e(s), f(1-t+st))$ is a fibre map. There is a similar map for $D_X \times \nu_Y$, which agrees with the first on $\nu_X \times \nu_Y$. Hence we get a fibre map $\nu_X * \nu_Y \to \nu_{X\times Y}$. Now ν_X restricts from a fibration ν'_X over N. ν'_Y and $\nu'_{X\times Y}$ are defined similarly, and we have a fibre map $\nu'_X * \nu'_Y \to \nu'_{X\times Y}$. There is an initial point map $\nu'_{X\times Y} \to N\times M_1 \cup N_1 \times M$, which is a homotopy equivalence. $\nu'_X * \nu'_Y \to N\times M_1 \cup N_1 \times M$ via the composition is likewise a homotopy equivalence. Hence by Dold [7], $\nu'_X * \nu'_Y \to \nu'_{X\times Y}$ is a fibre homotopy equivalence. Hence so is $\nu_X * \nu_Y \to \nu_{X\times Y}$. Q.E.D.

Now $X\times Y$ is a Spivak space iff $\nu_{X\times Y}$ is spherical and, if $Z \subseteq X\times Y$ is a piece of the triad structure, $\nu_{X\times Y}|Z \cong \nu_Z$. If X and Y are both Spivak pairs, the lemma shows the result easily. If $X\times Y$ is a Spivak triad, $(X\times \partial Y, \partial X\times \partial Y)$ is a Spivak pair. By the lemma, the normal fibration is $\nu_X * \nu_{\partial Y}$. Since its fibre has the homology of a sphere, the fibres of both ν_X and $\nu_{\partial Y}$ must have the homology of spheres. Since $\nu_X * \nu_{\partial Y}|X\times p$, $p \in \partial Y$ is equivalent to $\nu_X * (n)$ for $n = \dim \nu_{\partial Y}$, and since $\nu_X * \nu_{\partial Y}|\partial X\times \partial Y = \nu_{\partial X\times \partial Y}$, as $(X\times \partial Y, \partial X\times \partial Y)$ is a Spivak pair, one sees $\nu_X * (n)|\partial X \cong \nu_{\partial X} * (n)$, so $(X,\partial X)$ is a Spivak pair. A similar argument shows $(Y,\partial Y)$

is a Spivak pair. □

Theorem 6: Let X be a Spivak n-ad, and let N be a regular neighborhood for X. If (D(X),C(X)) is a simple cwation for this normal fibration, there is a proper map of (n+1) - ads $g:N \to D(X)$ such that the composition $N \to D(X) \to X$ is a proper homotopy inverse for $X \subseteq N$. If [N] and [D(X)] are the fundamental classes for N and D(X) respectively, $g_*[N] = [D(X)]$.

g is a homotopy equivalence of (n+1)-ads (not necessarily a proper homotopy equivalence). g is however properly (dim N - dim X-1) connected.

Remarks: If [X] lives in k-dimensional homology, then the normal fibration has a simple cwation if $\dim N - k \geq 3$.

Proof: To be momentarily sloppy, let D(X) denote the total space of the normal disc fibration for X. Since $X \subseteq N$ is a proper homotopy equivalence, pick an inverse $N \to X$. Pull D(X) back over N. It is also a disc fibration and so has a section (see Dold [7] Corollary 6.2). Map $N \to D(X)$ by the section followed by the map into D(X). Under the composition $N \to D(X) \to X$, we just get our original map. But now we can take the map from the total space of the fibration to the cwation. Letting D(X) be the disc cwation again, we get a map $N \to D(X)$ so that the composition $N \to D(X) \to X$ is our original map. The map $N \to D(X)$ is easily seen to be proper and is g .

g is a homotopy equivalence of $(n+1)$-ads by construction. $g_o: N \to D(X)$ is also a proper homotopy equivalence ($g: N \to D(X)$, the map of n-ads is not necessarily a proper homotopy equivalence). The following diagram commutes

$$\begin{array}{ccc} H^*(N,\partial N) & \xleftarrow{g^*} & H^*(D(X),C(X)) \\ \downarrow \cap[N] & & \downarrow g_*[N] \\ H^{\ell.f.}_{N-*}(N) & \xrightarrow{(g_o)_*} & H^{\ell.f.}_{N-*}(D(X)) \end{array}$$

$\cap[N]$, g^* , and $(g_o)_*$ are all isomorphisms, so $g_*[N]$ is also an isomorphism. Therefore, $g_*[N] = \pm [D(X)]$, and we may orient N so that $g_*[N] = [D(X)]$.

The map $C(X) \to X$ is properly q-connected, where the normal spherical fibration has dimension q . This is easily seen from the fibration sequence $S^q \to S(\xi) \to X$, where $S(\xi)$ is the total space of the normal spherical fibration, by noticing that

$$\begin{array}{ccc} \Delta(S(\xi) : \pi_k) & \longrightarrow & \Delta(C(X) : \pi_k) \\ & \searrow \quad \swarrow & \\ & \Delta(X : \pi_k) & \end{array}$$

commutes, where $\Delta(S(\xi): \pi_k)$ is formed from the groups $\pi_k(S(\xi)|X - C, \hat{p})$, where $\hat{p} \in S(\xi)$ covers $p \in X$ (i.e. just pick one $\hat{p}$ for each base point in X). The horizontal map is an isomorphism since $C(X)$ is a cwation.

The first vertical map is an isomorphism for $k < q$ and an epimorphism for $k = g$, so we are done.

The map $N_1 \subseteq N \to X$ is properly r-connected, where $r = (\dim N - \dim X - 1)$. This is seen by showing that the map $N_1 \subseteq N$ is properly r-connected. But this is easy. If K is a locally compact complex with $\dim K \leq r$, any map of $K \to N$ deforms properly by general position to a map whose image lies in $N - X$. Hence $\Delta(N_1 : \pi_k) \to \Delta(N : \pi_k)$ is onto for $k \leq r$ and 1-1 for $k \leq r-1$.

Now $g_0 : N \to D(X)$ is a proper equivalence so the map is properly r-connected. Since $r \leq q$, $g: N_1 \to C(X)$ is properly r-connected. If X is a space, we are done. If $(X, \partial X)$ is a pair, the regular neighborhood is $(N: N_1, N_2)$ and the cwation is $(D(X): C(X), D(\partial X))$. $C(X) \cap D(\partial X) = C(\partial X)$. $g: N_1 \cap N_2 \to C(\partial X)$ is properly r-connected as it is an example of the absolute case. $N_2 \to D(\partial X)$ is a proper homotopy equivalence, hence properly r-connected. $g: N_1 \to C(X)$ and $g: N \to D(X)$ we saw were properly r-connected, so the case for pairs is done. For the n-ad case, just induct. □

We are now ready to define Poincaré duality spaces.

Definition: A Spivak n-ad X is a Poincaré duality n-ad iff the g of Theorem 6 is a proper homotopy equivalence of (n+1)-ads for some regular neighborhood.

Remarks: Apriori our definition depends on which regular neighborhood we have used in Theorem 6. In fact this is not really the case, as our next theorem demonstrates.

<u>Theorem 7</u>: Let X be a locally compact, finite dimensional CW n-ad. Then X is a Poincaré duality space iff X satisfies Poincaré duality with respect to $[\tilde{X}] \in H_n^{\ell.f.}(\tilde{X};Z)$ and with respect to a universal covering functor.

A pair $(X,\partial X)$ is a Poincaré duality pair iff ∂X is a Poincaré duality space and X satisfies Poincaré duality with respect to a universal covering functor and a class $[X] \in H_n^{\ell.f.}(\tilde{X},\overline{\partial X};Z)$ such that $\partial[\tilde{X}] = [\overline{\partial X}]$.

A similar result holds for n-ads.

<u>Proof</u>: Since $\cap[\tilde{X}] : \Delta^*(X,\sim) \to \Delta_{n-*}(X:\sim)$ an isomorphism implies $\cap[\tilde{X}] : H_c(\tilde{X}) \to H_{n-*}(\tilde{X})$ is an isomorphism, if X satisfies Poincaré duality then, by Theorem 2, X is a Spivak space. Similarly, by Theorem 3, we may show $(X,\partial X)$ is a Spivak space if ∂X is a Poincaré duality space and if $(X,\partial X)$ satisfies Poincaré duality. In both cases, the fundamental class, $[X]$, transfers up to give $\pm\ [\tilde{X}]$. Now look at

$$\begin{array}{ccccc}
\Delta^*(X,\partial X:\sim) & \xrightarrow{r^*} & \Delta^*(N,N_2:\sim) & & \\
 & & \Big\downarrow \cap[N] & & \\
 & & \Delta_{n+k-*}(N,N_1:\sim) & \xrightarrow{g_*} & \Delta_{n+k-*}(D(X),C(X):\sim) \\
 & & & & \Big\downarrow U_\nu\cap \\
 & & & & \Delta_{n-*}(X:\sim)
\end{array}$$

where $r\colon (N,N_2) \to (X,\partial X)$ is a proper homotopy inverse for $(X,\partial X) \subseteq (N,N_2)$, and U_ν is the Thom class for the normal fibration ν. By Theorem 4, the composition is just $\cap[X]$, and r^*, $\cap[N]$, and $U_\nu \cap$ are all isomorphisms. Hence $(X,\partial X)$ satisfies Poincaré duality iff g_* is an isomorphism. If g is a proper homotopy equivalence, g_* is clearly an isomorphism. If $(X,\partial X)$ satisfies Poincaré duality, and if $\dim N - \dim X \geq 3$, g_* is a proper homotopy equivalence by the Whitehead theorem. To see this, first note $\sim$ is a universal covering functor for both N and $D(X)$. Since $\dim N - \dim X \geq 3$, $N_1 \subseteq N$ and $C(X) \subseteq D(X)$ are at least properly 2-connected. Since ∂X is by hypothesis a Poincaré duality space, $g_*: \Delta_*(N_1:\sim) \to \Delta_*(C(X):\sim)$ is an isomorphism. By the connectivity of $N_1 \subseteq N$ and $C(X) \subseteq D(X)$, these groups are already the subspace groups for a wise choice of base points. By the Browder lemma $g_*:\Delta_*(N:\sim) \longrightarrow \Delta_*(D(X):\sim)$ is an isomorphism, and g is at least properly 2-connected, so the Whitehead theorem applies to show that g is a proper homotopy equivalence. $\square$

Remarks: Note that the proof shows that g must be a proper homotopy equivalence whenever $\dim N - \dim X \geq 3$.

We have seen that manifolds satisfy Poincaré duality with respect to any covering functor. The Thom isomorphism also holds for any covering functor. Hence it is easy to see

<u>Corollary 7.1</u>: A Poincaré duality n-ad satisfies Poincaré duality with respect to any covering functor. □

<u>Definition</u>: The torsion of the equivalence $\cap[X]: \Delta^*(X,\partial X:\sim) \to \Delta_{n-*}(X:\sim)$ is defined to be the torsion of the Poincaré duality space X ($\sim$ is the universal covering functor). Since $(D(X),C(X))$ is a simple cwation, and since $\cap[N]$ is a simple equivalence (Theorem 2.1.2), $\tau(X) = (-1)^{n+k}\tau(g)$, where $\tau(X)$ is the torsion of X and everything else comes from the diagram in the proof of Theorem 7. A simple Poincaré n-ad is one for which all the duality maps are simple isomorphisms.

Examples: By Theorems 2.1.1. and 2.1.2, any paracompact manifold n-ad is a simple Poincaré n-ad. There are also examples of Spivak spaces which are not Poincaré duality spaces. One such is the following. Let X be a finite complex whose reduced homology with integer coefficients is zero, but which is not contractable. (The dodecahedral manifold minus an open disc is such an example.) Look at $\mathring{C}(X \vee S^2)$, the open cone on $X \vee S^2$. The obvious map $R^3 = \mathring{C}(S^2) \to \mathring{C}(X \vee S^2)$ is seen to induce isomorphisms on H_* and H_c^*. Since R^3 is a Spivak space, so is $\mathring{C}(X \vee S^2)$. $\mathring{C}(X \vee S^2)$ is not a Poincaré duality space as $X \vee S^2$ is not a Poincaré duality space.

In the other direction, we have as an application of a theorem of Farrell-Wagoner [9]

<u>Theorem 8</u>: Let X be a locally compact complex with monomorphic ends. Then X is a Poincaré duality space iff X is a Spivak space. An analogous result is true for n-ads

<u>Corollary 8.1</u>: Let X be a Spivak n-ad. Then $X \times R^2$ is a Poincaré duality space.

<u>Proof</u>: We only prove X Spivak implies X Poincaré. If X has monomorphic ends, and if N is an s-r neighborhood with dim N-dim $X \geq 3$, ∂N has monomorphic ends. $C(X)$ also has monomorphic ends. The g of Theorem 6 is at least properly 2-connected. Hence by [9] we need only prove g induces isomorphisms on H_* and H_*^c. But $g_*[N] = [D(X)]$, and g on homology is an isomorphism since it is a homotopy equivalence. Since N and $C(X)$ are both Spivak spaces, Theorem 1 shows g induces isomorphisms on H_c^*.

To show the corollary, observe that if X is not compact, $X \times R^2$ has monomorphic ends. It is a Spivak space by Theorem 5, so, in this case, we are done. If X is compact, X is already Poincaré duality space, so the result will follow from the next theorem. □

<u>Theorem 9</u>: Let X be a Poincaré duality n-ad, and let Y be a Poincaré duality m-ad. Then $X \times Y$ is a Poincaré duality (n+m-1)-ad. If X or Y is compact, the converse is true.

<u>Proof</u>: From Lemma 2 we have

$C(X \times Y) = D(X) \times C(Y) \cup C(X) \times D(Y) \subseteq D(X) \times D(Y) = D(X \times Y)$

If N is an s-r neighborhood for X and if M is one for Y, $N \times M$ is one for $X \times Y$. Hence we have

$g \times f\colon N \times M \to D(X \times Y)$ is a map on (n+m-1)-ads. It is a proper homotopy equivalence if g and f are.

Now suppose X is compact. By Theorem 5, X is a Spivak n-ad, and hence a Poincaré duality n-ad. Since $g \times f$ is a proper homotopy equivalence, it induces isomorphisms on the proper homotopy groups. We claim $\Delta(N \times M : \pi_k) = \pi_k(N) \times \Delta(M:\pi_k)$ for N compact. This is easily seen by using the cofinal collection of compact subsets of $N \times M$ of the form $N \times C$, $C \subseteq M$ compact. A similar result computes $\Delta(D(X \times Y)\colon \pi_k)$. Since $g \times f$ and g induce isomorphisms, $f_*\colon \Delta(M:\pi_k) \to \Delta(D(Y):\pi_k)$ is an isomorphism. By inducting this argument over the various subspaces of $D(Y)$, f is seen to be a proper equivalence of (m+1)-ads. Hence Y is a Poincaré duality m-ad. □

<u>Theorem 10</u>: X a Poincaré duality n-ad implies $\tilde{X}$ is a Poincaré duality n-ad for any cover of X. If X is compact or if $\tilde{X}$ is a finite sheeted cover, then the converse is true.

<u>Proof</u>: Let N be an s-r neighborhood for X. Then $\tilde{N}$ is an s-r neighborhood for $\tilde{X}$, so $D(\tilde{X}) = \widetilde{D(X)}$. X a Poincaré duality n-ad implies $N \to D(X)$ is a proper homotopy equivalence of n-ads. But then so is $\tilde{N} \to \widetilde{D(X)}$, so $\tilde{X}$ is a Poincaré duality n-ad.

If $\tilde{X}$ is a Poincaré duality n-ad, X is a Spivak n-ad by the n-ad analogue of Theorems 1 and 2. Hence if X is compact, it is a Poincaré duality n-ad.

Now if $\tilde{X} \to X$ is finite sheeted and we know $\tilde{N} \to D(X)$ is a proper homotopy equivalence, we must show $N \to D(X)$ is a proper homotopy equivalence. But if $\dim N - \dim X \geq 3$ (which we may freely assume), this map is properly 2-connected. Since $\Delta(\tilde{N}: \pi_k) \to \Delta(N:\pi_k)$ is an isomorphism for $k \geq 2$ when $\tilde{N}$ is a finite sheeted cover, $N \to D(X)$ is seen to be a proper homotopy equivalence. Inducting the argument shows $N \to D(X)$ a proper homotopy equivalence of (n+1)-ads, so X is a Poincaré duality n-ad. □

Remarks: The full converses to Theorems 9 and 10 are false. Let X be any Spivak space which is not a Poincare duality space. Then $X \times R^2$ is a counterexample to the converse of Theorem 9 as it is a Poincaré duality space by Corollary 8.1. $X \times T^2$ is a counterexample to Theorem 10, since $X \times T^2$ is not a Poincaré duality space by Theorem 9, but its cover $X \times R^2$ is

<u>Theorem 11</u>: Let ξ be any spherical fibration of dimension ≥ 2 over a locally compact, finite dimensional CW n-ad X . Then X is a Poincaré duality n-ad iff $D(\xi)$ is a Poincaré duality (n+1)-ad.

<u>Proof</u>: By Theorem 4 or Corollary 4.1, we may assume X and $D(\xi)$ are Spivak ads, and we have the formula

$U_\xi \cap [\xi] = [X]$. Since the Thom isomorphism is valid for the Δ theory (see the appendix), $\cap[X]$ is an isomorphism iff $\cap[\xi]$ is an isomorphism. Since $\dim \xi \geq 2$, a universal covering functor for X induces one for $D(\xi)$. Theorem 7 now gives the desired conclusions. $\square$

Remarks: The torsions of the Poincare duality spaces occurring in Theorems 9, 10, and 11 can be "computed". In particular, $\tau(X \times Y) = A(\tau(X), \tau(Y))$ where A is the pairing $\zeta(X) \times \zeta(Y) \to \zeta(X \times Y)$ (see Lemma 1.5.23 and the preceding discussion). $\tau(\tilde{X}) = \text{tr}\ \tau(x)$, where $\text{tr}: \zeta(X) \to \zeta(\tilde{X})$. $\tau(D(\xi)) = (-1)^n \tau(D(\xi))^t$, where n is the dimension of the fundamental class of X , and t is the transpose operation on $\zeta(X)$. These formulas are not very hard to deduce and will be left to the reader.

We conclude this section by investigating the "uniqueness" of the Spivak normal fibration. We first prove

<u>Lemma 3</u>: Let $D(\xi)$ be a cwation for some spherical fibration ξ over a Poincare duality n-ad. If there is a stably parallelizable manifold (n+1)-ad N and a proper, degree one, homotopy equivalence $N \to D(\xi)$, then ξ is stably equivalent to the Spivak normal fibration.

Remarks: Given all the spherical fibrations over a Poincare duality n-ad X , we wish to determine which of these could be the normal fibration of some complex having the proper homotopy type as X . In the compact case, Spivak [36] showed that there was only one, the one with the reducible Thom space. Lemma 3 shows that if $D(\xi)$ has the degree one proper homotopy type of a stably parallelizable manifold, then ξ is the normal fibration for X. If ξ is the normal fibration for some complex Y, $D(\xi)$ has the degree one proper homotopy of a parallelizable manifold, so again there is one and only one candidate for a normal fibration.

Proof: If the equivalence were simple, N would be an s-r neighborhood and this would follow from Theorem 1. Now by Siebenmann [33], $N \times S^1 \to D(\xi) \times S^1$ is a simple equivalence. $D(\xi) \times S^1$ is a simple cwation for $\xi \times S^1$ over $X \times S^1$. $N \times S^1$ is an s-r neighborhood for $X \times S^1$. $\theta(N_1, N, N) \times S^1 \to N \times S^1$ makes the map $N_1 \times S^1 \subseteq N \times S^1$ into a fibration, so $\nu_X \times S^1$ is fibre homotopy equivalent to $\nu_{X \times S^1}$. But $\xi \times S^1$ is stably fibre homotopy equivalent to $\nu_{X \times S^1}$ by Theorem 1. Hence ν_X is stably ξ. □

Theorem 12: If $f : X \to Y$ is a proper homotopy equivalence between Poincare duality n-ads, then $f^* \nu_Y \cong \nu_X$.

Proof: Let $\xi = f^*(\nu_Y)$. Then

$$\begin{array}{ccccccc} D(\xi) & \longrightarrow & \xi & \longrightarrow & \nu_Y & \longrightarrow & D(\nu_Y) \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & X & \longrightarrow & Y & & \end{array}$$

commutes. The top horizontal row is a proper homotopy equivalence, as one can easily check by applying $\Delta(\ :\pi_k)$

to everything. Since $D(\nu_Y)$ has the degree 1 proper homotopy type of a parallelizable manifold, so does $D(\xi)$. Hence by Lemma 3, $\xi \cong \nu_X$. □

Spivak's identification of the normal fibration actually proves a stronger theorem. We can also prove this result as

<u>Theorem 13</u>: Let $f : X \to Y$ be a degree one map of Poincare duality n-ads. If there is a spherical fibration ξ over Y such that $f^*(\xi) \cong \nu_X$, then $\xi \cong \nu_Y$.

<u>Proof</u>:

$$\begin{array}{ccccccc} D(\nu_X) & \longrightarrow & \nu_X & \longrightarrow & \xi & \longrightarrow & D(\xi) \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & X & \longrightarrow & Y & & \end{array}$$

commutes, so it is not hard to show the top row is a degree one map. $U_{\nu_X} \cap [D(\nu_X)] = [X]$; $U_\xi \cap [D(\xi)] = [Y]$; and $f^*U_\xi = U_{\nu_X}$ and $f\,[X] = [Y]$. Hence the top row must take $[D(\nu_X)]$ to $[D(\xi)]$. $D(\nu_X)$ has the proper homotopy type of a parallelizable manifold (n+1)-ad, N , so there is a degree 1 map $g: N \to D(\xi)$. Since N is parallelizable, there is a topological microbundle over $D(\xi)$ which pulls back to the normal bundle of N (namely the trivial bundle). If $\dim \xi \geq 2$ (which we may always assume) then any pair $(D(\xi(Z), C(\xi(Z))$, for $Z \subseteq Y$ as part of the n-ad structure on Y , is properly 2-connected. Hence by the remarks following Theorem 3.1.2, we can find a parallelizable M and a degree one proper homotopy equivalence $M \to D(\xi)$. By Lemma 3, $\xi \cong \nu_Y$. □

Remarks: Logically Theorem 13 should follow Theorem 2 in Chapter 3. We do not use the result until we are past that point, so it does no harm to include it here.

The chief purpose of Theorem 13 is to severely limit the bundles which can occur in a surgery problem.

Section 3. The normal form for Poincaré duality spaces.

In order to get a good theory of surgery, one needs to be able to do surgery on Poincaré duality spaces; at least one must be able to modify fundamental groups. The results of this section show that Poincaré duality spaces look like manifolds through codimension 1. These results are a direct generalization of Wall [39] Section 2, expecially pages 220-221.

<u>Definition</u>: Let X be a Poincaré duality n-ad. Then, if $[X] \in H_n^{\ell.f.}$, X is said to have formal dimension n. (X is often said to be of dimension n.)

<u>Theorem 1</u>: Let X be a Poincaré duality space of dimension $n \geq 2$. Then X satisfies Dn. If X is a connected Poincaré duality m-ad, $m \geq 2$, of dimension $n \geq 3$, then X satisfies $D(n-1)$.

<u>Proof</u>: This follows easily from definitions and Theorem 1.6.2. □

Theorem 2: Let X be a Poincaré duality space of dimension n, $n \geq 4$. Then X has the proper homotopy type of Y, where Y is a Poincaré duality space which

is the union of two Poincare duality pairs $(Z,\partial H)$ and $(H,\partial H)$, where H is a smooth manifold of dimension n formed from a regular neighborhood in R^n of a given tree for Y by adding 1 handles along the boundary, and where Z is a subcomplex satisfying $D(n-2)$. The torsion of this equivalence may be made to assume any preassigned value. The map induced by inclusion $\Delta(H:\pi_1) \to \Delta(Y:\pi_1)$ is surjective.

<u>Proof</u>: Let $\hat{C}_*$ be the dual chain complex X. There is a chain map $\cap[X]: \hat{C}_* \to C_*(X)$. By Theorem 1.6.3, we can find a complex Y with $C_*(Y) = \hat{C}_*$ in dimensions greater than 3. $C_3(Y) = \hat{C}_3 \oplus$ junk, and the complex $Y^2 \cup$ junk satisfies $D2$.

Now we could have arranged things so that the only vertices of X were the vertices of the tree. This is seen as follows. First we claim we can find a subcomplex $V \subseteq X$ which contains all the vertices and such that $T \subseteq V$ as a proper deformation retract. We do this as follows. Let $\mathscr{A} = \{U|\ U$ is a 1-dimensional subcomplex of X, $T \subseteq U$, and U contains all the vertices of $X\}$. $\mathscr{A} \neq \emptyset$ as $X^1 \in \mathscr{A}$. $\mathscr{A}$ is ordered by inclusion. Let $U_1 \supset U_2 \supseteq \cdots$ be a totally ordered sequence in $\mathscr{A}$. Then $\cap U_i$ is also in $\mathscr{A}$. Let V be a minimal element of $\mathscr{A}$, which exists by Zorn. We claim $H_1(V) = 0$, so if not look at a cycle in V. At least one of the 1-simplexes of the cycle is not in T for T has no 1-cycles. Let $V_1 \subseteq V$

be all of V less one of the 1-simplexes in the cycle which is not in T. Then V_1 is a subcomplex, $T \subseteq V_1$, and V_1 contains all the vertices. This contradiction shows $H_1(V) = 0$. The inclusion $T \subseteq V$ is a proper 0-equivalence, and $\Delta(T:\pi_k) = \Delta(V:\pi_k) = 0$ for $k \geq 1$. Hence $T \subseteq V$ is a proper deformation retract.

Set $K = \overline{V - T}$, and look at X/K. The collapse map $X \to X/K$ is a proper homotopy equivalence. For a proof see [6] Proposition 2.11, page 220. Note that all the maps there may be taken to be proper. X/K has only the vertices of the tree for 0-cells.

Now, to return to our proof, we may assume $\hat{C}_n = C_0(X)$ has a generator for each vertex of the tree. $\hat{C}_{n-1}$ has a generator for each one cell of X. As in Wall [39] Corollary 2.3.2, each (n-1)-cell is incident to either two n-cells, or to the same n-cell twice. Look at an attaching map $S^{n-1} \to X^{n-1}$ for an n-cell. This can be normalized to take a finite, disjoint, collection of discs, onto the (n-1)-cells homeomorphically and to take the rest of S^{n-1} into the (n-2)-skeleton. Each (n-1)-cell eventually gets just two such discs mapped onto it. The n-discs together with the (n-1)-cells corresponding to the 1-cells of the tree are seen to form a regular neighborhood in R^n of the tree, and H is obtained from this by attaching 1-handles.

If Z is the part of Y in dimensions $\leq n-2$ (or is $Y^2 \cup$ junk if $n = 4$) $Y = Z \cup_{\partial H} H$ where H is formed from n-discs corresponding to the n-cells by attaching 1-handles as indicated by the (n-1)-cells. Actually, we want to form the mapping cylinder of $\partial H \to Z$ and then take the union along ∂H. Since H is a manifold, the result clearly is homeomorphic to Y. We denote by Z the mapping cylinder, so $Y = Z \cup_{\partial H} H$, and ∂H is a subcomplex of Z. Note that Z still satisfies D(n-2).

Now $Z \subseteq Y$ is at least properly 2-connected, for Z always contains the 2-skeleton of Y. Since $(H, \partial H)$ is a Poincaré duality space, Theorem 2.1.4 says $(Z, \partial H)$ satisfies Poincaré duality with respect to the covering functor induced from the universal covering functor for Y. But this is just the universal covering for Z as $Z \subseteq Y$ is properly 2-connected. ∂H is a Poincaré duality space, so Theorem 2.2.7 says $(Z, \partial H)$ is a Poincaré duality pair.

The statement about the torsion is contained in Theorem 1.6.3, so we finish by showing $\Delta(H:\pi_1) \to \Delta(Y:\pi_1)$ is onto. Our proof is basically Wall [39] Addendum 2.3.3, but is more complicated. We too will use the construction of Z and H via dual cell decomposition. In our original complex, there were 0-cells e_0^p, one for each p a vertex of T. There were 1-cells e_1^i satisfying $\partial e_1^i = g_i e_0^p - e_0^q$, where g_i is a loop at p. The g_i which occur generated $\Delta(Y:\pi_1)$. In the dual complex we have n-cells, e_n^p, and (n-1)-cells e_{n-1}^i with $\partial e_n^p = \sum_i (\pm g_i e_{n-1}^i) - \sum_j e_{n-1}^j$, where the sign is given by the local coefficients on Y , and where the sums run over all (n-1)-cells incident to e_n^p . The core 1-disc of the handle corresponding to e_{n-1}^i followed by the unique minimal path in T from the endpoint of the 1-disc to its initial point has homotopy class g_i . Hence $\Delta(H:\pi_1)$ is onto $\Delta(X:\pi_1)$. □

Corollary 2.1: Let X be a Poincaré duality space of dimension 3. Then X has the proper homotopy type of Y , where Y is the union of two Poincaré duality pairs $(Z,\partial H)$ and $(H,\partial H)$, where H is a regular neighborhood in R^3 of a given tree for X , and is a subcomplex of Y satisfying D2 . The torsion of this equivalence can be arbitrary.

Proof: Using the dual cell decomposition as before, let Z be the subcomplex of Y such that $\hat{C}_3 = C_3(Y,Z)$ and such that Z satisfies D_2 . $\hat{C}_3$ has one 3-cell for each vertex of the tree. Now there is a locally finite collection of paths from each n-cell to the vertex of the tree it represents.

Given H , a regular neighborhood of the tree in R^3 , we describe a map $\partial H \to Z$ which extends to a map $H \to Y$ such that the induced map $C(H,\partial H) \to C(Y,Z)$ is an isomorphism. Hence $Z \cup_{\partial H} H$ has the proper homotopy of Y and we will be done. The map is the following. H can be viewed as the connected sum of a collection of n-discs, one for each vertex of the tree, by tubes corresponding to the 1-cells of the tree. H can then be properly deformed to the subcomplex consisting of n-discs joined by the cores of the connecting tubes. ∂H under this deformation goes to a collection on (n-1)-spheres joined by arcs. Map the (n-1)-sphere to Z by the attaching map of the corresponding n-cell in Y . Map an arc between two such spheres to the paths to the tree, and then along the unique minimal path in the tree between the two vertices. This map clearly has the necessary properties. □

<u>Theorem 3</u>: Let $(X,\partial X)$ be a Poincaré duality pair of dimension n , $n \geq 4$. Then $(X,\partial X)$ has the proper homotopy type of a Poincaré duality pair $(Y,\partial Y)$ which is the union of a Poincaré duality pair $(Z, \partial H \cup \partial Y)$ and a Poincaré duality pair $(H,\partial H)$, where H is a regular neighborhood in R^n of any given tree for Y by adding 1-handles along the boundary, and Z is a subcomplex of Y satisfying D(n-1). The torsion of this equivalence may be given any preassigned value. $\Delta(H:\pi_1) \to \Delta(Y:\pi_1)$ is onto.

<u>Proof</u>: By Theorem 2 or Corollary 2.1, we may assume ∂X already looks like $K \cup M$, where M is a regular neighborhood for a tree of ∂X in R^{n-1} , and K satisfies D(n-2) .

Let $\hat{C}_*$ be the dual complex for $C_{n-*}(X)$. Then there is a chain map $\cap[X]: \hat{C}_* \to C_*(X,\partial X)$. We apply

Theorem 1.6.4 to find a complex Y with $C_*(Y) = C_*$ in dimensions greater than 3 and with $\partial X \subseteq Y$. $C_3(Y) = C_3 \cup \text{junk}$. Set L to be $Y^{(n-1)}$. Then $M \subseteq L$. Normalize the attaching maps for the n-cells as before. If $Z = Y^{(n-2)} \cup M$ ($Y^2 \cup \text{junk} \cup M$ if $n = 4$), then $Y = Z \cup H$ where H has the advertised description. Notice $\partial H \cap \partial X$ can be M if one likes. As before, $(H,\partial H)$ is a Poincaré duality pair, so one shows $(Z, \partial H \cup \partial X)$ is a Poincaré duality pair. $\partial H \cap \partial X = M$, so $\partial H = (\overline{\partial H - M},\partial M) \cup (M,\partial M)$ and $\partial X = (K,\partial M) \cup (M,\partial M)$. The rest of the theorem proceeds as in Theorem 2. $\square$

Appendix: *The cwation of a spherical fibration.*

We recall the definition. Let ξ be a spherical fibration over a finite dimensional, locally finite CW n-ad. Assume $\dim \xi \geq 2$. Let $S(\xi)$ be the total space. We seek an n-ad Y, a proper map $f : Y \to X$, and maps $S(\xi) \underset{h}{\overset{g}{\rightleftarrows}} Y$ which commute with the two projections. We also require that Y have the proper homotopy type of a locally compact, finite dimensional CW n-ad. $g \circ h$ must be properly homotopic to the identity, and $h \circ g$ must be fibre homotopic to the identity. We give Y a simple homotopy type by finding an equivalent CW complex for which the Thom isomorphism is simple.

We digress briefly to include a discussion of the Thom isomorphism. If $D(\xi)$ is the total space of the disc fibration associated to ξ, we define $\Delta(D(\xi):h:\sim)$ and $\Delta(D(\xi),S(\xi):h:\sim)$ to be the groups one gets by applying the Δ construction to the groups $h(\pi^{-1}(\widetilde{X-C}),\hat{p})$ for $D(\xi)$ and $h(\pi^{-1}(\widetilde{X-C}),\rho^{-1}(\widetilde{X-C}),\hat{p})$ for $(D(\xi),S(\xi))$, where p is a vertex of X, $X-C$ is a cofinal subcomplex of X, $\hat{p}$ is a lift of p into $\widetilde{D(\xi)_c}$, and $\pi : \widetilde{D(\xi)_c} \to \widetilde{X-C}$ and $\rho : \widetilde{S(\xi)_c} \to \widetilde{X-C}$ are the projections for the fibrations induced over $\widetilde{X-C}$ by restriction and pull back from $D(\xi)$ and $S(\xi)$ respectively.

Now the Thom class for ξ, U_ξ goes under $\widetilde{X-C} \to X-C \to X$ to the Thom class for $\widetilde{S(\xi)_c}$. If h is cohomology we modify the groups above in the obvious manner. We will denote by $\Delta_*(D(\xi):\sim)$ the $*$-th homology group with covering functor $\sim$. Δ^* is the cohomology theory. Then we have maps $U_\xi \cup : \Delta^*(D(\xi):\sim) \longrightarrow \Delta^{*+h}(D(\xi),S(\xi):\sim)$ and $U_\xi \cap : \Delta_{*+k}(D(\xi),S(\xi):\sim) \longrightarrow \Delta_*(D(\xi):\sim)$. They are easily seen to be isomorphisms. The maps h and g induce isomorphisms of $\Delta_*(S(\xi):\sim)$ and $\Delta_*(Y:\sim)$, with a similar result for cohomology (the reader should have no trouble defining $\Delta_*(S(\xi):\sim)$ or its cohomology analogue). We also get isomorphisms of $\Delta_*(D(\xi),S(\xi):\sim)$ and $\Delta_*(M_f,Y:\sim)$, again with a similar result in cohomology. Hence we can speak of a Thom isomorphism for the cwation.

We first prove that if we can find a cwation, we can give it a unique simple homotopy type. Let C be a CW n-ad the proper homotopy type of the cwation Y (C locally compact, finite dimensional).

$$\begin{array}{ccc} C & \xrightarrow{\rho} & Y \\ & {\scriptstyle f\circ\rho}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

where ρ is a proper homotopy equivalence with $f\circ\rho$ cellular. (It is easy to find such ρ.) Let τ_ρ denote the torsion of the corresponding Thom homology isomorphism. If $\lambda: K \to C$ is a proper homotopy equivalence, the Thom isomorphism associated to $\rho\circ\lambda$ has torsion $\tau_\rho + \rho_*\tau(\lambda)$ by Lemma 1.5.22. Since we may pick $\tau(\lambda)$ arbitrarily, we can find a ρ with $\tau_\rho = 0$.

Suppose now we have $\lambda : K \to Y$ with $\tau_\lambda = 0$, $f\circ\lambda$ cellular. Let $a : Y \to K$ be a proper homotopy inverse to λ. Then

$$\begin{array}{ccc} C & \xrightarrow{a\circ\rho} & K \\ & {\scriptstyle f\circ\rho}\searrow \quad \swarrow{\scriptstyle f\circ\lambda} & \\ & X & \end{array}$$

properly homotopy commutes. We get a proper homotopy equivalence of pairs $F:(M_{f\circ\rho},C) \to (M_{f\circ\lambda},K)$ such that

$F|C = a\circ\rho$, and
$$\begin{array}{ccc} M_{f\circ\rho} & \xrightarrow{F} & M_{f\circ\lambda} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$
commutes.

By Lemma 1.5.19, $M_{f\circ\rho} \to X$ and $M_{f\circ\lambda} \to X$ are simple, so $F : M_{f\circ\rho} \to M_{f\circ\lambda}$ is a simple equivalence. The torsion of F from $(M_{f\circ\rho},C)$ to $(M_{f\circ\lambda},K)$ is $\tau_\rho - \tau_\lambda = 0$, so by Theorem 1.5.1, the torsion of $a\circ\rho$ on the subspace groups is zero. But as $\dim \xi \geq 2$, $f\circ\rho$ and $f\circ\lambda$ are at least properly 2-connected. Hence the subspace groups with the induced covering functor are the absolute groups with the universal covering functor. Hence $a \circ \rho$ is a simple homotopy equivalence, so the simple homotopy type of a cwation is unique.

We now construct the promised Y. Notice first that we can replace X by any locally compact, finite dimensional CW complex of the same proper homotopy type. Hence we may as well assume X is a locally finite simplicial complex of finite dimension. This is seen as follows. By [11] Theorem 4.1 and Lemma 5.1, X is the union of A and B where A and B are the disjoint union of finite complexes. Each finite complex has the homotopy type of a finite simplicial complex, and if a subcomplex is already simplicial, we need not disturb it. Hence we get a locally finite simplicial complex Y and a map $f: X \to Y$ by making subcomplexes of the form $C \cap D$, $C \in A$, $D \in B$ simplicial, and then making C and D simplicial. Then $Y = A' \cup B'$ where $f: A \to A'$ and $f: B \to B'$ are proper homotopy equivalences. Also $f: E \to E'$ is a proper homotopy equivalence where $E = \{C \cap D \mid C \in A,\ D \in B\}$. The proper Whitehead Theorem shows f is a proper homotopy equivalence. X being what it is, we can subdivide X until we find open sets C_i such that $X - C_i$ and $\overline{C}_i$ are subcomplexes, each $\overline{C}_i$ is compact, and $\xi | \overline{C}_i$ is trivial. Furthermore, $\cup C_i = X$, the C_i are locally finite, and the C_i are indexed by the positive integers. We set $V_i = \bigcup_{j \leq i} C_j$. We can also find an increasing collection of open sets U_i such that $U_i \subseteq V_i - C_i$, $\overline{U}_i$ is compact, and $\bigcup_i U_i = X$.

We first construct spaces Y_i and maps g_i and f_i inductively so that

$$\text{A)} \qquad \begin{array}{ccc} \xi|\overline{V}_i & \xrightarrow{\;g_i\;} & Y_i \\ & {\scriptstyle \pi|\overline{V}_i} \searrow \quad \swarrow {\scriptstyle f_i} & \\ & \overline{V}_i & \end{array}$$

commutes.

Let $Y_1 = \overline{V}_1 \times S^k$, $k = \dim \xi \geq 2$. g_1 and f_1 exist since $\xi|V_1$ is trivial. f_1 is just projection. We now induct; i.e. we have

1) A space Y_i and maps g_i and f_i such that A) commutes.

2) g_i is a homotopy equivalence.

3) $Y_i = Y_{i-1} \cup_\rho \overline{C}_i \times S^k$ via some homotopy equivalence $\rho: Y_{i-1} \cap f_{i-1}^{-1}(\overline{V}_{i-1} \cap \overline{C}_i) \to (\overline{V}_{i-1} \cap \overline{C}_i) \to S^k$.

4) $g_{i-1}|f_{i-1}^{-1}(U_{i-1}) = g_i|f_{i-1}^{-1}(U_{i-1})$ $f_{i-1}|Y_{i-1} = f_i|Y_{i-1}$.

5) Let $\mathcal{A}_r = \{\overline{C}_{i_1} \cap \overline{C}_{i_2} \cap \cdots \cap \overline{C}_{i_r} \mid i_1 < i_2 < \cdots < i_r\}$. If $C \in \mathcal{A}_r$, g_i restricted to $f_i^{-1}(C \cap \overline{V}_i)$ is a homotopy equivalence.

Notice that Y_1, g_1, and f_1 satisfy (1-5). (Let $Y_0 = \emptyset$.)

If we can verify 1) - 5), we can construct Y as the increasing union of Y_i with identifications. g and f can be defined from the g_i and f_i respectively by 4).

Intuitively, Y has the proper homotopy type of a locally compact, finite dimensional complex, since it is covered by finite complexes, $\overline{C}_i \times S^k$, of bounded dimension in a locally finite fashion. For a better proof, see Proposition 1.

Now given Y_{i-1}, f_{i-1}, and g_{i-1}, we construct Y_i, f_i, and g_i.

By Dold [8], $\xi|\overline{V}_i$ can be gotten from $\xi|\overline{V}_{i-1}$ and $\xi|\overline{C}_i$ as follows. Over $\overline{C_i \cap V_{i-1}}$, we have an equivalence $\varphi : (\xi|\overline{V}_{i-1})|\overline{C_i \cap V_{i-1}} \longrightarrow (\overline{C_i \cap V_{i-1}}) \times S^k$. Let $H_1 = \xi|\overline{V}_{i-1}$, $H_2 = \overline{C}_i \times S^k$, and let $H_3 = \{(x,w) | x \in H_1 | \overline{C_i \cap V_{i-1}}$, $w \in ((\overline{C_i \cap V_{i-1}}))^I$, $\pi(x) = \pi(w(t))$ for all $t \in I$, $\varphi(x) = w(1)\}$. Then $\xi|\overline{V}_i \cong H_1 \cup H_3 \cup H_2$, where $H_1|\overline{C_i \cap V_{i-1}}$ is embedded in H_3 via $x \longrightarrow (x$, constant path at $\varphi(x))$. The embedding of $H_2|\overline{C_i \cap V_{i-1}}$ is harder to describe. Let φ' be the inverse to φ. Then $\varphi \circ \varphi'$ is fibre homotopic to the identity. Let ψ be a fibre homotopy between these two maps, with $\psi(\ ,0) = \mathrm{id}$. Then $H_2|\overline{C_i \cap V_{i-1}}$ is embedded in H_3 via $x \longrightarrow (\varphi'(x), \psi(x,t))$.

We must now define the ρ in 3). We are given

$$\text{B)}\qquad \begin{array}{ccc} H_1|(\overline{C_i \cap V_{i-1}}) & \xrightarrow{\ \varphi\ } & (\overline{C_i \cap V_{i-1}}) \times S^k \\ \downarrow g_{i-1} & & \downarrow \mathrm{id} \\ Y_{i-1}|f_{i-1}^{-1}(\overline{C_i \cap V_{i-1}}) & \dashrightarrow & (\overline{C_i \cap V_{i-1}}) \times S^k \\ \downarrow f_{i-1} & & \downarrow \mathrm{proj} \\ \overline{C_i \cap V_{i-1}} & \xrightarrow{\ \mathrm{id}\ } & \overline{C_i \cap V_{i-1}} \end{array}$$

.

We would like to fill in the dotted arrow with ρ so that the diagram actually commutes. To do this, we may have to alter φ within its fibre homotopy class, but this will not change our bundle.

Since g_{i-1} is a homotopy equivalence, it has an inverse, h. h may be assumed to be a fibre map, so $h \circ g_{i-1}$ is a fibre homotopy equivalence. Let G be its fibre homotopy inverse. Then $G \circ h \circ g_{i-1}$ is fibre homotopic to the identity. $g_{i-1} \circ (G \circ h)$ is homotopic to the identity.

Set $\rho = (id) \circ \varphi \circ (G \circ h)$. Then ρ is a fibre map so the bottom square commutes. Set $\varphi_1 = (id)^{-1} \circ \rho \circ g_{i-1}$. Then φ_1 is fibre homotopic to φ, and B) commutes with φ_1 in place of φ. ρ is a homotopy equivalence, so 3) is satisfied.

From now on, we assume φ chosen so that B) commutes with the ρ along the dotted arrow. Set $Y_i = Y_{i-1} \cup_\rho \overline{C}_i \times S^k$. f_i is defined by $f_i | Y_{i-1} = f_{i-1}$ and $f_i | \overline{C}_i \times S^k = \text{proj}$. B) assures this is well defined on the intersection.

g_i is unfortunately harder to define. $\xi | \overline{V}_i \cong H_1 \cup H_2 \cup H_3$, so let $\alpha : \xi | \overline{V}_i \to H_1 \cup H_2 \cup H_3$ be an equivalence. α may be chosen to be the identity on $\xi | U_{i-1}$. We define a map $h : H_1 \cup H_2 \cup H_3 \to Y_i$ as follows. $g | H_1 = g_{i-1}$. To define g on the other two pieces, look at ψ, the fibre homotopy between $\varphi \circ \varphi'$

and id. This can be extended to a fibre map of $(\overline{C}_i \times S^k) \times I \longrightarrow (\overline{C}_i \times S^k)$ since id: $(\overline{C_i \cap V_{i-1}}) \times S^k \longrightarrow (\overline{C_i \cap V_{i-1}}) \times S^k$ can clearly be extended. Let F be the fibre map which extends $\varphi \circ \varphi'$. Note F is fibre homotopic to the identity.

Now define $(g|H_3)(x,w) = g_{i-1}(x)$. Note our two definitions agree on $H_1 \cap H_3$. We could have defined $(g|H_3)(x,w) = w(1)$ equally well. We define $(g|H_2)(x) = F(x)$. If $x \in H_2 \cap H_3$, then $(g|H_3)(x) = (g|H_3)(\varphi'(x), \psi(x,t)) = \psi(x,1) = \varphi \circ \varphi'(x)$. $(g|H_2)(x) = F(x) = \varphi \circ \varphi''(x)$ by the definition of F. Hence g is well defined, and we set $g_i = g \circ \alpha$.

Now 4) clearly holds since $\alpha | f_{i-1}^{-1}(U_{i-1})$ is the identity. 1) holds as $g: H_1 \cup H_3 \cup H_2 \longrightarrow Y_i$ preserves fibres by construction. Hence we are left with showing 2) and 5).

For r sufficiently large, $C \in \mathcal{A}_r$ implies $C \cap \overline{V}_i = \emptyset$, since the collection $\{C_i\}$ is locally finite. We show 5) by downward induction on r, since if $C \cap \overline{V}_i = \emptyset$, 5) is obvious. Assume we have established the result for $r = k+1$. Let $C \in \mathcal{A}_k$. If $C \cap \overline{C}_i = \emptyset$, then $C \cap \overline{V}_{i-1} = C \cap \overline{V}_i$ and we are done since 5) holds for g_{i-1} and α is a fibre homotopy equivalence. If $C \cap \overline{V}_i = \overline{C}_i \cap \overline{V}_i$ we are done since F is a fibre map. So let $L = C \cap \overline{V}_{i-1}$, and let $K = C \cap \overline{C}_i$ with both K and L non-empty. $g_i | f_i^{-1}(L)$ is a homotopy equivalence, again since α is a fibre homotopy equivalence and $g_{i-1}|f_i^{-1}(L)$ is. $g_i|f_i^{-1}(K)$ is also a homotopy equivalence, again since F is a fibre homotopy equivalence. $K \cap L \subseteq \overline{V}_{i-1}$, and $K \cap L \in \mathscr{S}_{k+1}$. Hence $g_i|f_i^{-1}(K \cap L)$ is a homotopy equivalence. Therefore $g_i|f_i^{-1}(C)$ is a homotopy equivalence and we are done with 5).

For 2), note that g_{i-1} is a homotopy equivalence, so $g_i|Y_{i-1}$ is. $g_i|f_i^{-1}(\overline{C}_i)$ is since F is fibre homotopic to the identity. $g_i|f_i^{-1}(\overline{C_i \cap V_{i-1}})$ is a homotopy equivalence by 5). Hence g_i is a homotopy equivalence.

Therefore we have a space Y and maps

$$\begin{array}{ccc} S(\xi) & \xrightarrow{g} & Y \\ & \pi\searrow \quad \swarrow f & \\ & X & \end{array}$$

We claim g is a homotopy equivalence. Since by Milnor [22], $S(\xi)$ has the homotopy type of a CW complex, this is equivalent to showing g induces isomorphisms in homotopy. But $\pi_k(g) = \varinjlim_i \pi_k(g_i)$, and since $\pi_k(g_i) = 0$, $\pi_k(g) = 0$.

Let $h : Y \to S(\xi)$ be a homotopy inverse for g. By an easy argument like the one after diagram B, we may assume h preserves fibres and that $h \circ g$ is fibre homotopic to the identity. Notice that by construction $f^{-1}(x)$ is homeomorphic to a sphere of dimension $\dim \xi$. $\pi^{-1}(x)$ has the homotopy type of such a sphere. Since $h \circ g$ is fibre homotopic to the identity, $g_x : \pi^{-1}(x) \to f^{-1}(x)$ has a left inverse. As both spaces are spheres of dimension 2 or more, g_x is a homotopy equivalence.

Now in the terminology of Bredon [2], f is ψ-closed, and $f^{-1}(x)$ is ψ-taut, where ψ is the family of compact supports. (Note Y is locally compact, so ψ is paracompactifying, and then apply (d) on page 52 to show $f^{-1}(x)$ is ψ-taut. f is ψ-closed easily from the definition, which is on page 53, since X is Hausdorff.) Hence we have a Leray spectral sequence for the map $f : Y \to X$. We have the Serre sequence for $\pi: S(\xi) \to X$, and g induces a map between these two. g induces an isomorphism on the E_2 terms since it is a homotopy equivalence on each fibre. Hence $g : H_c^*(Y) \longrightarrow H_\varphi^*(S(\xi))$ is an isomorphism, where φ is the set of supports whose image in X is compact.

As $\dim \xi \geq 2$, $\pi^*: H_c^*(X) \longrightarrow H_\varphi^*(S(\xi))$ is an isomorphism for $* < 2$. Hence $f^*: H_c^*(X) \longrightarrow H_c^*(Y)$ is an isomorphism for $* < 2$, so $f^*: H_{end}^0(X) \longrightarrow H_{end}^0(Y)$ is an isomorphism, so f is a proper 0-equivalence.

We claim f is a proper 1-equivalence. To see this, note $f|C$ is a 1-equivalence for $C \in \mathscr{S}_r$ all $r \geq 1$. Now an easy van-Kampen induction shows f is a 1-equivalence when restricted to any union of $\overline{C}_i$'s. Hence f is a proper 1-equivalence.

Thus $g_\#: \Lambda(S(\xi):\pi_1) \longrightarrow \Lambda(Y:\pi_1)$ is an isomorphism as both groups are isomorphic, via $\pi_\#$ and $f_\#$, to $\Lambda(X:\pi_1)$.

Now we still have maps

$$\begin{array}{ccc} Y - f^{-1}(K_i) & \underset{g}{\overset{h}{\rightleftarrows}} & S(\xi | X-K_i) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X - K_i & \end{array} \qquad \text{where} \quad K_i = X - \bigcup_{j \geq i} \overline{C}_i$$

g restricted to each fibre is still a homotopy equivalence with inverse induced from h. For any cover, $\sim$, of $X - K_i$, we get

$$\begin{array}{ccc} \widetilde{Y - f^{-1}(K_i)} & \underset{\tilde{g}}{\overset{\tilde{h}}{\rightleftarrows}} & \widetilde{S(\xi | X - K_i)} \\ & {\scriptstyle \tilde{f}}\searrow \quad \swarrow{\scriptstyle \tilde{\pi}} & \\ & \widetilde{X - K_i} & \end{array}$$

where the covers on the top row are the induced covers from $\sim$ over $X - K_i$. $\widetilde{S(\xi | X - K_i)}$ is the same as $S(\tilde{\xi} | \widetilde{X - K_i})$, the spherical fibration induced from $\xi | X - K_i$ over $\widetilde{X - K_i}$. $\tilde{g}$ likewise induces a homotopy equivalence of fibres, so as before we get

$$h^* : H^*_{\varphi}(\widetilde{S(\xi | X - K_i)}, \widetilde{S(\xi | \partial(X - K_i))}) \longrightarrow H^*_c(\widetilde{Y - f^{-1}(K_i)}, \widetilde{\partial(Y - f^{-1}(K_i))})$$

is an isomorphism. A word about the existence of these covering spaces is in order. Since $X - K_i$ is a CW complex, its cover exists. The cover for $S(\xi | X - K_i)$ then also clearly exists. We claim $Y - f^{-1}(K_i)$ is semi-locally 1-connected, from which it follows that its cover also exists. To see our claim, observe $f : Y - f^{-1}(K_i) \longrightarrow X - K_i$ is a 1-equivalence. Given any point $y \in Y - f^{-1}(K_i)$, let $N \subseteq X - K_i$ be a neighborhood of $f(y)$ such that $\pi_1(N) \longrightarrow \pi_1(X - K_i)$ is the zero map.

Since $X - K_i$ is semi-locally 1-connected, such an N exists. Now $f^{-1}(N)$ is a neighborhood for y, and $\pi_1(f^{-1}(N) \to \pi_1(Y - f^{-1}(K_i))$ is also zero. Hence $Y - f^{-1}(K_i)$ is semi-locally 1-connected.

Therefore, $h^*: \Delta^*(S(\xi):\sim) \to \Delta^*(Y:\sim)$ is an isomorphism for any covering functor induced from one over X. Since f is a proper 1-equivalence, if we take a universal covering functor for X, we get one for Y. (The actual covering functor on Y is the following. Any $A \in \mathcal{C}(Y)$ is contained in a unique minimal $f^{-1}(X - K_i)$, so let the cover over A be induced from the cover over this space.)

$g^*: \Delta^*(Y:\sim) \to \Delta^*(S(\xi):---)$ is defined where $---$ is the covering functor indiced by g from $\sim$ over Y. $g^* \circ h^* = (g \circ h)^*: \Delta^*(S(\xi):\sim) \to \Delta^*(S(\xi):---)$ is an isomorphism as $\sim$ and $---$ are equivalent covering functors. Hence $h \circ g = (h \circ g)^* : \Delta^*(Y:\sim)\circlearrowleft$ is an isomorphism, so $h \circ g$ is a proper homotopy equivalence. $g \circ h$ is already a fibre homotopy equivalence, and it is not hard to change h until $h \circ g$ is properly homotopic to the identity and $g \circ h$ is fibre homotopic to the identity.

To finish we need only show Proposition 1 below. We first need

<u>Theorem 1</u>: Let Y be a locally compact, separable ANR. Then Y is properly dominated by a locally-finite simplicial complex.

<u>Proof</u>: Let α be an open covering of Y by sets where closure is compact. Since Y is metrizable, Y is paracompact, so we can assume α is locally finite.

We now apply Hu [15], Theorem 6.1, page 138, to get a locally finite simplicial complex X and maps $\varphi: X \to Y$ and $\psi: Y \to X$ with $\varphi \circ \psi$ α-homotopic to the identity, i.e. if H is the homotopy, for each $y \in Y$, there exists $U \in \alpha$ such that $H(y,t) \in U$ for all $t \in [0,1]$. By our choice of α, $\varphi \circ \psi$ is properly homotopic to the identity.

Now X is actually the nerve of some cover δ in the proof of Hu, Theorem 6.1. In the proof, we may take δ to be star-finite and locally finite. Then the nerve X is a locally finite simplicial complex, and the map $\varphi: X \to Y$ is proper. To see this last statement, it is enough to show $\varphi'(U)$ is contained in a compact subset of X for any $U \in \alpha$. Recall φ is defined by picking a point in each $V \in \delta$ and sending the vertex of the nerve which corresponds to V to our chosen point and then extending. Our extension satisfies the property that any simplex lies entirely in some element of α. So let U_1 be the union of all elements of α intersecting U. $\overline{U}_1$ is compact as α is locally finite, so let U_2 be the union of all elements of α intersecting $\overline{U}_1$. $\overline{U}_2$ is again compact, so there are only finitely many elements of δ which intersect U_2. Let $K \subseteq X$ be the subcomplex generated by these elements of δ . K is finite, hence compact, and $\varphi^{-1}(U) \subseteq K$. □

Corollary 1.1: Let Y be a locally compact, separable ANR , and suppose the covering dimension of Y , dim Y , is finite (see Hurewicz and Wallman [16] for a definition). Then Y is properly dominated by a locally finite simplicial complex of dimension dim Y .

Proof: Make the same changes in Hu [15] Theorem 6.1, page 164 that we made to the proof of Theorem 6.1, page 138. We get a simplicial complex P and a proper map $\varphi : P \to Y$ such that for any map $f : X \to Y$ with X a metric space of dimension $\leq$ dim Y , there exists a map $\psi : X \to P$ with $\varphi \circ \psi$ α-homotopic to f . Moreover, P has no simplices of dimension > dim Y . Apply this for $X = Y$, $f = \mathrm{id}$. □

Corollary 1.2. A locally compact, separable ANR of dimension $\leq n$ satisfies Dn .

Proof: By Corollary 1.1 and nonsense, it remains to show Y is homogamous. But an ANR is locally contractible (Hu [15], Theorem 7.1, page 96), and any metric space is paracompact, so Corollary 1.1.2.1 applies. □

Proposition 1: The space Y which we constructed has the proper homotopy type of a locally compact, finite dimensional CW complex.

Proof: We first show Y is a finite dimensional, locally compact, separable ANR. We then find a finite dimensional simplicial complex Z and a proper map $h : Z \to Y$ which is properly n-connected for any finite n. Since Y and Z both satisfy Dn for some finite n, h is a proper homotopy equivalence.

Step 1: Y is a finite dimensional, locally compact, separable ANR.

By Hu [15] Lemma 1.1, page 177, Theorem 1.2, page 178, and induction, each Y_i is an ANR. The induction is complicated by the necessity of showing

$Y_{i-1} \cap f_{i-1}^{-1}(\overline{V}_{i-1} \cap \overline{C}_i)$ is an ANR. Hence our induction hypothesis must be

a)$_k$ Y_k is an ANR

b)$_{k,r}$ $Y_k \cap f_k^{-1}(\overline{V}_k \cap C)$ is an ANR for all $C \in \mathscr{S}_r$.

One then shows that for some finite r, b)$_{k,r}$ holds vacuously. b)$_{k,s}$, $s > r$, and b)$_{k-1,r}$ imply b)$_{k,r}$, so we get b)$_{k,r}$ for all r. b)$_{k,1}$ and a)$_{k-1}$ imply a)$_k$, so we are done.

Since each Y_i is an ANR, each Y_i is a local ANR (Hu, Proposition 7.9, page 97). If Y is metrizable, Y is an ANR by Hu, Theorem 8.1, page 98. Now Y is T_1 and regular. To see this observe each Y_i is T_1 and regular since it is metrizable. Now if $U \subseteq Y$ is any compact set, there is a Y_i with $V \subseteq Y_i$ and V homeomorphic to U. With this result and the observation that Y is locally compact, it is easy to show Y is T_1 and regular. Y is locally compact because it has a proper map to the locally compact space X. Y is σ-compact since X is, so Y is second countable. Hence Y is metrizable (see Kelley [17] page 125) and separable.

We are left with showing Y has finite covering dimension. By Nagami [27] (36-15 Corollary, page 206), we need only show the small cohomological dimension with respect to the integers (Nagami, page 199) is finite (Y is paracompact since it is σ-compact and regular (see Kelley [17], page 172, exercise Y, a) and b))).

To compute $d(Y{:}\,Z)$, look at the map $f\colon Y \to X$. f is a closed, onto map. f is onto by construction, and f is closed since Y is the increasing union of compact sets $\{D_i\}$, $F \subseteq Y$ is closed iff $F \cap D_i$ is closed for all i, and $f(F \cap D_i)$ is closed since $F \cap D_i$ is compact and X is Hausdorff. We can find an increasing sequence of compact sets $\overline{V}_i$ such that $E \subseteq X$ is closed iff $E \cap \overline{V}_i$ is closed. Since f is proper, $D_i = f^{-1}(\overline{V}_i)$ has the expected properties. But $f(F \cap D_i) = f(F) \cap \overline{V}_i$ if $D_i = f^{-1}(\overline{V}_i)$, so f is closed. Hence by Nagami [27] (38-4 Theorem, page 216), $d(Y{:}Z) \leq \operatorname{Ind} X + k$, where k is the dimension of the bundle ξ. To see this, note $f^{-1}(x)$ is homeomorphic to S^k for all $x \in X$, so $d(f^{-1}(x){:}Z) = k$. Since X is paracompact and metrizable, Ind X = dim X = d(X:Z) = dimension of X as a CW complex (see Nagami 8-2 Theorem for the first equality; Nagami 36-15 Corollary shows the second; Nagami 37-12 Theorem and subdivision shows the third [this uses the fact that X is a regular complex]).

Step 2: There is a locally compact, finite dimensional CW complex Z and a proper map $h : Z \to Y$ which is properly n-connected for all n.

We define Z and h by induction; i.e. we have

1) a finite CW complex Z_i and a map $h_i : Z_i \to Y_i$

2) h_i is a homotopy equivalence

3) h_i restricted to $(f_i \circ h_i)^{-1}(C \cap \overline{V}_i)$ is a homotopy equivalence for all $C \in \mathscr{A}_r$, $r \geq 1$.

4) $h_i | (h_{i-1} \circ f_{i-1})^{-1}(U_{i-1}) = h_{i-1} | (h_{i-1} \circ f_{i-1})^{-1}(U_{i-1})$

5) $Z_i = Z_{i-1} \cup_\lambda \overline{C}_i \times S^k$ where $\lambda : Z_{i-1} \cap (h_{i-1} \circ f_{i-1})^{-1}(\overline{V}_{i-1} \cap \overline{C}_i) \to (\overline{V}_{i-1} \cap \overline{C}_i) \times S^k$ is a cellular homotopy equivalence.

If we can find such Z_i and h_i, we can find Z and $h : Z \to Y$. h is clearly proper. $h | (f \circ h)^{-1}(C) : (f \circ h)^{-1}(C) \to f^{-1}(C)$ is a homotopy equivalence by 3) for all $C \in \mathscr{A}_r$ $r \geq 1$, so $h | (f \circ h)^{-1}(D_i)$ is a homotopy equivalence where $D_i = \bigcup_{j \geq i} \overline{C}_i$. Thus h induces isomorphisms on H^0 and H^0_{end}, and $\Delta(h: \pi_s) = 0$ for $s \geq 1$. Hence we are done if we can produce Z_i and h_i.

We proceed by induction on i . $Z_1 = \overline{V}_1 \times S^k$ and $h_1 = id$. 1) - 5) are trivial, so suppose we have Z_{i-1} and h_{i-1} . We have

$$\begin{array}{c} Z_{i-1} \cap (f_{i-1} \circ h_{i-1})^{-1}(\overline{V}_{i-1} \cap \overline{C}_i) \\ \downarrow \\ Y_{i-1} \cap f_{i-1}^{-1}(\overline{V}_{i-1} \cap \overline{C}_i) \xrightarrow{\rho} (\overline{V}_{i-1} \cap \overline{C}_i) \times S^k \end{array}$$

Let ρ' be this composition. Deform ρ' to a cellular map as follows. For some $r \geq 0$, $C \in \mathscr{A}_r$ implies $C \cap \overline{C}_i = \varphi$. Now deform ρ' to a cellular map over each $C \cap \overline{C}_i \cap \overline{V}_{i-1}$ for $C \in \mathscr{A}_r$, all $r \geq 1$ and finally to a cellular map over $\overline{C}_i \cap \overline{V}_{i-1}$. Denote this map by λ .

Let $Z_i = Z_{i-1} \cup_\lambda (\overline{C}_i \times S^k)$. We can extend h_{i-1} to a homotopy equivalence $h_i\colon Z_i \to Y_i$ which leaves h_{i-1} fixed on $(f_{i-1} \circ h_{i-1})^{-1}(U_{i-1})$. h_i in fact can be chosen to be a homotopy equivalence on each $(f_i \circ h_i)^{-1}(C \cap \overline{V}_i)$ by extending inductively over the various $C \in \mathscr{A}_r$.

1) - 5) hold and we are done.

CHAPTER 3

The Geometric Surgery Groups

Section 1: The fundamental theorems of surgery

In this section we will prove three results which may be called the fundamental theorems of surgery. They constitute all the geometry needed to define surgery groups and to prove these groups depend only on the proper 1-type of the spaces in question. These results together with the s-cobordism theorem constitute the geometry necessary to give a classification of paracompact manifolds in a given proper homotopy class à la Wall [41], Chapter 10.

Let $\mathcal{C}$ denote either TOP, PL, or DIFF. If X is a locally finite, finite dimensional CW n-ad, and if ν is a $\mathcal{C}$-bundle over X, then $\Omega_m(X,\nu)$ is the space of cobordism classes of the following triples: a $\mathcal{C}$ manifold n-ad M, $\dim M = m$; a proper map of n-ads $f\colon M \to X$; a stable bundle map $F\colon \nu_M \to \nu$, where ν_M is the normal bundle of M and F covers f. Such a triple is called a normal map, and the cobordisms are called normal cobordisms.

<u>Theorem 1</u>: Given $\alpha \in \Omega_m(X,\nu)$, there is a representative (M,f,F) of α with f properly $[\frac{m}{2}]$-connected if X is a space. ([ ]) = greatest integer.)

For a pair $(X,\partial X)$, we have a representative

$((M,\partial M),f,F)$ with $f\colon M \to X$ properly $[\frac{m}{2}]$-connected; $f\colon \partial M \to \partial X$ properly $[\frac{m-1}{2}]$-connected; and the pair map $f\colon (M,\partial M) \to (X,\partial X)$ properly $[\frac{m}{2}]$-connected. If $\partial X \subseteq X$ is properly 0-connected, the map of pairs may be made properly homologically $[\frac{m+1}{2}]$-connected provided $m \geq 3$.

Proof: The proof follows Wall [40], Theorem 1.4. (See the remark following his proof.) We first remark that his Lemma 1.1 is equally valid in our case.

Lemma 1: Suppose M and X locally compact, finite dimensional CW complexes, $\psi : M \to X$ a map. Then we can attach cells of dimension $\leq k$ to M so that the resulting complex is locally finite and so that the map is properly k-connected.

Proof: We may assume ψ cellular by the cellular approximation theorem. Then the mapping cylinder of ψ is a locally compact, finite dimensional complex, and (M_ψ, M) is a CW pair. Set $M' = M_\psi^k \cup M$. Note then that M' is obtained from M by adding cells of dimension $\leq k$ and that the $M' \to M_\psi$ is properly k-connected. Q.E.D.

Now given a representative (N,g,G) for α, attach handles of dimension $\leq k$ to N to get $\psi : W \to X$ with $\partial W = N \cup M$, $\psi|N = g$, and with ψ covered by a bundle map which is G over N, and ψ is properly k-connected. The argument that we can do this is the same as for the compact case. Wall [41] Theorem 1.1 generalizes immediately to

Lemma 2: Given $\alpha \in \Omega_m(X,\nu)$ with any representative (M,f,F), any element of $\Delta(f\colon \pi_k)$ determines a proper regular homotopy class of immersions of a disjoint collection of $S^k \times D^{m-k}$ into M for $k \leq m-2 = \dim M-2$.

Proof: Precisely as in Wall, Theorem 1.1, we get a stable trivialization of the tangent bundle of M over each sphere S^k in our collection. Given any sphere S^k, we see in fact that there is an open submanifold $U \subseteq M$ such that we get a trivialization of the tangent bundle of U which agrees with the one for τ_M. In fact $U = f^{-1}$ (the disc bounding $f(S^k)$) will do (we have momentarily confused S^k with its image in M). Notice that we can pick such a collection of U's to be locally finite. Now apply Hirsch [14], Haefliger [12], or Lees [19] to immerse each S^k in its U with trivial normal bundle. This is a proper homotopy, so each α determines a proper map which immerses each sphere.

It is not hard to show any two such immersions which are properly homotopic are regularly properly homotopic. Q.E.D.

If there is an embedding in the proper regular homotopy class of α, we can attach a collection of handles by α and extend our map and bundle map over resulting trace of the surgeries. Notice that in an embedding, all the spheres have disjoint images, so we can certainly do the surgery. The map can be extended properly by construction, and one shows the bundle map extends precisely as in the compact case (Wall [41] Theorem 1.1).

Lemma 3: Given $\alpha \in \Omega_m(X,\nu)$ with any representative (M,f,F), we can do surgery on any element $\alpha \in \Delta(f:\pi_k)$ for $m > 2k$.

Proof: General position supplies us with an embedding. Q.E.D.

We now return to the proof of Theorem 1. By our lemmas, we see that if $m > 2k$, we can get W as advertized. Now W is obtained from M by adding handles of dimension $\geq (m+1)-k > k+1$, so $M \subseteq W$ is properly k-connected. Hence the map $M \to X$ is properly k-connected.

In the pairs case, given a representative, we first fix up the boundary as above. Then we can attach handles away from the boundary to get the absolute map fixed up.

The long exact homotopy sequence shows that the pair map is properly $[\frac{m}{2}]$-connected. If m is even, we are done. The case for $m = 2k+1$ follows Wall [41] Theorem 1.4.

We may assume that we have $f: (M,\partial M) \to (X,\partial X)$ connected up to the middle dimension on each piece. Let E be the disjoint union of the (k+1)-cells of $M_f - M$. Then we have a proper may $\partial E \to M_f$. Since ∂E is k-dimensional, and since (M_f,M) is properly k-connected, there is a proper homotopy of the attaching maps into M.

$\partial E = \bigcup_p S_p^k$, so embed these spheres in M with trivial normal bundle by Lemmas 2 and 3. Join each sphere to ∂M by a locally finite collection of tubes, one for each sphere (Since $H^0_{end}(X,\partial X) = 0$ by hypothesis, and since $M \to X$ is properly 1-connected (at least), and since $\partial M \to \partial X$ is properly 0-connected, $H^0_{end}(M,\partial M) = 0$ so we can do this. Note in fact that we need only disturb ∂M in a (pre-assigned) neighborhood of a set of base points.) By general position we may assume all these tubes disjoint ($m \geq 3$). Hence we get framed embeddings of a collection of disjoint D^k's . We may assume (by adding trivial discs if necessary) that the centers of our discs form a set of base points for M .

We claim that if we do these relative surgeries we will have killed $K_k(M,\partial M)$ without affecting our other conditions. Our proof of this claim is the same as Wall's. Let H denote the union of the handles, N_0 the constructed manifold, $f_0 : (N_0,\partial N_0) \to (X,\partial X)$ the resulting map. Note that $(N_0,\partial N_0) \to (M, H \cup \partial N_0)$ is a proper excision map. We can pick a set of base points for ∂M away from $\partial M \cap H$. As usual we can pick them so that they are a set of base points for $f : \partial M \to \partial X$. They are then also seen to be a set of base points for M, N_0, ∂N_0, and $H \cup \partial N_0$. With these base points and the above excision map we get an exact sequence $\Delta_k(H \cup \partial N_0,\partial M:M:\sim) \to \Delta_{k+1}(f:\sim) \to \Delta_{k+1}(f_0:\sim) \to \Delta_{k-1}(H \cup \partial N_0,\partial M:M:\sim)$.

Clearly the lower relative proper homotopy groups of f_o vanish. Notice $(H, H \cap \partial M) \to (H \cup \partial N_o, \partial M)$ is also a proper excision map. Since $(H, H \cap \partial M)$ is a collection of copies of $(D^k \times D^{k+1}, S^{k-1} \times D^{k+1})$, $\Delta_*(H, H \cap \partial M)$ is 0 except in dimension k. If we pick base points in H, $\Delta_*(H \cup \partial N_o, \partial M:-) = 0$ also except in dimension k(-here is any covering functor). Hence $\Delta_{k-1}(H \cup \partial N_o, \partial M:M:\sim) = 0$.

Let $g: M \to X$ denote f on M to distinguish it from f on $(M,\partial M)$. The collection of elements above generates $\Delta(g: \pi_{k+1})$. Clearly the composite $\Delta(g:\pi_{k+1}) \to \Delta_{k+1}(g:\sim) \to \Delta_{k+1}(f:\sim) \to \Delta_{k+1}(f_o:\sim)$ is the zero map. But by Hurewicz, the first map is an isomorphism, and the second map is onto since $\partial M \to \partial X$ is properly k-connected. Hence $\Delta_{k+1}(f:\sim) \to \Delta_{k+1}(f_o:\sim)$ is the zero map.

Now the last two paragraphs and our exact sequence show $\Delta_{k+1}(f_o) = 0$ as claimed. □

Remarks: Note throughout the proof that should $\partial X = \partial_1 X \cup \partial_2 X$ and $\partial M = \partial_1 M \cup \partial_2 M$, and if $\partial_2 M \to \partial_2 X$ is already properly r-connected, then we need attach no cells of dimension less than r to $\partial_2 M$ in our construction (provided $H^o_{end}(X, \partial_1 X) = 0$, otherwise to get this part of the result we must attach some k-cells in $\partial_2 M$). In particular, if $\partial_2 M \to \partial_2 X$ is a proper homotopy equivalence, we can do our construction away from $\partial_2 M$ (except possibly for the last step).

Theorem 2: Let $f: (M,\partial M) \to (X,\partial X)$ be a degree 1 normal map; i.e. a bundle over X and a bundle map over f are understood. Let $(X, \partial X)$ be a Poincare duality pair of formal dimension at least 6. Suppose $\partial X \subseteq X$ is a proper 1-equivalence. Then f is normally cobordant to $g: (N,\partial N) \to (X, \partial X)$ with g a proper homotopy equivalence. The torsion of $g: N \to X$ may have any preassigned value. The torsions of $g: \partial N \to \partial X$ and of g as a map of pairs is then determined.

Proof: The proof of the theorem divides into two cases.

Case 1: $\dim(X) = 2k$.

By Theorem 1, we can do surgery on f to make the map $f: M \to X$ k-connected, and to make the map $\partial f: \partial M \to \partial X$ (k-1)-connected (properly connected actually, but we shall be sloppy). Since $k \geq 3$, $f, \partial f$, and $\partial M \subseteq M$ are all (proper) 1-equivalences.

Now subdivide $(M,\partial M)$ until the chain map $C_*(M,\partial M) \to C_*(X,\partial X)$ is onto. $C_*(X,\partial X)$ is $C_*(X,\partial X:\wedge,F)$ for a collection of paths $\wedge$ and a lift functor F . The tree for X should come from a tree for ∂M , which we can clearly assume. $C_*(M, \partial M)$ is defined in the same way only with lift functor $f^{-1}F$. Let $D_*(f)$ be the kernel complex.

Then $H_r(D_*(f)) = 0$ for $r < k$ and $H^r(D_*(f)) = 0$ for $r > k$. Now Theorem 1.5.5 shows $H_k(D_*(f))$ is an s-free tree module. Doing surgery on trivial (k-1)-spheres in ∂M replaces M by its boundary connected sum with a collection of $(S^k \times D^k)$'s. Hence we may as well assume $H_k(D_*(f))$ is free and based. Let $\{e_i\}$ be a preferred basis for this module.

By the Namioka Theorem, $\Delta(f:{}_{k+1}) \to H_k(D_*(f))$ is an isomorphism. Thus the e_i determine classes in $\Delta(f:\pi_{k+1})$. These in turn determine a proper regular homotopy class of immersions $e_i:(D^k \times D^k, \partial D^k \times D^k) \to (M, \partial M)$. We claim the e_i are properly regularly homotopic to disjoint embeddings. It is clearly enough to show this for the restricted immersions $\bar{e}_i: (D^k, \partial D^k) \to (M, \partial M)$, for then we just use small neighborhoods of the $\bar{e}_i$ to get the e_i.

The proof for the $\bar{e}_i$ proceeds as follows. Let C_j be an increasing sequence of compact subsets of M with $\bigcup_j C_j = M$. Let C_j be such that any element of $\pi_1(M - C_j)$, when pushed into $\pi_1(M - C_{j-1})$, lies in the image of $\pi_1(\partial M \cap (M - C_{ij}))$ (compatible base points are understood). We can do this as $\partial M \subseteq M$ is a proper 1-equivalence.

We now proceed. Only a finite number of the $\bar{e}_i$ do not lie in $M - C_2$. Embed these disjointly by the standard piping argument.

Again, only finitely many $\bar{e}_i$ which do lie in $M - C_2$ do not lie in $M - C_3$. Put these in general position. The intersections and self-intersections can be piped into $\partial M \cap (M - C_1)$ without disturbing the $\bar{e}_i$ we embedded in the previous step. This follows from Milnor [24], Theorem 6.6, where we see that, to do the Whitney trick, we need only move one of the protagonists. Hence we can always leave the $\bar{e}_i$ from previous steps fixed.

Continuing in this fashion, we can always embed an $\bar{e}_i$ which lies in $M - C_j$, but not in $M - C_{j+1}$, in $M - C_{j-1}$. This gives us a proper regular homotopy and establishes our claim.

We next perform handle subtraction. Let N be obtained from M by deleting the interiors of the images of the e_i. Let U be the union of the images of the e_i. Let $\partial N = N \cap \partial M$.

By our construction, there is a chain map $C_*(U \cup \partial M, \partial M) \to D_*(f)$ such that

$$\begin{array}{ccccccccc} 0 & \to & C_*(U \cup \partial M, \partial M) & \to & C_*(M,\partial M) & \to & C_*(M, U \cup \partial M) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & D_*(f) & \to & C_*(M,\partial M) & \to & C_*(X,\partial X) & \to & 0 \end{array}$$

chain homotopy commutes. $C_*(U \cup \partial M, \partial M)$ has homology only in dimension k where it is $H_k(D_k(D_*(f)))$. The map $C_*(U \cup \partial M, \partial M) \to D_*(f)$ gives this isomorphism in homology by construction.

Hence $C_*(M, U \cup \partial M) \to C_*(X,\partial X)$ is a chain equivalence. Now $(N,\partial N) \subseteq (M, U \cup \partial M)$ is a proper excision map, so $g: (N,\partial N) \to (X,\partial X)$ is a proper homotopy equivalence from N to X. It induces proper homology isomorphisms on $\partial N \to \partial X$ and is thus a proper equivalence there since $\partial X \subseteq X$ is 1-connected. Hence g is a proper homotopy equivalence of pairs. By adding an h-cobordism to ∂N, we can achieve any torsion we like for the map $g: N \to X$. To compute the other two torsions is now a standard exercise. We record merely the result. Let $g': (N,\partial N) \to (X,\partial X)$ and $\partial g: \partial N \to \partial X$ be the other two maps.

$\tau(g') = (-1)^{\dim X}\tau(g)^t$ and $(\partial g) = \tau(g) - (-1)^{\dim X}\tau(g)^t$.

Case 2: $\dim (X) = 2k+1$.

This time, Theorem 1 permits us to suppose that f induces k-connected maps $M \to X$ and $\partial M \to \partial X$, and moreover we may assume $K_k(M,\partial M) = 0$. Hence we get a short exact sequence of the modules $0 \to K_{k+1}(M,\partial M) \to K_k(\partial M) \to K_k(M) \to 0$. ($K_*(M)$ is the tree of modules which is the kernel of the map $H_*(C(M:\Lambda,f^{-1}F)) \to H_*(C(X:\Lambda',F))$. The other K-groups are defined similarly).) Theorem 1.5.5 now tells us each of these modules is s-free. As before we can perform surgery on trivial (k+1)-spheres in ∂M to convert all of the above modules to free modules. Again we can get a locally finite collection of immersions $\bar{e}_i: (D^{k+1},\partial D^{k+1}) \to (M,\partial M)$ representing a basis of $K_{k+1}(M,\partial M)$.

We can no longer modify the $\bar{e}_i$ by a proper regular homotopy to get disjoint embeddings (we could do this if $\partial M \subseteq M$ were properly 2-connected) but by the same sort of argument as in the first part, we can modify the $\bar{e}_i$ until $\bar{e}_i|\partial D^{k+1}$ is a collection of disjoint embeddings.

The rest of the proof is the same as Wall's. We have represented a basis of $K_{k+1}(M,\partial M)$ by framed, disjoint embeddings $S^k \to \partial M$. Attach corresponding (k+1)-handles to M , thus performing surgery. Let U be the union of the added handles, and let $(N,\partial N)$ be the new pair. Since our spheres are null homotopic in M , M is just replaced (up to proper homotopy type) by M with (k+1)-spheres wedged on in a locally finite fashion. Hence $K_k(N)$ is free, with a basis given by these spheres.

Dually, the exact sequence of the triple $\partial N \subseteq \partial N \cup U \subseteq N$, reduces, using excision, to

$$0 \to K_{k+1}(N,\partial N) \to K_{k+1}(M,\partial M) \to K_k(U,U \cap \partial N{:}M) \to K_k(N,\partial N) \to 0.$$

The map $K_{k+1}(M,\partial M) \to K_k(U,U \cap \partial N{:}N)$ is seen to be zero since it factors as $K_{k+1}(M,\partial M) \to K_k(\partial M) \to K_k(U{:}M) \to K_k(U,U \cap \partial N{:}M)$, and $K_k(U{:}M)$ is zero. (Note that in this composition, $K_k(\partial M)$ should be a subspace group, but such a group is isomorphic to the absolute group in our case.) Since $K_k(U,U \cap \partial N{:}M)$ is free, so is $K_k(N,\partial N)$ and $K_{k+1}(N,\partial N) \cong K_{k+1}(M,\partial M)$.

The attached handles correspond to a basis of $K_{k+1}(M,\partial M)$, so the map $K_{k+1}(N) \to K_{k+1}(M,\partial M)$ is an epimorphism, since $K_{k+1}(N)$ is free and based on a set of generators for $K_{k+1}(M,\partial M)$ and the map takes each basis element to the corresponding generator. But $K_{k+1}(M,\partial M)$ is free on these generators, so this map is an isomorphism. Hence $K_{k+1}(N) \to K_{k+1}(N,\partial N)$ is an isomorphism.

Now, by Poincaré duality, $K^k(N,\partial N) \to K^k(N)$ is an isomorphism. The natural maps $K^k(N,\partial N) \to (K_k(N,\partial N))^*$ and $K^k(N) \to (K_k(N))^*$ are isomorphisms by Corollary 1.5.4.2 since all the modules are free. Hence the map $K_k(N) \to K_k(N,\partial N)$ is an isomorphism. Thus $K_k(\partial N) = 0$, so f restricted to ∂N is a proper homotopy equivalence.

Next choose a basis for $K_k(N)$ and perform surgery on it. Write P for the cobordism so obtained of N to N' say. Consider the induced map of degree 1 and Poincaré triads $(P: N \cup (\partial N \times I), N') \to (X \times I: X \times 0 \cup \partial X \times I, X \times 1)$. We will identify $N \cup (\partial N \times I)$ with N. In the exact sequence

$$0 \to K_{k+1}(N) \to K_{k+1}(P) \to K_{k+1}(P,N) \xrightarrow{d} K_k(N) \to K_k(P) \to 0$$

the map d is by construction an isomorphism. Hence $K_k(P) = 0$ and $K_{k+1}(N) \to K_{k+1}(P)$ is an isomorphism.

The dual of d is $K_{k+1}(N,\partial N) \to K_{k+1}(P,N')$, so it is an isomorphism (the map is the map induced by the inclusion). Now, since f on ∂N is a proper homotopy equivalence, $K_{k+1}(N) \to K_{k+1}(N,\partial N)$ is an isomorphism. $K_{k+1}(N) \to K_{k+1}(P)$ is an isomorphism, so $K_{k+1}(P) \to K_{k+1}(P,N')$ is an isomorphism.

Thus in the sequence

$$0 \to K_{k+1}(N') \to K_{k+1}(P) \to K_{k+1}(P,N') \to K_k(N') \to 0$$

we have $K_{k+1}(N') = K_k(N') = 0$, so $N' \to X$ is a proper homotopy equivalence. $\partial N' \to \partial X$ is the same as $\partial N \to \partial X$ (i.e. we did nothing to ∂N as all our additions were in the interior of N) and therefore is a proper homotopy equivalence. Hence we have an equivalence of pairs. The statement about torsions is proved the same way as for Case 1. □

Remarks: Note that our proof is still valid in the case $\partial X = \partial_1 X \cup \partial_2 X$ provided $\partial_1 M \to \partial_1 X$ is a proper homotopy equivalence (of pairs if $\partial_1 X \cap \partial_2 X \neq \emptyset$) and $\partial_2 X \subseteq X$ is a proper 1-equivalence ($(X:\partial_1 X, \partial_2 X)$ should be a Poincare triad). The proof is word for word the same after we note that $K_i(\partial_2 M) \to K_i(\partial M)$ is always an isomorphism and that we may attach all our handles away from $\partial_1 M$. By induction, we can prove a similar theorem for n-ads, which is the result we needed to prove Theorem 2.2.13.

Our approach to surgery is to consider the surgery groups as bordism groups of surgery maps. To make this approach work well, one needs a theorem like Theorem 3 below.

<u>Definition</u>: Given a Poincare duality n-ad X, a surgery map is a map $f: M \to X$ where M is a $\mathcal{C}$-manifold n-ad, f is a degree 1 map of n-ads, and there is a bundle ν over X and a bundle map $F: \nu_M \to \nu$ which covers f.

Given a locally finite CW n-ad K with a class $w_1 \in H^1(K;Z_2)$, we say $M \xrightarrow{f} X \xrightarrow{g} K$ is a surgery map over (K, w_1) provided g is a map of n-ads with $g^* w_1$

equal to the first Stiefel-Whitney class of X, and provided f is a surgery map.

Two surgery maps over (K, w_1) are said to be bordant (over (K, w_1)) if there is a surgery $(n+1)$-ad $W \xrightarrow{F} Y \xrightarrow{G} (K \times I, w_1)$ which is one of the surgery maps on $K \times 0$ and the other on $K \times 1$.

Theorem 3: Let $M \xrightarrow{f} X \xrightarrow{g} K$ be a surgery map over (K, w_1), a 3-ad. Suppose the formal dimension of X is at least 6. Then, if $f|\partial_1 M$ is a proper homotopy equivalence, and if $\partial_2 K \subseteq K$ is a proper 1-equivalence, we can find another surgery map $N \xrightarrow{h} Z \xrightarrow{i} K$ over (K, w_1) with h a proper homotopy equivalence of 3-ads, and with i bordant over (K, w_1) to g so that over $\partial_1 K \times I$ the bordism map is $\partial_1 K \to \partial_1 X$ crossed with I.

Proof: If $\partial_2 X \subseteq X$ were a proper 1-equivalence we could finish easily using Theorem 2. The proof then consists of modifying X and $\partial_2 X$ to get this condition. The idea is to do surgery first on $\partial_2 X$ (and then on X) to get $\partial_2 X \to \partial_2 K$ a proper 1-equivalence (similarly for $X \to K$) and then show that we can cover these surgeries on $\partial_1 M$ (and M).

Look at the map $g\colon \partial_2 X \to \partial_2 K$. By Theorem 2.3.2, $\partial_2 X$ can be replaced by $L \cup H$, where H is a manifold and L satisfies $D(n-3)$, where n is the formal dimension of X. This replacement does not alter the bordism class in which we are working. Let w_1 also denote the restriction of $w_1 \in H^1(K;Z_2)$ to $\partial_2 K$. Let ν be the line bundle over $\partial_2 K$ classified by w_1. Let $g: H \to \partial_2 K$ denote the induced map.

Then $\tau_H \oplus g^*\nu$ is trivial, for H has the homotopy type of a 1-complex so the bundle is trivial iff its first Stiefel-Whitney class vanishes (and $w_1(\tau_H \oplus g^*\nu) = 0$ by construction). Hence we can find a bundle map $F: \nu_H \to \nu$.

By Theorem 1, we can add 1 and 2 handles to H to get W with $\partial W = H \cup H' \cup \partial H \times I$ and a map $G: W \to \partial_2 K$ with $G|H = h$ and $G|H'$ a proper 1-equivalence. Let $Y = L \times I \cup W$ by gluing $\partial H \times I$ to $L \times I$ via the map $\partial H \to L$ crossed with I. $(Y{:}L \cup_{\partial H} H, L \cup_{\partial H} H' \cup L \times I)$ is a Poincare duality triad. This follows since $(L,\partial H)$ is a Poincare duality pair and Y is $(L,\partial H) \times I$ glued to the manifold triad $(W{:}\ H, \partial H \times I, H')$ along $\partial H \times I$. $(L,\partial H) \times I$ is a Poincare duality triad by Theorem 2.2.9, and we can glue by Theorem 2.1.3 and Theorem 2.2.7.

Let $Z = L \cup_{\partial H} H'$. We have a map of $Y \to K \times I$ given by $L \to K$ crossed with I on $L \times I$ and by $W \to K \times I$ on W. We claim the restriction $Z \to \partial_2 K \times 1$ is a proper 1-equivalence.

To this, note first that

$$\begin{array}{ccc} \partial H & \subseteq & H' \\ \cap| & & \cap| \\ L & \subseteq & Z \end{array}$$

is a push out.

$\partial H \subseteq L$ is properly 1-connected by construction (see Theorem 2.3.2). It follows from a Mayer-Vietoris argument that $H' \subseteq Z$ induces isomorphisms on H^0_{end} and H^0.

Since $\Delta(\partial H:\pi_1) \to \Delta(L:\pi_1)$ is onto, it follows from a van-Kampen argument that $\Delta(H':\pi_1) \to \Delta(Z:\pi_1)$ is onto.

Now consider $H' \subseteq Z \to \partial_2 K$. The composite is a proper 1-equivalence by construction. The first map is properly 1-connected, as we saw in the last paragraph. It then follows that $Z \to \partial_2 K$ is a proper 1-equivalence.

It is easy to extend our bundle ν over all of Y. Wall [41] pages 89-90 shows how to cover our surgeries back in $\partial_2 M$. One changes $f: \partial_2 M \to \partial_2 X$ through a proper homotopy until it is transverse regular to all our core spheres in $H \subseteq \partial_2 X$. The inverse image of a core sphere back in $\partial_2 M$ will be a collection of disjoint spheres, and Wall shows that, if we do surgery correctly on these spheres, then we can extend all our maps and bundles. Hence we get $F: P \to Y$ and a bundle map $\nu_p \to \nu$, where ν is the extended ν over Y.

Thus our original problem $M \to X \to K$ is normally cobordant over (K, w_1) to a problem for which $\partial_2 X \to \partial_2 K$ is a proper 1-equivalence. We have not touched $\partial_1 M \to \partial_1 X$, so we still have that this map is a proper homotopy equivalence. In fact the part of ∂P over $\partial_1 M$ is just a product.

Now use Theorem 2.3.3 on X and proceed as above to get a problem for which $X \to K$ is a proper 1-equivalence. Note that we need never touch ∂X so $\partial_1 X \to \partial_1 K$ is still a proper homotopy equivalence and $\partial_2 X \to \partial_2 K$ is still a proper 1-equivalence. $\square$

Section 2: Paracompact surgery-patterns of application.

It has been noted by several people (see especially Quinn [29] or [30] that the theorems in section 1, the s-cobordism theorem, and transverse regularity are all the geometry one needs to develop a great deal of the theory of surgery.

We define surgery groups as in Wall [41] Chapter 9. Let K be a locally compact CW n-ad, and let $w \in H^1(K;Z_2)$ be an orientation. An object of type n over (K,w) is a surgery map (see section 1) over $s_n K$ for which, if $M \longrightarrow X \longrightarrow s_n K$ is the surgery map. $\emptyset\colon \partial_n M \to \partial_n X$ is a proper homotopy equivalence of n-ads.

We write $(\emptyset,f) \sim 0$ to denote the existence of a surgery map over $(s_n K,w)$ such that ∂_{n+1} is $(\emptyset,f)$; i.e. if $W \rightarrow Z \rightarrow s_{n+1}s_n K$ is the surgery map, $\partial_{N+1}W \rightarrow \partial_{N+1}Z \rightarrow s_n K$ is our original problem; and such that ∂_n is a proper homotopy equivalence of (n+1)-ads. $(\emptyset,f) \sim (\emptyset_1,f_1)$ provided $(\emptyset,f) + -(\emptyset_1,f_1) \sim 0$, where $+$ denotes disjoint union, and $-(\emptyset_1,f_1)$ denotes the same object but with the reverse orientation. Write $L^h_m(K,w)$ for the group of objects of type n and dimension m (i.e. m is the dimension of M) modulo the relation $\sim$. One checks $\sim$ is an equivalence relation and that disjoint union makes these sets into abelian groups.

If we require the torsions of all the homotopy equivalences in the above definitions to be 0 , we get groups $L_m^s(K,w)$. If $c \subseteq \zeta(K)$ is a subgroup closed under the involution induced by the orientation w , then we get groups $L_m^c(K,w)$ by requiring all torsions to lie in $c(\zeta(K)$ is Siebenmann's group of simple homotopy types; see Chapter 1, section 5, or [33]).

Theorem 1: Let $\alpha \in L_m^c(K,w)$, $n+m \geq 6$. Then if $M \xrightarrow{\varphi} X \xrightarrow{f} K$ is a representative of α with f a proper 1-equivalence, $\alpha = 0$ iff there is a normal cobordism $W \to X \times 1$ with $\partial_- W \to X \times 0$ our original map φ, and $\partial_+ W \to X \times 1$ a proper homotopy equivalence of n-ads with torsion lying in c .

Proof: Standard from Theorem 1.2, by doing surgery on the boundary object. □

Theorem 2: $\cdots \to L_m^c(\partial_n K,w) \to L_m^c(\delta_n K,w) \to L_m^c(K,w) \to L_{m-1}^c(\partial_n K,w) \to \cdots$ is exact.

Proof: A standard argument. □

Theorem 3: If $f\colon K_1 \to K_2$ is a proper map of n-ads we get an induced map $L_m^c(K_1, f\, w) \to L_m^{c'}(K_2,w)$ where $f_\#(c) \subseteq c'$, $f_\#\colon \zeta(K_1) \to \zeta(K_2)$. If f is a proper 1-equivalence, the induced map is an isomorphism for $c = f_\#^{-1}(c')$.

Proof: The induced map is easily defined by $M \to X \to K_1$ goes to $M \to X \to K_1 \xrightarrow{f} K_2$. For the last statement, if $m \geq 5$ this is just Theorem 1.3 if K_1 and K_2 are 1-ads.

If K_1 and K_2 are n-ads, an induction argument shows the result for $n+m \geq 6$.

The result is actually true in all dimensions and a proof can be given following Quinn's proof in the compact case (see [29] or [30]). We will not carry it out here. □

Theorem 4: Let K be a 1-ad, and let $M^m \xrightarrow{\varphi} X \xrightarrow{f} K$ be a surgery map over (K,w) with φ a proper homotopy equivalence and with f a proper 1-equivalence. Suppose given $\alpha \in L^c_{m+1}(K,w)$, $m \geq 5$, and suppose the torsion of φ lies in c. Then there is an object of type 1, $W \to X \times I \to K$, over (K,w) with $\partial W = M \cup N$, $N \to X \times 1$ a proper homotopy equivalence whose torsion also lies in c, and such that the surgery obstruction for this problem is α.

Proof: The proof is basically Quinn's (see [29]). Given α, there is always an object of type 1, $P \to Z \to K$, whose obstruction is $(-\alpha)$. (We may always assume ∂P and ∂Z are non-empty by removing a disc from Z and its inverse image from P, which we can modify to be a disc.) $M \times I \to X \times I \to K$ is also an object of type 1 over K.

Take the boundary connected sum of Z and $X \times I$ by extending $\partial Z \# X \times 0$ (we may always assume X and Z are in normal form so we may take this sum in their discs). Similarly we may extend $\partial P \# M \times 0$. We get a new object of type 1, $P \#_{M\times 0} M \times I \to Z \#_{X\times 0} X \times I \to K$.

By the proof of Theorem 1.3, we may do surgery on

$Z \#_{X\times 0} X\times I$ until the map of it to K is a proper 1-equivalence, and we may cover this by a normal cobordism of $P \#_{M\times 0} M\times I$. In doing this, we need never touch $M\times 1$ or $X\times 1$. Let $P' \to Z' \to K$ denote this new object of type 1. Note that it still has surgery obstruction $(-\alpha)$.

Now using Theorem 1.2, we can do surgery on $\emptyset: P' \to Z'$ where Z' is considered to be a triad $(Z'; X\times 1$, any other boundary components). $\emptyset$ restricted to the other boundary components is a proper homotopy equivalence, so we may do surgery leaving them fixed ($X\times 1 \subseteq Z'$ is a proper 1-equivalence). Let W be the normal cobordism obtained over $M\times 1$. Then $W \to X\times 1\times I$ is a surgery map, $\partial_- M \to X\times 1\times 0$ is our old map, and $\partial_+ M \to X\times 1\times 1$ is a proper homotopy equivalence. We can make all our torsions lie in c, and then the surgery obstruction for $W \to X\times I \to K$ must be α. $\square$

<u>Definition</u>: Let $\mathcal{S}_{\mathcal{C}}(X)$, for X a Poincaré duality space of dimension n, be the set of all simple, degree 1, homotopy equivalences $\emptyset: N^n \to X$ (N a $\mathcal{C}$-manifold) modulo the relation $\emptyset \sim \psi$ iff there is a $\mathcal{C}$-homeomorphism h such that

$$\begin{array}{ccc} N & \xrightarrow{\emptyset} & \\ \downarrow h & & X \\ M & \xrightarrow{\psi} & \end{array}$$

properly homotopy commutes.

A similar definition holds for X a Poincaré n-ad.

Theorem 5: There is an exact structure sequence

$$\cdots \to [\Sigma X, F/\mathcal{C}] \to L^s_{m+1}(X,w) \to \mathcal{S}_{\mathcal{C}}(X) \to [X,F/\mathcal{C}] \xrightarrow{\theta} L^s_m(X,w),$$

where w is the first Stiefel-Whitney class of the Poincare duality space X with the dimension of $X \geq 5$. We also insist that the Spivak normal fibration of X lift to a $\mathcal{C}$-bundle. By exactness we mean the following. First of all, $\mathcal{S}_{\mathcal{C}}(X)$ may be empty, but in any case, $\theta^{-1}(0)$ is the image of $\mathcal{S}_{\mathcal{C}}(X)$. If $\mathcal{S}_{\mathcal{C}}(X)$ is not empty, then $L^s_{m+1}(X,w)$ acts on it, and two elements of $\mathcal{S}_{\mathcal{C}}(X)$ which agree in $[X,F/\mathcal{C}]$ differ by an element of this action. The sequence continues infinitely to the left. (ΣX is the ordinary suspension of X.)

Proof: See Wall [41], Chapter 10. □

Theorem 6: Let $-$ be the involution defined on $\zeta(K)$ in Chapter 1, section 5. Define $A_m(K,w) = H^m(Z_2,\zeta(K))$, where $\zeta(K)$ is made into a Z_2-module by the involution $-$. If K is an n-ad, then $\cdots \to A_{m+1}(K,w) \to L^s_m(K,w) \to L^h_m(K,w) \to A_m(K,w) \to \cdots$ is exact for $m+n \geq 6$.

Proof: The map $L^s \to L^h$ is just the forgetful map. The map $L^h \to A$ just takes the torsion of the part of the boundary that was a homotopy equivalence and maps it into A (if the homotopy equivalence is over more than one component, sum the torsions). The map $A \to L^s$ takes a proper homotopy equivalence $M^m \to X$ whose torsion hits an element in A_{m+1}, and maps it to the obstruction to surgering the map to a simple homotopy equivalence. See

Shaneson [31] for the details of proving these maps well-defined and the sequence exact. □

<u>Corollary 6.1</u>: If $A^c_m(K,w) = H^m(Z_2,c)$,

$$\cdots \to A^c_{m+1}(K,w) \to L^s_m(K,w) \to L^c_m(K,w) \to A^c_m(K,w) \to \cdots$$

is exact for $m+n \geq 6$. □

We now produce our major computation.

<u>Theorem 7</u>:[1] Let K have a finite number of stable ends, $\varepsilon_1,\ldots,\varepsilon_n$, and let π_1 of each end and $\pi_1(L)$ be finitely generated, finitely presented. Then K^2 has the proper 2-type of a finite (n+1)-ad $L \cup (\bigcup_{i=1}^{n} \partial_i L \times [0,\infty))$, and $L^s_m(K,w) = L^c_m(L,w)$, where, if K is an ℓ-ad, L additionally is an $(\ell+n)$-ad. c denotes simple homotopy equivalence over L , with any permissible torsion over each $\partial_i L$; i.e. we have an exact sequence

$$\cdots \to \bigoplus_{i=1}^{n} L^{c_i}_m(\pi_1\varepsilon_i,w) \to L^s_m(\pi_1(L),w) \to L^s_m(K,w) \to$$

$$\bigoplus_{i=1}^{n} L^{c_i}_{m-1}(\pi_1\varepsilon_i,w) \to \cdots \text{ , where } c_i = \ker(Wh(\pi_1\varepsilon_i) \to Wh(\pi_1(L))).$$

$\ell + m \geq 7$.

<u>Proof</u>: The map $L^c_m(L,w) \to L^s_m(K,w)$ is given by $M \to X \to L$ goes to

$$M \cup (\bigcup_{i=1}^{n} \partial_i M \times [0,\infty)) \to X \cup (\bigcup_{i=1}^{n} \partial_i X \times [0,\infty)) \to L \cup (\bigcup_{i=1}^{n} \partial_i L \times [0,\infty)).$$

Siebenmann's thesis [32] shows this map is a monomorphism. To show that the map is onto we can assume

[1]Note added in proof: Compare Maumary, <u>The open surgery obstruction in odd dimensions</u>. Notices Amer. Math. Soc. 17 (number 5) p. 848.

$W \xrightarrow{\phi} Z \longrightarrow K$ is a surgery map and that Z is a manifold using Theorem 4 (this representation theorem is also needed to show injectivity). By Siebenmann [32], we can assume Z is collored; i.e. $Z = N \cup (\bigcup_{i=1}^{n} \partial_i N \times [0,\infty))$. By making ϕ transverse regular to the $\partial_i N$, we get a problem over L, say $V \to N \to L$. We claim

$$V \cup (\bigcup_{i=1}^{n} \partial_i V \times [0,\infty)) \to N \cup (\bigcup_{i=1}^{n} \partial_i N \times [0,\infty))$$ has the same

surgery obstruction as $W \to Z$. But this is seen by actually constructing the normal cobordism using Siebenmann's concept of a 1-neighborhood and some compact surgery. □

<u>Corollary 7.1</u>: We can improve $\ell + m \geq 7$ to $\ell + m \geq 6$.

<u>Proof</u>: Using recent work of Cappell-Shaneson [5], one can get a modified version of Siebenmann's main theorem. One can not collar a 5-manifold, but one can at least get an increasing sequence of cobordisms whose ends are $\partial_i N \# S^2 \times S^2 \# \cdots \# S^2 \times S^2$. This is sufficient. □

Actually, one would hope that these surgery groups would be periodic, just as the compact ones are. This is actually the case, but the only proof I know involves describing surgery in terms of algebra. This can be done, but the result is long and will be omitted.

We briefly consider splitting theorems. The two-sided codimension 1 splitting theorem holds; i.e. if W

has the simple homotopy type of $Z = (X,\partial X) \cup (Y,\partial X)$ with $\partial X \subseteq X$ a proper 1-equivalence, then the map $W \to Z$ can be split. The proof is the same as for the compact case. Hence we also get codimension greater than or equal to 3 splitting theorems for proper submanifolds. In fact, most of Wall [41] Chapter 11 goes over with minor modifications.

We are unable to obtain a one-sided splitting theorem in general, due to a lack of a Farrell fibering theorem in the non-compact case.

We also note in passing that one could define surgery spaces as in [29] and [30]. We then get the same basic geometric constructions; e.g. assembly maps and pullback maps. We have nothing new to add to the theory, so we leave the reader the exercise of restating [29] so that it is valid for paracompact surgery spaces.

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