2-LOCAL COBORDISM THEORIES

LAURENCE R. TAYLOR

1. *Introduction*

We give new proofs of the principal results of Thom **[11]**, Wall **[12]**, and Browder–Liulevicius–Peterson **[3]** on the structure of various cobordism theories at the prime 2. We improve the principal results of Browder–Liulevicius–Peterson by removing their hypothesis that certain cohomology groups are finite. The proofs use classical facts about $H_*(BO)$, $H_*(BSO)$ and the Steenrod algebra, together with an idea of J. Cohen **[6]**. Cohen's idea was to observe that for an homology theory E and certain spectra X, $E_*(X)$ may be quite easy to calculate. We can then use the Atiyah–Hirzebruch spectral sequence to try to calculate $E_*(\text{pt.})$, which appears in E^2 of the AHss.

2. *Unoriented cobordism*

We need the following three facts.

1. $H_*(MO)$ is a polynomial algebra with one generator in each positive dimension. This follows from the Thom isomorphism theorem and Borel's calculation of $H^*(BO)$ **[2]**. All homology and cohomology groups without indicated coefficients are with Z_2 coefficients.

2. $H_*(MO) = MO_*(HZ_2)$ since both are the homotopy of $MO \wedge HZ_2$. We use Adams's notation for spectra **[1]**.

3. $H_*(HZ_2) = Z_2[\xi_1, \xi_2, \ldots]$ where $\dim \xi_k = 2^k - 1$ **[9]**.

Consider the Atiyah–Hirzebruch spectral sequence for $MO_*(HZ_2)$.

$$E^2_{p,q} = MO_q \otimes H_p(HZ_2)$$

and the edge homomorphism $MO_p(HZ_2) \to H_p(HZ_2)$ is the map

$$MO \wedge HZ_2 \xrightarrow{u \wedge \mathrm{id}} HZ_2 \wedge HZ_2$$

where $u : MO \to HZ_2$ is the Thom class in $H^0(MO)$, **[1]**. The map $u \wedge \mathrm{id}$ is onto in homotopy which is verified by using the lemma below to show that $\mathrm{id} \wedge u$ is onto in homotopy.

The AHss is multiplicative since both MO and HZ_2 are ring spectra. Since all the differentials vanish on $E^r_{0,q}$ and on $E^r_{p,0}$, the spectral sequence collapses.

Since $H_*(HZ_2)$ is polynomial, the map $MO_*(HZ_2) \to H_*(HZ_2)$ is split as a ring map. Hence there is a map of rings extending the splitting

$$\psi : MO_* \otimes H_*(HZ_2) \to MO_*(HZ_2).$$

The ring $MO_*(HZ_2)$ is filtered to produce the AHss and $MO_{*-p} \otimes H_p(HZ_2)$ lands in the pth filtration under ψ. With the obvious filtration on the left, ψ induces an isomorphism of associated grades and is therefore an isomorphism. We have proved

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THEOREM 1. *MO_* is a polynomial algebra with one generator in each dimension not equal to 2^k-1.*

$MO_* \to H_*(MO)$ is monic since it is the other edge homomorphism in the AHss. $H_*(MO)$ is a Z_2-vector space, so this map is split monic. It factors through

$$MO_* \to H_*(MO; Z)$$

which must also be split monic.

THEOREM 2. *MO is a product of HZ_2's.*

Proof. Homotopy is a summand of integral homology if and only if all the k-invariants are trivial, [**8**; Corollary 1.3].

To state our lemma, consider sequences $I = (i_1, ..., i_r, 0, ...)$ such that

$$i_1 \geqslant ... \geqslant i_r > 0.$$

We can order such sequences by $(i_1, ...) > (j_1, ...)$ if and only if $i_1 > j_1$; or $i_1 = j_1$ and $i_2 > j_2$; or $i_1 = j_1$, $i_2 = j_2$ and $i_3 > j_3$; etc. To $I = (i_1, ..., i_r, 0, ...)$ we can associate the monomial $w_I = w_{i_1} ... w_{i_r}$ in $H^*(BO)$ and the element

$$Sq^I = Sq^{i_1} ... Sq^{i_r}$$

in the Steenrod algebra. We say w_I is bigger than w_J if and only if $I > J$. Let $U \in H^\circ(MO)$ be the Thom class and $\Phi : H^*(MO) \to H^*(BO)$ the Thom isomorphism.

LEMMA. *If I is admissible (i.e. if $i_k > 2i_{k+1}$, all k)*

$$\Phi(Sq^I U) = w_I + \text{smaller monomials}.$$

Proof. The proof is an easy induction on r using admissibility, the Cartan formula, and the Wu relations [7]. It is done in [**11**].

The lemma proves $H^*(HZ_2) \to H^*(MO)$ monic since $H^*(HZ_2)$ has a vector space basis Sq^I, I admissible [**5**]. We used the dual statement.

3. *Oriented cobordism*

$H_*(MSO)$ is a polynomial algebra with one generator in each dimension greater than 1, [**2**].

$H_*(HZ) \cong Z_2[x, y_2, y_3, ...]$ where $\dim x = 2$, $\dim y_k = 2^k-1$. To see this, recall that $H_*(HZ)$ is the kernel of the derivation, d, on $H_*(HZ_2)$ defined by $d(\xi_k) = \xi^2{}_{k-1}$. This kernel is generated as a polynomial algebra by $b_1{}^2$ and b_k, $k > 1$, where b_k is the conjugate of ξ_k.

$H_*(MSO) \to H_*(HZ)$ is onto where $MSO \to HZ$ is the Thom class. This follows from the lemma as before.

$MSOZ_2$ is the ring spectrum for the cobordism theory of manifolds whose w_1 is the mod 2 reduction of an integral class. We can use the AHss to calculate

$$(MSOZ_2)_*(HZ) = H_*(MSOZ_2; Z) = H_*(MSO) = \pi_*(MSO \wedge HZ \wedge M_2)$$

where M_2 is the Moore spectrum of type Z_2. Mimicking Section 2, we have

THEOREM 3. *$(MSOZ_2)_*$ is a polynomial algebra over Z_2 with one generator in each dimension not equal to $2^k - 1$ or to 2.*

$H_*(MSO, Z)$ has no elements of order 4, [**2**]. $\tilde{H}_*(HZ; Z_4)$ has no elements of order 4, [**5**]. This can be seen directly if we observe that the Bockstein, β, satisfies $\beta(b_k) = b^2{}_{k-1}$. E^2 of the Bockstein spectral sequence is generated by the $b_k{}^2$, so the higher Bocksteins vanish for dimensional reasons.

Let $E^r{}_{p,q}$ be the E^r term of the AHss for $(MSOZ_4)_*(HZ)$. Let $F^r{}_{p,q}$ be the E^r term of the AHss for $(MSOZ_2)_*(HZ)$.

If G is an abelian group, define $\rho(G) = \dim_{Z_2} G \otimes Z_2$. One can see that

$$\rho(E^2{}_{p,q}) = \rho(F^2{}_{p,q}),$$

which in turn equals $\rho(F^\infty_{p,q})$ since $E^2 = E^\infty$ for $(MSOZ_2)_*(HZ)$, as the reader who has actually carried out the proof of Theorem 3 has seen.

$$\rho(H_i(MSO)) = \sum_{k=0}^{i} \rho(F^\infty{}_{k,\,i-k})$$

since all the extensions are split. $\rho(E^\infty{}_{p,q}) \leqslant \rho(E^2{}_{p,q})$ and

$$\rho(H_i(MSO; Z_4)) \leqslant \sum_{k=0}^{i} \rho(E^\infty{}_{k,\,d-k}).$$

Since $\rho(H_i(MSO)) = \rho(H_i(MSO; Z_4))$, $\rho(E^2{}_{p,q}) = \rho(E^\infty{}_{p,q})$. Since $E^2{}_{p,q}$ has no elements of order 4 for $p > 0$, there can be no differentials. Hence

$$(MSOZ_4)_* \to H_*(MSO; Z_4)$$

is monic and therefore, by the universal coefficient theorem [**1**; Prop. 6.6, p. 200], so is $MSO_* \otimes Z_4$. But this implies that MSO_* has no elements of order 4 and, if $Z_{(2)}$ denotes rationals with odd denominators, $(MSOZ_{(2)})_*$ is a direct summand of $H_*(MSOZ_{(2)}; Z)$. Hence we have

THEOREM 4. *All the k-invariants of $MSOZ_{(2)}$ are trivial.*

4. *Super cobordism theories*

Definition. A graded ring R_* is an $l-r$ Hopf algebra if R_* is a left and a right coalgebra comodule over the dual of the Steenrod algebra. We require that the dual algebra, which is both a left and a right module over the Steenrod algebra, be a right-left algebra as in [**4**; page 50]. Moreover, the coalgebra structure should make R_* into a cocommutative Hopf algebra.

$H_*(MO)$ and $M_*(MSO)$ are two examples.

A super O theory is a connective ring spectrum MH, whose homology is an $l-r$ Hopf algebra, and a map of ring spectra $MO \to MH$ which induces an $l-r$ Hopf algebra map on homology.

The only examples we know come from Thom spectra associated to various "bundle" theories. We have spaces $BH(n)$ and maps $g_n : BO(n) \to BH(n)$ and $h_n : BH(n) \to BF_{(2)}(n)$, which is the classifying space for n-dimensional, 2-local, spherical fibrations with cross section. $h_n g_n$ should be the usual map. The h_n give Thom spaces $MH(n)$ and Thom isomorphisms with Z_2 coefficients. We have a stabilization map $BH(n) \to BH(n+1)$. The two obvious squares involving $BO(n)$ and $BF_{(2)}(n)$ should commute up to weak homotopy. We further postulate a Whitney sum $BH(n) \times BH(m) \to BH(n+m)$ so that the obvious squares involving the $BO(n)$ or the $BF_{(2)}(n)$ commute up to weak homotopy. Finally we require that (1) should commute up to weak homotopy.

$$\begin{tikzcd}
 & BH(n+1) \times BH(m) \arrow[dr] & \\
BH(n) \times BH(m) \arrow[ur] \arrow[r] \arrow[dr] & BH(n+m) \arrow[r] & BH(n+m+1) \\
 & BH(n) \times BH(m+1) \arrow[ur] &
\end{tikzcd} \qquad (1)$$

(1) guarantees that the $MH(n)$ fit together to form a ring spectrum, MH, and that the $BH(n)$ fit together to form a weak H-space, BH. We assume that BH is weakly homotopy associative. $H_*(MH) \cong H_*(BH)$ as algebras. $H_*(BH)$ is a Hopf algebra and a left comodule over the dual of the Steenrod algebra, so $H_*(MH)$ is also. The usual left comodule structure of $H_*(MH)$ becomes a right one by using the conjugation in the dual of the Steenrod algebra. $H^*(MH)$ is a right-left algebra by Theorem 8.5 of [4] and the proof of the principal result of [7]. Hence $H_*(MH)$ is an $1-r$ Hopf algebra.

Since $h_n g_n$ is the standard map, we get a map of ring spectra $MO \to MH$ which is easily seen to induce an $l-r$ Hopf algebra map. Thus MH is a super O theory.

For any super O theory we have

THEOREM 5. *MH is a product of HZ_2's. There exists a Z_2-vector space C_* and isomorphisms $MH_* \to MO_* \otimes C_*$ and $H_*(MH) \to H_*(MO) \otimes C_*$. If the image of $H_*(MO)$ in $H_*(MH)$ commutes with all of $H_*(MH)$, then C_* becomes a ring and the above maps are ring isomorphisms.*

Notice that we have required no finiteness hypothesis on $H_*(MH)$ and so we can apply Theorem 5 to some of the "bundle" theories of Quinn [10]. If BH is weakly homotopy commutative, $H_*(MH)$ is commutative.

Proof. Brown and Peterson [4] produce a map $H^*(MO) \to H^*(MH)$ which can be de-dualed to get a map $r : H_*(MH) \to H_*(MO)$. We can do this since $H_*(MO)$ is finite in each dimension. The needed result from linear algebra is that, if

$$T : \mathrm{Hom}_F(F^n, F) \to V^*$$

is a linear map, then there exists a linear map $S : V \to F^n$ with $T = S^*$. S is defined by the equation $\pi_i \circ S = T(\pi_i)$, where $\pi_i : F^n \to S$ is the ith co-ordinate projection.

r is a map of coalgebras and left and right comodules. To see that it is a ring map, note that both ways of going from $H_*(MH) \otimes H_*(MH)$ to $H_*(MO)$ are maps of coalgebras and left and right comodules. Since there is only one such map from $H_*(MH) \otimes H_*(MH) \to H_*(MO)$ [4; Corollary 8.6], r is a ring map. This uniqueness also shows that r splits $H_*(MO) \to H_*(MH)$.

Just as in part 2, $MH_* \otimes H_*(HZ_2) \cong H_*(MH)$, but now only as abelian groups. Still, MH is a product of HZ_2's. Let C_* be $H_*(MH)$ modulo the subgroup R_*, where R_* is the subgroup generated by all elements of the form $m \cdot h$, where $h \in H_*(MH)$ and $m \in H_i(MO)$ with $i > 0$. The map $\phi : H_*(MH) \to H_*(MO) \otimes C_*$ is given by $H_*(MH) \to H_*(MH) \otimes H_*(MH) \to H_*(MO) \otimes C_*$. Split the projection to C_* so that $C_* \to H_*(MH) \to \tilde{H}_*(MO)$ is zero. The structure map $H_*(MO) \to H_*(MH)$ and the product give a map $H_*(MO) \otimes C_* \to H_*(MH)$ and the composite with ϕ can be checked to be an isomorphism.

Any element in $H_*(MH)$ can be written as $c+\sum_i m_i h_i$ where c is something from the splitting of C_*, m_i is from $H_*(MO)$ with $* > 0$. Since $H_0(MH) = C_0$, induction on the grading proves that the image of C_* generates $H_*(MH)$ as an $H_*(MO)$ module. Hence ϕ is an isomorphism. If the image of $H_*(MO)$ in $H_*(MH)$ commutes with $H_*(MH)$, R_* becomes a two-sided ideal. Hence C_* is a ring and ϕ becomes a ring isomorphism.

The reader can check that C_* is always a coalgebra and a right and left comodule over the dual of the Steenrod algebra. ϕ can be seen to be a map of $l-r$ Hopf algebras. This recovers all of the Browder–Liulevicius–Peterson results on the structure of C_*.

The map $MH_* \to H_*(MH) \to C_*$ is also onto since $H_*(HZ_2) \to H_*(MH)$ can be picked to factor through $H_*(MO)$. Splitting this gives a map $MO_* \otimes C_* \to MH_*$ and, as before, the image of C_* generates over MO_*. The map $MH_* \to MO_* \otimes C_*$ is given by $MH_* \to H_*(MH) \to H_*(MO) \otimes C_* \to MO_* \otimes C_*$. The composite $MO_* \otimes C_* \to MH_* \to MO_* \otimes C_*$ is again checked to be an isomorphism, and the rest of the proof follows easily.

A super SO theory is a connective ring spectrum MSH, whose homology is an $l-r$ Hopf algebra, and a map of ring spectra $MSO \to MSH$ which induces an $l-r$ Hopf algebra map on homology. Further we require that Sq^1 is zero on $H^0(MSH)$.

This last condition guarantees that the map $H_*(MSH) \to H_*(MO)$ factors through $H_*(MSO)$. We can now analyse $MSHZ_2$ as above. We leave the details to the reader.

We finish with

THEOREM 6. *All the k-invariants of* $MSHZ_{(2)}$ *are* 0.

Proof. The sphere spectrum S is the unit for the ring spectra $MSOZ_{(2)}$ and $MSHZ_{(2)}$. The map $S \to MSOZ_{(2)}$ factors through $HZ_{(2)}$ by Theorem 4.

$$MSH \wedge S \to MSH \wedge HZ_{(2)} \to MSH \wedge MSOZ_{(2)} \to MSH \wedge MSHZ_{(2)} \to MSHZ_{(2)}$$

shows that $(MSHZ_{(2)})_*$ is a summand of $H_*(MSH : Z_{(2)})$. But

$$H_*(MSH; Z_{(2)}) = H_*(MSHZ_{(2)}; Z)$$

since both are the homotopy of $MSH \wedge M_{(2)}$, where $M_{(2)}$ is the Moore spectrum of type $Z_{(2)}$.

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Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556
U.S.A.