Unoriented geometric functors

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Abstract. Farrell and Hsiang [2, p. 102] noticed that [5] implies that the geometric surgery groups defined in [6, Chapter 9] do not have the naturality Wall claims for them. Augmenting Wall's definitions using spaces over $\mathbb{R}\mathbb{P}^{\infty}$ and line bundles they fixed the problem.

The definition of geometric Wall groups involves homology with local coefficients and these also lack Wall's claimed naturality.

One would hope that a geometric bordism theory involving non-orientable manifolds would enjoy the same naturality as that enjoyed by homology with local $\mathbb{Z}$ coefficients. A setting for this naturality entirely in terms of local $\mathbb{Z}$ coefficients is presented in this paper.

Applying this theory to the example of non-orientable Wall groups restores much of the elegance of Wall's original approach. Furthermore, a geometric determination of the map induced by conjugation by a group element is given as well as a discussion of further cases beyond the reach of [5].

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1 A review of local coefficients

A *local coefficient system* is a functor from the path groupoid, $\Pi(X)$, of a space X to some category, [4, page 58]. Two coefficient systems are *equivalent* if there is a natural transformation between the two functors. In the case of interest here the category has one object, the group $\mathbb{Z}$, and the morphisms are $\text{Aut}(\mathbb{Z}) = \{\pm 1\}$. Such a local coefficient system will be called a $\mathbb{Z}^{t}$-system.

Definition 1.1. For any space X, let $\mathcal{Z}^{t}(X)$ denote the category whose objects are $\mathbb{Z}^{t}$-systems on X and whose morphisms are the natural transformations between them.

The data for such a system on a space X can be packaged as a function $\Lambda : \Pi(X) \to \{\pm 1\}$ which is a homomorphism of groupoids. One such system is the *trivial* $\mathbb{Z}^{t}$-system which assigns $+1$ to every path. A natural transformation, or morphism, or just map, between Λ_0 and Λ_1 is a function $\zeta : X \to \{\pm 1\}$ such that for every path $\lambda \in \Pi(X)$,

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$$\Lambda_0(\lambda) \cdot \Lambda_1(\lambda) = \zeta(\lambda(0)) \cdot \zeta(\lambda(1)).$$

Given any $\mathbb{Z}^t$-system Λ on X check that for any based loop L in X, $\Lambda(L)$ is independent of the base point and defines a homomorphism $H_1(X; \mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$. Define the *twist* of Λ to be the corresponding element $\omega(\Lambda) \in H^1(X; \mathbb{Z}/2\mathbb{Z})$. Furthermore, check that two $\mathbb{Z}^t$-systems Λ_0 and Λ_1 on X are equivalent if and only if $\omega(\Lambda_0) = \omega(\Lambda_1)$. If $\omega(\Lambda)$ is trivial, Λ is said to be *untwisted*.

Here is a description of all $\mathbb{Z}^t$-systems on a space X. Pick a set of base points $\{b_i\}$, one in each path component of X. For each point $x \in X$ there is a unique i such that x and b_i can be joined by a path. For each x pick one path λ_x joining x to b_i. Use the constant path to join b_i to b_i. This choice is called a *polarization* of $\Pi(X)$ and can be characterized as a function $s : X \to \Pi(X)$ such that $s(x)(0) = x$, $s(x)(1) = b_i$ and $s(b_i)$ is constant.

Lemma 1.2. *Fix any $\omega \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ and any polarization s of $\Pi(X)$. Then any function $f : X \to \{\pm 1\}$ such that $f(b_i) = 1$ for all i can be uniquely extended to a $\mathbb{Z}^t$-system Λ such that $\omega(\Lambda) = \omega$ and $f = \Lambda \circ s$.*

Proof. Given a path λ from x_0 to x_1, both points lie in a path component with base point b_i. There exists a unique $\gamma \in \pi_1(X, b_i)$ such that γ is equivalent to $s(x_0) \cdot \gamma \cdot s(x_1)^{-1}$. Define $\Lambda(\lambda) = f(x_0) \cdot \omega(\gamma) \cdot f(x_1)$. □

Given a subspace $A \subset X$ and a polarization $s_A : A \to \Pi(A)$, construct a polarization $s_{X, s_A} : X \to \Pi(X)$ as follows. Pick base points $\{b_j\} \in X$. If $a_i \in A$ is one of the base points in A, let $s_{X, s_A}(a_i)$ be any path from a_i to the appropriate b_j. If $x \in A$, define $s_{X, s_A}(x) = s_A(x) \cdot s_{X, s_A}(a_i)$ where x is joined in A to a_i. If $x \in X - A$ then choose any path from x to the appropriate b_j using the constant path whenever possible.

Proposition 1.3. *Let $A \subset X$ and suppose Λ_A is a $\mathbb{Z}^t$-system on A. If there exists a class $\omega \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ which restricts to $\omega(\Lambda_A)$, then there exists Λ_X on X which restricts to Λ_A and which satisfies $\omega(\Lambda_X) = \omega$.*

Proof. Fix a polarization s_A of A and a polarization s_{X, s_A} of X. Define $f : X \to \{\pm 1\}$ as follows. If $x \in A$, define $f(x) = \Lambda_A(s_A(x))$ and if $x \in X - A$ let $f(x) = 1$. Let Λ_X be the $\mathbb{Z}^t$-system constructed by Lemma 1.2 using ω for the twist. □

Note that if ζ is a morphism, so is $-\zeta$. Indeed the sign can be switched or not on each path component so the set of morphisms between two $\mathbb{Z}^t$-systems on X is an $H^0(X; \mathbb{Z}/2\mathbb{Z})$-torsor. An element $\mathfrak{c} \in H^0(X; \mathbb{Z}/2\mathbb{Z})$ is equivalent to a homomorphism $H_0(X; \mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ and hence to a function $X \to \{\pm 1\}$ which is constant on path components. If ζ is a morphism, let $\mathfrak{c} \bullet \zeta$ be defined by $\mathfrak{c} \bullet \zeta(x) = \mathfrak{c}(x) \cdot \zeta(x)$ for all $x \in X$.

Proposition 1.4. *Let $\iota : A \subset X$ and suppose Λ_0 and Λ_1 are $\mathbb{Z}^t$-systems on X with $\omega(\Lambda_0) = \omega(\Lambda_1)$. Suppose $\zeta_A : \iota^*(\Lambda_0) \to \iota^*(\Lambda_1)$ is given. Then ζ_A extends to $\zeta_X : \Lambda_0 \to \Lambda_1$ provided one of the properties below holds.*

(1) $H^0(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\iota^*} H^0(A; \mathbb{Z}/2\mathbb{Z})$ *is onto.*

(2) *Pick base points $\{a_i\}$ for A and for each path component of X pick a base point b_j to be one of the a_i. Pick paths λ_i from a_i to the appropriate b_j. For each i require $\Lambda_0(\lambda_i) \cdot \Lambda_1(\lambda_i) = \zeta_A(\lambda_i(0)) \cdot \zeta_A(\lambda_i(1))$.*

Proof. Since the twists are the same, there exists $\zeta_X' : \Lambda_0 \to \Lambda_1$ which restricts to $\zeta_A' : \iota^*(\Lambda_0) \to \iota^*(\Lambda_1)$. Let $\mathfrak{c}_A \in H^0(A; \mathbb{Z}/2\mathbb{Z})$ denote the element such that $\zeta_A = \mathfrak{c}_A \bullet \zeta_A'$. If (1) holds, pick $\mathfrak{c}_X \in H^0(X; \mathbb{Z}/2\mathbb{Z})$ which restricts to $\mathfrak{c}_A$ and check $\zeta_X = \mathfrak{c}_X \bullet \zeta_X'$.

For case (2), pick base points $\{b_j\}$ for X so that each $b_j = a_{i_j}$. Since $\omega(\Lambda_0) = \omega(\Lambda_1)$ choose an equivalence $\zeta_X : \Lambda_0 \to \Lambda_1$. With a bit of care, further require that $\zeta_X(a_{i_j}) = \zeta_A(a_{i_j})$ for all j. It follows from the hypotheses that $\zeta_X|_A = \zeta_A$. $\square$

Proposition 1.5. *Let Λ_0 be a $\mathbb{Z}^t$-system on X and let Λ_A be a $\mathbb{Z}^t$-system on A where $\iota : A \to X$ is the inclusion. Let $\zeta : \Lambda_A \to \iota^*(\Lambda_0)$ be a map. Then there exists a $\mathbb{Z}^t$-system Λ_1 on X such that Λ_1 restricted to A is equal to Λ_A and ζ extends to a map $\zeta : \Lambda_1 \to \Lambda_0$.*

Proof. Pick a polarization $s_X : X \to \Pi(X)$ and let $f_0 : X \to \{\pm 1\}$ be $\Lambda_0 \circ s_X$. Define $f_1 : X \to \{\pm 1\}$ by $f_1(x) = f_0(x) \cdot \zeta(x)$ if $x \in A$ and 1 otherwise. Then Λ_1 is the $\mathbb{Z}^t$-system from Lemma 1.2. $\square$

2 The category $\mathcal{Z}^t(\mathbf{Top})$

A map $f : X_0 \to X_1$ induces a functor $\Pi(X_0) \to \Pi(X_1)$ and hence a functor $f^* : \mathcal{Z}^t(X_1) \to \mathcal{Z}^t(X_0)$. Composition and identity behave correctly, so there is a functor from the category of topological spaces and continuous functions to the category of categories which takes X to the category $\mathcal{Z}^t(X)$ and f to the functor f^*.

Define a category $\mathcal{Z}^t(\mathbf{Top})$ whose objects are all pairs (X, Λ_X) for X a topological space and $\Lambda_X \in \mathcal{Z}^t(X)$. A morphism $(X_0, \Lambda_{X_0}) \to (X_1, \Lambda_{X_1})$ is a pair (f, ζ) where $f : X_0 \to X_1$ is any map and $\zeta : \Lambda_{X_0} \to f^*(\Lambda_{X_1})$ is any natural transformation.

Composition is given by $(g, \psi) \circ (f, \zeta) = (g \circ f, f^*(\psi) \circ \zeta)$. The identity for (X, Λ) is given by $(1_X, 1_\Lambda)$.

Definition 2.1. The category $\mathcal{Z}^t(\mathbf{Top})$ has a pairing resembling a product. Given $(X_i, \Lambda_{X_i}) \in \mathrm{Obj}(\mathcal{Z}^t(\mathbf{Top}))$, $i = 0, 1$, define a new object

$$(X_0 \times X_1, \Lambda_{X_0} \bullet \Lambda_{X_1}).$$

Recall $\Pi(X_0 \times X_1) = \Pi(X_0) \times \Pi(X_1)$ so define

$$\Lambda_{X_0} \bullet \Lambda_{X_1}(\lambda_0 \times \lambda_1) = \Lambda_{X_0}(\lambda_0) \cdot \Lambda_{X_1}(\lambda_1).$$

This construction is not a categorical product since there is no projection onto X_0 unless Λ_{X_1} is untwisted. But if $(g_i, \zeta_i) : (Y, \Lambda_Y) \to (X_i, \Lambda_{X_i})$ are given, there is a morphism

$$(g_0 \times g_1, \zeta_0 \bullet \zeta_1) : (Y, \Lambda_Y) \to (X_0 \times X_1, \Lambda_{X_0} \bullet \Lambda_{X_1})$$

Definition 2.2. The category $\mathcal{Z}^t(Top)$ has an involution: it is the identity on objects and sends (f, ζ) to $(f, -\zeta)$.

Remarks 2.3. Given a bundle over X orient each fibre and associate a $\mathbb{Z}^t$-system as follows. Given a path in X, use the homotopy lifting property to get a map from the fibre over the initial point to the fibre over the terminal point of the path. Assign $+1$ to this path if the map is orientation preserving, -1 otherwise. This is a $\mathbb{Z}^t$-system and the twist is the first Stiefel-Whitney class of the bundle. There is no preferred choice of orientations if the bundle is non-orientable and even if it is, there are two choices over each path component of X. Similar remarks work for spherical fibrations.

3 Homotopy functors

The remaining sections describe functors out of subcategories of $\mathcal{Z}^t(Top)$ or its opposite. Many of these functors have a homotopy invariance which is described next.

Definition 3.1. Let $\mathcal{Z}^t(\mathcal{T})$ be a subcategory of $\mathcal{Z}^t(Top)$ and let $\theta_{[0,1]}$ denote the trivial system on $[0, 1]$. Say $\mathcal{Z}^t(\mathcal{T})$ *has homotopies* provided the following properties hold.

(1) If $(X, \Lambda) \in \mathrm{Obj}(\mathcal{Z}^t(\mathcal{T}))$ then

$$(X \times [0, 1], \Lambda \bullet \theta_{[0,1]}) \in \mathrm{Obj}(\mathcal{Z}^t(\mathcal{T})).$$

(2) For $i = 0$ and $i = 1$ the morphisms

$$(\iota_i, 1_\Lambda) : (X, \Lambda) \to (X \times [0, 1], \Lambda \bullet \theta_{[0,1]})$$

are in $\mathcal{Z}^t(\mathcal{T})$ where ι_i is the evident inclusion. Since $\iota_i^*(\Lambda \bullet \theta_{[0,1]})$ is identical to Λ, 1_Λ is a morphism between the relevant $\mathbb{Z}^t$-systems.

(3) Let $(f_i, \zeta_i) : (X, \Lambda_X) \to (Y, \Lambda_Y)$, $i = 0, 1$ be two maps in $\mathcal{Z}^t(\mathcal{T})$ whose f_i are homotopic. For any homotopy $F : X \times [0, 1] \to Y$ require the existence of a map $\zeta_F : \Lambda_X \bullet \theta_{[0,1]} \to F^*(\Lambda_Y)$ such that (F, ζ_F) is a map in $\mathcal{Z}^t(\mathcal{T})$ and the composition $(F, \zeta_F) \circ (\iota_0, 1_{\zeta_X})$ is (f_0, ζ_0).

(4) If $(f, \zeta_i) : (X, \Lambda_X) \to (Y, \Lambda_Y)$ $i = 0, 1$ are two morphisms with the same f, then there exists $\mathfrak{c} \in H^0(X; \mathbb{Z}/2\mathbb{Z})$ such that $\zeta_0 = \mathfrak{c} \bullet \zeta_1$. Require that $(1_X, \mathfrak{c} \bullet 1_{\Lambda_X})$ is a map in $\mathcal{Z}^t(\mathcal{T})$.

Remarks 3.2. The category $\mathcal{Z}^t(Top)$ has homotopies. The map ζ_F required by (3) is unique. If $r : X \times [0, 1] \to X$ denotes the evident projection, r is a homotopy from 1_X

to itself. The $\mathbb{Z}^t$-system $r^*(\Lambda) = \Lambda \bullet \theta_{[0,1]}$ so if ζ_r is the identity, $(r, 1_{\Lambda \bullet \theta_{[0,1]}}) \circ (\iota_i, 1_\Lambda) = (1_X, 1_\Lambda)$ for both $i = 0$ and $i = 1$.

Definition 3.3. A functor Ω_* defined on $\mathcal{Z}^t(\mathcal{T})$ is a *homotopy functor* provided $\mathcal{Z}^t(\mathcal{T})$ has homotopies and

$$\Omega_*(\iota_0, 1_\Lambda) \circ \Omega_*(r, 1_{\Lambda \bullet \theta_{[0,1]}}) = 1_{\Omega_*(X \times [0,1], \Lambda \bullet \theta_{[0,1]})}$$

There is a similar definition for functors out of $\mathcal{Z}^t(\mathcal{T})^{op}$.

Lemma 3.4. *If Ω is a homotopy functor the two maps*

$$\Omega_*(\iota_i, 1_\zeta) : \Omega_*(X, \Lambda) \to \Omega_*(X \times [0,1], \Lambda \bullet \theta_{[0,1]})$$

$i = 0, 1$ *are equal.*

Proof. The map $\Omega_*(r, 1_{\Lambda \bullet \theta_{[0,1]}})$ is an isomorphism inverse to both the maps $\Omega_*(\iota_0, 1_\Lambda)$ and $\Omega_*(\iota_1, 1_\Lambda)$. By uniqueness of inverse, $\Omega_*(\iota_0, 1_\Lambda) = \Omega_*(\iota_1, 1_\Lambda)$. $\qquad\square$

Even for a homotopy functor homotopic maps need not induce the same map. To see the problem, let $F : X \times [0,1] \to Y$ be a homotopy between $f_i : X \to Y$ so $f_i = F \circ \iota_i$. Let $\zeta_F : \Lambda_X \bullet \theta_{[0,1]} \to F^*(\Lambda_Y)$ be the map such that $(F, \zeta_F) \circ (\iota_0, 1_{\Lambda_X})$ is (f_0, ζ_0). Then let $(F, \zeta_F) \circ (\iota_1, 1_{\Lambda_X}) = (f_1, \zeta')$ so by 3.4 $\Omega_*(f_0, \zeta_0) = \Omega_*(f_1, \zeta')$.

The problem is that ζ' may not be ζ_1. Since ζ' and ζ_1 are maps between the same two $\mathbb{Z}^t$-systems, there exists a $\mathfrak{c}_F \in H^0(X; \mathbb{Z}/2\mathbb{Z})$ such that $\zeta' = \mathfrak{c}_F \bullet \zeta_1$. Then by (4),

$$\Omega_*(f_0, \zeta_0) = \Omega_*(f_1, \zeta_1) \circ \Omega_*(1_X, \mathfrak{c}_F \bullet 1_{\Lambda_X}).$$

It remains to identify $\mathfrak{c}_F$.

Theorem 3.5. *For all $x \in X$*

$$\mathfrak{c}_F(x) = \Lambda_Y(F(x \times t)) \cdot \zeta_0(x) \cdot \zeta_1(x)$$

Proof. By definition $\mathfrak{c}_F(x) = \zeta_1(x) \cdot \zeta'(x)$. Moreover $\zeta'(x) = \zeta_F(x \times 1)$ and $\zeta_0(x) = \zeta_F(x \times 0)$. Also by definition,

$$\zeta_F(x \times 0) \cdot \zeta_F(x \times 1) = F^*(\Lambda_Y)(x \times t) \cdot (\Lambda_X \bullet \theta_{[0,1]})(x \times t).$$

But $(\Lambda_X \bullet \theta_{[0,1]})(x \times t) = r^*(\Lambda_X)(x \times t) = \Lambda_X(r(x \times t)) = 1$ since $r(x \times t)$ is a constant path. Similarly $F^*(\Lambda_Y)(x \times t) = \Lambda_Y(F(x \times t))$. $\qquad\square$

Corollary 3.6. *Suppose X is path connected and there exists a point $x \in X$ such that $f_0(x) = f_1(x)$. Then $F(x \times t)$ is a loop L in Y and $\Lambda_Y(F(x \times t)) = \omega(L)$.*

If X is path connected, then $\mathfrak{c}_F = \pm 1$ and $\Omega_*(1_X, \mathfrak{c}_F \bullet 1_{\Lambda_X})$ is the identity if $\mathfrak{c}_F = +1$ and is the map induced by the involution on $\mathcal{Z}^t(Top)$ if $\mathfrak{c}_F = -1$ (2.2).

Definition 3.7. Any additive category $\mathscr{A}$ has an involution which is the identity on objects and sends f to $-f$ in the abelian group of morphisms between any two objects. If Ω_* takes values in $\mathscr{A}$, say Ω_* *preserves the involution* provided $\Omega_*(f, -\zeta) = -\Omega_*(f, \zeta)$, provided both $(f, \pm\zeta) \in \mathcal{Z}^t(\mathcal{T})$. There is a similar definition if Ω_* is a functor out of $\mathcal{Z}^t(\mathcal{T})^{op}$.

4 Twisted homology and twisted cohomology

There is a detailed treatment of this material in [7, Chapter VI] but here is a quick review. Twisted homology and cohomology come from modifying the singular chain complex $S_*(X)$. The chain groups are the same, but the boundary is changed. Recall that $\partial = \sum_i (-1)^i \partial_i$. If Λ is a $\mathbb{Z}^t$-system, $\partial^\Lambda = \sum_i (-1)^i \partial_i^\Lambda$ where ∂_i^Λ is defined as follows. Given a singular r-simplex $\sigma \to X$, $\partial_i \sigma$ is a $(r-1)$-simplex. Define $\partial_i^\Lambda \sigma$ to be $\partial_i \sigma$ multiplied by Λ applied to the path obtained by applying σ to the straight line from the barycenter of σ to the barycenter of $\partial_i \sigma$. This can be checked to be a chain complex $S_*^\Lambda(X)$. Homology and cohomology with twisted coefficients is defined as the homology or cohomology of S_*^Λ.

Given a natural transformation $\zeta : \Lambda_0 \to \Lambda_1$, define a chain map $S(\zeta)_* : S_*^{\Lambda_0}(X) \to S_*^{\Lambda_1}(X)$ by sending σ to $\pm\sigma$ where ± 1 is the value of the natural transformation applied to the barycenter of σ. This chain map induces an isomorphism on twisted homology and cohomology groups.

It follows that S_*^Λ and the twisted homology and cohomology groups are functors out of $\mathcal{Z}^t(Top)$ or $\mathcal{Z}^t(Top)^{op}$. The twisted homology and cohomology are homotopy functors. Furthermore S_*^Λ, H_* and H^* all preserve involution (2.2).

The Alexander-Whitney diagonal map induces a chain map

$$S_*^{\Lambda_{X_0}}(X_0) \otimes S_*^{\Lambda_{X_1}}(X_1) \to S_*^{\Lambda_{X_0} \bullet \Lambda_{X_1}}(X_0 \times X_1)$$

so the usual products have twisted versions: given two systems, Λ_0 and Λ_1, there is a bilinear cap product

$$\cap : H^r(X; \mathbb{Z}^{\Lambda_0}) \otimes H_{r+s}(X; \mathbb{Z}^{\Lambda_1}) \to H_s(X; \mathbb{Z}^{\Lambda_0 \bullet \Lambda_1}).$$

5 Poincaré duality spaces

A remark that the author has found helpful is that *non-orientable manifolds can not be oriented*. Without paying enough attention to naturality one can come away with a vague feeling that they can: see the remarks at the end of this section.

If X is path-connected, there can be at most one class of local coefficients Λ and one integer m such that $H_m(X; \mathbb{Z}^\Lambda) \cong \mathbb{Z}$ and such that, if $[X]$ is a generator,

$$(5.1) \quad \cap[X] : H^r(X; \mathbb{Z}^{\Lambda'}) \to H_{m-r}(X; \mathbb{Z}^{\Lambda \bullet \Lambda'})$$

is an isomorphism for all r and all Λ'.

If the first Stiefel-Whitney class $w_1(M) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ of a connected, closed, compact manifold or Poincaré duality space is used as the twist, (5.1) holds. When $w_1(M) = 0$ write $H_*(M; \mathbb{Z})$ for homology with trivial $\mathbb{Z}^t$-system. The choice of a generator in $H_m(M; \mathbb{Z})$ is often called an orientation and M is said to be oriented. Given two oriented connected, closed, compact manifolds M_1^m and M_2^m and a map $f : M_1 \to M_2$, f has a degree.

In the non-orientable case the $\mathbb{Z}^t$-system is non-trivial and even in the orientable case untwisted but non-trivial systems can appear. Whenever the $\mathbb{Z}^t$-system is non-trivial, the class $[M] \in H_m(M; \mathbb{Z}^\Lambda)$ will be called a *fundamental class*. Given a map $f : M_0 \to M_1$ which preserves the first Stiefel-Whitney class, there is an induced map

$$(f, \zeta)_* : H_m(M_0; \mathbb{Z}^{\Lambda_{M_0}}) \to H_m(M_1; \mathbb{Z}^{\Lambda_{M_1}})$$

and $(f, \zeta)_*$ has a degree, but if ζ is replaced by $-\zeta$, the degree switches sign.

Remarks 5.2. The Farrell-Hsiang repair of Wall's problem can be viewed as checking that one can fix a $\mathbb{Z}^t$-system over $\mathbb{RP}^\infty$ and always use the pull-back local system. Whitehead [7, Chapter VI] also discusses naturality using pull-back systems. This results in the following conundrum. Consider the projection $p : S^{2n} \to \mathbb{RP}^{2n}$ and the antipodal map α on S^{2n}: α commutes with p so there is a commutative triangle of twisted homology groups starting with any Λ on $\mathbb{RP}^{2n}$ with non-trivial twist. The induced system on S^{2n} is untwisted, but not trivial and α ends up having degree $+1$. This is all correct but unsettling.

In the approach taken here, one can work with the trivial system on S^{2n}, in which case α has degree -1 but now one must choose an isomorphism between the system induced by p and the trivial one, say ζ. Then the projection maps are (p, ζ) and $(p, -\zeta)$, α has degree -1 and commutativity still holds. But this shows that $\mathbb{RP}^{2n}$ can not be "oriented" by orienting S^{2n} and using "p_*" to pick out a fundamental class.

Remarks 5.3. Here is another approach to "orientation". Twisted homology satisfies excision, so $H_m(M, M - p; \mathbb{Z}^\Lambda) = H_m(D^m, S^{m-1}; \mathbb{Z}^\Lambda|_{D^m})$. Since D^m is simply-connected, $H_m(D^m, S^{m-1}; \mathbb{Z}^\Lambda|_{D^m}) \cong \mathbb{Z}$.

There is always a map $H_m(M; \mathbb{Z}^\Lambda) \to H_m(M, M - p; \mathbb{Z}^\Lambda)$ and if $\omega(\Lambda) = w_1(M)$, a fundamental class $[M] \in H_m(M; \mathbb{Z}^\Lambda)$ picks out a generator in $H_m(D^m, S^{m-1}; \mathbb{Z}^\Lambda|_{D^m})$. Pick the unique morphism from the trivial system to $\mathbb{Z}^\Lambda|_{D^m}$ which is $+1$ at the point p. This gives an orientation on D^m.

However, whenever Λ is not trivial, given any cover of M by disks there will be disks in the cover for which the selected orientation depends on p.

6 Geometric bordism

As a warm up exercise, consider unoriented bordism. For the spaces in the category $\mathcal{Z}^t(\mathcal{T})$ take $BO \times [0, 1]^k$ for all integers $k \geq 0$. For the $\mathbb{Z}^t$-systems take all the ones with non-trivial twist and all their morphisms, and for the maps of spaces take all homotopy equivalences. Fix a $\mathbb{Z}^t$-system Λ_{BO} on $BO \times [0, 1]^k$ with non-trivial twist.

Hence an object in this "enhanced" bordism theory is a manifold M with a $\mathbb{Z}^t$-coefficient system Λ_M, a generator $[M] \in H_m(M; \mathbb{Z}^{\Lambda_M})$, a map $v : M \to BO \times [0,1]^k$ classifying the stable normal bundle and a natural isomorphism $\zeta : \Lambda_M \to v^*(\Lambda_{BO})$.

The definitions for a manifold with boundary present no problems, but manifolds should bound which are not actually equal to a boundary but merely "equivalent" to one. This involves a diffeomorphism of M with an actual boundary and an identification of fundamental classes. Since there is no natural choice of ζ, this requires making $(M, v, \mathfrak{c} \bullet [M], \mathfrak{c} \bullet \zeta)$ equal to $(M, v, -[M], -\zeta)$. Alternatively one can use Proposition 1.5 to put a $\mathbb{Z}^t$-system on $M \times [0,1]$ which alters the sign of ζ at the two ends.

Bordism defines a homotopy functor on $\mathcal{Z}^t(\mathcal{T})$ to abelian groups which preserves orientation.

Remark 6.1. These definitions can be repeated with BSO. Oriented bordism has elements of infinite order, so why are not similar elements present in the unoriented case?

To answer this last question, recall that BO has a homotopy $F : BO \times [0,1] \to BO$ which is the identity at both ends and such that for each $b \in BO$, $F(b \times [0,1])$ is a loop which is not null homotopic. Corollary 3.6 implies that the identity map has order 2 so every element in the bordism group has order 2.

In the oriented case no such homotopy exists.

The case of $Pin^{\pm}$ structures is interesting from this point of view as well. The bundles are not orientable, but $BPin^{\pm} = BSpin \times K(\mathbb{Z}/2\mathbb{Z}, 1)$ so there is a homotopy like that for BO. However the $Pin^{\pm}$ bordism groups are not always of exponent 2. Here the problem is that the homotopy is not a homotopy of lifts over BO.

7 Geometric surgery

The definition of Wall's geometric surgery groups mimic the unoriented bordism example. Start with a reference n-ad K and fix a $\mathbb{Z}^t$-coefficient system, Λ_K. An object in the bordism theory is a Poincaré $(n+1)$-ad X, a manifold $(n+1)$-ad M, $\mathbb{Z}^t$-coefficient systems Λ_X on X and Λ_M on M, classes $[M] \in H_m(M; \mathbb{Z}^{\Lambda_M})$ and $[X] \in H_m(X; \mathbb{Z}^{\Lambda_X})$, maps $g : M \to X$ and $f : X \to K$, and finally, isomorphisms $\zeta : \Lambda_X \to f^*(\Lambda_K)$ and $\psi : \Lambda_M \to g^*(\Lambda_X)$ such that $(g, \psi)_*$ has degree 1. The map g should be covered by a normal map and should have some sort of equivalence on part of the boundary depending on exactly which variant of the Wall group is being constructed.

It is better to denote the resulting bordism groups by $L_*(K, \Lambda_K)$ instead of just recording the twist as Wall does. Since the torsion is irrelevant to these discussions it is suppressed.

With these minor alterations in the definition, the material in Wall's Chapter 9 goes through with no difficulty, except for one caveat: the resulting bordism groups are homotopy functors on the full subcategory of $\mathcal{Z}^t(Top)$ whose spaces are the homotopy type of CW complexes with a finite 2-skeleton.

Similar results hold for many other variations of surgery theory in the non-orientable case including the LS groups of [6, Chapter 11], the Cappel-Shaneson Γ groups [1], and many others.

8 Surgery groups

The philosophical import of Wall's Chapter 9 is that any "geometric bordism theory" which has a π-π theorem depends only on the fundamental group.

More explicitly, if K is path-connected, let $u : K \to K(\pi_1(K), 1)$ be a classifying map for the universal cover. The map u induces an isomorphism

$$H^1(K(\pi_1(K), 1); \mathbb{Z}/2\mathbb{Z}) \to H^1(K; \mathbb{Z}/2\mathbb{Z})$$

so the set of twists for the two spaces coincide. There is an explicit model for $K(\pi, 1)$ due to Segal [3] denoted $B\pi$ and there is a particular $\mathbb{Z}^t$-system Λ_ω with twist $\omega \in H^1(B\pi; \mathbb{Z}/2\mathbb{Z})$.

The explicit model is Segal's classifying space of a small category where the category is the group in question. There is one 0-simplex which will be the base point. Each k-simplex for $k > 0$ is a sequence of k group elements. If G is the group, let BG denote the classifying space. Recall that BG is a CW complex and it has the homotopy type of a CW complex with finite 2 skeleton if and only if G is finitely generated and finitely presented.

The space BG has an explicit polarization, s_G, defined as follows. Each point $x \in BG$ lies in the interior of a unique simplex and there is always the straight line path to the initial vertex of this simplex. This gives a path from $x \in BG$ to the base point. Denote the path by $s_G(x)$.

Given a twist $\omega \in H^1(BG; \mathbb{Z}/2\mathbb{Z})$ define Λ_ω to be the $\mathbb{Z}^t$-system on BG from Lemma 1.2 with polarization s_G, function $f : BG \to \{\pm 1\}$ identically 1 and twist ω.

If $\psi : H \to G$ is a group homomorphism then the induced map $B\psi : BH \to BG$ is piecewise linear on each simplex so

$$
\begin{array}{ccc}
BH & \xrightarrow{s_H} & \Pi(BH) \\
{\scriptstyle B\psi}\Big\downarrow & & \Big\downarrow{\scriptstyle \Pi(B\psi)} \\
BG & \xrightarrow{s_G} & \Pi(BG)
\end{array}
$$

commutes. It follows that $(B\psi)^*(\Lambda_\omega) = \Lambda_{B\psi^*(\omega)}$ so

$$(B\psi, 1_{\Lambda_{(B\psi)^*(\omega)}}) : (BH, \Lambda_{(B\psi)^*(\omega)}) \to (BG, \Lambda_\omega) \in \mathcal{Z}^t(Top).$$

Since $\Lambda_{(B\psi)^*(\omega)}$ is just the identity map, write $\mathbf{B}(\psi, \omega)$ for the map $(B\psi, 1_{\Lambda_{(B\psi)^*(\omega)}})$. Note that for group homomorphisms, $\psi_0 : G_0 \to G_1$ and $\psi_1 : G_1 \to G_2$, $\mathbf{B}((\psi_1 \circ \psi_0), \omega) = \mathbf{B}(\psi_1, (B\psi_0)^*(\omega)) \circ \mathbf{B}(\psi_0, \omega)$.

Segal's construction has an additional property. Any $g \in G$ gives an automorphism of G, $c_g : G \to G$ defined by $c_g(h) = g^{-1}hg$. This inner automorphism is homotopic to the identity since there is a natural transformation between the two functors, namely multiplication by $g \in G$. Let $C_g : BG \times [0, 1] \to BG$ be the homotopy from 1_{BG} at 0 to c_g at 1. The homotopy is not base point preserving unless g is the identity: in fact the path $C_g(t)$ is just the 1-simplex g.

Definition 8.1. A *surgery theory* is a homotopy functor Ω on a subcategory $\mathcal{Z}^t(\mathcal{T})$ of $\mathcal{Z}^t(Top)$ with the following closure property. Whenever (K, Λ) is an object in $\mathcal{Z}^t(\mathcal{T})$, then for any $u : K \to B\pi_1(K)$ which induces an isomorphism on π_1 the pair $(B\pi_1(K), \Lambda_{(u^*)^{-1}(\omega(\Lambda))})$ is in $\mathcal{Z}^t(\mathcal{T})$ and for at least one map $\zeta : \Lambda \to u^*(\Lambda_{(u^*)^{-1}(\omega(\Lambda))})$ the pair (u, ζ) is a map in $\mathcal{Z}^t(\mathcal{T})$. A surgery theory *only depends on* π_1 provided

$$\Omega_*(u, \zeta) : \Omega_*(K, \Lambda) \to \Omega_*(B\pi_1(K), \Lambda_{(u^*)^{-1}(\omega(\Lambda))})$$

an isomorphism.

Remark 8.2. Wall's geometric surgery groups are a surgery theory on the full subcategory of $\mathcal{Z}^t(Top)$ whose spaces have the homotopy type of CW complexes with finite 2-skeleton. They preserve the involution. If $* > 4$, then the Wall geometric surgery groups only depend on π_1.

Theorem 8.3. *Suppose* Ω_* *is a surgery theory on* $\mathcal{Z}^t(\mathcal{T})$ *and that* G *is a group such that* $(BG, \Lambda_\omega) \in \mathcal{Z}^t(\mathcal{T})$. *Then*

$$\Omega_*(\mathbf{B}(c_g, \omega)) : \Omega_*(BG, \Lambda_\omega) \to \Omega_*(BG, \Lambda_\omega)$$

is the identity if $\omega(g) = +1$ *and is* $\Omega_*(1_{BG}, -1_{\Lambda_\omega})$ *if* $\omega(g) = -1$. *If* Ω_* *preserves involution, then* $\Omega_*(\mathbf{B}(c_g, \omega))$ *is minus the identity if* $\omega(g) = -1$.

Proof. Apply Theorem 3.5 and Corollary 3.6: note $\zeta_0(x) = \zeta_1(x)$. □

Remark 8.4. Theorem 8.3 is the analogue of the main result in [5] for the geometric Wall groups: it implies that the surgery obstruction map [6, Cor. 9.4.1, p. 90] is natural.

Remark 8.5. This material also implies the geometric version of the conjecture in [5] concerning the map induced by conjugation. Without hypotheses the conjecture is false so some explanation may be helpful. Consider the pair case, $\iota : H \subset G$. Then BH is a subspace of BG and $\iota^*(\Lambda_\omega) = \Lambda_{\iota^*(\omega)}$. Let $\zeta : \Lambda_{\iota^*(\omega)} \to \iota^*(\Lambda_\omega)$ be the identity. The data for the pair should be written $\Omega_*((BG, \Lambda_\omega), (BH, \Lambda_{\iota^*(\omega)}), \zeta)$.

(1) Any homomorphism of pairs $\psi : (G_0, H_0) \to (G_1, H_1)$ induces a map of long exact sequences.

(2) If $g \in H$ then conjugation by g induces an automorphism of the entire long exact sequence of the pair and is multiplication by $\omega(g)$.

(3) If $g \in G - H$, then $g^{-1}Hg$ must be H before c_g is even a map of pairs. Fix $\psi : H \to H$. Then there are examples of groups G with elements g such that $g^{-1}Hg = H$ and the induced map on H is ψ. Hence nothing can be said in general about the map induced by conjugation on a pair if $g \notin H$.

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