ON THE GENERA OF KNOTS

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We wish to study knots, i.e. PL locally-flat embeddings $k : S^{2n-1} \subset S^{2n+1}$ and more specifically, their genera.

To define these genera we must first define a class of pairs of manifolds, say $\mathcal{S}(k)$, depending on k. $(M,W) \in \mathcal{S}(k)$ iff

(a) M is a compact orientable PL manifold with $\partial M = S^{2n+1}$

(b) W is a compact, PL, orientable, locally-flat submanifold of M with $W \cap \partial M = \partial W = S^{2n-1}$: $\partial W \subset \partial M$ is the knot k.

(c) the fundamental class of W determines, via the embedding, a class in $H_{2n}(M,\partial M,Z)$: this class is 0.

Define $g_\gamma(k) = \frac{1}{2} \min (\beta_n(W) + \beta_{n+}(M) + |\mathrm{Sign}(M)|)$ where (M,W) runs over all elements of $\mathcal{S}(k)$: Betti numbers are denoted by β and $\mathrm{Sign}(M)$ is the signature of M.

Define another genus, $g_s(k)$, to be $\frac{1}{2} \min \beta_n(W)$ where W runs over all PL manifolds such that $\pi_i(W) = 0$ for $0 \leq i < n$ and there is an embedding $W \to D^{2n+2}$ such that $(D^{2n+2},W) \in \mathcal{S}(k)$.

Clearly $g_\gamma(k) \leq g_s(k)$. Equally clearly these genera depend only on the concordance class of k.

For any knot, Levine [Le 1] defines a set of Seifert matrices. Let A be any one of them. If a is the dimension of A, A induces a bilinear form λ on Z^a by the formula $\lambda(x,y) = xAy^*$ where $*$ denotes transpose.

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Let $z(A)$ be the maximal dimension of a null-space of λ: a null-space of λ is a subspace $N \subset Z^a$ such that $\lambda(x,y) = 0$ for all $x, y \in N$. Define $m(k)$ to be $\frac{1}{2}(a-z(A))$. Lemma 1, stated at the end of section 2, shows $m(k)$ to be well-defined and to depend only on the concordance class of k. Notice that a Seifert matrix for k is null-cobordant in Levine's sense [Le 1] iff $m(k) = 0$.

Our first result is

Theorem 1 : $m(k) \leq g_\gamma(k)$.

Levine's techniques in [Le 1] and theorem 1 suffice to prove

Theorem 2 : If $n \geq 2$, $m(k) = g_s(k) = g_\gamma(k)$.

A great deal can be said even if $n = 1$. An immersion of a surface into S^3 will be called a Seifert ribbon if the immersion has only disjoint, simple, ribbon singularities (Fox [Fo 2] p. 72). The ribbon genus of a knot, $g_r(k)$, is the minimal genus of an orientable, compact, Seifert ribbon whose boundary is k. Fox's proof [Fo 1] that a ribbon knot is slice generalizes to show $g_s(k) \leq g_r(k)$. With these preliminaries completed we have

Theorem 3 : Let $k : S^1 \subset S^3$ be a knot which has A for a Seifert matrix. Then there exists a knot k_1 such that

(a) k_1 has A for a Seifert matrix

(b) $m(k_1) = g_r(k_1)$.

Theorem 3 could be proved by using the ideas in Fox [Fo 2], but we prefer to give a proof using a method of some independent interest.

Note that if $n \geq 2$ we have calculated any reasonable candidate for the special genus, and, if $n = 1$, we have given the best possible lower bound that one can get from a Seifert matrix. The results of Casson-Gordon [CG 1] show that the inequality $m(k) \leq g_s(k)$ can be strict.

$m(k)$ is not easy to compute but lower bounds for it are available. For any complex number of norm one, ξ, Levine [Le 1] defines a signature $\sigma_\xi(k)$. It is easy to see $\frac{1}{2}|\sigma_\xi(k)| \leq m(k)$. This, together with theorem 1, gives all the lower bounds for $g_s(k)$ to be found in [Mu 1], [Tr 1] or [KT 1].

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§ 2. The proofs of Theorem 1 and Lemma 1.

We recall the classical construction of the double branched cover of M along W. W is a proper $(W \cap \partial M = \partial W)$, orientable, locally-flat, codimension-two, PL submanifold of the orientable PL manifold M^m. If $H_{m-2}(W,\partial W;Z) \to H_{m-2}(M,\partial M;Z)$ is zero we can find $X \subset M$ with $X \cap \partial M = \partial_- X$; $\partial X = \partial_- X \cup W$; $\partial_- X \cap W = \partial W$; and X is a locally-flat, orientable, codimension-one submanifold of M.

$M(W)$, the double branched cover of M along W, is obtained by splitting M along X and glueing two copies of the resulting manifold together. This split manifold is the closure of M minus a regular neighborhood of X. It has the same homotopy type as $M - X$, so we denote it by $M - X$. With this abuse of notation firmly fixed we continue. The interesting part of the boundary of $M - X$ is just two copies of X glued along their copies of W. The involution on $M(W)$ just flips the two copies of $M - X$ and interchanges the two copies of X in the interesting part of the boundary.

Now given a knot $k : S^{2n-1} \subset S^{2n+1}$, as a special case of the above discussion we get $F \subset S^{2n+1}$ with $\partial F = S^{2n-1}$ being the knot k. In D^{2n+2} let $F \times I$ be embedded in a collar of ∂D so that $f \times 0 \subset S^{2n+1}$ is our original copy of F. Let $F^\wedge$ denote $F \times 1 \cup \partial F \times I$. $D(F^\wedge)$ denotes the double branched cover of D^{2n+2} along $F^\wedge$.

Recall the following construction due to Kauffman [Ka 1]. Given any element in $H_n(F^\wedge;Z)$ it comes from a unique element in $H_{n+1}(D^{2n+2} - (F\times I), F^\wedge;Z)$. Since $D(F^\wedge)$ is just two copies of $D^{2n+2} - (F\times I)$ glued together, we can glue two chains for our element in $H_{n+1}(D^{2n+2} - (F\times I), F^\wedge)$ together so as to get an element in $H_{n+1}(D(F^\wedge);Z)$. Kauffman shows this construction defines a homomorphism $\kappa : H_n(F^\wedge) \to H_{n+1}(D(F^\wedge))$ which is an isomorphism when homology is taken with rational coefficients.

The intersection form on $H_{n+1}(D(F^\wedge);Q)$ defines, via κ, a symmetric (if n is odd: skew-symmetric if n is even) form on $H_n(F^\wedge;Q)$. Intersection defines a non-singular, skew-symmetric (if n is odd: symmetric if n is even) form on $H_n(F^\wedge)$. If we pick a basis for $H_n(F^\wedge)$ and get a Seifert matrix A, Kauffman further shows that $A - A^*$ is the skew-symmetric form and $A + A^*$ is the symmetric form. Hence the intersection form on $H_{n+1}(D(F^\wedge);Q)$ is non-singular so $\partial D(F^\wedge)$ is a rational homology sphere. We conclude this paragraph with an important remark. Notice that $z(A)$ is the maximal dimension of a subspace on which both of our forms vanish: as it were, a simultaneous null-space.

To prove theorem 1 we show that for any $(M,W) \in \mathcal{S}(k)$, $m(k) \leq \frac{1}{2}\beta_n(W) + \frac{1}{2}(\beta_{n+1}(M) + |\mathrm{Sign}(M)|)$. To show this, consider $N = M \cup D^{2n+2}$ glued along their boundary S^{2n+1}. Let $K = F^{\wedge} \cup W$ and notice K is a codimension-two, locally-flat, orientable submanifold of N. Since $H_{2n}(W,\partial W;Z) \to H_{2n}(M,\partial M,Z)$ is 0, $H_{2n}(K;Z) \to H_{2n}(N;Z)$ is also 0. Transversality gives us an $X \subset N$ with $\partial X = K$ and all other properties needed to form the double branched cover, $M(K)$, by splitting M along X. It is possible to find an X such that $X \cap D^{2n+2} = F \times I$, and we do so.

If $\Delta(X)$ denotes the double of X, Kauffman's construction produces a homomorphism $\delta : H_{n+1}(M-X,\Delta(X)) \to H_{n+1}(M(K))$. There are maps $H_{n+1}(D^{2n+2} - (F\times I),F^{\wedge}) \to H_{n+1}(M-X,\Delta(X))$ and $i_* : H_{n+1}(D(F^{\wedge})) \to H_{n+1}(M(K))$ which are both induced by the obvious inclusions. We have an isomorphism $H_n(F^{\wedge}) \xleftarrow{\partial} H_{n+1}(D^{2n+2} - (F\times I),F^{\wedge})$ and the following diagram commutes

$$\begin{array}{ccc}
H_n(F^{\wedge};Q) & \xrightarrow{\kappa} & H_{n+1}(D(F^{\wedge});Q) \\
\downarrow & & \downarrow i_* \\
H_{n+1}(M-X,\Delta(X);Q) & \xrightarrow{\delta} & H_{n+1}(M(K);Q)
\end{array}$$

Since $\partial D(F^{\wedge})$ is a rational homology sphere, i_* is a monomorphism which preserves the intersection form. Also note that $H_n(F^{\wedge}) \to H_{n+1}(M-X, (X)) \xrightarrow{\partial} H_n(\Delta(X))$ is the map induced by the inclusion $F^{\wedge} \subset K \subset \Delta(X)$.

Our remark of three paragraphs above suggests that we hunt for a simultaneous null-space. A null-space for the intersection form on $H_n(F^{\wedge};Q)$ is just given by $\ker s$, where $s : H_n(F^{\wedge};Q) \to H_n(X;Q)$ is the map induced by inclusion. Note $\ker s$ goes to 0 under the map $H_n(F^{\wedge};Q) \to H_n(\Delta(X);Q)$ and so we can construct a cummutative diagram

$$\begin{array}{ccc}
\ker s & \longrightarrow & H_n(F^{\wedge};Q) \\
\downarrow & & \downarrow \\
H_{n+1}(M-X;Q) & \longrightarrow & H_{n+1}(M-X,\Delta(X);Q)
\end{array}$$

Our goal is to locate a null-space for our other form on $H_n(F^\wedge;Q)$ inside of $\ker s$. The map

$$\ker s \to H_{n+1}(M-X;Q) \to H_{n+1}(M-X,\Delta(X);W) \xrightarrow{\delta} H_{n+1}(M(K);Q)$$

preserves this second form and also admits a nice description. To wit, let $j : M - X \to M(K)$ denote one of the inclusion maps and let $\tau : M(K) \to M(K)$ denote the involution associated with the double branched cover. $\tau j : M - X \to M(K)$ then is the other inclusion. The map

$$\alpha : H_{n+1}(M-X;Q) \to H_{n+1}(M-X,\Delta(X);Q) \to H_{n+1}(M(K);Q)$$

is $j_* + \tau_* j_*$. From this description it follows that if $V \subset H_{n+1}(M-X;Q)$ is a null-space for the intersection form on $H_{n+1}(M-X;Q)$ then $\alpha(V)$ is a null-space for the intersection form on $H_{n+1}(M(K);Q)$. Moreover, if $\beta : H_{n+1}(M-X;Q) \to H_{n+1}(M;Q)$ is the map induced by the inclusion, V is a null-space iff $\beta(V)$ is a null space for the intersection form on $H_{n+1}(M;Q)$

Let the composite $\ker s \to H_{n+1}(M-X;Q) \to H_{n+1}(M;Q)$ be denoted by ν. Let N be a maximal null-space for $\nu(\ker s)$. Then for a Seifert matrix, A, associated to the spanning surface F we have $z(A) \geq \dim \nu^{-1}(N) = \dim(\ker \nu) + \dim N$. We leave to the reader the task of demonstrating theorem 1 from the above facts plus the two estimates

(a) $\dim \ker s \geq \beta_n(F) - \frac{1}{2}\beta_n(K) = \frac{1}{2}(\beta_n(F) - \beta_n(W))$

(b) $\dim N \geq \dim \nu(\ker s) - \frac{1}{2}(\beta_{n+}(M) + |\mathrm{Sign}(M)|)$.

To see (a) note Image $\{H_n(F^\wedge;Q) \to H_n(X)\} \subset$ Image $\{H_n(K) \to H_n(X)\}$ and this latter vector space has dimension $\frac{1}{2}\beta_n(K)$. The result follows.

To see (b) consider $N \subset \nu(\ker s) \subset H_{n+1}(M;Q)$. We can take a maximal null-space of $H_{n+1}(M;Q)$ containing N, say B. Then elementary quadratic form theory shows

$$\dim B = \tfrac{1}{2}(\beta_{n+1}(M) - |\mathrm{Sign}(M)|).$$

Moreover $\dim B \leq \dim N + \beta_{n+1}(M) - \dim \nu(\ker s)$ since $B \cap \nu(\ker s)$ is a null-space containing N and hence is N. The estimate in (b) now follows easily.

We begin the proof and statement of lemma 1. The notion of cobordism of matrices was defined by Levine [Le 1] and we insist that all matrices have property ε ([Le 1] p.231) before we speak of them as being cobordant. If A is a matrix with property ε, define $m(A) = \frac{1}{2} \dim A - z(A)$. Then $m(k) = m(A)$ for A a Seifert matrix for k. Since any two Seifert matrices are cobordant ([Le 1]), lemma 1 shows $m(k)$ is a well-defined concordance invariant of k.

We now prove

Lemma 1 : Let A and B be cobordant matrices. Then $m(A) = m(B)$.

Proof : If $\oplus$ denotes block sum, it is easy to prove $M(X \oplus Y) \leq m(X) + m(Y)$. Since it is enough to prove $m(A \oplus N) = m(A)$ if N is null-cobordant, it is enough to prove $m(A) \leq m(A \oplus N)$. Lemma 1 of [Le 1] is just a proof of this for the case $m(A \oplus N) = 0$. The proof there adapts easily to cover this generalization.

§ 3. The proofs of Theorems 2 and 3

The proof of theorem 2 follows Levine [Le 1], lemmas 4 and 5. First replace k by a simple knot with an $(n-1)$-connected spanning surface F.

We then get a Seifert matrix corresponding to a choice of basis for $H_n(F;Z)$. By lemma 1 we can find a basis for $H_n(F;Z)$ so that, if $x_1, \dots, x_r$ are the first $r = \frac{1}{2}\beta_n(F) - m(k)$ basis elements, then the x_i span a null-space for our form λ.

If $n \geq 2$, Levine's argument in lemma 5 of [Le 1] shows that we can produce $W \subset D^{2n+2}$ with boundary k, where W is obtained from F by surgery on $x_1, \dots, x_r$. Hence $\pi_*(W) = 0$, $\pi^* < n$, and $H_n(W;Z)$ is the free abelian group of rank $2m(k)$. This proves theorem 2.

If $n = 1$, the first two paragraphs of this section are still correct. Moreover, we can complete $x_1, \dots, x_r$ to a basis for $H_1(F;Z)$ which is a symplectic basis for the intersection form. It is therefore possible to represent $x_1, \dots, x_r$ by disjoint embedded circles in F. To see this last statement observe that we can find some symplectic basis for $H_1(F;Z)$ in which the first $r \leq \frac{1}{2}\beta_1(F)$ basis elements are represented by disjoint embedded circles. There is a symplectic matrix taking these r generators to $x_1, \dots, x_r$. But every symplectic matrix is induced by a homeomorphism of F (see e.g. p. 178 [MKS 1]) so $x_1, \dots, x_r$ are represented by disjoint embedded circles.

Since $F \subset S^3$, these circles give rise to an r-component link, $L_r \subset S^3$, such that each component of L_r links every other component with linking number 0. If L_r is a slice link in the strong sense then we can finish just as in the case $n \geq 2$.

Let us pause to improve this last statement a bit. L_r is called a ribbon link in the strong sense if each component of L_r bounds an immersed disc so that the singular set consists of disjoint simple ribbon singularities.

Claim : If L_r is a ribbon link in the strong sense then the ribbon genus of k is $m(k)$.

Proof : By theorem 1, $g_r(k) \geq m(k)$ so if we can just construct a Seifert ribbon with $\beta_1(W) = 2m(k)$ we are done. Our goal is to do the surgery on the x_i .

It is easy to embed $L_r \times [0,1] \subset S^3$ so that $L_r \times [0,1] \cap F = L_r \times 0$. Since L_r is a ribbon link in the strong sense we can find a collection of r immersed discs, say $D_r \subset S^3$ s.t. $\partial D_r = L_r \times 1$ and D_r has only disjoint simple ribbon singularities. We next improve $D_r \cap F$ which, at the moment, may be terrible. Note first we can assume $D_r \cap L_r \times [0,1] = L_r \times 1$.

Since F is a 2 manifold with boundary, F has a 1 dimensional spine. We can move D_r just a little so that D_r is transverse to the spine. $D_r \cap$ spine is a finite number of points which we can assume miss all the singularities of D_r. Moreover, we do not move ∂D_r and $D_r \cap L_r \times [0,1]$ is still $L_r \times 1$.

Now by shrinking F toward its spine if necessary we can assume that $D_r \cap F$ consists of disjoint simple ribbon singularities and $D_r \cap L_r \times [0,1] = L_r \times 1$.

It is easy to see that the normal bundle of $D_r \cup L_r \times [0,1]$ in S^3, when restricted to $L_r \times 0$, is just the normal bundle to $L_r \times 0$ in F. Use this normal bundle to push off another copy of $D_r \cup L_r \times [0,1]$ and call it $D_r' \cup L_r' \times [0,1]$.

Let $B \subset F$ be the band between $L_r \times 0$ and $L_r' \times 0$. Then $(\overline{F-B}) \cup (D_r \cup L_r \times [0,1]) \cup (D_r' \cup L_r' \times [0,1])$ is our surface. It clearly has the correct genus and it is easy to see that it is immersed with simple ribbon singularities. This proves the claim.

Our last major hurdle is to describe an operation we call tying a link into the bands of k. This operation is not well-defined but it is useful. To begin, we choose a spanning surface F for k and represent a symplectic basis for $H_1(F;Z)$ by a set of canonical curves, i.e. embedded circles representing the basis which are disjoint unless the two elements in $H_1(F;Z)$ have intersection ± 1. In this case the two circles intersect in one point.

We have represented F as a disc with $\beta_1(F)$ bands. If we order this basis for H (F;Z) we have a Seifert matrix, A. Call the i^{th} band the band through which the circle representing the i^{th} basis element passes.

In the i^{th} band choose an arc, α_i, cutting the band. Think of the knot and its spanning surface as lying in the lower hemisphere of S^3. It is easy to find an isotopy of S^3 so that each α_i is brought up into the upper hemisphere, D_+^3. Each α_i has a neighborhood in F^2 that looks like D^2. If we write D_+^3 as $D^2 \times [0,1]$ it is easy to arrange things so that

after our isotopy $D_+^3 \cap F = \bigcup_{i=1}^{\beta_1(F)} B^2 \times \frac{1}{i}$, where, if $D^2 = [-1,1] \times [0,1]$,

$B^2 = [-\frac{1}{2},\frac{1}{2}] \times [0,1]$. In D_+^3 we just have $\beta_1(F)$ unlinked, unknotted, untwisted bands.

We wish to replace these bands by some twisted, knotted, and linked bands. To describe this, we consider a framed link in D_+^3. Such a link consists of an ordered set of disjoint embedded arcs in D_+^3 with $\partial D_+^3 \cap (i^{th}\text{ arc}) = \partial(i^{th}\text{ arc}) = (0 \times 0 \times \frac{1}{i}) \cup (0 \times 1 \times \frac{1}{i}) \in D_+^3$ and we have an integer associated to each arc. Such a link has a matrix, B, defined by b_{ij} = linking number of i^{th} arc with j^{th} arc if $i \neq j$ and b_{ii} = integer associated to i^{th} arc.

Given such a link, we get a collection of bands by thickening up the arcs. It is possible to do this so that, if B_i denotes the image of the i^{th} thickened arc, B_i is homeomorphic to D^2; $B_i \cap B_j = \phi$ unless $i = j$, $\partial D_+^3 \cap B_i = \partial B_i = ([-\frac{1}{2},\frac{1}{2}] \times 0 \times \frac{1}{i}) \cup ([-\frac{1}{2},\frac{1}{2}] \times 1 \times \frac{1}{i})$; B_i twists b_{ii} times (i.e. $\frac{1}{2} \times [0,1] \times \frac{1}{i}$ links $0 \times [0,1] \times \frac{1}{i}$ with linking number b_{ii}).

Suppose our framed link had $\beta_1(F)$ components. In D_+^3 we can replace our unlinked, unknotted, untwisted bands by

$\bigcup_{i=1}^{\beta_1(F)} B_i$. This gives a new knot k_1. k_1 has a spanning surface, F_1, with

$\beta_1(F_1) = \beta_1(F)$. On F_1 we have an embedded symplectic basis which is ordered. With respect to this basis and ordering, the Seifert matrix for k_1 is just $A + B$.

If we have a framed link with $r < \beta_1(F)$ components we still do the operation by bringing only the first r of the bands into D_+^3. If $\overline{B}$ denotes the $\beta_1(F) \times \beta_1(F)$ matrix with $\overline{b}_{ij} = b_{ij}$ if $i, j \leq r$ and $\overline{b}_{ij} = 0$ if $i > r$ or $j > r$, then the Seifert matrix for k_1 is $A + \overline{B}$.

We describe a special case of the above which is the only case we need. As before we have our knot, k, with spanning surface F and canonical curves Suppose the first r of these curves span a null space of the Seifert form.

As above, isotope the α_i, $i = 1,\ldots, r$, into D_+^3. Our embedded circles in F now become a link in D_-^3 which we frame by requiring each arc to have 0 twist. Reflect this link through S^2 to get a framed link in D_+^3. Doing our operation as above we get a new knot k_1. We say k_1 is obtained from k by isotropic reflection.

We now have:

Theorem 3 : Let k be a knot with spanning surface F. Suppose $m(k) = \frac{1}{2}\beta_1(F) - r$. Then we can find r disjoint embedded circles in F representing a null-space for the Seifert form. If we do an isotropic reflection using these r circles, the resulting knot k_1 satisfies

(i) $g_r(k_1) = m(k_1)$

(ii) k_1 and k have a common Seifert matrix.

Proof : Complete our r circles to an embedded sympletic basis for $H_1(F;Z)$. Look at k_1, the knot obtained by the isotropic reflection. k_1 and k have the same Seifert matrix and if L_r is the link of interest for k, $L_r \# -L_r$ is the link for k_1. $L_r \# -L_r$ is the link obtained from L_r by mirror reflection and then joining each component in L_r to the corresponding component in $(-L_r)$ by a straight band. $L_r \# -L_r$ is a ribbon link in the strong sense so we are done.

We conclude with the following observation.

Suppose k_1 and k_2 have cobordant Seifert matrices. Then there exist knots k_3 and k_4 such that $k_1 \# -k_3$ and $k_4 \# -k_2$ are ribbon knots and such that k_4 is obtained from k_3 by isotropic reflection.

Proof : Let $k_3 = k_1 \# (-k_2 \# k_2) = (k_1 \# -k_2) \# k_2$. Since $m(k_1 \# -k_2) = 0$, we can do an isotropic reflection so that it becomes a ribbon knot. Hence we can do an isotropic reflection on $(k_1 \# -k_2) \# k_2$ to get k_4 with $k_4 = (\text{ribbon knot}) \# k_2$. Hence $k_4 \# -k_2$ is a ribbon knot.

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