# Five lectures on Topological Field Theory

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## Contents



## 1. Introduction to TQFT and examples

Topological quantum field theories arose in physics as a 'baby' (zero-energy) sector of honest quantum field theories, showing an unexpected dependence on the large-scale topology of space-time, and in mathematics as an intriguing organizing structure for certain brave new topological or differential invariants of manifolds, which could not be captured by standard techniques of algebraic topology. (We will see a reason for that.)

*(1.1) Definition.* The original axioms of Atiyah and Witten [W], inspired by Graeme Segal's axioms of Conformal Field theory, defined an $D$-dimensional TQFT as a *symmetric monoidal functor*

$$Z : (\mathscr{B}ord_D^{or}, \coprod) \to (Vect, \otimes).$$

Here, $\mathscr{B}ord_D^{or}$ is the category whose objects are compact oriented manifolds of dimension $(D-1)$, and morphisms are oriented $n$-dimensional bordisms. A bordism is assumed to have an incoming and an outgoing boundary, but this choice is also indicated by comparing the boundary orientation on each component with the independent orientation. Thus, for an interval with positive endpoints necessarily has one incoming and one outgoing end.

*1.2 Remark.* In some precise formulations, the manifolds come embedded in a very large Euclidean space ($\mathbb{R}^\infty$), and bordisms embed in an extra 'time' dimension. We will ignore this structure.

The bordism category has a *symmetric monoidal structure* (an associative and commutative multiplication functor $\coprod : \mathscr{B}ord_D^{or} \times \mathscr{B}ord_D^{or} \to \mathscr{B}ord_D^{or}$) defined by disjoint union. The category of vector spaces has a similar multiplication, defined by the tensor product. (Because of the additional linearity properties of $Vect$, and the bi-linearity of $\otimes$, we call this a *tensor structure.*)

*(1.3) Baby example $D = 1$.* The vector spaces $Z(+), Z(-)$ are assigned to the point with the two orientations. The left arc $\subset$ gives a morphism $Z(\subset) : \mathbb{C} = Z(\emptyset) \to Z(+) \otimes Z(-)$, and the right arc a pairing $Z(+) \otimes Z(-) \to \mathbb{C}$. These two maps establish a perfect duality between $Z(+)$ and $Z(-)$, forcing them in particular to be finite-dimensional. (Otherwise, the identity map on $Z(+)$ would not sit in $Z(+) \otimes Z(+)^\vee$.) So $Z$ is described by $Z(+) = V$; $Z(S^1) = \dim V$ and all operations involve the standard expansion and contraction tensors in $V \otimes V^\vee$.

Remarkably enough, this baby example contains the germ of Lurie's *Cobordism Hypothesis*, which classifies (fully extended) TQFTs in terms of the datum $Z(+)$, which must satisfy a finiteness hypothesis expressed in terms of dualities.

*1.4 Remark* (The unoriented case). Even the $D = 1$ situation is interesting if we abandon orientations. In that case, we must have $Z(+) = Z(-)$ since there is only one point, and at this stage we have a vector space *with non-degenerate symmetric bilinear form.* In higher dimension, our choice of oriented, as opposed to *framed* manifolds[1] will enforce a strong restriction on our theories, related to the *Calabi-Yau* condition of complex geometry (triviality of the canonical bundle).

*(1.5) Finite gauge theory.* This theory $Z_F$, assigned to a finite group $F$, is easy to construct in any dimension. On the downside, it only detects fundamental groups.[2]

To a closed $D$-manifold $M$, $Z_F$ assigns the count of principal $F$-bundles on $M$, weighted down by their automorphisms. This is also $\#Hom(\pi_1(M), F)/\#F$. Thus, for $D = 1$, $Z_F(S^1) = 1$.

For a closed $D - 1$-manifold $N$, $Z_F(N)$ is the space of functions on isomorphism classes of $F$-bundles.

It should be easy to guess now that $Z_F(M) : Z_F(\partial^- M) \to Z_F(\partial^+ M)$ is the linear map whose matrix entry relating $F$-bundles $F^- \to \partial^- M$ and $F^+ \to \partial^+ M$ counts the $F$-bundles on $M$ restricting to the specified bundles on the two boundaries. The count is weighted by automorphisms that are trivial on the outgoing boundary $\partial^+ M$. Checking that this gives a TQFT (that is, composition of bordisms maps to composition of linear maps) is an exercise.

*1.6 Remark* (Twistings). This theory does not use orientations, but a *twisted* variant does. Choose a group cohomology class in $\tau \in H^D(BF; \mathbb{C}^\times)$; this defines a class $\tau_P \in H^n(M; \mathbb{C}^\times)$ for every principal $F$-bundle $P \to M$. In the top dimension, we define $Z_F^\tau(M)$ by weighting $P$ by $\int_M \tau_P$ in the count. It takes a bit more thought to see that $\tau$ defines a 1-dimensional character of the automorphism group of each principal bundle on an $D - 1$-dimensional manifold $N$. (Hint: use an automorphism to define a bundle on $S^1 \times N$, and use $\tau$ to extract in this way a complex number for each automorphism.) In defining $Z_F^\tau(N)$, we now *delete all lines corresponding to bundles with non-trivial automorphism actions.* Equivalently, we retain only the invariant part under automorphisms.
*Problem:* describe explicitly the matrix coefficients of the $Z_F^\tau$ defined by a $D$-manifold with boundary. (For help, see the next remark.)

*1.7 Remark* (Correspondence diagram). The TQFT is secretly defined by a *correspondence of groupoids*, which will appear again in Lecture 2. Namely, the groupoid of $F$-bundles on $M$ maps, by restriction of bundles, to the groupoids of bundles on $\partial^- M$ and $\partial^+ M$. Call the maps $p_\pm$. We can identify $Z_F(\partial^\pm M)$ with $H^0$ of the corresponding groupoid, and the map $Z_F(M)$ is the 'push-pull' composition $(p_+)_* \circ p_-^*$. (We must convene that the push-forward $(p_+)_*$ along a groupoid weights each point by dividing by the automorphism group.)

This story extends to the twisted case, and furnishes a baby example of a path integral in mathematical physics, a beloved (if non-rigorous) technique to construct quantum field theories. The role of the (exponentiated) classical action is played by $\tau$.

*(1.8) Dimension 2 and Frobenius algebras.* It is possible to classify TQFTs as defined for $D = 2$; the result, folklore for a while, was written up rigorously by L. Abrams [A].

**1.9 Theorem.** *A 2-dimensional oriented TQFT is equivalent to the datum of a commutative Frobenius algebra structure on a finite-dimensional space $Z(S^1)$.*

The notion introduced describes a unital associative algebra $A$ with a *trace* $\theta : A \to \mathbb{C}$ inducing a non-degenerate pairing $A \times A \to \mathbb{C}$ by $a \times b \mapsto \theta(ab)$. The trace condition, $\theta(ab) = \theta(ba)$, becomes important in the non-commutative case, and will appear for extended TQFTs in 2D. An example of a commutative Frobenius algebra is the cohomology of a compact oriented manifold, with the cup-product ant the integration map as a trace. (When deforming the cup-product by counting holomorphic curves in a projective manifold, this example becomes relevant to Gromov-Witten theory.)

The geometric representation of the operations is well-known: the multiplication is represented by the pair of pants, mapping $Z(S^1) \otimes Z(S^1) \to Z(S^1)$, the unit by the disk with outgoing boundary, and the trace by the outgoing disk. To see amy the theorem holds takes a bit more work. A Frobenius algebra contains a distinguished element, the *Euler class* $\alpha$, the vector output by a torus with a single outgoing circle. Then, a closed genus $g$ surface computes $\theta(\alpha^g)$, while a surface with incoming and outgoing boundaries computes the product of the inputs, times $\alpha^g$, co-multiplied into the outputs.

[1] The distinction is invisible in dimension 1.

[2] However, a sophisticated enhancement lurks in the background, which detects higher homotopy information and relates to a categorified notion of the group algebra.

The special case when the algebra is semi-simple is worth noting. Then, $A = \bigoplus \mathbb{C}P_i$, for projectors $P_iP_j = \delta_{ij}P_i$, with traces $\theta_i := \theta(P_i)$ non-zero complex numbers. These numbers, up to order, determine the isomorphism class of the Frobenius algebra. Then, $\alpha = \sum_i \theta_i^{-1}P_i$, and $Z(\Sigma_g) = \sum_i \theta_i^{1-g}$.

*(1.10) Finite group gauge theory.* This toy quantum theory can be computed by following the physicist's path integral ideas, since integration only involves counting. Isomorphism classes of $F$-principal bundles on the circle are the conjugacy classes in $F$, which have the natural basis of characters. Computation of the pair of pants shows that the product is computed by convolution of characters on the group. So the projectors are the class functions $(\#F)^{-1}\dim V \cdot \chi_V$, for all irreducible representations $V$. The unit is the delta-function at the origin, which indeed is $(\#F)^{-1}\sum_V \chi_V \cdot \dim V$. The trace is the evaluation at the identity, weighted down by the number $\#F$ of automorphisms of the trivial bundle on the disk. So the projector traces are $\theta_V = \dim^2 V/(\#F)^2$, and by comparing the two computations for the genus $g$ partition function we get

$$(\#F)^{2g-2}\sum_V (\dim V)^{2-2g} = \#\{u_1, u_2, \ldots, u_{2g} \in F \mid [u_1,u_2]\cdot[u_3,u_4]\cdot\cdots\cdot[u_{2g-1},u_{2g}] = 1\}/\#F$$

which in genera 0 and 1 encodes some classical identities form the character theory of finite groups. (Which ones?)

*(1.11) Finite groupoid theories* The finite gauge theories generalize to capture more of the homotopy of manifolds. For this, we re-think the finite group $F$ in terms of its classifying space $BF$. Recall that this is the quotient by $F$ of a contractible space with free $F$-action. (An example you all know is $B\mathbb{Z}/2 \cong \mathbb{RP}^\infty = S^\infty/(\mathbb{Z}/2)$.) This $BF$ is an *Eilenberg-MacLane space*, a space with a single non-vanishing homotopy group: in this case, $\pi_1 = F$. The isomorphism classes of principal bundles on $M$ are the connected components $[M, BF]$ of the space of maps to $BF$. To follow a physics analogy, we are quantizing $BF$-valued fields, but the 'path integral' counts components (with automorphisms, and weighted by the action in the twisted case).

Similarly, the space of states for the $(D-1)$-manifold $N$ is the space of locally constant functions on $BF$-valued fields on $N$. In the twisted case, we get a (flat) line bundle on this space of maps and we are taking the locally constant sections.

We can now replace $BF$ by a more general target space $X$. Finiteness of the resulting theory turns out to require finiteness of the homotopy groups of $X$. In addition, homotopy groups above $\pi_D$ do not affect $[M, X]$ if $\dim M \le D$, so we may take $X$ with finite homotopy up groups, up to dimension $D$. We can build a TQFT by declaring $Z_X(N^{D-1})$ to be the vector space of locally constant functions on $\mathrm{Map}(N;X)$, and letting $Z_X(M)$ count the components $[M,X]$ with some weights. The weight is a little tricky to work out: for $m : M \to X$, it is the alternating product of the order of the homotopy groups of the $m$-component of the space of maps. (Exercise, show that no other choice will give a good definition.) This theory can also be twisted by an 'action' class $\tau \in H^D(X;\mathbb{C}^\times)$.

*1.12 Remark.* The homotopy groups of $\mathrm{Map}(M;X)$ can be computed using a spectral sequence, from $H^*(M;\pi_*X)$. The computation, however, involves some finer information about $M$ and $X$, namely the cup-product on $M$ and the way the homotopy groups of $X$ are layered together (the so-called *Postnikov k-invariants*.) So the TQFT described above is sensitive to more information than the homology of $M$ and the homotopy groups of $X$. For an easy example,[3] recall that $\pi_2(S^2) \cong \pi_3(S^2) \cong \mathbb{Z}$ and $\pi_4(S^2) \cong \mathbb{Z}/2$. A space with the same homotopy groups in dimension up to 4 is $\mathbb{CP}^\infty \times S^3$. There are plenty of maps from $M = \mathbb{CP}^2$ to the latter space: maps to $\mathbb{CP}^\infty$ are classified by the degree of the line bundle they pull back, or equivalently, by the induced map $\pi_2\mathbb{CP}^2 \to \pi_2\mathbb{CP}^\infty$ (both groups equal to $\mathbb{Z}$). None of these interesting maps come from a map into the $S^2 \subset \mathbb{CP}^\infty$, because the cup-square of the area form $\sigma$ on $S^2$ is zero, but the cup-square $H^2 \to H^4$ on $\mathbb{CP}^2$ is not. So $\sigma$ pulls back to zero under any map $\mathbb{CP}^2 \to S^2$. We will encouter this quadratic obstruction again in a different guise.

*(1.13) Yang-Mills theory in* 2*D*. The final variation on gauge theory, which will occupy next lecture, pertains to a *compact* gauge group $G$. The story just told, about counting principal bundles, requires

[3] Albeit with infinite homotopy groups, in order to stay with familiar spaces; one can collapse the homotopy groups mod $n$ to get finite examples

interpretation. One way to proceed is to interpret 'principal bundles' as 'flat principal bundles', that is, bundles with flat connections. (For discrete groups, principal bundles have a natural flat structure.)

As before, the moduli space of isomorphism classes of flat bundles on $S^1$ is identified with the space $G/G^{ad}$ of conjugacy classes, so vector space associated to the circle should be that of *class functions on the group*, again carrying the natural basis of characters. There is also a moduli space $F(\Sigma; G)$ of flat bundles over a closed surface of genus $g$, which is a compact real-analytic space of real dimension $2(g-1) \cdot \dim(G)$.[4] The variety is usually singular, and can be described with reference to $\pi_1$ as

$$F(\Sigma, G) = \left\{(u_1, u_2, \ldots, u_{2g}) \in G^{2g} \,|\, [u_1, u_2] \cdot \cdots \cdot [u_{2g-1}, u_{2g}] = 1\right\}/G \tag{1.14}$$

with the group acting by simultaneous conjugation. Smooth varieties (of different dimensions) arise when we set the commutator is set instead to lie in some specified, general conjugacy class; these are relevant for computing the TQFT for surfaces with boundary. More interestingly, $\pi_1$ of a surface with an outgoing boundary component being free, the moduli space is $G^{2g}/G^{ad}$, and we should be studying the 'product of commutators' map $G^{2g} \to G$, all conjugated by $G$; and the generalizations of $F$ just mentioned are the fibers of this map.

'Counting' bundles is not an option, but an alternative stems from the observation that these moduli spaces have natural volume elements coming from a *symplectic form*, which does not require any structure on $\Sigma$. (See next lecture.) If we convene to integrate against this volume instead of counting, we almost obtain a 2D TQFT, whose genus $g$ partition function is the symplectic volume of the space (1.14). 'Almost' refers to the fact that we obtain an infinite-dimensional vector space $Z(S^1)$, albeit with a natural basis (the characters of $G$). The underlying Frobenius algebra was computed by Witten [W2]) and is strikingly similar to the finite group story: he finds a projector $P_V$ for each irreducible character of $G$, with trace $\theta_V = \mathrm{vol}(G)^{-2} \dim^2 V$. (The Riemannian volume of $G$ is computed using a conjugation-invariant metric on $\mathfrak{g}$; the same metric is used to define the symplectic form.) From here, Witten finds the *symplectic volume formula*

$$\int_{F(\Sigma;G)} \exp(\omega) = \#Z(G) \cdot \mathrm{vol}(G)^{2g-2} \sum_V (\dim V)^{2-2g}$$

which is a convergent series for simple groups, in genus $\geq 2$.

*1.15 Remark.* The factor $\#Z(G)$, the order of the center of $G$, seems out of place, when compared to the finite story. But in fact, central elements define automorphisms of any flat bundle, and generically they are the only automorphisms. So $F$ is generically an orbifold with stabilizer $Z(G)$, and we should have divided by that when integrating.

*(1.16) Variant: Cohomological Field theories.* A rather famous class of $2D$ field theories enhances the structure we discussed by remembering that surfaces have diffeomorphisms. They are the *$A$- and $B$-models of Mirror symmetry*, which count (pseudo-)holomorphic curves in compact symplectic manifolds, reps. describe operations on coherent sheaves on complex manifolds, and produce invariants valued in characteristic classes of surface bundles.

In the simplest formulation, we retain the vector space $Z(S^1)$ associated to the circle, but asking that the linear maps $Z(S^1)^{\otimes p} \to Z(S^1)^{\otimes q}$ induced by surfaces with $p$ inputs and $q$ outputs 'vary cohomologically' in families. That is, if $B$ is a family of a bundle of such surfaces $\Sigma$, we want $Z(\Sigma)$ to define a class in

$$H^*\left(B; Hom\big(Z(S^1)^{\otimes p}, Z(S^1)^{\otimes q}\big)\right)$$

The $H^0$ part is the original linear map, so the 'variation on the base' does not refer to a continuous dependence on points $b \in B$, but a coupling to homology classes. (This is the only sensible definition in the topological context, where we like to compute homotopy-invariant quantities.)

There are many variants of this notion (see [C, T]), but the one to flag here is the *Cohomological Field theory* defined by Kontsevich and Manin [M], where we allow the surfaces to acquire nodes, as in the Lefschetz fibrations of algebraic geometry. This means that, locally near a singularity, the family

[4] If $G$ is a simple group; work it out in general!

of curves is described by the family $((x, y) \to t = xy$, with a local coordinate $t$ on the base, and $t = 0$ describing the nodal locus.

The connection with complex geometry is quite fundamental: the classifying space for the diffeomorphism group $\mathrm{Diff}(\Sigma)$ of a surface (with marked points, if desired) is homotopy equivalent to the moduli orbifold of complex structures on the same surface. (This holds exactly in the hyperbolic cases, excluding, that is, genus 0 with one or two marked points, and genus 1 with no marked points.) If we impose a *stability* condition on our nodal surfaces — all irreducible components should be hyperbolic — then there are also universal classifying spaces for Lefschetz fibrations, the much-studied *Deligne-Mumford moduli spaces* $\overline{M}_g^n$ of algebraic geometry. The indexes $g$ and $n$ refer to the genus and number of marked points (which may be thought of as tiny boundary circles, in the context of TQFT), and the bar indicates that these spaces are compactifications of the moduli $M_g^n$ of smooth algebraic curves. So the CohFT is determined by classes in $H^*\left(\overline{M}_g^{p+q}; Hom\big(Z(S^1)^{\otimes p}, Z(S^1)^{\otimes q}\big)\right)$, which are subject to *factorization conditions* describing the restriction of these classes to boundary strata, and expressing in this context the fact that composition of surfaces leads to composition of linear maps.

*1.17 Remark.* The homotopy equivalence of $B\mathrm{Diff}(\Sigma)$ with the moduli orbifold of smooth Riemann surfaces of topological type $\Sigma$ is not a mystery. We can realize $BDiff$ as the moduli orbifold of metrics on a surface, modulo diffeomorphisms. But, a complex structure is the same as a metric up to a conformal rescaling, and the space of rescalings is contractible. So the moduli of metric and conformal surfaces are equivalent. The more difficult result lurking in the background is that the components of the group Diff are contractible in the hyperbolic case, so that $B\mathrm{Diff} \sim B\pi_0\mathrm{Diff}$.

The spaces $\overline{M}_g^n$ have a beautiful local structure, with normal-crossing boundary divisors labelled by the manners in which curves can acquire a node: think topologically of pinching a simple closed loop on the surface into a self-intersection. The combinatorics of the strata encodes a sophisticated algebraic structure ('cyclic operad', see [GK]), closely related to that of a CohFT.

The notion of Cohomological Field theory was motivated by the desire to encode the structure of *Gromov-Witten invariants* of a Kähler (or symplectic) manifold $X$, which count holomorphic curves with prescribed incidence conditions. The Frobenius algebra we get by ignoring the higher classes on the $\overline{M}$ is the famous *quantum cohomology*, a (commutative) deformation of the cup-product on $H^*(X)$. The Kontsevich-Manin axioms include some constraints specific to the GW situation, most importantly pertaining to the grading. The general strategy of understanding the full structure of GW invariants stumbled upon a fatal obstacle: the cohomology of the spaces $\overline{M}_g^n$ remains unknown to this day. Understanding the structure of CohFTs then is similar to studying modules over an unknown ring! A key motivation for the *extended TQFTs* we will discuss in Lectures 3-5 was Kontsevich's program to classify TQFTs algebraically from minimal data, from which the Gromov-Witten invariants could be reconstructed. This program is still under development, and just beginning to bear fruit.

One great success of the theory concerns the *genus zero* part of the story — interesting enough geometrically, for it counts (trees of) rational curves in algebraic varieties. Note that, while the space of *smooth* genus zero curves with marked points is simple enough (configurations of distinct points in $\mathbb{P}^1$), its Deligne-Mumford compactification $\overline{M}_0^n$ is far from trivial, and its boundary divisors intersect along interesting patterns. The cohomology of this space was completely understood by Keel [K], who gave an explicit presentation in terms of boundary divisors, modulo linear and quadratic relations. (The relations are easy to guess by studying fibers of the forgetful maps $\overline{M}_0^n \to \overline{M}_0^4 \cong \mathbb{P}^1$.)

A more specialized but important success followed ideas of Givental, who investigated Gromov-Witten theory of manifolds for which the quantum cohomology, the deformed $H^*(X)$, was semi-simple. Such is the case for $\mathbb{P}^n$, where the usual cohomology ring becomes $\mathbb{C}[\omega]/\omega^{n+1} = q$, and generally for all toric manifolds. (The parameter $q$ in the theory is used to separate the count of holomorphic curves according to degree.) Based on experimental evidence, Givental conjectured that all GW invariants — which you recall are counting holomorphic curves of various genera and degrees — are uniquely determined from the quantum cohomology alone (and the grading information). In other words, there is a unique way to extend a 'naive' TQFT as in §1.8 to a full CohFT, and there is an explicit (recursive) formula for this extension. The conjecture was confirmed [T]; but there seems to be no hope of extending the result without the semi-simplicity assumption, which is very restrictive. ('Most' target manifolds will not meet that; only when faced with an abundance of rational curves that we can hope to deform the nilpotent cup-product on $H^*(X)$ enough to make it semi-simple.) One of Kontsevich's motivations in studying extended TQFTs was to find an analogous structural result for the Gromov-Witten theory of general varieties.

## 2. Two-dimensional gauge theory

This example has been a favorite ever since Witten [W2] surprised the gauge theory community by computing the theory explicitly, using localization arguments in path integrals. This gives a TQFT computation of all integrals over the moduli space $F(\Sigma; G)$ of flat $G$-connections on a surface $\Sigma$, in terms of an explicitly computed Frobenius algebra. Assume for convenience that $G$ is simple and simply connected; the dimension of $F(\Sigma; G)$ is then $2d = 2\dim G \cdot (g-1)$.

The universal flat bundle over $\Sigma \times F(\Sigma; G)$ is classified by a map $u$ to the classifying space $BG$ (defined up to homotopy). Recall that $H^*(BG; \mathbb{C})$ is a polynomial ring, isomorphic to $(\mathrm{Sym}\mathfrak{g}^*)^G \cong (\mathrm{Sym}\mathfrak{t}^*)^W$ with generators $\phi_2, \ldots, \phi_\ell$ of degrees $2m_i+2$, with the exponents $m_i$ of the group $(2, 3, \ldots, n$ for $\mathrm{SU}(n)$.) The slant products $\beta\backslash u^*\phi_i$ (integrals of $u^*\phi_i$ over $\beta$) with elements $\beta$ of a basis of the homology of $\Sigma$ define classes of degrees $2m_i + 2, 2m_i + 1, 2m_i$. Atiyah and Bott [AB] proved that $H^*F(\Sigma; G)$ is generated by these classes, whenever $F(\Sigma; G)$ is smooth.[5]

The classes $[\Sigma]\backslash u^*\phi_i$, of degree $2m_i$ are the most 'interesting' for integration, not easily reduced to elementary computations. As an example, from the *quadratic Casimir* $\phi_2$ corresponding to the invariant quadratic form $\langle , \rangle$ on $\mathfrak{g}$, we obtain the symplectic form $\omega$ alluded to in Lecture 1. (There is a preferred normalization, called a *basic* quadratic form, in which the shortest non-zero $\log(1)$ in $\mathfrak{g}$ has square-length 2.) For two tangent vectors $\delta A, \delta B$ at a principal bundle $P$, represented by connection forms on the trivial bundle, we have $\omega(\delta A, \delta B) = \int_\Sigma \langle A, B \rangle$.

Witten's formula reads as follows. Specialize to SU(2), and choose a polynomial $Q$. For each $k \in \mathbb{Z}_+$, let $\xi(k, t)$ be the formal power series (in $t$) representing the unique critical point of the function in $\xi$ (with parameters $h, k, t$)

$$F(\xi; h, k, t) := \frac{1}{2}(\xi - k)^2 + t \cdot \frac{h}{2\pi^2} \cdot Q(\pi\xi/h) \tag{2.1}$$

in which $t$ is treated as a formal variable, so $\xi(k, t)$ is computed by $t$-series expansion around the minimum $\xi(k, 0) = k$. A special case of Witten's formula then says

$$\int_{F(\Sigma;G)} \exp\left\{h\omega + t \cdot [\Sigma]\backslash Q(u^*\phi_2)\right\} = \#Z(G) \cdot h^{3g-3} \mathrm{vol}(G)^{2g-2} \sum_{k>0} \left[\frac{1 + tQ''(\xi(k, t))/2h}{\xi(t, k)^2}\right]^{g-1}$$

Of course, $\#Z(G) = 2$ for SU(2). The scaling factor $h^{3g-3}$ should really be absorbed in the volume, as we should rescale the basic metric by $h$. (For SU(2), $\mathrm{vol}(G) = 1/\pi\sqrt{2}$ in the basic metric, $h^{3/2}/\pi\sqrt{2}$ rescaled.) We recognize on the rightthe genus $g$ partition function for the semi-simple (but infinite-dimensional) Frobenius algebra $\bigoplus \mathbb{C}P_k$, with one projector $P_k$ for each positive integer, of trace

$$\mathrm{vol}_h(G)^{-2} \frac{\xi(t, k)^2}{1 + tQ''(\xi(k, t))/2h} \tag{2.2}$$

They reduce to the values of §1.13 when $h = 1$ and $t = 0$, as they should.

To get the most general integrals, we choose several polynomials $Q_i$ and couple them to independent formal variables $t_i$; the $\partial^n/\partial_1 \ldots \partial_n$-derivative of the analogous formula computes

$$\int_{F(\Sigma;G)} \exp[h\omega] \wedge ([\Sigma]\backslash Q_1(u^*\phi_2)) \cdots \wedge ([\Sigma]\backslash Q_n(u^*\phi_2)).$$

*2.3 Remark.* The most general formula sor a simple group $G$ incorporates all other generators of $H^*F(\Sigma; G)$; $Q$ will be an invariant polynomial on the Lie algebra $\mathfrak{g}$, which we may restrict to the Cartan subalgebra $\mathfrak{t}$; the role of the integers $k$ is played by the highest weights of $G$, shifted by the Weyl vector $\rho$. The numerator in the Frobenius structure constants (2.2) is the Hessian determinant of the multi-variable function $F$ of (2.1); $\xi(t, k)^2$ is replaced by the volume of the adjoint orbit of $\xi(t, k)$ in $\mathfrak{g}$.

[5] There is a corresponding statement for the singular case, in terms of equivariant cohomology.

*(2.4) Interpretation in $K$-theory.* Actually, interpreting and understanding Witten's formulas is no easy task. For example, most moduli spaces are singular, and the characteristic classes above do not live on them. (The exception concerns $\mathrm{SU}(n)$-connections with central holonomy prime to $n$, at some specified point on $\Sigma$.) It turns out that the spaces do have preferred (orbifold) desingularizations; etc. Here I will discuss an interpretation in terms of (twisted) $K$-theory, which allows for a topological description and computation of the field theory.[6]

For starters, recall the Hirzebruch-Riemann-Roch theorem for a holomorphic line bundle $E \to X$ over a complex (projective) manifold:

$$\mathrm{Ind}(X; E) := \sum\nolimits_k (-1)^k \dim H^k(X; \mathscr{O}(E)) = \int_X \mathrm{ch}(E)\mathrm{Td}(X) \tag{2.5}$$

where $H^k$ denotes $k$th cohomology with coefficients in the sheaf of holomorphic sections $\mathscr{O}(E)$ (equal to the Dolbeault cohomology with coefficients in $E$), and the *Chern character* $\mathrm{ch}(E)$ and *Todd class* $\mathrm{Td}(X)$ are characteristic classes of $E$ and of the tangent bundle of $X$, which can be expressed in terms of the Chern roots. Specifically,

$$\mathrm{ch}(E) = \mathrm{Tr}\left(\exp R_E/2\pi\mathrm{i}\right), \qquad \mathrm{Td}(X) = \det \frac{R_{TX}/2\pi\mathrm{i}}{1-\exp(R_{TX}/2\pi\mathrm{i})}$$

where $R$ denotes the curvature form of some connection on the respective bundle. An alternative definition, if $E$ is a sum of line bundles $L_i$, is $\mathrm{ch}(E) = \sum \exp c_1(L_i)$, and a similar maneuver, but with a product, works for Td.[7]

Actually, the additive map Ind, from holomorphic vector bundles on $X$ to $\mathbb{Z}$, can be extended to all topological bundles, and defines a linear map $\mathrm{Ind} : K(X) \to \mathbb{Z}$ from the Grothendieck group $K(X)$ of complex topological vector bundles. (This group is called the (even) topological $K$-theory of $X$, and supplies an example of an exotic, or generalized, cohomology theory.) Formula (2.5) gives a factorization of this map via $\mathrm{ch} : K(X) \to H^{ev}(X)$, which is a ring homomorphism, followed by integration against the Todd class.

*2.6 Remark.* It turns out that $K(X) \otimes \mathbb{Q} \cong H^{ev}(X; \mathbb{Q})$ as algebras, but viewing the Index map as a trace defines a Frobenius algebra structure on $K$ different from that on $H^*$.

*(2.7) Integration from the index.* Assume for now that $E$ is a line bundle with Chern class $\omega$. There is a way to extract the symplectic volume of $X$ as an asymptotic of the index:

$$\mathrm{Ind}(X; E^{\otimes n}) = n^{\dim X} \int_X \exp(\omega) + O(n^{\dim X - 1}),$$

making the Todd class disappear in the leading term. A similar trick can be used for any vector bundle $E$, but requires, instead of the tensor power, the use of the *$n$th Adams operation* $\psi^n$, defined by $\psi^n L = L^{\otimes n}$ for a line bundle $L$ and imposing additivity. There is in fact an expression for $\psi^n$ in terms of exterior powers, but it involves signs, so it is only valid in the Grothendieck group $K(X)$ and not meaningful in the category of vector bundles. At any rate, if we regard the computation of $\psi^n$ as known, we see by linearity of $\psi^n$ that integration over $X$ can be recovered as an asymptotic of the index:

$$\mathrm{Ind}(X; \psi^n E) = n^{\dim X} \int_X \mathrm{ch}(E) + O(n^{\dim X - 1}).$$

*(2.8) Index formulas on $F(\Sigma; G)$.* The point of this long preamble is the following: the integration formulas over $F(\Sigma; G)$ can be recovered from index formulas. However, the index formulas, unlike integration, comes from a genuine TQFT, based on a finite-dimensional Frobenius algebra. Moreover, the index formulas can be derived purely geometrically, from correspondence diagrams with moduli spaces of flat connections, as in the case of gauge theory with a finite group.

[6]Based on joint work with Freed and Hopkins [FHT], and with Woodward [TW].

[7]By the so-called *splitting principle* of topological $K$-theory, this special case suffices to define ch and Td, once we take into account the behavior under sums: $\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$, $\mathrm{Td}(E_1 \oplus E_2) = \mathrm{Td}(E_1) \cdot \mathrm{Td}(E_2)$.

A word of philosophy may or may not help — the index formulas represent a version of gauge theory for the *loop group* go $G$. Loop groups have a representation theory reminiscent of that of compact Lie groups, but the theory carries a fundamental discrete parameter, the *level*; and there are only *finitely many* representations at a fixed level. Thus, the theory has some features of the representation theory of *finite* groups.

The easiest story pertains to the $K$-theory of the analogues of symplectic volumes. For integral $h$, $\exp(h\omega)$ is the Chern character of a line bundle $\mathscr{O}(h)$ on $F(\Sigma, G)$. Now, a choice of complex structure of $\Sigma$ gives a complex analytic (and in fact projective algebraic) structure on the variety $F(\Sigma, G)$: this is a deep theorem of Narasimhan and Seshadri, which identifies the latter with the moduli of (poly)stable holomorphic $G$-bundles on $\Sigma$. It is easier to show that the line bundles $\mathscr{O}(h)$ admit holomorphic structures, and these are in fact unique when the group $G$ is simple.[8]

The index of $\mathscr{O}(h)$ has a very nice interpretation, thanks to the

**2.9 Theorem** (Kumar-Narasimhan). *The higher cohomology $H^{>0}(F(\Sigma, G); \mathscr{O}(h))$ vanishes if $h \geq 0$.*

These indexes thus measure the dimensions of vector spaces. These spaces, the *conformal blocks*, have been much studied. A formula for their dimension was conjectured for SU(2) by E. Verlinde:

$$\dim H^0(F(\Sigma, G); \mathscr{O}(h)) = (2h+4)^{g-1} \sum\nolimits_{k=1}^{h+1} \left(2 \sin \frac{k\pi}{h+2}\right)^{2-2g}$$

it was proved, for general $G$, in the work of numerous authors. Witten's symplectic volume formula can be obtained from the asymptotics of Verlinde's. I will not reproduce the derivation here, see [TW], §5 for the general story. My goal, instead, is to explain why these numbers (and their generalization to indexes of vector bundles) are controlled by a 2-dimensional TQFT.

*(2.10) The Verlinde ring.* The 2-dimensional TQFT controlling this indexes is a semi-simple Frobenius algebra, called the *Verlinde ring* $V(G; h)$. (The extension that includes vector bundles is less well known, and was indicated in [T2].) I only want to flag here that the projectors in the controlling Frobenius algebra are in natural correspondence with the *projective, positive energy representations* of the smooth loop group $LG$ of $G$, with projective co-cycle determined by the Chern class $h$ of the line bundle. There is indeed a deep connection between loop group representations and the Verlinde ring; for example, the Verlinde ring is the Grothendieck $K$-group of the category of said representations, and there is a *tensor structure* on that category, the *fusion product*, which induces the product in the Verlinde ring; there is a distinguished *vacuum representation* which plays the role of the unit, and the Frobenius trace of a general representation extracts the multiplicity of the vacuum. You notice the formal resemblance with the gauge theory of a finite group. But this takes us away from the story.

*2.11 Remark.* From the TQFT point of view, the numbers we associate to surfaces are dimensions of vector spaces $H^0$. Remembering the spaces themselves suggests the possibility of a $3D$ TQFT, with numbers assigned to 3-manifolds. Indeed, this *Chern-Simons theory* has been constructed rigorously [RT]. The subject leads to deep connections between conformal field theory, loop group representations and 3-dimensional topology.

*(2.12) Twisted $K$-theory, a crash course.* We need one more ingredient to describe the TQFT controlling the Verlinde numbers and their generalizations (to be described). Constructing twisted $K$-theory rigorously, especially the equivariant version, would take a course on its own, but the idea is easy enough. The famous Serre-Swan theorem asserts that vector bundles over a compact Hausdorff space $X$ are precisely the projective modules over the ring $C^0(X)$ of continuous functions. This offers a purely algebraic definition of $K(X)$, as the corresponding Grothendieck group of projectives.

When a compact group $G$ acts on $X$, we can define the equivariant $K$-group $K^0_G(X)$ as the Grothendieck group of vector bundles which carry a lifting of the $G$-action.

There is again an algebraic description. We can form the *crossed product algebra* $G \ltimes C^0(X)$ (these are functions on $G \times X$, with the convolution product on $G$, and the intertwining action of $G$ on $X$).

[8]For U(1), $F(\Sigma, G)$ is the Jacobian of $\Sigma$, over which holomorphic line bundles vary in continuous families.

For example, if $G$ is finite, an element in $G \ltimes C^0(X)$ can be expressed as a sum $\sum_{g\in G} g\cdot\varphi_g$, with $\varphi_g \in C^0(X)$, with multiplication

$$\Big(\sum_{g\in G} g\cdot\varphi_g\Big)\cdot\Big(\sum_{h\in G} h\cdot\psi_h\Big) = \sum_{k\in G} k\cdot\sum_{gh=k}(h^{-1})^*\varphi_g\cdot\psi_h,$$

where for $u\in G$, $(u^*\varphi)(x) := \varphi(u^{-1}(x))$. In other words, we act on the function when moving a group element across.

The Serre-Swan theorem in this context equates $K_G(X)$ with the Grothendieck group of projective modules over $G \ltimes C^0(X)$.

Imagine now a bundle on algebras over $X$, locally isomorphic to the constant bundle $\mathbb{C}$. Actually, such a bundle would have to be a product bundle, but that is because isomorphism is the wrong notion for algebras, and should be replaced by *Morita equivalence*. The simplest example is a bundle of matrix algebras. Such a bundle need not be globally trivial, or even Morita equivalent to the trivial bundle. Indeed, for a *projective* bundle $\mathbb{P}\to X$, the associated bundle of matrix algebras is well-defined.[9] It need not be the endomorphism algebra of a globally defined vector bundle $E\to X$. Indeed, the obstruction to this is precisely that of lifting $\mathbb{P}$ to a vector bundle, $\mathbb{P}=\mathbb{P}E$. From the short exact sequence

$$1\to \mathrm{GL}(1)\to\mathrm{GL}(n)\to\mathbb{P}\mathrm{GL}(n)\to 1$$

we get the portion of long exact sequence

$$\cdots\to H^1(X;\mathrm{GL}(1))\to H^1(X;\mathrm{GL}(n))\to H^1(X;\mathbb{P}\mathrm{GL}(n))\to H^2(X;\mathrm{GL}(1))$$

which locates the obstruction in $H^2(X;\mathrm{GL}(1)$. We are in topology and are using continuous coefficients, so this is $H^3(X;\mathbb{Z})$. In the literature, this is called the *Dixmier-Douady class* of the *gerbe* defined by our projective bundle.

The classes we get are $n$-torsion, as the skilled among you will see by comparing with the sequence

$$1\to\mu_n\to\mathrm{SL}(n)\to\mathbb{P}\mathrm{GL}(n)\to 1$$

which places the obstruction in $H^2(X;\mu_n)$. However, there is a good infinite-dimensional version of this construction using projective Hilbert bundles, where the trick of comparing with $\mathrm{SL}(n)$ fails, and indeed one can show that *any class in* $H^3(X;\mathbb{Z})$ *is realized by a gerbe*, unique up to a certain equivalence. The analogue of the bundle of matrix algebras in the Hilbert story is the bundle of *compact* endomorphisms.

All in all, for each class $[\tau]\in H^3(X;\mathbb{Z})$, we can define a twisted $K$-group ${}^\tau K(X)$ as the Grothendieck group of projective modules over the sections of the bundle of matrix algebras defined by $\tau$.

Modulo technical difficulties which have been resolved in a number of ways [AS], the story extends literally to spaces $X$ with compact group action: a class in $H^3_G(X;\mathbb{Z})$ defines a $G$-equivariant bundle of matrix algebras over $X$ (infinite-dimensional ones, unless the class happens to be torsion), and (morally) we define the twisted $K$-group ${}^\tau K^0_G(X)$ as the Grothendieck group of finitely generated projective modules.[10]

*2.13 Remark* (Chern character computation of ${}^\tau K$).

- There is an odd $K$-group $K^1(X)$; a cheating definition is $K^1(X)=K^0(S^1\times X, X)$. This is really the definition of $K^{-1}$, and we are using the *Bott periodicity theorem* $K^i\cong K^{i+2}$.
- For a compact finite-dimensional space $X$, we have an *isomorphism* $\mathrm{ch}: K^0(X)\otimes\mathbb{Q}\to H^{ev}(X;\mathbb{Q})$. There turns out to be a matching isomorphism $\mathrm{ch}: K^1(X)\otimes\mathbb{Q}\to H^{odd}(X;\mathbb{Q})$. Let now $(C*(X;\mathbb{Q}),\delta)$ be the algebra of rational cochains on $X$, with differential $\delta$. (At the price of switching to real coefficients, we can use de Rham's differential forms.) Let also $\tau$ be a 3-co-cycle representing the twisting class. It is easy to describe twisted $K$-theory in this language: ${}^\tau K^{0|1}$

[9] This is because $\mathrm{End}(E)$ is canonically defined from $\mathbb{P}E$ alone.

[10] A technical change is required in the definition of $K$-theory in the infinite-dimensional case, because compact operators form a *non-unital* algebra; see [AS].

are isomorphic, via a twisted Chern character, to the cohomology of $C^*(X;\mathbb{Q})$ with modified differential $\delta + \tau\wedge$. Note that $\delta\tau = \tau\wedge\tau = 0$ confirms that $(\delta+\tau\wedge)^2 = 0$, so this is a complex.

Alas, this easy model does not help with *equivariant* twisted $K$-theory; see the more complicated story of the delocalized Chern character in [**?**].

*(2.14) Twisted $K_G(G)$.* Let $G$ be a compact, simple, simply connected Lie group. Then, $H^3_G(G;\mathbb{Z})$, the group of twistings of $K$-theory, is canonically isomorphic to $\mathbb{Z}$. (There is a preferred generator, giving a positive definite form on the Lie algebra.) It turns out that the *levels*, or projective co-cycles of loop group representations are also parametrized by the integers.[11] The key theorem of [FHT] is an isomorphism of Frobenius rings

$$^{\tau}K_G^{\dim G}(G) \cong V(G;h)$$

The twisting $\tau$ is $h+c$, with the shift $c$ in the level (the *dual Coxeter number*) depending on the group; it is equal to $n$ for $\mathrm{SU}(n)$. Already, $c=2$ has made an appearance in Verlinde's formula for $\mathrm{SU}(2)$.

We already alluded to the multiplication on $V(G;h)$; the one on $^{\tau}K_G^{\dim G}(G)$ is the *Pontrjagin product*, induced by the multiplication $G\times G\to G$. Again, the finite group gauge theory should come to mind, if we think of convolution of characters; but this time, instead of $G$-invariant functions on $G$, we are dealing with $G$-equivariant vector bundles ('twisted' by $\tau$). This replacement of complex-valued function with vector bundles — vector-space valued functions — is an instance of categorification, and hints at the (true) fact that the Verlinde theory we describe is a $3D$ theory in disguise.

The Frobenius trace on $^{\tau}K_G^{\dim G}(G)$ is a bit less obvious; see (3) below. For the following theorem, recall that the stack of flat $G$-connections on the circle is equivalent to the stack $G/G$, with $G$ acting on itself by conjugation. A compact oriented surface $\Sigma$ with incoming boundary $\partial^-\Sigma$ and outgoing one $\partial^+\Sigma$ defines a correspondece of *stacks*

$$\begin{tikzcd} & F(\Sigma;G) \arrow[dl, "\pi_-"'] \arrow[dr, "\pi_+"] & \\ (G/G)^p \cong F(\partial^-\Sigma;G) & & F(\partial^+\Sigma;G)\cong (G/G)^q \end{tikzcd}$$

Those of you who dislike stacks must choose a base-point in (each connected component of) $\Sigma$ and in each boundary component, and paths from the $\Sigma$-base point to each of the others. The stacks are then presented as a product of many copies of $G$, one holonomy for each path and each boundary circle, divided by the simultaneous action of $G$ at all base-points. (The drawback of proceeding in this manner is that you must then prove that the operation defined below is independent of your choice of base-points or paths; this is avoided by defining $K$-theory and the operations on stacks and representable morphisms thereof.)

*(2.15) Exercises.*

(i) By starting with a presentation and simplifying as possible, show that the correspondence diagram induced by a pair of pants with two incoming and one outgoing circle is

$$\begin{tikzcd} & (G\times G)/G^{\mathrm{diag}} \arrow[dl, "\mathrm{Id/diag}"'] \arrow[dr, "m"] & \\ G/G\times G/G & & G/G \end{tikzcd}$$

with all groups acting by conjugation, $G^{\mathrm{diag}}$ representing the diagonal subgoup, and $m$ the multiplication map on the group. So the operation is restricting from $G\times G$ to $G^{\mathrm{diag}}$-equivariance, and then multiplication in the group.

You should have done this exercise when studying the finite gauge theory!

[11] Projective $LG$-cocycles are always classified topologically by $H^3_G(G;\mathbb{Z})$.

(ii) Show that the correspondence diagram induced by a cylinder with two outgoing cicles is the anti-diagonal inclusion $G/G \to G/G \times G/G$.

**2.16 Theorem** ([FHT, TW]). *(i) There is a 2-dimensional TQFT based on the space* ${}^{h+c}K_G^{\dim G}(G)$.

*(ii) A surface with $p$ inputs and $q > 0$ outputs induces the linear map*

$$(p_+)_* \circ p_-^* : {}^\tau K_G^{\dim G}(G)^{\otimes p} \to {}^\tau K_G^{\dim G}(G)^{\otimes q}.$$

*(iii) The bilinear form underlying the Frobenius structure is given by the cylinder with two outgoing ends, and is non-degenerate.*

*(iv) The partition function for a closed surface computes the index of $\mathscr{O}(h)$ over $F(\Sigma; G)$.*

The statement above conceals many fine points discussed in [FHT], such as how to handle the twistings in pullbacks and push-forwards, as well as the $K$-theory orientations on $F$, which must be tracked with care when $G$ is not simply connected.

*(2.17) Generalizations.* The theory extends to incorporate the $K$-theoy analogue of the Atiyah-Bott generating cohomology classes on $F(\Sigma; G)$. We refer to [T2, TW] for details; one subtlety *not* present for line bundles appears in statement (iv), where the moduli space of flat connections must be replaced with the moduli stack of holomorphic $G_{\mathbb{C}}$-bundles.

Here I just give the formula of $K$-theory integration on the moduli of *Higgs bundles*, leaving the asymptotic derivation of the integration formula as an exercise for the reader. (The integration formula was found with physics arguments by Moore, Nekrasov and Shatashvili [MNS].)

The moduli stack $\mathfrak{M} := \mathfrak{M}(\Sigma; G)$ of all holomorphic $G_{\mathbb{C}}$-bundles over $\Sigma$ has $F(\Sigma; G)$ as an associated GIT quotient, meaning

$$F(\Sigma; G) = \mathrm{Proj}\left(\bigoplus\nolimits_h \Gamma(\mathfrak{M}; \mathscr{O}(h))\right)$$

Under certain conditions (after a large $\mathscr{O}(h)$ twist), indexes of vector bundles over $\mathfrak{M}$ and $F(\Sigma; G)$ agree; this often allows us to dispense with the stack in the story of the index and the associated TQFT, as we did for line bundles.

Now, $\mathfrak{M}$ has a cotangent stack $T^*\mathfrak{M}$; ordinarly, this would be a *differential graded* stack, but in genus 2 or more, the dg structure vanishes and $T^*\mathfrak{M}$ is an ordinary Artin stack, locally presentable as a quotient of a locally complete intersection variety by a reductive group. Using the same line bundle $\mathscr{O}(h)$, we can define an associated moduli space $H(\Sigma; G)$, the moduli of semi-stable Higgs bundles, which is a partial compactification of $T^*F(\Sigma; G)$. There is a similiar story equating indexes of bundles over $T^*\mathfrak{M}$ and $H$, which applies in particular to $\mathscr{O}(h)$ for $h > 0$; so again people who dislike stacks can avoid them in the story. Non-compactness of $H$ gives infinite answer to index questions, but the $\mathbb{C}^*$-scaling action on the fibers of $T^*\mathfrak{M}$ can be used to render the answers finite. Indexes will not be numbers, but power series in $q \in \mathbb{C}^*$, labelling the dimensions of weight spaces. With these prelimiaries, we are ready for the

**2.18 Theorem** (following [TW]). *For $G = \mathrm{SU}(2)$,*

$$\mathrm{Ind}(H(\Sigma; G); \mathscr{O}(h)) = (2h+4)^{g-1} \sum\nolimits_{k=1}^{h+1} \left(2 \sin \frac{k_q \pi}{h+2}\right)^{2-2g} \cdot \left(1 + \frac{2q}{h+2} \frac{1 - q\cos\frac{2\pi k_q}{h+2}}{(1+q^2) - 2q\cos\frac{2\pi k_q}{h+2}}\right)^{g-1}$$

The points $k_q = k + qk_1 + q^2k^2 + \dots$, with $k = 1, 2, \dots, h+1$ are the power series solutions of the equation

$$k_q + \frac{1}{2\pi \mathrm{i}} \log\left(1 - qe^{-2\pi \mathrm{i} k_q/(h+2)}\right) - \frac{1}{2\pi \mathrm{i}} \log\left(1 - qe^{2\pi \mathrm{i} k_q/(h+2)}\right) = k$$

This is a TQFT over the power series ring $\mathbb{C}[\![q]\!]$. At $q = 0$ we get Verlinde's formula, as we should, since $q^0$ counts the sections which are constant along the fibers of $T^*\mathfrak{M}$ and therefore come from $\mathfrak{M}$. There is a generalization to all compact groups; other vector bundles can also be included, but then we must integrate over $T^*\mathfrak{M}$ unless the line bundle twist is large.

## 3. Extended TQFT and Higher Categories

A basic flaw in the Atiyah-Witten definition is the restriction to co-dimension 1 boundaries. While this keeps the story clean and simple, it makes it impossible to compute the TQFT by cutting up the manifold into simple pieces; the cases D=1,2 were special, and the next best classification theorem ($D = 3$, Reshetikhin-Turaev [RT]) requires the use of circles and cutting in co-dimension 2.

*Extended TQFTs* are, intuitively, functors from $\mathscr{B}ord_D^{or}$ with enough structure to allow the cutting of manifolds into simple pieces with corners of all co-dimensions, and a reconstruction of the invariant. This is a backwards way of putting it, and it soon enough becomes clear that we need to replace our two tiers of structure in the bordism category — $(D-1)$-dimensional objects and $D$-dimensional morphisms — with $(D+1)$ tiers of structure, going down to points.

*(3.1) Higher categories.* The various ways of encoding such an algebraic structure, with $D$ tiers of morphisms layered over objects, allowing multi-dimensional compositions, are known as *$D$-categories.* A prime example is the bordism $D$-category $Bord_D^{or}$, in which (oriented) 0-manifolds are the objects, oriented 1-manifolds with boundary are morphisms between 0-manifolds, 2-manifolds with corners are "2-morphisms" and so forth. It quickly emerges that the algebraic rules of the game are not as clearly set anymore (in fact, many sets of rules are imaginable), so here is a first inductive

**3.2 Definition.** A *strict $D$-category* is a category in which all sets $Hom(x, y)$ of morphisms have the structure of $(D-1)$-categories, and the compositions $Hom(x, y) \times Hom(y, z) \to Hom(x, z)$ are strict bi-functors of $(D-1)$-categories. Composition is strictly associative.[12] A *functor* $\phi$ between $D$-categories is a functor of underlying categories, such that the induced maps on morphisms $\phi_* : Hom(x, y) \to Hom(\phi x, \phi y)$ are functors of $D-1$-categories.

We are concerned with ($\mathbb{C}$-)linear categories, which at the top two layers reproduce vector spaces and linear maps. It is also customary to require the categories to admit finite direct limits, at least (arbitrary limits are sometimes convenient). There is a 'unit object' in the world of $D$-categories for any $D$, unit in the sense that everything is a module over it, much like every $\mathbb{C}$-vector space is a module over $\mathbb{C}$. Among linear categories, the unit is the category $Vect$ of finite-dimensional vector spaces. (In a $\mathbb{C}$-linear category, every object can be tensored with a finite-dimensional vector space, and this is functorial both in the object and in the vector space.) Next is the strict 2-categories of linear categories, linear functors and natural transformations, followed by the strict 3-category of strict 2-categories, etc. If direct limits are assumed to exist, we ask for the linear functors to be right exact, that is, to preserve them.

*3.3 Remark.* Aother familiar one-step 'de-looping' of the category of vector spaces of linear maps, instead of the 2-category of linear categories, is the 2-category $\mathscr{A}lg$ of algebras, bi-modules and intertwining maps. One can embed "fully faithfully" this into linear categories by sending every algebra to its category of modules. (Observe that for two algebras $A, B$, any right-exact functor from $A$-modules to $B$-modules is induced by tensoring over $A$ with a $B-A$-bimodule.) This is a *non-strict* 2-category: composition of morphisms — tensoring of bi-modules — is defined only up to natural isomorphism.

*(3.4) Strict versus lax categories.* Experience with categories should suggest the flaw in the strict definition: when in a category, one should ask for 'well-behaved isomorphism' rather than equality. Good behavior depends on the problem at hand, for instance, functoriality under some operations. For example, it seems wrong to require the composition operation to be associative on the nose — instead we should ask for an *associator* $\alpha$, a natural isomorphism of the two functors from $Hom(x, y) \times Hom(y, z) \times Hom(z, w)$ to $Hom(x, w)$ obtained by composing in different orders. There is then a natural condition on this associator (Stasheff's pentagon identity), which suffices to allow us to work with 'associativity up to coherent associators' just as we would with strict associativity. Similarly, we could ask for functors between 2-categories to preserve the composition of 1-morphisms only up to coherently chosen 2-morphisms. As we progress in categorical depth, there are more identities that we could relax to 'coherent' isomorphisms. Any systematic listing of the associative and commutative

[12] So that 'equal' really means equal and not canonically isomorphic.

data, and coherence conditions on it, will lead to a theory of *lax* $D$-categories. Much current work in higher categories is motivated by the search for spelling out methodical, but convenient and practical ways to encode the lax data and its coherence conditions; see [B, L, R] and many others.

Intuition is not completely reliable. It turns out that there is no problem for $D = 2$: any lax 2-category can be 'strictified'; but we run into trouble beyond that, as we will see below. Ignoring this trouble for a moment, let us discuss a simple

*(3.5) 2D gauge theory.* This example will be useful before discussing the general theory of duality in the next lecture. We construct a 2-functor from the unoriented bordism 2-category $\mathscr{B}ord_2$ to $\mathscr{A}lg$.

To the point we associate the group algebra $A := \mathbb{C}\langle F\rangle$, of linear combinations $\sum_{f\in F} a_f \cdot e_f$, with $a_f \in \mathbb{C}$ and multiplication $e_f \cdot e_{f'} = e_{ff'}$ imitating the group one. $A$-modules are representations of $F$. Note that $A \cong A^{op}$ by the anti-involution $f \leftrightarrow f^{-1}$; so that for instance there is no distinction between $A - A$ bimodules and $A^{\otimes 2}$-modules. (This does not apply to the *twisted* version of gauge theory, see Remark 3.10.)

Now to any interval we associate the bi-module $A$, which could be a left or a right $A^{\otimes 2}$-module, or $A - A$ bimodule, according to the endpoints: both outgoing, both incoming or one of each. Note the natural isomorphism

$$A \otimes_A A \cong A, \quad \text{expressing the composition} \quad \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id}. \tag{3.6}$$

To a closed circle, we associate the space $A \otimes_{A\otimes A} A$. This computation is forced upon us by the presentation of the circle as the composition $\subset \circ \supset$, and identifies $Z_F(S^1)$ with the space of class functions on $F$, as before.

Having defined the bottom two tiers, consider a surface with corners. To read it as a 2-morphism, we must supply more data, such as labelling corners as sources or targets, orienting edges compatibly, and labeling them as source or target as well. Listing all possibilities and the corresponding maps quickly becomes cumbersome; the good construction is again inspired by the physiscist's path integrals.

Let then $\Sigma$ be a surface with corners, read as a 2-morphism between an incoming and an outgoing part of its boundary. Those boundary parts, $\partial_\pm\Sigma$, are 1-morphisms with the *same* source and target. That is, the corners on $\partial_-\Sigma$ must be matched with corners on $\partial_+\Sigma$ by some part $B$ of the boundary.[13] We now note that the space associated to a 1-morphism (1-manifold with boundary) is $H^0$ of the groupoids $Bun_F$ of $F$-bundles, but trivialized on the boundary. Similarly, the operation $Z_F(\Sigma)$ is given by $(\pi_+)_* \circ \pi_-^*$ in the familiar correspondence diagram

$$\begin{tikzcd} & Bun_F(\Sigma, B) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ Bun_F(\partial_+\Sigma, B_+) & & Bun_F(\partial^-\Sigma, B-) \end{tikzcd}$$

*(3.7) The correspondence 2-category.* What we did was to factor the gauge TQFT from $\mathscr{B}ord_2 \to \mathscr{A}lg$ via a 2-category $\mathscr{C}or_2$ of *correspondences* of finite groupoids. Objects in $\mathscr{C}or_2$ are finite groupoids $X_\bullet$. This notation must be conceived to include the sets $X_0$ of objects, $X_1$ of morphisms, the source and target maps $X_1 \rightrightarrows X_0$, the identity section $X_0 \to X_1$, and the composition law. A map of groupoids $X_\bullet \to Y_\bullet$ is a pair of maps on objecs and morphisms, compatible with the structures.

1-morphisms in $\mathscr{C}or_2$ are correspondences $Y_\bullet \leftarrow X_\bullet \to Z_\bullet$, and 2-morphisms between $X_\bullet, X'_\bullet$ are again correspondences $X_\bullet \leftarrow C_\bullet \to X'_\bullet$, compatible with the projection to $Y_\bullet \times Z_\bullet$. Composition of 1- and 2-morphisms is the *homotopy fiber product*: this is the naive fiber product obtained after replacing one of the maps with a *fibration* — a morphism where each arrow lifts, once a lifting of the source or target has been found.

*(3.8) Example.* The point is $* \rightrightarrows *$. The groupoid $BF$ (finite group $F$) is described as $F \rightrightarrows *$. The morphism from $* \to BF$ is not a fibration, but can be converted to one after replacing $* \rightrightarrows *$ with the equivalent groupoid $F \times F \rightrightarrows F$, representing the translation $F$-action on $F$. The homotopy fibre product $* \times_{BF} *$ is then $F^3 \rightrightarrows F$, and is equivalent to the set $F$ (with no non-trivial morphisms). Verify this! □

[13] In open-closed TQFT, those boundary intervals are variously called *free* or *constrained.*

*3.9 Remark.* This lax 2-category is strictifiable because every morphism of groupoids can be functorially converted to a fibration.

Assigning to a manifold the groupoid of principal $F$-bundles over it gives a 2-functor from $\mathscr{B}ord_2$ to $\mathscr{C}or_2$. The task is to continue by defining a 2-functor from $\mathscr{C}or_2$ to $\mathscr{A}lg$.

Now, to each object in $\mathscr{C}or_2$ we assign the *path algebra* of the groupoid. To a correspondence $Y_\bullet \leftarrow X_\bullet \rightarrow Z_\bullet$, we assign the sum, over objects in $Y_0 \times Z_0$, of the functions of the homotopy fiber of $X$ there. This is a module over the path algebra of $Y_\bullet \times Z_\bullet$. There is a more conceptual description: if $f : X_\bullet \rightarrow Y_\bullet$ is a fibration of groupoids, then $f_*\mathbb{C}$, the direct image of the constant sheaf, is a flat vector bundle over $X_\bullet$ — a bundle with a composable lifting of the arrows — and so its sections over $X_0$ give a module for the path algebra of $Q$. One should picture here a submersion $f : X \rightarrow Y$ of manifolds, for which the cohomologies $R^i f_*\mathbb{C}$ along the fibers are vector bundles with a canonical flat connection.

*Example.* Show that the path algebra of $BF$ is $\mathbb{C}\langle F\rangle$. Starting from the map $* \rightarrow BF$, show that we produce the regular representation of $F$. For $G \subset F$ and the induced map $BG \rightarrow BF$, show that we get the *induced representation*, consisting of functions on $F/G$. □

Finally, for a correspondence $C_\bullet$ between correspondences, we define a linear map between spaces of functions by the push-pull construction: the matrix coefficient relating to functions counts points mapping to both, weighted down by automorphisms relative to the second map. This construction is already familiar from unextended finite gauge theory.

*3.10 Remark* (Twisted gauge theory). A class $[\tau] \in H^2(BF;\mathbb{C}^\times)$ defines a central extension of $F$ by $\mathbb{C}^\times$. Think of a central extension as a line bundle over the group, with a compatible multiplication on lines: that is, isomorphisms $\alpha_{f,g} : L_f \otimes L_g \rightarrow L_{fg}$, satisfying the coherence expressed by commutativity of the square:

$$\begin{tikzcd}
L_f \otimes L_g \otimes L_h \arrow[r, "\mathrm{Id}\otimes\alpha_{g,h}"] \arrow[d, "\alpha f{,}g\otimes\mathrm{Id}"] & L_f \otimes L_{gh} \arrow[d, "\alpha_{f,gh}"] \\
L_{fg} \otimes L_h \arrow[r, "\alpha_{fg,h}"'] & L_{fgh}
\end{tikzcd}$$

Of course, all the lines can be identified with $\mathbb{C}$, and then the $\alpha_{f,g}$ become $\mathbb{C}^\times$-valued group cocycles.

The twisted group algebra ${}^\tau\mathbb{C}\langle F\rangle$ is defined as the space of sections of this line bundle, and it carries an obvious multiplication lifting the product of group elements. Its modules are the $\tau$-projective representations of $F$. But now, the opposite algebra is ${}^{-\tau}\mathbb{C}\langle F\rangle$. The twisted gauge theory is defined on oriented manifolds, but does not factor through the unoriented bordism category.

The reader is encouraged to construct the twisted gauge theory for oriented manifolds; the fundamental cycle of a surface rel boundary, and Stokes' theorem for cohomology in $\mathbb{C}^\times$, should make an appearance.

*(3.11) Inadequacy of strict categories.* A precise problem can be identified when restricting to $D$-groupoids, categories where all morphisms are invertible. It is assumed that any sensible theory of higher groupoids is equivalent to the theory of homotopy types in topology; specifically, $D$-groupoids should correspond to $D$-types, topological spaces with vanishing homotopy groups above dimension $D$. (Declaring that homotopy of maps is an equivalence relations is akin to declaring all morphisms to be invertible in a higher category.) So, for any good definition of $D$-category, restriction to groupoids should produce homotopy $D$-types. However, the strict inductive definition is faulty beyond $D = 2$:

**3.12 Proposition.** *A connected homotopy type $X$ can be represented by a strict groupoid if and only if its $k$-invariants beyond $k_2$ vanish. In other words, $X$ is the classifying space of a group which is the extension of $\pi_1(X)$ by a topological abelian group.*

*In particular, if $X$ is simply connected, it must be equivalent to the classifying space of a topological abelian group, that is, a product of Eilenberg-MacLane spaces.*

Most simply connected homotopy types are not of that form: indeed, restricting to such spaces erases nearly all of homotopy theory. For example, from the 2-sphere we discussed earlier, we can produce a homotopy 3-type by killing the homotopy groups above 3 and keeping just $\pi_2 = \pi_3 = \mathbb{Z}$. (This can be done by attaching cells of increasing dimensions.) But we have seen that the result differs from the product 3-type $\mathbb{CP}^\infty \times K(\mathbb{Z};3)$, having tested maps from $\mathbb{CP}^2$ into the two spaces.

The proof of the theorem imitates the proof of commutativity of $\pi_2$ of a topological space: the key lemma that a space with two strictly commuting associative multiplications is in fact strictly commutative, and the two multiplications agree. In the world of strict 3-groupoids, this proves strict commutativity of the first loop space, and entails the vanishing of the Whitehead bracket below.

*(3.13) The quadratic map $\pi_2 \to \pi_3$.* We give an interpretation of the Postnikov $k$-invariant $k_3 \in H^4(K(\pi_2(X), \pi_3(X))$ of a space with $\pi_2, \pi_3$ only. The standard argument for commutativity of $\pi_2(X)$ and higher, for any space $X$ conceals higher operations on homotopy groups, the first of which is the *Whitehead bracket*, a collection of bilinear maps $\pi_m(X) \times \pi_n(X) \to \pi_{m+n-1}(X)$. These are best seen by using the isomorphism $\pi_m(X) \cong \pi_{m-1}(\Omega X)$, for the space $\Omega X$ of based loops in $X$. The commutator map $\Omega X \times \Omega X \to \Omega X$ can be deformed so as to squash the subspace $\Omega X \times \{1\} \cup \{1\} \times \Omega X$ to the identity. The resulting composition, $S^{m-1} \times S^{n-1} \to \Omega X \times \Omega X \to \Omega X$, then squashes to factor through

$$S^{m+n-2} = S^{m-1} \times S^{n-1} / (S^{m-1} \times \{*\} \cup \{*\} \times S^{n-1}).$$

This gives a class in $\pi_{m+n-2}(\Omega X) \cong \pi_{m+n-1}(X)$.

The bilinear Whitehead bracket $\pi_2 \times \pi_2 \to \pi_3$ has a quadratic refinement. This is because one can compute the generator of $\pi_3(S^2)$, represented by the famous *Hopf fibration*, to be *one-half* the Whitehead square of $1 \in \pi_2(S^2)$. For any space $X$, one gets a quadratic map $\pi_2(X) \to \pi_3(X)$ by pre-composing any $\alpha : S^2 \to X$ with the Hopf map $S^3 \to S^2$. We have

**3.14 Theorem.** *A space $X$ with only two homotopy groups, $\pi_2(X)$ and $\pi_3(X)$, is completely determined by these groups, and by the quadratic map $\pi_2 \to \pi_3$.*

Homotopy theory tells us that the space is completely determined by the Postnikov invariant $k_3$. A result of MacLane asserts the correspondence of such classes with quadratic maps $\pi_2 \to \pi_3$, by the construction we just described.

*(3.15) A braided tensor category from $X$.* Now let us produce a more rigid incorporation of $k_3$ in the form of a *braided tensor category.*

**3.16 Definition.** A *tensor category $T$* is a ($\mathbb{C}$-)linear category with a bi-linear multiplication functor $m : T \times T \to T$, containing a unit object 1 (let us take it to be strict, for simplicity; so $m(1, x) = x = m(x, 1)$) and an associator $\alpha_{x,y,z} : m(m(x, y), z) \to m(x, m(y, z))$ which allows us to 'move parentheses' in multiplication. The associator satisfies a coherence identity for four objects (Stasheff's *pentagon identity*), which says that the choice of order of parenthesis moves is irrelevant.[14]

The product $m(x, y)$ is often denoted by $x \otimes y$. Informally, a braided tensor category is a tensor cat with a first order of commutativity. Call $\tau$ the transposition automorphism on the square $B \times B$ of a category.

**3.17 Definition.** A *braided tensor category $B$* is a tensor category equipped with a $b$i-multiplicative braiding isomorphism of functors $\beta : m \to m \circ \tau$. That means, for each pair $x, y$, an isomorphism $\beta(x, y) : x \otimes y \to y \otimes x$, functorial in $x, y$. The category is *symmetric* if $\beta^2 = \mathrm{Id}$.

Bi-multiplicativity means that for any three objects $x, y, z$, the diagrams commute:

$$\begin{tikzcd} x \otimes y \otimes z \arrow[rr, "\beta(x,y\otimes z)"] \arrow[dr, "\beta(x,y)\otimes\mathrm{Id}_z"'] & & y \otimes z \otimes x \\ & y \otimes x \otimes z \arrow[ur, "\mathrm{Id}_y\otimes\beta(x,z)"'] & \end{tikzcd} \qquad \begin{tikzcd} x \otimes y \otimes z \arrow[rr, "\beta(x\otimes y,z)"] \arrow[dr, "\mathrm{Id}_x\otimes\beta(y,z)"'] & & z \otimes x \otimes y \\ & x \otimes z \otimes y \arrow[ur, "\beta(x,z)\otimes\mathrm{Id}_y"'] & \end{tikzcd}$$

We have omitted the associators $\alpha$ (set them to Id).

[14] *MacLane's coherence theorem* ensures that we can replace $T$ by an equivalent category in which $\alpha \equiv 1$; but this may require enlarging the category, or breaking some extra structure, such as continuuity.

*3.18 Remark.* One way to think of it is that the tensor product of $n$ objects in a braided category carries a natural action of the braid group on $n$ strands. In the symmetric case, this action factors through the symmetric group.

*Examples.*

(i) The category of representations of a group, with the tensor product over $\mathbb{C}$, is braided and in fact symmetric.

(ii) The category of modules over a commutative algebra $A$, with $\otimes_A$, is also symmetric. If $A$ is not commutative, it is generally not a tensor category.

(iii) It is more difficult to produce a non-symmetric braided category, it you have not seen one. A famous example comes from *quantum groups*. Below, I will give a topological example.

Given now $X$ with two homotopy groups $\pi_2, \pi_3$, let $B_X$ be the category of flat vector bundles on the second loop space $\Omega^2 X$, with support on finitely many components. The choice of base-point of $X$ will not matter. We are thus choosing a representation of $\pi_3$ for each element of $\pi_2$. This category is even an abelian and semi-simple if the homotopy groups are finite.

Let $m : \Omega^2 X \times \Omega^2 X \to \Omega^2 X$ denote the multiplication, and define a tensor structure on $B_X$ by $U \odot V := m_*(U \boxtimes V)$, the fiberwise cohomology along $m$.

**3.19 Proposition.** *The above defines a braided tensor structure on $B_X$. The braiding is symmetric if $k_3 = 0$.*

*Proof.* The associativity property of $\odot$ should be clear; let us just indicate the braiding. The standard argument for the commutativity of $\pi_2$ gives a fixed homotopy between the multiplication $m$ and its transpose, exploiting the fact that we are in a double loop space. (Realize $\Omega^2 X$ as maps from the square to $X$, sending the boundary to the base-point; then two adjacent squares $\square\square$, representing the product in $\Omega^2 X$, can be moved past each other by clockwise rotation.) In this way, we link $\odot$ to $\odot \circ \tau$ by a one-parameter family of maps. The direct image bundle carries a flat connection, following this along the interval gives an isomorphism between $U \odot V$ and $V \odot U$. $\square$

*Exercise.* Compute the category and braiding in simple examples, such as the 2-sphere.

*3.20 Remark.* The braiding can be symmetric even for some non-zero $k_3$. In fact, $k_3$ contains finer information, a *ribbon structure* on $B_X$, which *trivializes the Serre functor* on the category and helps ensure that $X$ defines a 4-dimensional TQFT for oriented manifolds.

*(3.21) Finite homotopy types.* We conclude by indicating the construction of the enhanced gauge theory of §1.11 as an extended TQFT, following the example of gauge theory. Except for the language layered with morphisms of all orders, this does not differ from the extended gauge theory in §3.5. In fact, the twisted gauge theory of Remark 3.10 was secretly a theory for a 2-groupoid.

Let $X$ be a space with finite homotopy groups and fix the dimension $D > 0$ of the intendet TQFT. Homotopy groups above $D$ will not play a role and will be truncated. We seek a (lax) functor from $\mathscr{B}ord_D$ to the $D$-category $\mathscr{C}at_{D-1}$ of lax linear $(D-1)$-categories in two steps:

(i) First we construct the functor $\mathscr{B}ord_D \to \mathscr{C}or_D$ to the $D$-category of correspondences of topological spaces with finitely many, finite homotopy groups. This is analogous to the classical field theory, over which physicists perform the path integral to quantize.

(ii) Next, we construct the quantization functor $Q : \mathscr{C}or_D \to \mathscr{C}at_{D-1}$

The first part is unambiguous and every manifold $M$, representing a morphism of whatever layer, gets sent to the space of maps to $X$.

Now, since we did not commit to a definition of lax higher categories, we cannot actually perform the second step, and the details of the construction are model-dependent. One construction, using the notion of *m-algebras* (algebras with layers of compatible multiplications) was indicated in [FHTL]. Another construction can be given in terms of the "Blob complex" of Morrison and Walker [MW]. In both cases, adding the top layer of the theory (linear maps beteween vector spaces, or numbers) is the more delicate step, and uses the finiteness of $X$. However, we indicate the common idea; this is to construct the $m$-linearization functor of a space, which is a linear $m$-category, in a way that takes homotopy fiber products to tensor products. (A zero-category is a vector space; a $(-1)$-category would be a complex number, I guess.)

The 0-categorification of $Y$ is the space of locally constant functions on $Y$.

The 1-categorification is the category of vector bundles with flat connection.

The 2-categorification is the 2-category of linear category bundles with flat connection (locally constant sheaves of linear categories)

And so forth.

The key observation is that one can define the *cohomology of a space with coefficients in a locally constant sheaf of $m$-categories* and obtain an $m$-category. This allows us to define 'wrong-way maps' for bundles of higer categoreis, and thus convert correspondences into functors. The finiteness conditions on $X$ are needed to stay within the world of *dualizable objects* (next lecture), which are needed for the consistency of the TQFT.

The braided tensor category of §3.15 is a model for the 3-categorification of the space $X$; we will see in teh last lecture why a braided tensor category is an object of 3-categorical nature: it has a "3-category of modules".

## 4. The cobordism hypothesis in dimensions 1 and 2

The "cobordism hypothesis" is the classification of fully extended $D$-dimensional TQFTs with values in an arbitrary symmetric monoidal $D$-category. It was formulated (not completely precisely) by Baez and Dolan [BD]: roughly speaking, such a theory should be completely determined by an object in the $D$-category, which is assigned to a point; the object must satisfy certain *conditions* and carries additional *structure.* This was made precise and proved by Lurie [L]. The key insight behind his formulation (distinguishing it from other variants, such as [MW]) was to separate the *conditions* from the *structure.*

This is accomplished by passing to the *framed bordism $D$-category $\mathscr{B}ord_D^{fr}$*, to be defined below. In this case, the object assigned to the point is subject to the *full dualizability* condition, which generalizes the finite-dimensionality of vector spaces in the case $D = 1$. There is no extra structure; the latter only appears when attempting to factor a framed theory via manifolds with other structure (orientation, spin, or any topologically defined structure on the tangent bundle). Again, speaking loosely, to extend a framed theory to the category of manifolds whose tangent bundle reduces topologically, from the orthogonal group, to a group $G \to \mathrm{O}(D)$, requires the object $Z(*)$ to be made $G$-equivariant. More precisely, it requires exhibiting $Z(*)$ as a $G$-fixed object within the space of all dualizable objects in the target $D$-category, with respect to the canonical $\mathrm{O}(D)$ action on the space of sub objects.

Let us start with a general definition[15]

**4.1 Definition.** A *$D$-framing* on a smooth $k$-manifold $M$ (where $k \le D$) is a framing (trivialization) of the bundle $\mathbb{R}^{D-k} \oplus TM$: that is, a bundle isomorphism $\mathbb{R}^{D-k} \oplus TM \cong \mathbb{R}^D$ over $M$.

The $D$-category $\mathscr{B}ord_D^{fr}$ has $D$-framed points as objects, and its $k$-morphisms are the compact $D$-framed manifolds with corners. For $k = D$, we take $D$-manifolds, modulo boundary-fixing diffeomorphisms and homotopy of the framings.

*4.2 Remark.* (i) It is important that the extra $\mathbb{R}$'s in the framing *need not* align with the first summands of $\mathbb{R}^D$. A $D$-framing is similar to, but slightly finer than, a *stable framing*, obtained by letting $D$ large for fixed $M$; the notions agree when $\dim M < D-1$ because of the stabilization of the homotopy groups of SO.

(ii) However, the extra $\mathbb{R}$ summands on a component $B$ of the $k$-dimensional boundary of $M$ are the inward or outward normals to $B$ in $M$, the direction chosen according to the direction of the morphism. This makes it clear that a manifold with corners can be read as a morphism in many ways. See, for instance, the ways of reading a square in Fig. 1.

[15] which conceals some fine points, as we see in the remark which follows it; but one may unravel it gradually.

(iii) As soon as $D \geq 1$, there are, up to isomorphism, exactly two $D$-framed points $+,-$ in $\mathscr{B}ord_D^{fr}$, distinguished by determinant sign of the framing $\mathbb{R}^D \cong \mathbb{R}^D \oplus T(\text{point})$. The isomorphism is realized by any homotopy from one framing to another in the same orientation class, which we regard as a $D$-framed interval, a 1-morphism in $\mathscr{B}ord_D^{fr}$ The opposite homotopy provide the inverse isomorphism (up to invertible 2-morphisms, etc.).

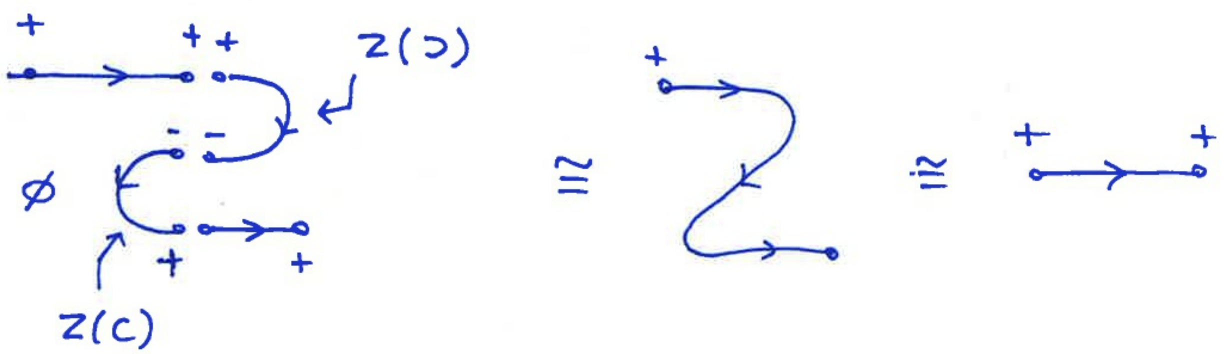
(iv) The automorphism group of each point is homotopy equivalent to $\Omega\mathrm{SO}(D)$.

*4.3 Remark.* Modding out by diffeomorphisms in the top dimension should look wrong to the experienced among you: this is because diffeomorphism groups have non-trivial topology, which is lost in our definition. For example, one could never capture the structure of Gromov-Witten theory, which relies on the topology of the $\overline{M}_g^n$, with this definition. Really, each top-dimensional manifold should be replaced by a space of morphisms with homotopy type the respective $BDiff$.[16] The preferred, *professional* way to accomplish the same involves the notion of $(\infty, D)$-category, which has interesting $k$-morphisms for all values $k$, but all of them are required to be isomorphisms above $D$. The homotopy type of $BDiff$ is then captured in the higher automorphism groups of $D$-morphisms.

*(4.4) Duals and 1D framed TQFTs.* Framings and orientations agree for $\mathbb{R}$-bundles, up to a contractible space of choices. Oriented theories, you recall, comprise the vector spaces $Z(+), Z(-)$ assigned to the two oriented points, and the maps $Z(\subset): \mathbb{C} \to Z(-) \otimes Z(+)$, $Z(\supset): Z(+) \otimes Z(-) \to \mathbb{C}$. These are subject to the conditions that the compositions

$$Z(+) \xrightarrow{\mathrm{Id}\otimes Z(\subset)} Z(+) \otimes Z(-) \otimes Z(+) \xrightarrow{Z(\supset)\otimes \mathrm{Id}} Z(+),$$
$$Z(-) \xrightarrow{Z(\subset)\otimes \mathrm{Id}} Z(-) \otimes Z(+) \otimes Z(-) \xrightarrow{\mathrm{Id}\otimes Z(\supset)} Z(-)$$

are the respective identities. The conditions are seen geometrically by gluing the two half-circles at the respective end: we obtain an 'S' or 'Z' which are diffeomorphic to the identity intervals:




**4.5 Definition** (Internal duals in a monoidal category)**.** Given a pair of objects and morphisms as above in a monoidal category (with the unit object replacing $\mathbb{C}$), $Z(-)$ will be called *right dual* to $Z(+)$, and $Z(+)$ *left dual* to $Z(-)$. $Z(\subset)$ and $Z(\supset)$ are called the *unit* and *trace* of the adjunction.

*4.6 Remark.* (i) The left dual of a given object, if it exists, is determined up to canonical isomorphism, along with the unit/trace.

(ii) Strenghening the above, a morphism between dualizable objects has an *adjoint morphism* relating the duals, in the opposite direction.

(iii) Invertible objects are always dualizable, and their inverses can be taken as their duals.

(iv) In a *symmetric* monoidal category, left and right duality are the same condition; we can then denote the dual of $x$ by $x^\vee$. But we will run into the left/right distinction later, when considering morphisms in a higher category.

(v) Dualizable vector spaces are precisely the finite-dimensional ones. Finitely generated projective modules over a commutative ring are dualizable, but these are not the only dualizables.

(vi) (Hom definition) An equivalent definition of adjunction is the existence of a bi-functorial isomorphism $Hom(Z(+) \otimes x, y) \cong Hom(x, Z(-) \otimes y)$.

[16]Plus the space of framings rel boundary.

*(4.7) Cobordism hypothesis in* $1D$. Call an object of a symmetric monoidal category *dualizable* if it does have a dual. We have just seen that *functors* $Z : \mathscr{B}ord_1^{fr} \to \mathscr{C}$ *to some symmetric monoidal category are classified, up to isomorphism, by* $Z(+)$ *which is a dualizable object in* $\mathscr{C}$.

It is a worthy exercise to show that any natural transformation of functors $Z \to Z'$ is an *isomorphism*, in particular comes from an isomorphism $f : Z(+) \to Z'(+)$. (Show that the companion morphism $g : Z(-) \to Z'(-)$ must be inverse to the adjoint morphism $f^\vee$.)

*(4.8)* $\mathrm{O}(1)$ *action on dualizable objects.* The orthogonal group $\mathrm{O}(D)$ acts on $\mathscr{B}ord_D^{fr}$ by changing the framings.[17] It therefore acts on framed TQFTs, and therefore on dualizable objects in a symmetric tensor category. The action on objects is obvious: it sends each dualizable object to its dual.

Now, when does the theory factor through unoriented manifolds? In that case, $+ = -$ so we may take $Z(+) = Z(-)$. Then, $Z(\supset)$ is a non-degenerate bilinear form on $Z(+)$. The absence of an orientation on $\supset$ forces this to be symmetric, so the theory is determined by a vector space with a non-degenerate quadratic form.

Here is the intrinsic description. The group $\mathbb{Z}/2$ acts on the *groupoid* $G_{>0}Vect$ of finite-dimensional complex vector spaces and isomorphisms, sending vector spaces to their duals and linear isomorphisms to their inverse duals. This is the action of $\mathrm{O}(1)$ on dualizable objects in $Vect$.

**4.9 Proposition.** *The fixed-point category for this is the groupoid of vector spaces with non-degenerate symmetric bilinear form. In particular, unoriented TQFTs are classified by* $\mathrm{O}(1)$*- fixed points among the dualizable objects in* $Vect$.

First, we say that a (discrete) group $G$ acts on a category $\mathscr{C}$ if, for each element $g \in G$, we are given an autofunctor $F_g : \mathscr{C} \to \mathscr{C}$, and for all $g, h$ we are given an isomorphism of functors $\alpha_{g,h} : F_g \circ F_h \xrightarrow{\sim} F_{gh}$, such that the $\alpha$'s in a triple $g, h, k$ satisfy an obvious coherence relation which give a unique isomorphism $F_g \circ F_h \circ F_k \cong F_{ghk}$. It is also convenient to assume that $F_1 = \mathrm{Id}$.

The *fixed-point category* $\mathscr{C}^G$ is, by definition, that whose objects are tuples $(x, \varphi_g|_{g\in G})$ where the $\varphi_g \in \mathrm{Isom}(x, F_g(x))$ satisfy an (obvious) coherence constraint with respect to composition, which ignoring the $\alpha$'s writes $\varphi_{gh} = \varphi_g \circ F_h(\varphi_h)$. Morphisms in $\mathscr{C}^G$ come from $\mathscr{C}$, but are compatible with the $\varphi$.

It is now a good exercise to prove the proposition: there is only one $\varphi : V \to V^\vee$ to specify, for the non-trivial element $\varepsilon$ of $\mathrm{O}(1)$, and the symmetry condition comes form the relation $\varepsilon^2 = 1$.

*(4.10) The cobordism hypothesis in* $2D$. We need the notion of adjoints (the analogue of duals) for morphisms in a higher category. (Fortunately, all higher definitions will be the same.) The key example pertains to functors between categories: $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ form an adjoint pair, with $F$ left and $G$ right adjoint, if we are supplied with a bi-functorial isomorphism $Hom_{\mathscr{C}}(x, Gy) = Hom_{\mathscr{D}}(Fx, y)$. Clearly, the right adjoint of $F$ is unique up to canonical isomorphism, if it exists; we write $G = F^\vee, F = {}^\vee G$.

Taking $y = Fx$ supplies a *unit morphism* $\varepsilon : \mathrm{Id}_{\mathscr{C}} \to G \circ F$ and taking $x = Gy$ gives a *trace* $\theta : F \circ G \to \mathrm{Id}_{\mathscr{D}}$. With the help of $\varepsilon$ and $\theta$, we can state the adjunction condition without involving objects in $\mathscr{C}$ and $\mathscr{D}$.

**4.11 Definition.** A morphism $F : c \to d$ in a 2-category is *left adjoint to* $G : d \to c$ if there exist a *unit* 2*-morphism* $\varepsilon : \mathrm{Id}_c \to G \circ F$ and a trace $\theta : F \circ G \to \mathrm{Id}_d$, such that the following compositions are the respective identity 2-morphisms:

$$F = F \circ \mathrm{Id}_c \xrightarrow{\mathrm{Id}_F \otimes \varepsilon} F \circ G \circ F \xrightarrow{\theta \otimes \mathrm{Id}_F} \mathrm{Id}_d \circ F = F,$$
$$\mathrm{Id}_c \circ G = G \xrightarrow{\varepsilon \otimes \mathrm{Id}_G} G \circ F \circ G \xrightarrow{\mathrm{Id}_G \otimes \theta} G \circ \mathrm{Id}_d = G$$

$G$ is then said to be *right adjoint to* $F$.

A morphism is *dualizable* if it has right and left adjoints, which in turn have right and left adjoints, ad infinitum.

[17] This is a *homotopical* action: that is, it factors through the *homotopy type* of the group and is not sensitive to the Lie structure; see Remark 4.17.

*4.12 Remark.* (i) The nonsensical but concise formula "$\theta \circ \varepsilon = 1$" can be made meaningful exactly in the two ways above.

(ii) We can revert to a *Hom*-definition of adjunction: $F_* : Hom_{\mathscr{C}}(\bullet, x) \rightleftarrows Hom_{\mathscr{C}}(\bullet, y) : G_*$, the induced functors between category-valued functors on $\mathscr{C}$, should be adjoint (functorially in $\bullet$).

(iii) There appear to be infinitely many conditions contained in the definition of dualizability. However, in the application to TQFTs below, the conditions will be finite in number. This is because the left and right adjoints will differ by an invertible functor (the *Serre automorphism*), and the checking can stop once we find that and its inverse: all further adjoints are guaranteed.

(iv) Note that if $\varepsilon, \theta$ are isomorphisms, then so are $F$ and $G$.

For any 2-category $\mathscr{C}$, let $G_{>1}\mathscr{C}$ ("groupoid above 1-morphisms") be the 2-category retaining all objects and morphisms, but only the *invertible* 2-morphisms.[18] There is a similar definition $G_{>k}\mathscr{C}$ for any $D$-category with $D > k$.

**4.13 Definition.** An object $x$ of a symmetric monoidal $(\infty, 2)$-category is *fully dualizable* if:

- It is dualizable in $G_{>1}\mathscr{C}$;
- Its unit and trace for duality are dualizable.

*4.14 Remark.* We are not asking for futher dualities, because only isomorphisms are present above 2. Insisting on more duality forces everything, including $x$, to be invertible. (Develop 4.12.iv.)

**4.15 Theorem** (Classification of framed TQFTs)**.** *A 2D framed, extended TQFT $Z$ with values in a 2-category $\mathscr{C}$ (or $(\infty, 2)$-category) is determined by $x = Z(+)$, which can be any fully dualizable object.*

*Natural transformations between two theories $Z_x$ and $Z_y$ form a (2-)groupoid (or $\infty$-groupoid) which is equivalent to $Hom(x, y)$ in the category $G_{>0}\mathscr{C}$ (obtained from $\mathscr{C}$ by retaining only the invertible 1- and 2-morphisms).*

*(4.16) The Serre twist.* Let $x$ be a fully dualizable object with dual $x^\vee$, and call $u, ev$ the unit, resp. trace of that duality. Let also $S : Z(+) \to Z(+)$ be the *Serre automorphism*, induced by a clockwise twist in the framing along an interval. We will see that, in $\mathscr{B}ord_2^{fr}$,

- $(S \boxtimes \mathrm{Id}_-) \circ u$ is the right adjoint of $ev$;
- $(S^{-1} \boxtimes \mathrm{Id}_-) \circ u$ is the left adjoint of $ev$;

The other adjoints are determined from here by algebra, since $S^{-1} = S^\vee$ (left and right adjoint). In particular, this proves half of the cobordism hypothesis, namely that $Z(+)$ must be fully dualizable. The other, more difficult half, is the statement that $\mathscr{B}ord^{fr}$ is the *free* 2-category generated by one fully dualizable object, in other words, there are no extra relations. However, armed with the Serre automorphism, we can state the classification of oriented and Spin TQFTs.

*4.17 Remark.* An equivalent description of $S$ starts from the observation that $Aut(+) \sim \Omega\mathrm{SO}(2) \sim \mathbb{Z}$, and $S$ is the negative generator. The $\mathbb{Z}$-action on a fully dualizable object captures the topological action of SO(2) on *the space of all fully dualizables*: because SO(2) is connected, $gx \cong x$ for any fully dualizable $x$; a choice of isomorphism arises from any choice of path from 1 to $g$, and is not sensitive to deformation of the path. However, attempting to straighten out these isomorphisms coherently to trivialize the action on the orbit of $x$ is obstructed precisely by the automorphism $S$, arising from the non-trivial loop in SO(2).

*(4.18) Oriented and r-spin theories.* We can now state the condition for a framed theory to factor through the oriented bordism category. In fact, for each $r$, we can enquire about TQFTs for surfaces and circles with *r-Spin structure*, with is a chosen $r$th root of the tangent bundle (assumed to be oriented). For $r = 1$, we have oriented surfaces, and for $r = 2$, traditional Spin structures.

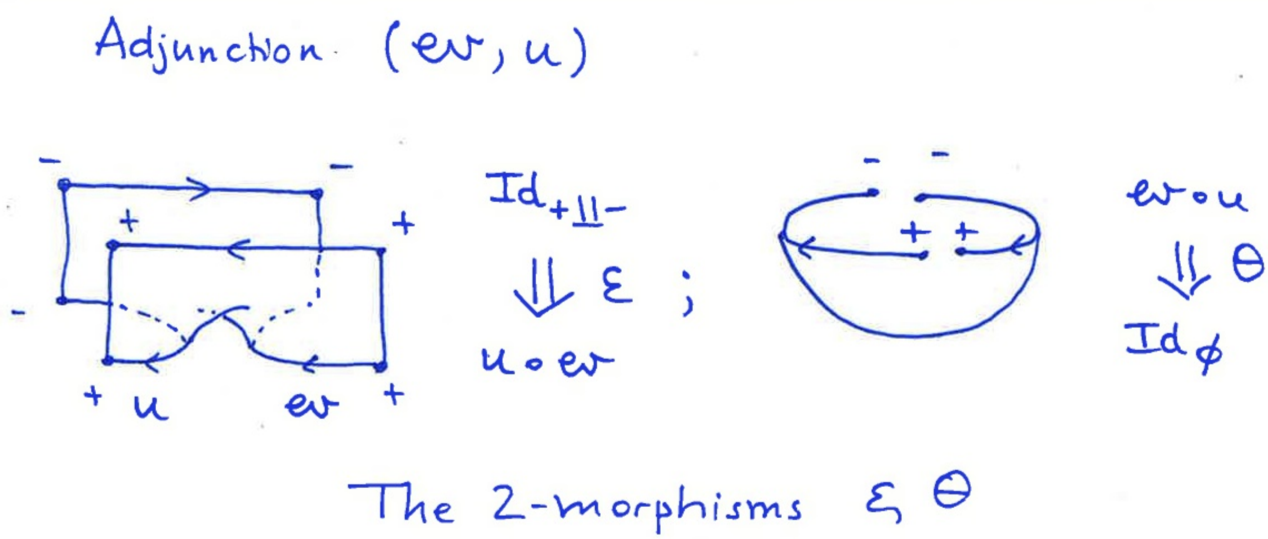
**4.19 Theorem.** *Factorizations of a framed theory $Z : \mathscr{B}ord_2^{fr} \to \mathscr{C}$ through the r-Spin category $\mathscr{B}ord_2^{r-\mathrm{Spin}}$ correspond to isomorphisms between $S^r$ and the identity automorphism of $Z(+)$.*

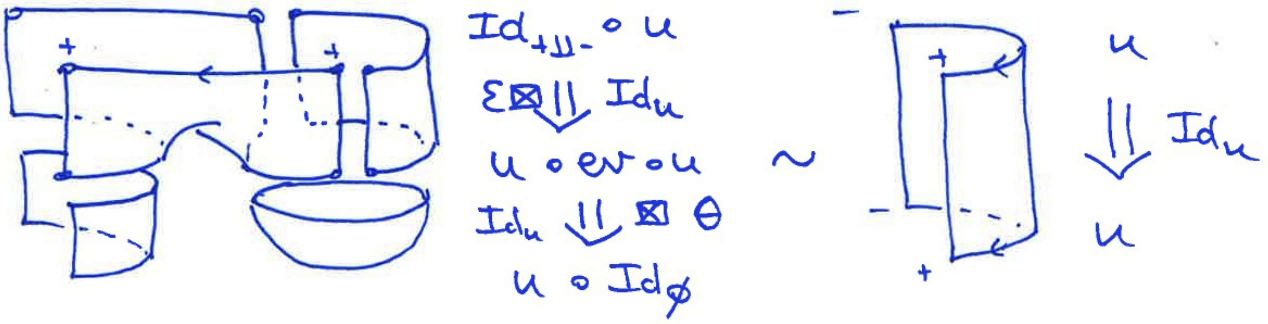
[18] If we start with an $(\infty, 2)$-category, we get an $(\infty, 1)$-category this way.

A given isomorphism can always be rescaled by $\lambda \in \mathbb{C}^\times$. In that case, one can show that the invariant for a closed surface gets rescaled by $\lambda^{2-2g}$. We can see that in teh case of semi-simple Frobenius algebras, the invariant $\sum \theta_i^{1-g}$ is entirely traceable to a choice of Serre automorphism, which we can scale independently for each projector

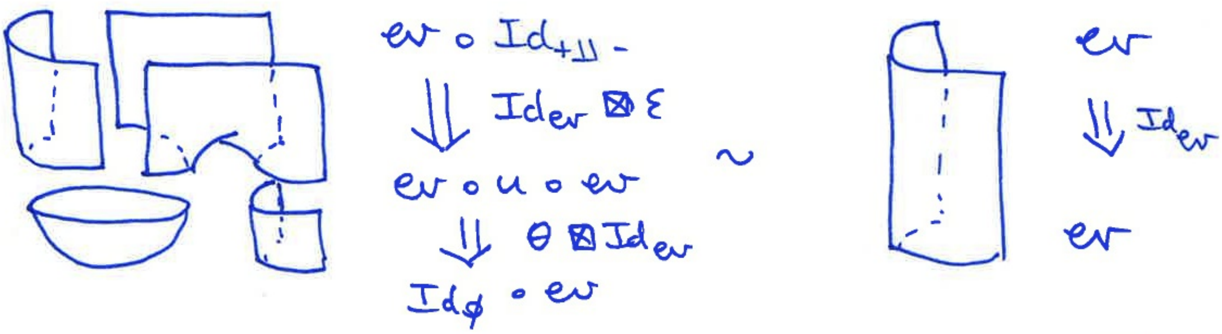
*(4.20) Adjunction in pictures: handles in the oriented category.* We will spell out the 2-duality data and relations geometrically, in terms of standard handles and handle cancellations. This happened already for $D = 1$, when converting 'Z' to the interval, but it was a bit too obvious to notice.

The pictures are easier if we assume that $S = \mathrm{Id}_+$, when the TQFT factors through oriented surfaces. In that case, $u$ and $ev$ are each other's left and right adjoints; the units and traces for adjunction are the standard handles in the topology of surfaces, and "$\theta \circ \varepsilon = 1$" is the handle cancellation relation. in this case, $ev$ and $u$ are each other's left and right adjoints.






The relation $(\mathrm{Id}_u \boxtimes \theta) \circ (\varepsilon \boxtimes \mathrm{Id}_u) = \mathrm{Id}_u$

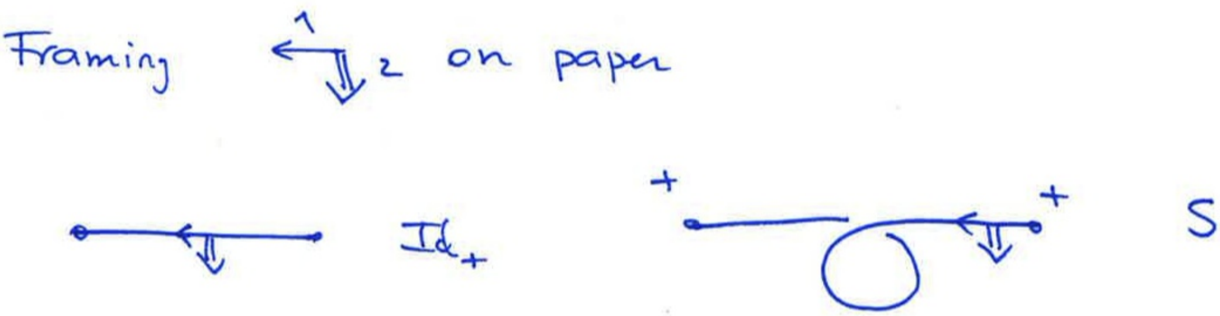



The relation $(\theta \boxtimes \mathrm{Id}_{ev}) \cdot (\mathrm{Id}_{ev} \boxtimes \varepsilon) = \mathrm{Id}_{ev}$

1-morphisms are read right-to-left, 2-morphisms defined by the saddle and disk are read top to bottom. Note that the identities '$\theta \circ \varepsilon = 1$" are the standard cancellation of a 1-handle (saddle) by a 2-handle, familiar form Morse theory.

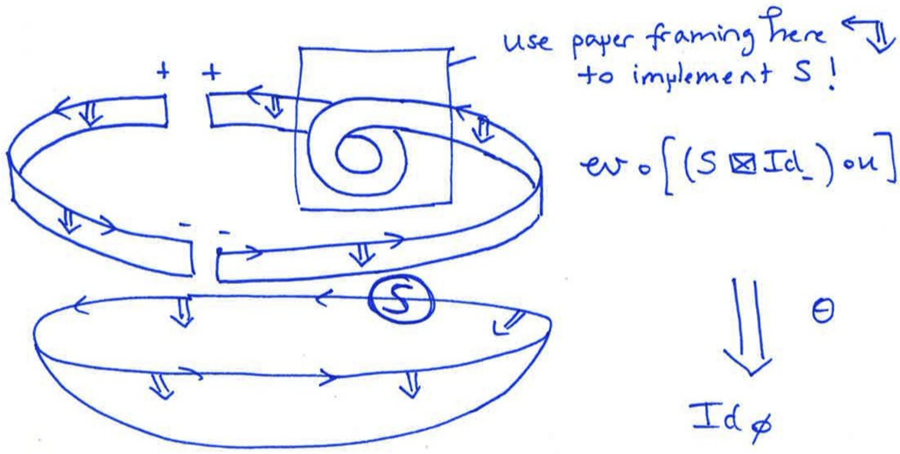
The opposite adjunction $(u, ev)$ involves the same pictures, but read from bottom to top; also, the names $\varepsilon$ and $\theta$ are swapped. Now, $\varepsilon$ is a 0-handle, and $\theta$ is a cancelling 1-handle.

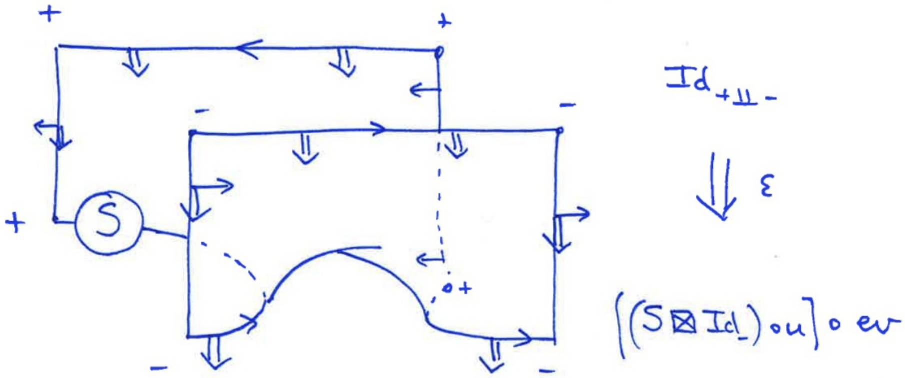
*(4.21) Framed handles.* A few attempts should convince you that it is not possible to place 2-framings on the 1-dimensional handles $\supset$ and $\subset$ to make *both* adjunctions $(ev, u)$ and $(u, ev)$ work. In fact, the most natural framings to choose in defining $u$ and $ev$ — the product (tangent, normal) — do not allow either adjunction: the product framing on the boundary of a disk does not extend over the interior. Instead, we must use the $S$-functor, given by a clockwise framing twist:




The diagram is read right-to-left; following the paper framing gives a clockwise twist compared to the product framing.

Now we can extend the boundary framings to the disk and saddle to exhibit the adjunction pair $(ev, (S \boxtimes \mathrm{Id}_-) \circ u)$:




The same handle cancellation identities apply to give $\theta \circ \varepsilon = 1$. The other adjunction, $((S^{-1} \boxtimes \mathrm{Id}_-) \circ u, ev)$, is obtained by reading the pictures up. However, we must also flip the sign of the normal copi of $\mathbb{R}$ on the boundaries to match our source-target conventions, and that changes the framing twist effected by the picture to its opposite (counterclockwise).

*(4.22) Adjunction: algebraic conditions.* To spell out the adjunction conditions, we specialize to the case of the 2-category $\mathscr{A}lg$ of algebras, bimodules and intertwiners. That is, $Hom(A, B)$ is the category of $B - A$ bimodules, and composition is the tensor product.[19] Actually, one can prove that a fully dualizable algebra must be semi-simple; more generally, a (compactly generated) abelian category is fully dualizable (within linear categories) iff it it semi-simple. Even this is instructive, for we will identify the automorphism $S$, and relate a trivialization of $S$ to a *Frobenius* or *Calabi-Yau* structure on the algebra. More interesting examples arise one uses differential graded algebras and their derived categories of modules; coherent sheaves on a projective manifold are an example we review below.

Every $A \in \mathscr{A}lg$ is dualizable, with dual the opposite algebra $A^\circ$. Indeed, let $A^e := A \otimes A^\circ$; we can then take for $u$ and $ev$ the space $A$, viewed as $A^e - \mathbb{C}$, respectively $\mathbb{C} - A^e$ bimodules. From this, we compute $Z_A(S^1) = A \otimes_{A^e} A$, known as the *zeroth Hochschild homology* of $A$.

The $Hom$ definition is more convenient for computing adjoints. The left adjoint ${}^\vee u$ of $u$ is a $\mathbb{C} - A^e$ bimodule satisfying

$$Hom_{A^e}(M, A \otimes_{\mathbb{C}} N) = Hom_{\mathbb{C}}({}^\vee u \otimes_{A^e} M, N) \tag{4.23}$$

for any $A^e$-module $M$ and $\mathbb{C}$-vector space $N$. (Actually, this should be compatible with any right $B$-module structure on $N$, but the latter fact will come for free.) Taking first $M = A^e$ and $N = \mathbb{C}$, but then enforcing it to any $N$, leads to the conditions

$$A = Hom_{\mathbb{C}}({}^\vee u, \mathbb{C}), \quad \dim_{\mathbb{C}} {}^\vee u < \infty;$$

[19] The conversion to the 2-category of linear categories, or differential-grades categories, is not difficult. The delicate step involves the definition of the monoidal structure, the tensor product of categories; see [G] for of abelian categories. In general, the categories of modules over algebras $A$ and $B$ tensor together to the category of $A \otimes B$-modules.

so that $\dim A < \infty$ and ${}^\vee u = A^\vee := Hom(A, \mathbb{C})$, the dual vector space to $A$. In addition, to get (4.23) for all $M$ requires left exactness of $M \mapsto A^\vee \otimes_{A^e} M$, that is, $A^e$-flatness of $A^\vee$.
*Exercise:* Show that, as a consequence, the category of finite-dimensional $A$-modules is semi-simple, and therefore $A$ is semi-simple. (Conclude exactness of $Hom$ in $A$-modules.)

Since $ev^\vee = (S \boxtimes \mathrm{Id}) \circ u$, $ev = {}^\vee u \circ (S^{-1} \boxtimes \mathrm{Id})$ and we conclude that $S$ is the $A - A$ bimodule $A^\vee$. For the left adjoint ${}^\vee ev$,

$$Hom_{A^e}({}^\vee ev \otimes_{\mathbb{C}} M, N) = Hom_{\mathbb{C}}(M, A \otimes_{A^e} N)$$

from which $M = \mathbb{C}, N = A^e$ gives $A = Hom_{A^e}({}^\vee ev, A^e)$. Moreover, we need flatness and finite presentation of $A$ over $A^e$ to recover the general isomorphism. The other adjoints offer no new information or constraints:

$$u^\vee = Hom_{A^e}(A, A^e) = {}^\vee ev, \qquad ev^\vee = A^\vee = {}^\vee u.$$

*4.24 Remark.* (i) It is no accident that $S$ is the *vector space* dual of $A$; see the next section.

(ii) Working with a differential graded algebra instead and using quasi-isomorphisms in lieu of equalities, finite-dimensionality of $A$ becomes finiteness of the homology of $A$, and is usually called *compactness*; while finiteness of $A$ over $A^e$ is *smoothness.* This is because they match the respective properties in the case of schemes. With that language, fully dualizable dga's are the compact, smooth ones; this had been already flagged in work of Kontsevich and Soibelman.

*(4.25) Oriented TQFTs from Frobenius algebras.* A trivialization of the Serre functor is an $A^e$-module isomorphism $A \cong A^\vee$. The image of 1 gives a linear map $t : a \to \mathbb{C}$ which induces a symmetric non-degenerate trace $a \times b \mapsto t(ab) = t(ba)$, making $A$ into a non-commutative Frobenius algebra. By means of $t$, we can identify the dual space to $Z_A(S^1) = ev \circ u$, namely $Hom(A \otimes_{A^e} A; \mathbb{C})$, with $Z(A) = Hom_{A^e}(A, A)$, the center of $A$. Therefore, 2-dimensional, extended, oriented TQFTs correspond precisely to semi-simple, not necessarily commutative, Frobenius algebras.

*(4.26) Finite gauge theory revisited.* Let $A$ be the group ring $\mathbb{C}\langle F \rangle$ of a finite group; we choose the trace $t$ to pick out the coefficient of 1. Explicitly, the pairing is

$$t(\varphi \cdot \psi) = \sum\nolimits_{f \in F} \varphi(f)\psi(f^{-1}).$$

I claim that $Z$ is the *$F$-gauge theory for surfaces with corners* (which in fact is an an unoriented TQFT). The theory is obvious on 1-manifolds: send the interval to $A = \mathbb{C}\langle F \rangle$ and the circle to $Z(A)$, the class functions on $F$. It is again defined on surfaces by 'counting $F$-bundles, weighted down by automorphisms'; the difference from the $(1, 2)$-theory of Lecture 1 is that bundles are now *trivialized* at the corners (and on the identity segments bounding a 2-morphism).
*Exercise:* Make the definition above precise, and check that it gives a functor from $\mathscr{B}ord_2^{\mathrm{O}(2)}$ to $\mathscr{A}lg$.

*4.27 Remark.* The trace $t$ has a natural extension to the entire category $\mathrm{Rep}(F)$ of $F$-representations; that is, we define $t_V = (\#F)^{-1} \cdot \mathrm{Tr}_V : \mathrm{End}^G(V) \to \mathbb{C}$, for any $V$. It has the property that $t(\varphi \circ \psi) = t(\psi \circ \varphi)$ for any pair with opposite sources and targets, and defines a non-degenerate pairing between $Hom^G(V, W)$ and $Hom^G(W, V)$ for all $V, W$. It is an example of a *Calabi-Yau structure* on $\mathrm{Rep}(F)$, a trivialization of the Serre functor, which we examine next in the context of varieties. It also furnishes an example of *open-closed theory with* $\mathrm{Rep}(F)$ *as category of branes.*

*(4.28) The Serre functor on a scheme.* In studying derived categories of coherent sheaves on a projective manifold $X$, Bondal and Orlov flagged the role of the *Serre automorphism* $S : \mathscr{C} \to \mathscr{C}$, characterized in any linear category by the condition that we should have a bi-functorial isomorphism

$$Hom(x, S(y)) = Hom(y, x)^\vee.$$

If it exists, $S$ is unique up to canonical isomorphism. If $S$ is invertible, it forces the $Hom$ spaces to be isomorphic to their double duals, hence finite-dimensional; this is a strong condition. If we also assume compact generation of $\mathscr{C}$ (a very mild finiteness condition) it follows that $\mathscr{C}$ is a 2-dualizable object in the 2-category of linear categories, so defines a framed $TQFT$. Moreover, the Serre functor defined above agrees with the Serre framing twist.

In the case of the derived category, taking for $Hom$ the zeroth group $\mathrm{Ext}^0(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ between complexes of sheaves,[20] the classical duality theorem of Serre identifies $S$ with tensoring with $K_X[\dim X]$, the canonical bundle shifted into degree $(-\dim X)$. The following is a remarkable application of the cobordism hypothesis; the Calabi-Yau case had been proved, in a variant, by Kontsevich and Costello.

**4.29 Theorem.** *The derived category $D^bCoh(X)$ of (bounded complexes of) coherent sheaves on a projective manifold $X$ is fully dualizable. The Serre functor is the clockwise framing twist. Defining an oriented TQFT from the category $X$ requires a* Calabi-Yau *structure on $X$, a trivialization of the canonical bundle.*

When $X$ is a collection of points, we are back to semi-simple algebras.

*(4.30) The vector spaces associated to the circle.* Let us understand $Z(S^1)$. When $D = 1$, the circle computes a number, the dimension of the vector space. So we are generalizing the notion of dimension to linear categories, and obtaining a vector space for an answer.

In the framed context, there are many circles $S^1_n$, one for each integer, counting the clockwise winding number of the 2-framing around the circle. So our TQFT will have a space of states, $Z(S^1_n)$, for each $n \in \mathbb{Z}$. Let us use 0 for the framing of the circle bounding the standard unit disk in $\mathbb{R}^2$. If we construct $S^1_n$ by attaching two half-circles together, these circles differ by inserting powers of the Serre automorphism before the gluing. With our convention, $Z(S^1_0) = ev \circ (S^{-1} \boxtimes \mathrm{Id}_-) \circ u$, and $Z(S^1_1) = ev \circ u$, the direct analogue of the dimension.

**4.31 Theorem.** *When $Z(+) = D^bCoh(X)$, $Z(S^1_n) = H^*(X; K^{\otimes n} \otimes \Lambda^*(T_X))$.*

*4.32 Remark.* For $n = 0$, this is the Hochschild cohomology of the manifold $X$. For $n = 1$, it is the Hochschild homology. Cohomology has a natural ring structure, and homology is a module over it (as are all other spaces). The algebra is the obvious one described by the TQFT, as the pair of pants defines a distinguished morphism $S^1_0 \coprod S^1_0 \to S^1_0$: picture two smaller disks embedded in a larger one, all with the standard framing in $\mathbb{R}^2$.

There is no Frobenius algebra structure away from the Calabi-Yau case, because the trace, the outgoing framed disk, comes from the map $S^1_2 \to \emptyset$. There is however a pairing $S^1_1 \coprod S^1_1 \to S^1_2$, and from there to $\emptyset$, giving a perfect (but not always symmetric) bilinear pairing on Hochschild homology.

## 5. Cobordism hypothesis in general dimension

The required definitions and the statement of the CH in any dimension should not come as a surprise. Let $\mathscr{C}$ be a symmetric monoidal $(\infty, D)$-category.

**5.1 Definition** (Lurie)**.** An object $x$ in $\mathscr{C}$ will be called *fully dualizable* if:

- it is dualizable in $G_{>1}\mathscr{C}$,
- Its unit and trace for duality are dualizable in $G_{>2}\mathscr{C}$,
- The unit and trace for dualities established above are dualizable in $G_{>3}\mathscr{C}$,
- and so forth until we reach $D$-morphisms, which are *not* required to be dualizable.

**5.2 Theorem.** *Functors $Z : \mathscr{B}ord^{fr}_D \to \mathscr{C}$ are determined up to isomorphism by fully dualizable objects $Z(+)$. More precisely, the $(\infty, D)$-category of such functors and natural transformations is a groupoid, and is equivalent to the full sub-groupoid of fully dualizable objects in $\mathscr{C}$, with isomorphisms[21] as the only morphisms.*

*The group $\mathrm{O}(D)$ acts on such functors $Z$, and therefore on the space of fully dualizable objects in $\mathscr{C}$, by changing the framing. A factorization of $Z$ via a category $\mathscr{B}ord^G_D$ of manifolds with $G \to \mathrm{O}(D)$ structure on the tangent bundle correspond to a $G$-fixed point structure on the underlying object $Z(+)$. If $G \to \mathrm{SO}(D)$, this is a trivialization of the $\Omega\mathrm{SO}(D)$-action on $Z(+)$.*

[20] The grading on the complexes makes this group tricky, as it contains a lot of cohomology! It is perhaps better to use the direct sum of all the Ext's in its stead.

[21] These will be *weak isomorphisms*, meaning, invertible up to higher morphisms.

Remember that, as seen in the cases of $D = 1, 2$, making the object $x$ into a fixed point for the group action is a combination of conditions and data. For $D = 1$ and $G = \mathrm{O}(1)$, we had the datum of a non-degenerate quadratic form. For $D = 2$ and $G = \mathrm{SO}(2)$, the obstruction to $x$ becoming a fixed point is its Serre automorphism; the datum of an isomorphism $S \cong \mathrm{Id}_x$ makes $S$ into a fixed point.

*(5.3) Reduction to co-dimension* 2. The number of conditions for full dualizability of an object seems to proliferate exponentially with the dimension; this would make exploitation of the theorem nearly impossible. I shall briefly explain below why this is not the case; in fact, if we ignore framings and focus on oriented theories, we have essentially seen the maximal complexity at $D = 2$.

Two observations underlie this simplification:

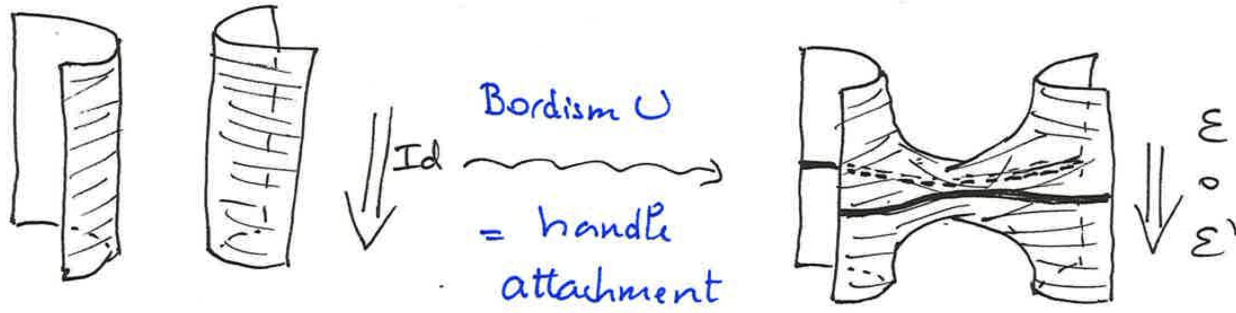
- All the unit-trace morphisms to be supplied correspond to cubes, and all relations take the geometric form of gluing two cubes along a common face;
- Boundary structure in co-dimension greater than two may be ignored.

As a result, we may assume that our cubes of morphisms are standard 'handles' $D^p \times D^q$, with boundary decomposed as $D^p \times S^{q-1} \coprod_{S^{p-1}\times S^{q-1}} S^{p-1} \times D^q$. That is, we are dealing with a $(p+q)-$morphism relating two $p+q-1-$morphisms from $\emptyset$ to the $(p+q-2)-$endomorphism $S^{p-1} \times S^{q-1}$ of the empty manifold, so a $3-$layered object. The duality relations "$\theta \circ \varepsilon = 1$" to be checked will be the *standard handle cancellation relations* of surgery theory.

The impact of this observation is that, to demonstrate full dualizability of an object, it suffices to provide the $k$-morphisms associated to all handles of all dimensions through $k$, and check the handle cancellation relations.

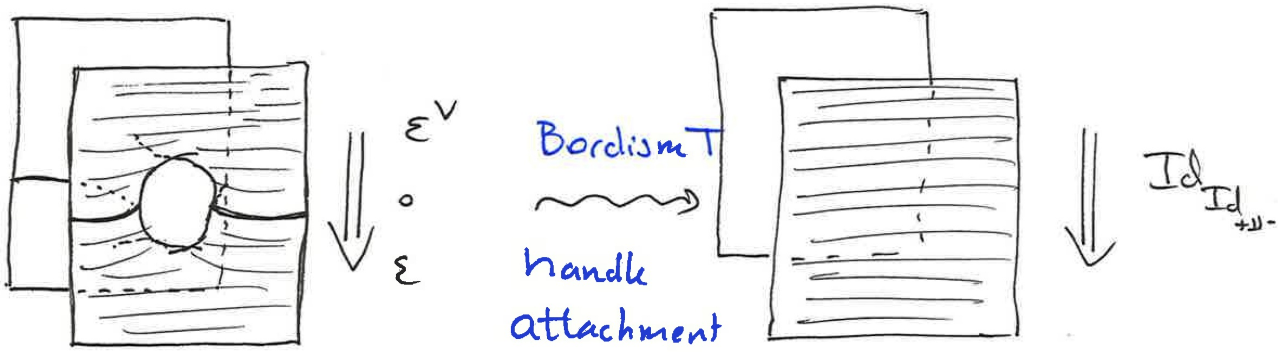
Framed theories requires framed handles, in which case a presentation also involves coming to grips with generators of the homotopy groups of $\mathrm{SO}(D)$; this adds to the complexity of framed theories in higher dimension.

*(5.4) Three-dimensional examples.* Let us illustrate these ideas in dimension 3, ignoring framings and retaining only orientations. The oriented category is especially easy, because the adjoint of a morphism is always represented by the 'opposite bordism', meaning the same manifold with opposite orientation, read backwards as a bordims. (The frame-reversing convention requires more care.) I will exhibit the saddle $\varepsilon^\vee$ as the right adjoint of $\varepsilon$, which was the upside-down saddle giving the unit $\mathrm{Id} \to u \circ ev$ (*Caution:* apologies, $\varepsilon$ and $\varepsilon^\vee$ labels are switched in the pictures):

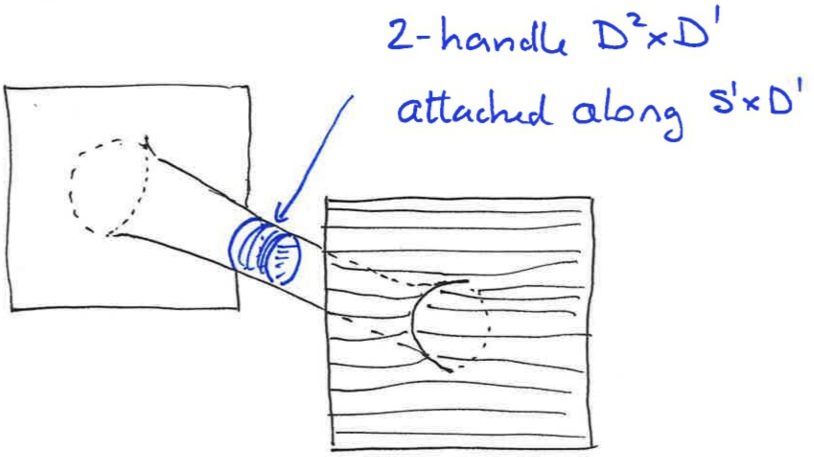




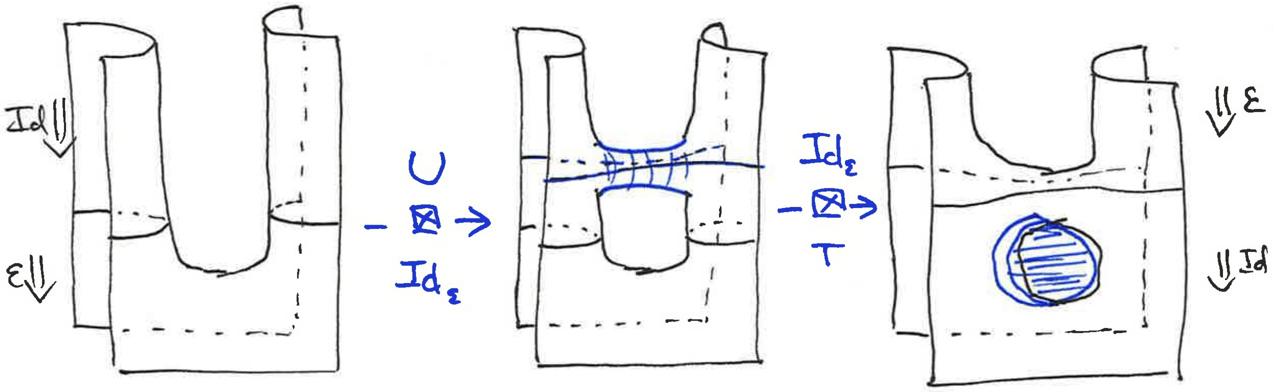
The bordism is the standard 1-handle attachment familiar from Morse theory.
The trace morphism $T : \mathrm{Id} \to \varepsilon \circ \varepsilon^\vee$ is a 2-handle attachment,




and requires stretching the connecting tube a bit to see:

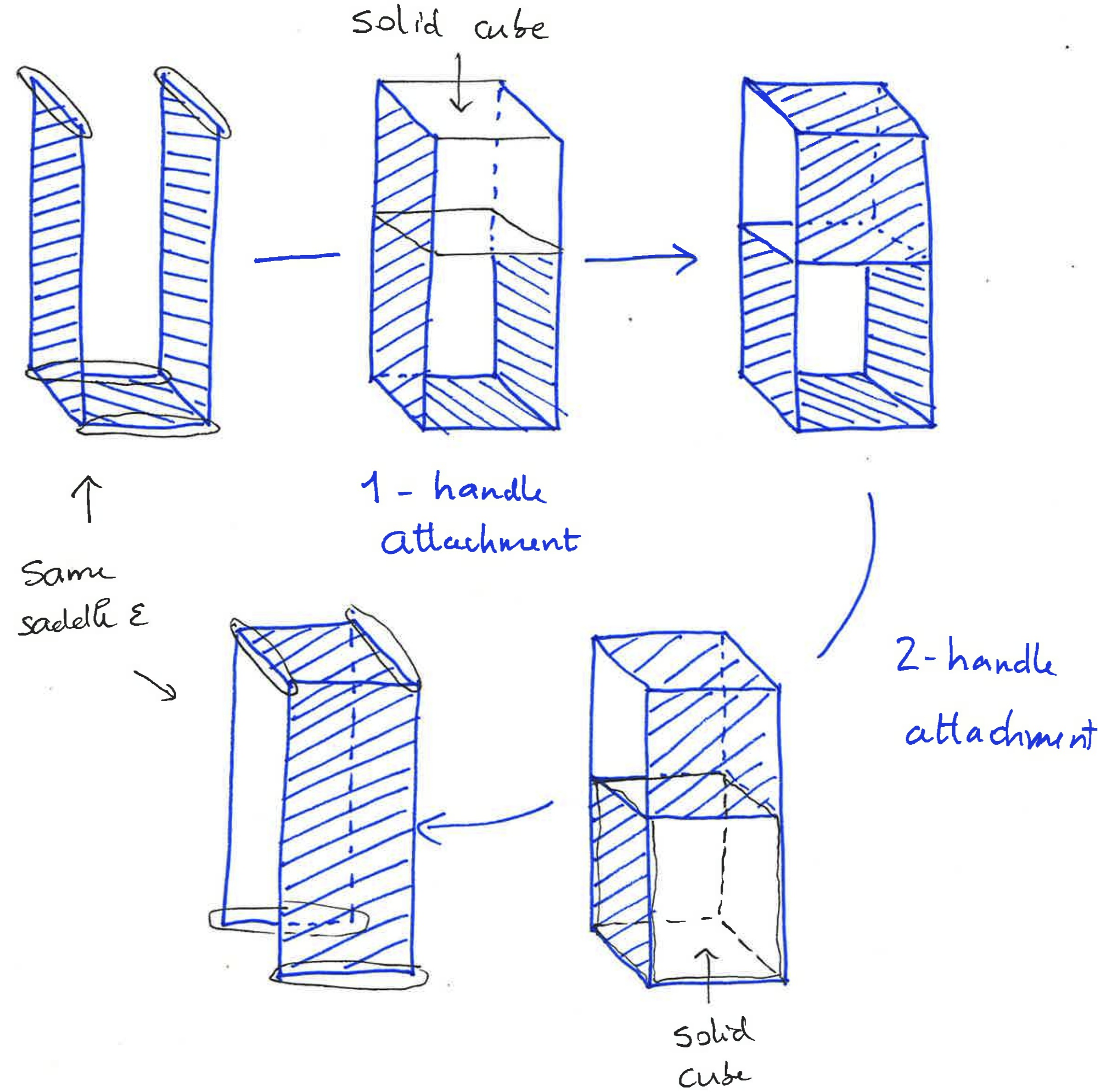


Here is one of the relations "$T \circ U = 1$", namely $(\mathrm{Id}_{\varepsilon^\vee} \boxtimes T) \circ (U \boxtimes \mathrm{Id}_{\varepsilon^\vee}) = \mathrm{Id}_{\varepsilon^\vee}$:




This is the standard cancellation of a 1-handle by a 2-handle.

Rewriting the diagrams in cubular form illustrates an inductive formulation, which allows one to reduce the duality morphisms to handles in any dimension.




The 1-morphisms for which the blue surfaces are 2-morphisms are circled in black in the initial and final picture: they are $u \circ ev$ at the top and Id at bottom, both from $+ \coprod -$ to itself. The initial and final angular saddles represent the same 2-morphism $\varepsilon$ in the bordism category. The two cubes to be inserted represent a 1-handle attachment and its a 2-handle cancellation. The generalization to any dimension arises by replacing the top square, $I \times I$, by $I^p \times I^q$.

I will omit the discussion which reduces us to boundaries of co-dimension 2: note that the cubes above still have co-dimension 3 vertices. The argument is an inductive application of the cobordism hypothesis. Morally, recall from Morse theory that any manifold can be built from the empty set by attaching handles, and all operations involve manifolds with co-dimension 2 corners only. This might suggest a classification of TQFTs which go only 2 layers deep, and avoid categories beyond 2 altogether. In fact, a partial result in this direction does exist in 3 dimension, the *Reshetikhin-Turaev theorem* [RT], classifying 2-deep $3D$ TQFTs with a semi-simple category associated to the circle; but the task it seems hopeless in general, because the spheres used for handle attachment can be very complicated. (Already in dimension 4, the entire complexity of differentiable 4-manifolds is concealed in links in $S^3$.) In fact, all the complexity gets pushed into understanding manifolds with embededded spheres in their boundary. However, if we are allowed to chop up the boundaries, and their further boundaries, into handles, then this 'codimension 2' intuition becomes correct, and develops into a recursive procedure for checking of the relations for full dualizability.

*(5.5) Tensor and module categories.* We now move to a 3-dimensional illustration of the cobordism hypothesis, which allows the construction of interesting $3D$ TQFTs. This is upcoming work by Douglas, Schommer-Pries and Snyder, and the author is most grateful for their explanations.

We defined a tensor category $\mathscr{T}$ as a linear category with bilinear monoidal structure.[22] Recall a minor complexity: a priori, associativity of the tensor product is moderated by an associator $\alpha_{x,y,x} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$, subject to a pentagon identity. We are in luck, though, because *Mac Lane's coherence theorem* allows us to replace $T$ by an equivalent category with strict associativity.

A *module category* $\mathscr{M}$ over $\mathscr{T}$ is a linear category with a bilinear multiplication functor $T \times M \to M$, satisfying the obvious constraints that define an action of $\mathscr{T}$ on $\mathscr{M}$. (Again, it turns out that we can modify $\mathscr{M}$ to obtain a strict module category, where the coherence isomorphisms are identities [G].)

*(5.6) Tensor product of categories.* [EGNO, G] The definition for abelian categories (due to Deligne) is borrowed from algebra: for a right $\mathscr{T}$-module $\mathscr{M}$ and a left one $\mathscr{N}$, $\mathscr{M} \boxtimes_{\mathscr{T}} \mathscr{N}$ is co-representing the functor sending a linear category $\mathscr{C}$ to $\mathscr{T}$-bilinear, right exact bi-functors $\mathscr{M} \times \mathscr{N} \to \mathscr{C}$. Thus defined, the tensor product of abelian categories is represented by an abelian category [G].

*5.7 Remark.* The tensor product of abelian categories over $Vect$ is already a bit subtle: for instance, $\mathscr{M} \boxtimes_{Vect} \mathscr{N}$ has more objects than just the direct sums $m \boxtimes n$, with $m \in \mathscr{M}$ and $n \in \mathscr{N}$; we must close this under quotients (co-kernels). The resulting category is abelian. If $\mathscr{M}$ and $\mathscr{N}$ are the categories of $A$- and $B$-modules, then $\mathscr{M} \boxtimes \mathscr{N}$ is the category of $A \times B$-modules.

The upshot of the definitions should be the construction of a (lax) symmetric monoidal 3-category, with tensor categories as object, bi-module categories as morphisms, tensor product as composition, bi-module functors as 2-morphisms and finally natural transformations between the latter as 3-morphisms. Key ingredients for the construction are spelt out in [G]. Full details are promised in upcoming work of Douglas, Schommer-Pries and Snyder, mentioned below.

**5.8 Definition.** A tensor category is *rigid* if every object has left and right duals.

Recall that we can make the left and right duals into functors on $\mathscr{T}$; they are then mutually inverse. Rigid tensor categories have some particularly good properties [EGNO]; in particular, if $\mathscr{T}$ is abelian, then the multiplication $\otimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ is bi-exact, as are the internal dualization functors.

*(5.9) 2-dualizability.* Every tensor category $\mathscr{T}$ is 1-dualizable, with dual the *multiplicatively opposite* category $\mathscr{T}^{mo}$, the same category with opposite tensor product; the argument repeats the one for algebras, and $u, ev$ are represented by $\mathscr{T}$ as $\mathscr{T}^e$-module. We can also mimic the algebra computations for 2-duality from Lecture 4. We need $\mathscr{T}$ to be dualizable as a linear category, and dualizable as a $\mathscr{T}^e$-module. The Serre functor is the dual of $\mathscr{T}$ as linear category, which we denote $\mathscr{T}^\vee$. 2-dualizability

[22] Some sources, such as [EGNO], impose additional conditions.

requires this to be an invertible bimodule (the analogue of a line bundle), with inverse the bimodule category of (right exact) functors $Hom_{\mathscr{T}^e}(\mathscr{T};\mathscr{T}^e)$. Furthermore, tensoring with $\mathscr{T}$ over $\mathscr{T}^e$ should be an exact operation on categories (commute with filtered limits and colimits); this means that tensoring over $\mathscr{T}$ should be exact.

We can identify $\mathscr{T}^\vee$ more precisely. If all objects in $\mathscr{T}$ are compact — such is the case in finite modules over a Noetherian ring — then $\mathscr{T}^\vee$ can be identified with the *opposite* linear category $\mathscr{T}^\circ$, by sending an object $a^\circ \in \mathscr{T}^\circ$ to the co-represented functor $x \mapsto Hom(a,x)$. We shall spell out the bimodule structure in Theorem 5.17 below, in the rigid case.

*(5.10) Drinfeld center and Hocschild homology.* The categories associated to framed circles $S^1_n$ are the tensor products $\mathscr{T} \otimes_{\mathscr{T}^e} (\mathscr{T}^\vee)^{\otimes n-1}$ with the $(n-1)$st power $\mathscr{T}^\vee \otimes_{\mathscr{T}} \cdots \otimes_{\mathscr{T}} \mathscr{T}^\vee$ of Serre. For $n \le 0$ we must use the inverse bi-module of $\mathscr{T}^\vee$, but we can also exploit the duality of $\mathscr{T}$ over $\mathscr{T}^e$, rewrite the tensor product as a $Hom$ and present the spaces for $n \le 0$ as $Hom_{\mathscr{T}^e}((\mathscr{T}^\vee)^{\otimes -n}, \mathscr{T})$. Here, $Hom$ stands for linear, right exact functors compatible with the $\mathscr{T}^e$-action. Note that compatibility carries data, not just conditions, as we will see in a moment.

The category $Hom_{\mathscr{T}^e}(\mathscr{T},\mathscr{T})$ for the 0-framed circle plays a distinguished role, as it has a natural tensor structure under composition. It is equivalent to the *Drinfeld center* of $\mathscr{T}$, the category of pairs $(x,\beta)$ where $x \in \mathscr{T}$ and $\beta$ is a *half-brading*, or an isomorphism between the functors $x\otimes$ and $\otimes x$ of left and right tensoring with $x$. This comes with a natural *braiding*, bi-functorial isomorphism $x \otimes y \xrightarrow{\sim} y \otimes x$, from the automorphism information carried by $x$. This braiding is usually not symmetric.

*5.11 Remark.* The braiding tries to be 'maximally non-symmetric'. for example, if $\mathscr{T}$ is the tensor category of representations of a finite abelian group $F$, its center is the tensor category of reps on $F \times F^\vee$, but with a Heisenberg-like braiding defined from the natural pairing of $F, F^\vee$.

Any finite group $F$ has a 'categorized group ring' which is the category of vector bundles on $F$, with convolution as the tensor structure. The center of this is the tensor category of $F$-equivariant vector bundles on $F$, with the convolution structure and an interesting braiding. This is the categorized analogue of class functions on $F$.

*5.12 Remark.* In the 3-dualizable case, there will be only two categories, going with the two 3-framings on the circle, $S^1_{even}$ and $S^1_{odd}$; namely, $Z(\mathscr{T}) = Hom_{\mathscr{T}^e}(\mathscr{T},\mathscr{T})$ and $\mathscr{T} \otimes_{\mathscr{T}} \mathscr{T}$.

*(5.13) Fusion categories.* There is no complete classification of 3-dimensional extended TQFTs based on tensor categories resembling the one in $2D$, by semi-simple (not necessarily commutative) algebras. There are some constraints; for instance, $\mathscr{T}$ is abelian, then full dualizability requires the Drinfeld center (and its twisted form, Hochschild homology) to be *semi-simple.*

*5.14 Remark.* Semi-simplicity of the center is a strong constraint (although automatic for rigid categories). For an example, compute the Drinfeld center of the "$2 \times 2$ upper triangular matrix algebra over $Vect$", the category $Vect^{\oplus 3}$, with tensor product imitating the multiplication of matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

However, a beautiful class of extended $3D$ TQFTs comes form the following result of Douglas, Schommer-Pries and Snyder. It is based on the notion of *fusion category* that has been the subject of much research (Etingof, Ocneanu *et al*). The wording similarity with the 2-dimensional story is a bit deceptive, as the structure of a fusion category is substantially more involved than that of a complex semi-simple algebra.

**5.15 Definition.** A *fusion category* is a semi-simple rigid tensor category with finitely many simple isomorphism classes.

*Example:* The 'categorified group ring' of vector bundles on a finite group $F$. This generates the $3D$ gauge theory with group $F$. The tensor structure can also be twisted by a co-cycle $\tau \in H^3(BF;\mathbb{C}^\times)$ which appears as an associator.
*Example:* The category of representations of a *small quantum group* at a root of unity (the category of the big quantum group, modded out by representations of quantum dimension zero) is a fusion

category. The associated 3-dimensional field theory computes *Turaev-Viro theory*, whose closed 3-manifold invariants are the square norms of the famous Chern-Simons invariants.[23]

The conditions are strong enough to imply the good behavior of the tensoring operation, as the following shows:

**5.16 Theorem** (Etingof-Nykshich-Ostrik)**.** *For any right and left semi-simple module categories $\mathscr{M}, \mathscr{N}$ over a fusion category $\mathscr{F}$, the tensor product $\mathscr{M} \boxtimes_{\mathscr{F}} \mathscr{N}$ is semi-simple. It is also exact in $\mathscr{M}$ and $\mathscr{N}$.*

So we can define a 'small' 3-category of fusion categories, semi-simple bimodule categories, functors and natural transformations. In the setting of fusion categories, we also have a duality which allows us interpret tensor products as functor categories. In particular, this establishes the semi-simplicity of the Drinfeld center and its twisted version.

**5.17 Theorem** (Douglas, Schommer-Pries, Snyder)**.** *Every multi-fusion category $\mathscr{F}$ is a fully dualizable objects of the* 3*-category $\mathscr{T}cat$. The category $Z_{\mathscr{F}}(S^1_0)$ is the Drinfeld center of $\mathscr{F}$, with its natural tensor structure. The Serre functor is the double dual in $\mathscr{F}$.*

Because $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$ and no longer $\mathbb{Z}$ as for SO(2), so the Serre automorphism of any 3-dualizable object must square to 1. As a consequence, the authors get an enlightening proof of a recent theorem of Etingof-Nyshnich-Ostrik about fusion algebras.

**5.18 Corollary** (Etingof-Nykshich-Ostrik)**.** *In any fusion category, there is a canonical isomorphism of the quadruple dual with the identity functor.*

The Serre automorphism is *a priori* a bi-module category; let us spell out its identification with the double right dual, in the rigid case. This only assumes 2-dualizability, so it does not rely on semi-simplicity of the category $\mathscr{T}$. Returning to the identification $\mathscr{T}^\vee = \mathscr{T}^\circ$, we have

$$x \otimes a^\circ \otimes y = \left({}^\vee y \otimes a \otimes x^\vee\right)^\circ$$

from the *Hom*-duality properties of $x^\vee, {}^\vee y$. Now use left duals to identify $\mathscr{T}$ with $\mathscr{T}^\vee$: $a \mapsto ({}^\vee a)^\circ$. The relation becomes

$$x \otimes ({}^\vee a)^\circ \otimes y = \left({}^\vee y \otimes a \otimes x^\vee\right)^\circ = \left[\left({}^\vee (x^{\vee\vee} \otimes a \otimes y\right)\right]^\circ$$

All in all, Serre is $\mathscr{T}$ as a $\mathscr{T} - \mathscr{T}$ bimodule, but the left tensor action is twisted by the double right dualization: this is the bimodule implementation of the double dual.

*5.19 Remark.* (i) If $S$ is not isomorphic to Id, then the TQFT can be defined for surfaces with a Spin structure. A rigid category is called *pivotal* if the double dual is isomorphic to the identity. In the fusion case, a pivotal structure allows us to pass from Spin surfaces to oriented surfaces. To go on to 3-manifolds and pass from framed to oriented, we need what is called a *spherical* structure, a pivotal structure in which the trivialization of the doulbe dual squares to the canonical trivialization of the quadruple dual.

(ii) The condition that Serre should be a tensor functor, rather than a bimodule is closely related to the existence of internal duals: a $\mathscr{F} - \mathscr{F}$ bimodule $\mathscr{S}$ is implemented by a tensor automorphism iff $\mathscr{S}$ is equivalent to $\mathscr{F}$ as a left and as a right $\mathscr{F}$-module separately. The tensor automorphism it implements is the composition of these separate 'straightening' isomorphisms. In the rigid case, both straightening isomorphisms between $\mathscr{F}$ and $\mathscr{F}^\vee$ are the internal duality functors.

[23]There is at present no construction of the Chern-Simons invariants for general non-abelian $G$, although a there is a promising program by Bartels, Henriques and Douglas. The generating tensor cateogory analogue is far from discrete, though, and is described in terms of von Neumann algebras. For torus groups, a $C^*$ Hopf-like algebra appears, [FHTL].

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