A Topological Study of Textile Structures. Part I: An Introduction to Topological Methods

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Abstract This paper proposes a new systematic approach for the description and classification of textile structures based on topological principles. It is shown that textile structures can be considered as a specific case of knots or links and can be represented by diagrams on a torus. This enables modern methods of knot theory to be applied to the study of the topology of textiles. The basics of knot theory are briefly introduced. Some specific matters relating to the application of these methods to textiles are discussed, including enumeration of textile structures and topological invariants of doubly-periodic structures.

Key words doubly-periodic interlacing structures, enumeration of textile structures, isotopic invariant, knot theory, topology of textiles, torus diagram, unit cell

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Modern computer-aided design methods in combination with sophisticated technology can deliver a diverse range of textiles for a number of various applications which include clothing, domestic, medical and technical textiles, and composite materials.

Throughout the history of textiles, their development progressed along two main paths: one was the production of fibers, natural or synthesized, as primary building blocks; the other was design and manufacture of new textile structures. Both of these paths obviously relied on the development of new technologies and appropriate machinery. The performance characteristics of textiles depend on the properties of the constituting material and the properties of the structure. Well-known examples of this dependence are the differences in the mechanical behavior of the single jersey and 1 × 1 rib, leno weave and plain weave [1–3].

Ultimately, the aim of the design procedure for the textile material is to find an optimal combination of fiber/yarn properties and structure that would provide the best performance characteristics of the end-use product.

Structure is the most essential characteristic that enables different textile materials to be distinguished. In the context of this study, the term ‘structure’ refers to binding patterns of interlacing threads in knitted and woven fabrics without considering any internal structural features of the threads involved. Structural characteristics of fabrics depend on the mutual position of constituting threads where the geometry is a derivative of position. For example, no continuous change in the geometrical parameters such as curvature, diameter or distance between the threads, made without breaking and self-intersection of the threads can ever transform a plain weave fabric into sateen (Figure 1(a), (b)) because the mutual position of the threads has been set in weaving.

There are many well-developed numerical characteristics of geometrical, physical and mechanical properties of fibers, yarns and fabrics and appropriate testing methods for measuring such properties. This makes it possible to establish quantitative relationships between properties of fibers/yarns on one hand and properties of fabrics on the other. Another factor that often plays a very significant

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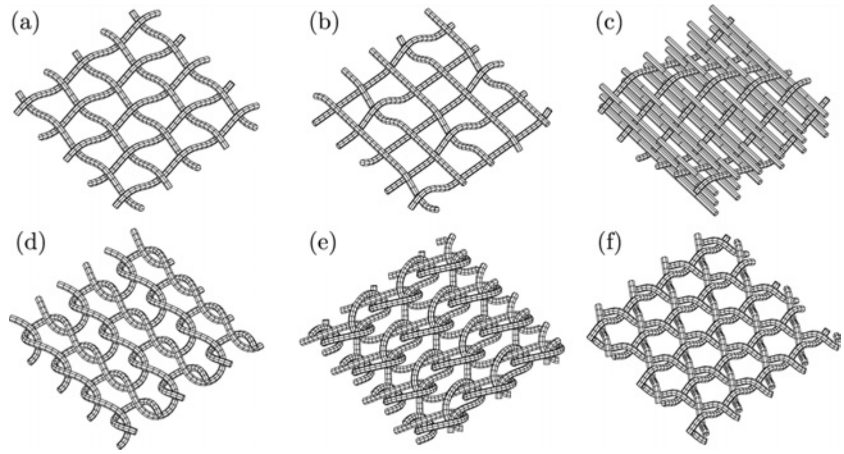

Figure 1 Examples of textile structures: plain weave (a); sateen (b); multi-layered woven fabric (c); single jersey (d); warp knit (e); triaxial woven fabric (f).

role in defining a fabric's properties is its structure. For example, plain woven fabric, single jersey and 1 × 1 rib all made from the same yarn obviously have different structure and they display different properties in tensile tests. At the present state of art in the research of structure-properties relationship this structure-related difference in behavior can be explained at a qualitative level, but not quantitatively because there is no universal numerical parameter that can be used to describe structural characteristics of all fabrics.

The very first step that should be made towards establishing quantitative structure-properties relationship of textiles is to generate a universal mathematical method and criteria that will be able to classify textile fabrics into classes according to their structure, i.e. to distinguish whether in mathematical terms two fabrics are structurally different or not. The same method may then be used for producing a universal numerical parameter that will be able to characterize structure of all fabrics.

In mathematical terms, structural properties of textile fabrics are nothing else but their *topological properties*. It is therefore reasonable to use topology as a specific branch of mathematics for description of structural features of textiles and classification of textile structures.

There have been many publications in textile and composite materials science studies which, in one way or another, mentioned *topology of textiles* [4–13], but the term 'topology' has been used mostly just as a synonym of the term 'structure'. There have been very few real attempts to apply topological methods to textiles. For example, results presented by Liebscher and Weber [14, 15] were limited by the generation of specific structural elements which can be used for the coding of structures such as weft knitted fabrics and wire netting. A series of papers by the members of the Itoh Laboratory [16–18] used elements of knot theory for the computer representation of knitted fabrics. Papers on combinatorial analysis of woven fabrics mainly concerned the enumeration of weaves using conditions defining the integrity of the fabric [19, 20].

Textile structures are extremely diverse [21–24], which is why it would not be possible to compose an exhaustive 'list' of all possible structures; on the other hand, there are relatively few basic textile structures. There have been many attempts to classify basic textiles according to the methods of their manufacture and structural features; one of the remarkable examples of such classification is presented in monograph by Emery [23]. It is important to note that all textiles were either discovered empirically or were the products of advanced technology like triaxial woven fabric (Figure 1(f)) [25]. There have been no attempts to describe *all possible* textile structures in a systematic way starting from the simplest.

The main aim of this series is to show that the topological classification of textiles can be built using methods which are employed in knot theory.

Knot theory, which is a part of modern topology, studies position-related properties of idealized objects which are similar to the textile structures, i.e. knots, links, and braids. These properties do not change by continuous deformations of the objects and it is these properties that, in application to textiles, can be called *structural properties*. Knot theory has at its disposal powerful mathematical tools which have been used in numerous applications in theoretical physics and pure mathematics [26]. However, despite an obvious similarity between textile structures and knots, links, and braids, textile structures have never been the subject of systematic topological studies.

This series will concentrate on the application of topology, in particular the theory of knots and links, to the problem of description and technology-independent classification of textile structures. Part I considers existing methods of description of textile materials and introduces new methods of representation based on knot theory. In Part II, topological invariants in application to textiles will be developed.

Classification and Description of Textile Structures

The majority of textile materials have regular structures produced by a pattern (unit cell) of interlaced threads repeating at regular intervals in two transversal directions. It is these regular structures that will be the main focus of this study.

There are four main industrial methods of manufacturing textile materials with regular structure from yarns and threads which provide fabric integrity by the mechanical interlocking of the threads:

- interweaving – used for manufacturing of woven fabrics;
- interlooping or intermeshing – used for knitted fabrics, fishing nets and machine-made laces;
- intertwining and twisting – a specific method used for making bobbinet fabrics and braids;
- combination of methods used for woven and knitted fabrics, for example, Malimo knitting-through system [27], which has been designed for production of 'nonwoven' materials.

In addition to these methods, there are many manual techniques used in creative crafts, such as macrame, plaits, and lace making [28–30].

Classification of woven and knitted fabrics is generally based on the idea of 'complexity' of the repeated part of the fabric; this often (but not necessarily always) implies that greater complexity refers to larger repeats. At the same time, this classification inevitably depends on the technique of manufacture of a given material and/or the use of specific machinery.

Traditionally, different textile structures are defined by specific terms where each one refers to a structure of certain kind. For example, in weaving the term 'sateen' means a 'weft-faced weave in which the binding places are arranged with a view to producing a smooth fabric surface, free from twill' (Figure 1(b)), whereas definition of a 'single jersey' refers to a 'fabric consisting entirely of loops which are all meshed in the same direction' (Figure 1(d)) [31].

The woven fabrics are classified in terms of mutual location of threads in space. This classification includes:

- basic weaves which include plain weave, twill, satin (sateen), hopsack, and leno weaves;
- derivatives of the basic weaves that can be obtained either by introducing additional warp and/or weft intersections or by combining several basic patterns;
- complex weaves which include jacquard and multi-layered fabrics.

Recent developments in weaving technology made it possible to produce new woven structures such as 3D fabrics [32, 33].

Classification of the knitted fabrics also recognizes patterns in order of increasing complexity and takes into account the technology of knitting. Thus, according to the technology used, knitted fabrics are classified into two distinctive groups as follows:

- weft or warp knitted;
- single or double faced.

On the other hand, knitted structures are classified on the basis of the shape of elementary parts which are loops and their derivatives. These fabrics, similar to the woven fabrics, are divided into basic and derivative patterns as follows:

- basic structures which in weft knitting are plain (or single jersey), rib, interlock, and purl; in warp knitting they are chain, tricot, and atlas;
- derivatives of the basic structures produced by combining loops, floats, tucks, and transferred loops;
- derivatives of the basic structures using additional threads (laying-in technique).

Nonwoven materials are classified by their method of manufacture. These structures may be based on a combination of weft or warp knitted fabrics with reinforcing threads in warp, weft, and bias directions [34] or produced directly from layers of bonded fibers; the latter structures will not be considered in this study.

Methods of mathematical description of the structure of woven and knitted fabrics are mainly based on the representation of their topology by a matrix where each individual element corresponds to a structural element of the fabric. For example, single-layered woven fabrics are coded using binary matrices [35, 36], where individual matrix entries represent warp and weft intersections. Conventionally, if the warp is above the weft then such intersection is coded as 1, and 0 otherwise. For more complicated fabrics based on multi-layered weaves, a non-binary coding system has been used [37]. It has been shown that even in the case of multi-layered woven fabrics (see Figure 1(c)), it is possible to represent their structure by binary matrices [38].

Obviously, in the case of knitted fabrics it is not possible to use binary coding because it is necessary to represent at least four different elements of knitted structure which are loop, float, tuck, and transferred loop. All four of these basic elements may be situated on the face or on the back of the fabric which requires at least eight different coding symbols. The coding of complex structures would require a more extended system [22, 24]. An attempt to create a generic description of weft knitted fabrics using such a system has been made by Grishanov et al. [39], where the fabric was considered as a 'text' in which 'letters' of a specific alphabet represented individual structural elements. This made it possible to use a formal grammar approach for the

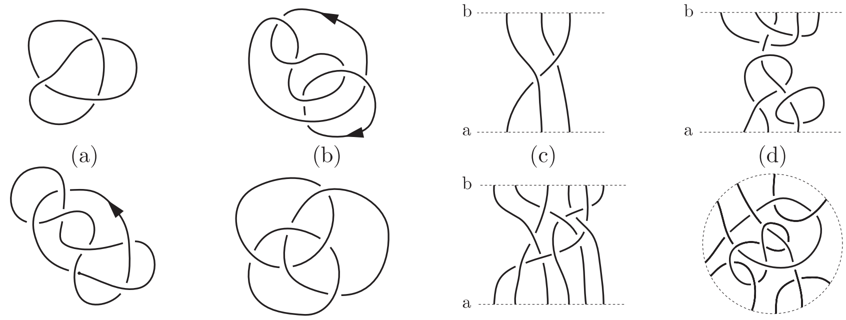

Figure 2 Basic objects studied in knot theory: knots (a); links (b); braids (c); tangles (d).

formulation of rules which define structurally coherent and technologically feasible combinations of loops and their derivatives. In a similar way, other workers have used a set of ‘stitch symbols’ each representing a basic simple pattern of knitting [16–18]. Using these symbols arranged into a matrix, it is possible to describe the structure of fairly large and complex knitted patterns.

The above-mentioned methods of description and classification of textiles have their own advantages, but their application is limited to the given class of structures produced by appropriate technology. For example, in the case of woven fabrics, they can be classified in a simple way by enumerating all binary matrices which satisfy specific conditions corresponding to the fabric integrity [19, 20].

Thus, it can be seen that, at present, there is no systematic classification of textiles based on a universal principle. General technology-independent methods of representation and classification of textile structures can be built using topological methods, in particular, those employed in knot theory. A universal method of characterisation of all textile structures based on topological invariants will pave the way for a systematic study of structure-properties relationship of textiles.

Basics of Knot Theory

The following sections introduce basic topological objects and methods that will be used in application to textiles. Terms and definitions discussed below generally follow the lines of those presented, for example, in studies by Prasdov and Sossinsky, Cromwell, and Adams [40–42].

Knots, links, braids, and tangles

Knot theory studies topological properties of one-dimensional objects in space, such as knots, links, braids, and tangles. All these objects can be thought of as made up of infinitely thin threads which can be continuously deformed without breaking and without having self-intersections. Thus, knot theory does not consider the physical properties of such objects but only those that relate to the mutual position of constituting ‘threads’ in space.

In mathematics, a *knot*, unlike its common understanding, is a *closed* smooth curve (Figure 2(a)), which is impossible to untie without cutting. Formally a knot $K \subset R^3$ is an embedding of the circle S^1 into R^3. Several knots $K_1, K_2, \ldots$, usually inter-chained together, create a *link* $L = K_1 \cup K_2 \cup \ldots$ (Figure 2(b)); each knot is called a *component* of the link.

A *braid* is a set of strictly ascending threads with end points fixed on two parallel lines *a* and *b* (Figure 2(c)); the points on the lines should be positioned exactly one under the other.

A *tangle* is a generalization of knots, links, and braids. A tangle is an arbitrary set of threads in space with fixed end points [41, 43]. Different types of tangles can be considered, for example, tangles with end points fixed on a sphere (Figure 2(d)).

All components of a link (knot, tangle) can be given a direction which is identified by arrows; this defines *orientation* of the link (Figure 2(a), (b)).

Isotopy

From the topological point of view, two knots K_1 and K_2 are equivalent ($K_1 \sim K_2$) if one of them can be transformed into the other in space by a *continuous* deformation without self-intersections and breaking. Such a deformation is called an *isotopic deformation* or simply an *isotopy*. All knots (or links) which are isotopic to one given knot (or link) form an *isotopic class*.

A knot that is isotopic to a circle is called a *trivial knot* or *unknot* (Figure 3). Similar to this, a link is called trivial if it

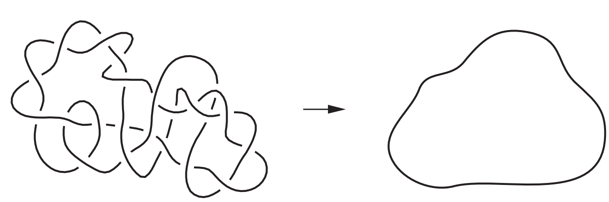

Figure 3 An isotopy of the unknot.

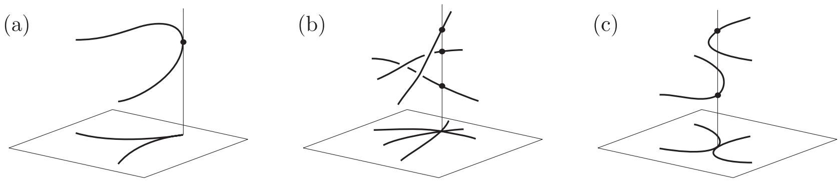

Figure 4 Prohibited projections.

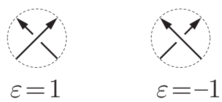

Figure 5 The crossing sign definition.

can be split into disjoined circles. Sometimes it is not easy to understand that two knots are isotopic (see Figure 3).

Link diagrams and Reidemeister theorem

Graphically knots and links are represented by means of *diagrams* which are projections on a plane with additional information at each crossing about which 'thread' is on the top (Figure 2). Any projection should have a finite set of crossings and satisfy conditions which are known as the *conditions of general position*:

- neither of the tangents to the link must be parallel to the projection direction (Figure 4(a));
- not more than two different points of the link must be projected onto the same point on the plane (Figure 4(b));
- projections of tangents to two arcs at each crossing point must be different (Figure 4(c)).

These conditions can always be achieved by a small isotopic deformation of relevant parts of the link. A projection that satisfies all these conditions is called a *regular projection* of the link.

Double points of regular projections are called *crossings*. Each double point is associated with two different parts of the link in space. By labeling mutual positions of the arcs at each crossing point as over-pass and under-pass, a *diagram* of the link can be obtained (see Figure 2 for example).

In an oriented diagram, all crossings are identified as positive (the crossing sign ε = 1) or negative (ε = –1), according to Figure 5.

An isotopic transformation of a link and the corresponding continuous transformation of its diagram at some point may lead to the violation of the conditions of general position described above. These singularities can be overcome by elementary moves Ω_1, Ω_2, and Ω_3, known as *Reidemeister moves*; they are shown in Figure 6. Reidemeister proved that two diagrams represent equivalent links if and only if they can be transformed into each other by a finite sequence of Reidemeister moves and plane isotopies; this statement is known as the Reidemeister Theorem.

For example, the first knot in Figure 7 and the 'figure-eight' knot (last in Figure 7) are isotopic (belong to the same isotopic class) because they can be deformed into each other using Reidemeister moves Ω_1 and Ω_2.

Topological invariants

There are two fundamental problems in knot theory. First is to answer the question of whether two knots or links are different or are the same; the second is triviality, i.e. whether or not a link is equivalent to a collection of disjoined circles (unknots). The answers to these questions may be found with the aid of topological invariants.

A *knot invariant*, *f*, is a function from the set of knots to some other set whose value depends only on the equivalence class of a knot:

$$K_1 \sim K_2 \Rightarrow f(K_1) = f(K_2).$$

Figure 6 Reidemeister moves.

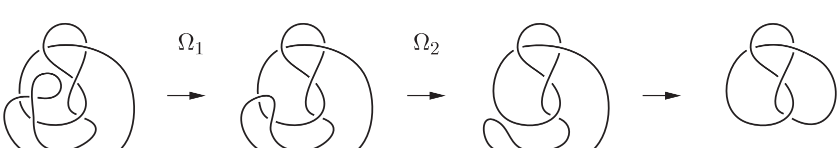

Figure 7 An isotopy of the figure-eight knot.

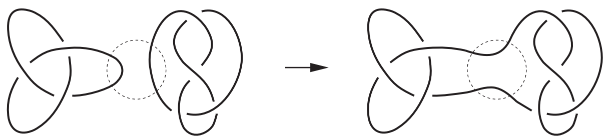

Figure 8 Connected sum operation for knots.

The target set can be, for example, the set of integers, **Z**, or a set of polynomials. Invariants for links and tangles are defined in the same way.

The Reidemeister Theorem enables invariants to be calculated using link diagrams. Any isotopic invariant obviously should take the same value for all diagrams which can be transformed into one another by Reidemeister moves.

A number of invariants has been proposed which include basic numerical, polynomial and finite type invariants (or Vassiliev invariants) [40, 41, 43], but none of these has been proven to distinguish between all knots or links. The question of whether Vassiliev invariants are able to distinguish between all possible knots remains an open problem in knot theory.

Classification and tabulation of knots and links

Classification of knots and links is based on the *crossing number*, which is the minimum possible number of crossings in any diagram of a knot *K*; this is denoted as $cn(K)$. This classification includes only *prime knots*, i.e. those that cannot be obtained by a composition (so called *connected sum*) of other knots, as illustrated in Figure 8.

All prime knots and links with the crossing numbers up to six are presented in Figure 9. Knots and links are denoted as M_n and $M_n{}^k$, respectively, where M is the crossing number, n is the ordinal number of knot (or link) within the group with M crossings, and k is the number of components.

The number of knots and links grows exponentially with the increase of *cn*. Currently, all knots with up to 16 crossings have been tabulated; this 'table' includes 1,701,936 prime knots [44].

If there is a diagram of a given knot in which over-passes and under-passes alternate, then such a knot is called *alternating*. Alternating knots and links form an important subclass of all knots and links. In Figure 9, all knots and links are alternating except for link $6_3{}^3$, but it is known [45] that the fraction of alternating knots dramatically decreases as the crossing number increases. Nevertheless, diagrams of alternating knots are used as a basis for construction of all possible knots with a given crossing number.

The general algorithm for enumeration of knots and links includes the following steps:

1. Generate all possible non-equivalent alternating diagrams.
2. For each diagram generated at Step 1, consider all 2^{cn} possible diagrams which can be obtained from the diagram in question by assigning over-passes and under-passes to its crossings in all possible combinations.
3. Select non-isotopic knots from the set obtained at Step 2 using knot invariants.

Modern enumeration methods are discussed in more detail by Hoste [45].

Topological Representation of Textile Structures

Let us now consider regular textile structures from the topological point of view using approaches proposed by Grishanov et al. [46]. For this it is necessary to assume that the constituting threads are smooth spatial curves and can be deformed without breaking or passing through each other, or themselves, during deformation. In this case, textiles become similar to the objects studied in knot theory.

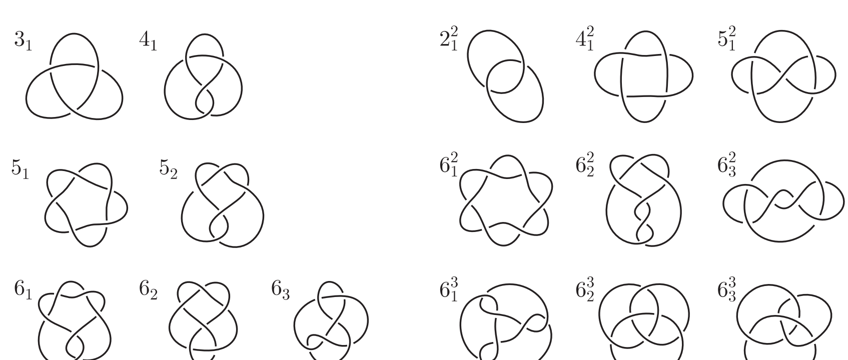

Figure 9 Table of prime knots and links with up to 6 crossings.

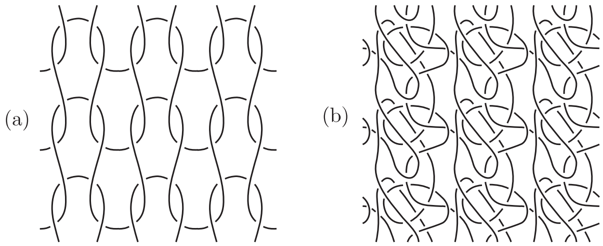

Figure 10 An isotopic deformation of single jersey.

At the same time textile structures are specific because they can be considered as structures that extend infinitely in two perpendicular directions in a periodic manner. Thus, textile structures can be defined as particular case of *doubly-periodic structures*; they will be further referred to for short as *2-structures*. It will be shown that 2-structures can be considered as specific types of knots and links.

Isotopy

First of all it is necessary to define an isotopy notion for 2-structures. Let isotopy for 2-structures be any continuous deformation that preserves their main property which is double periodicity. These deformations include homogeneous extensions, shear deformations, translations, rotations in space and, in addition to this, periodical deformations with the period equal to that of the structure.

Planar diagrams and torus diagrams of 2-structures

Similar to diagrams of knots and links, it is possible to define diagrams of 2-structures as planar projections which satisfy the conditions of general position described previously (see Link Diagrams and Reidemeister Theorem section). In fact, textile scientists have been using such diagrams for representation of fabrics (see, for example, Figure 10(a)).

A planar diagram inherits its properties from the structure, i.e. a planar diagram of a 2-structure is also doubly-periodic. Using planar isotopy, it is always possible to transform a planar diagram so that its period (unit cell) will be the unit square [46].

A unit cell of a given planar diagram completely defines the topology of the 2-structure. Therefore, instead of an infinite planar diagram containing an infinite number of crossings, it is convenient to consider a *torus diagram*, which can be obtained by joining the opposite sides of the unit cell of the planar diagram, as shown in Figure 11.

In this way, a diagram with a finite number of crossings can be constructed which is similar to ordinary link diagrams but 'drawn' on the standard torus $\boldsymbol{T}^2$. Formally, it is possible to consider a torus diagram as a diagram of a link embedded into the space defined as $\boldsymbol{T}^2 \times \boldsymbol{R}^1$, where $\boldsymbol{T}^2$ is the standard torus. Alternative ways for representing such links will be considered later (see Alterative Representations of Textile Structures section).

Let us define Reidemeister moves on the torus in the conventional way, as shown in Figure 6. It is possible to state that two 2-structures S_1 and S_2 are isotopic ($S_1 \sim S_2$) if their torus diagrams D_1 and D_2 can be transformed into each other by means of Reidemeister moves Ω_1, Ω_2, and Ω_3 and isotopies on the surface of the torus.

However, if two torus diagrams D_1 and D_2 are not equivalent in the above-mentioned sense, then this does not necessarily mean that two corresponding structures S_1 and S_2 are different. The problem is that the unit cell can be chosen in an infinite number of ways. Let us consider this in more detail.

Unit cells and torus diagram twists

A point lattice can be associated with any planar diagram of a 2-structure (Figure 12(a)). Without loss of generality, it can be assumed that the lattice is the integer point lattice $\boldsymbol{Z}^2$. In this case, a unit square can be considered as a unit cell of the structure. But in fact it can be seen that any lattice parallelogram of unit area or its parallel translated copy can be taken as a unit cell (Figure 12(b)). Such a par-

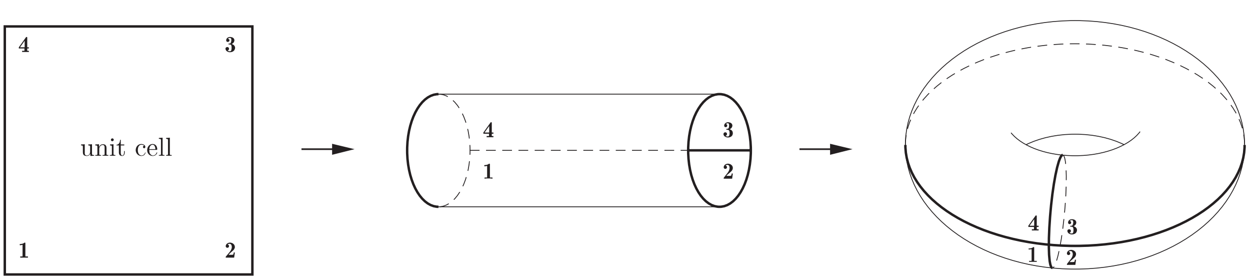

Figure 11 On construction of torus diagram.

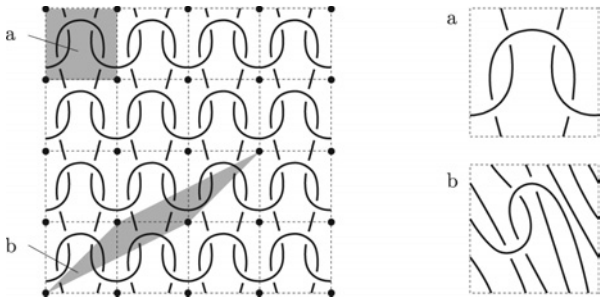

Figure 12 A planar diagram with the associated point lattice and two equivalent unit cells, (a) and (b).

allelogram contains the same number of crossings as the original unit square; the original doubly-periodic planar diagram can be re-constructed from this parallelogram.

Let us consider a lattice parallelogram of unit area with its opposite vertices at (0, 0) and $(\pm p, q)$, $p, q \geq 1$ (Figure 13(a)). It is known from the theory of numbers that any pair of co-primes (p, q) corresponds to a unique lattice parallelogram of unit area with the opposite vertices at (0, 0) and (p, q). The opposite is also true, i.e. if a lattice parallelogram of unit area has its vertices at (0, 0) and $(\pm p, q)$, then p and q are co-primes (see, for example, the study by Hardy and Wright [47]).

Since any pair of co-primes p and q defines exactly two unit cells with the origin at (0, 0) and the opposite vertex either at (p, q) or $(-p, q)$, there is an infinite number of torus diagrams corresponding to the same 2-structure; these diagrams should be considered as equivalent to one another.

It can be shown [46] that any torus diagram of a 2-structure can be obtained from an arbitrarily chosen torus diagram by a sequence of full revolution twists of the torus along its meridians and longitudes (Figure 14); these twists are known in geometric topology as *Dehn twists* [41]. Let us cut the torus along one of the meridians, twist one of the cut edges by 360°and then rejoin the edges again. This procedure will be further referred to as *meridional twist* (Figure 14(b)). *Longitudinal twist* can be defined in a similar way this time cutting, twisting and joining the torus along one of its longitudes (Figure 14(c)).

Figure 13 illustrates the correspondence between torus twists and transitions from one unit cell to another. Let $\boldsymbol{v}1 = (p_1, q_1)$ and $\boldsymbol{v}_2 = (p_2, q_2)$ be the basic vectors of the unit cell with the opposite vertices at (0, 0) and (p, q). One meridional twist (see Figure 13(b)) translates the unit cell vertex (p, q) by the vector $+\boldsymbol{v}_1$; the vertices of this new cell will be (0, 0), (p_1, q_1), $(p + p_1, q + q_1)$, and (p, q). Numbers $p + p_1$ and $q + q_1$ are again co-primes and thus define a parallelogram of unit area, i.e. a unit cell. In the same way, a longitudinal twist results in a unit cell with its vertex translated by the vector $+\boldsymbol{v}_2$ (Figure 13(c)).

Thus, the equivalence of torus diagrams of 2-structures should be considered up to torus twists, or in other words, the equivalence of planar diagrams of 2-structures should be considered up to linear transformations of the plane that map the integer lattice $\boldsymbol{Z}^2$ into itself. It is now possible to formulate a statement which is similar to the Reidemeister Theorem as follows:

Two doubly-periodic structures S_1 and S_2 are isotopic if and only if their torus diagrams D_1 and D_2 can be obtained from each other by a sequence of Reidemeister moves Ω_1, Ω_2, and Ω_3, isotopies on the torus surface, and torus twists.

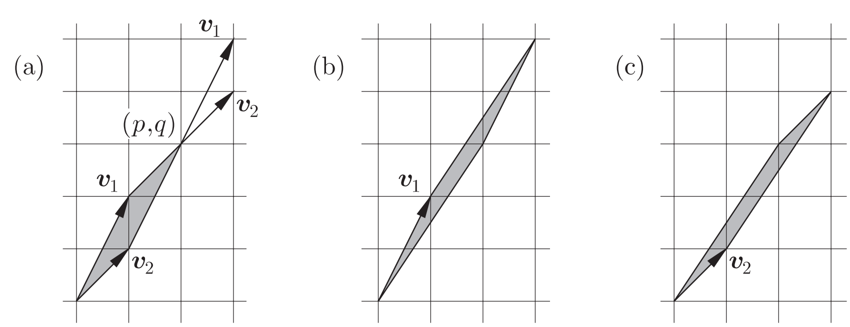

Figure 13 Unit cell transformation in torus twists: original unit cell (a); meridional twist (b); longitudinal twist (c).

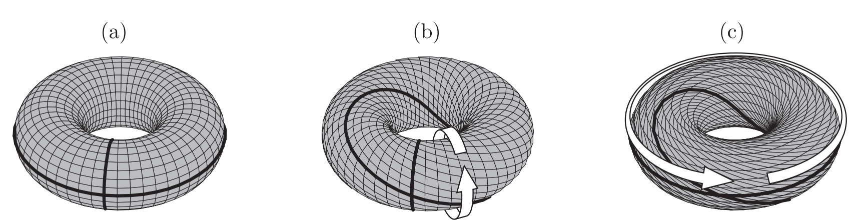

Figure 14 Torus twists: original torus (a); meridional twist (b); longitudinal twist (c).

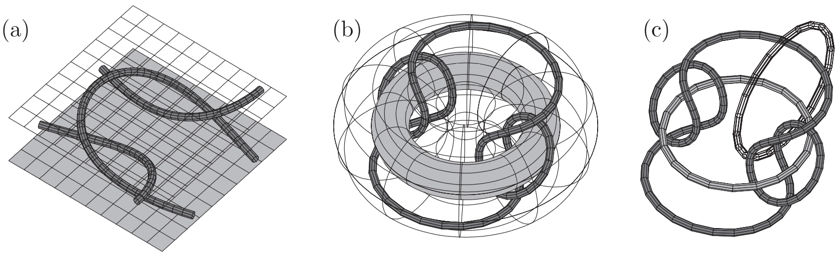

Figure 15 Representation of the single jersey by a composite link.

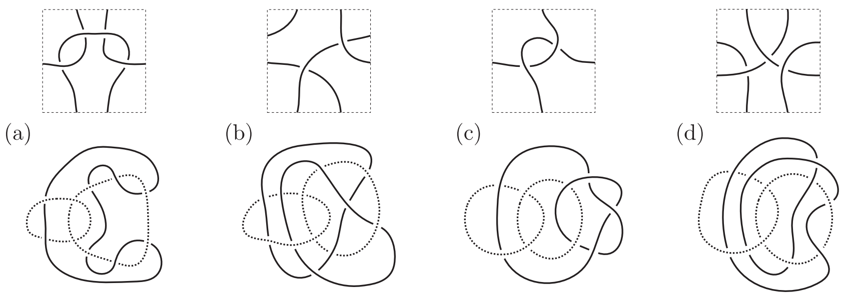

Figure 16 Composite link representation for some 2-structures: single jersey (a); plain weave (b); simple linking (c); simple looping (d) [23].

The proof of this proposition follows the proof of the classic theorem with respect to Reidemeister moves and uses the above-mentioned correspondence between twists and the unit cell choices.

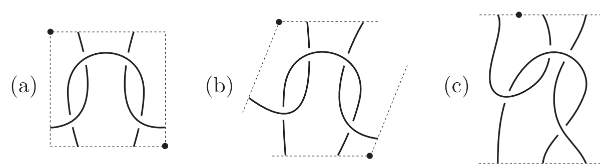

Figure 17 Representation of unit cell by a tangle with a marked point.

Alternative representations of textile structures

It has been shown above that a three-dimensional unit cell of a 2-structure can be associated with a link in the space $\boldsymbol{T}^2 \times \boldsymbol{R}^1$ (Figure 15). This link can be imagined as enclosed between two nested tori (Figure 15(b)). This imposes restrictions on isotopic deformations of the link because it cannot intersect the surfaces of the tori. It can be seen that these topological restrictions are exactly the same as those that are created by a system of two chained circles, known as the Hopf link (Figure 15(c)). Thus, it is possible to represent a unit cell of a 2-structure as a composite link which includes the two components of the Hopf link; examples for some simple 2-structures are shown in Figure 16.

Another representation is based on one-to-one transformation of the unit cell into a tangle of special type with one marked point (Figure 17).

The advantage of representations described above is that in this way conventional topological objects, such as links and tangles, can be obtained and then studied by standard methods of knot theory. The composite link mentioned above, however, has a number of extra crossings, and, furthermore, the type of the link very much depends on the choice of the unit cell. The representation by a tangle also suffers from the latter disadvantage. For this reason, further analysis of textile structures will be represented by torus diagrams as described earlier (see Planar Diagrams and Torus Diagrams of 2-Structures section) although, for simplification, only planar diagrams of the minimum repeat (unit cell) will be shown.

Invariants

It is not difficult to propose some simple topological invariants for 2-structures which are similar to those used for the conventional knots and links. Crossing number, $cn(S)$, of a 2-structure S which is the minimum possible number of crossings in any torus diagram of the 2-structure S, or the number of torus diagram components, $\mu(S)$, can be considered as examples of such basic invariants.

It is important to note that any invariant of 2-structures must be held not only in Reidemeister moves applied to the torus diagram, but also in twists as described in the section above. This implies that any invariant of a 2-structure does not depend on the choice of the unit cell. The functions $cn(S)$ and $\mu(S)$, obviously, comply with this requirement.

Since doubly-periodic structures have never been the subject of topological studies, it is necessary to construct

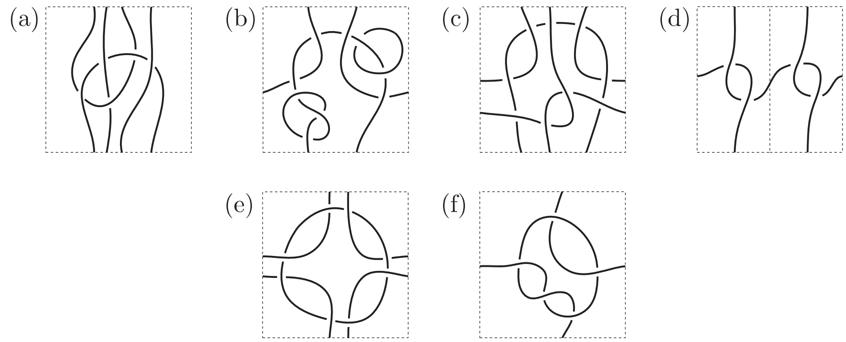

Figure 18 Examples of non-prime (a), (b), (c), (d) and non-textile (e), (f) 2-structures.

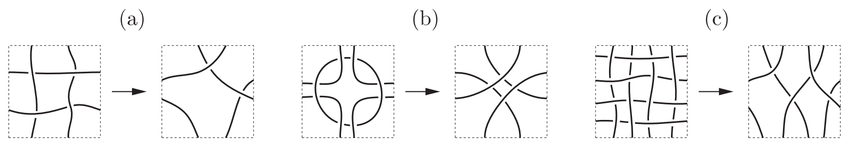

Figure 19 Traditional repeats and their equivalent minimal unit cells: plain weave (a); chain-mail (b); twill (c).

invariants which would be able to discriminate between such structures. Latest advances in topology have led to the development of methods which can be used for the generation of polynomial and finite type invariants for knots and links in arbitrary three-dimensional manifolds. Some of these methods can be adapted for the case of 2-structures.

Some numerical and polynomial invariants for 2-structures and, in particular, for textile structures, will be discussed in Part II.

Enumeration of 2-structures

It is known that there are millions of knots and links which have up to 16 crossings. It should be expected that the number of different 2-structures of the same complexity will be of similar order of magnitude. Not all 2-structures, however, belong to the set of textile structures, for example, structures which cannot be produced by known technology must be excluded. Even in this case, the number of remaining structures will be much greater than the number of basic structures which are known and commonly used in the textile industry. It is anticipated that exhaustive enumeration of 2-structures will enable new types of textiles to be found.

The problem of classification of 2-structures in terms of torus diagrams can be solved for a given value of crossing number, by enumerating all torus diagrams which are non-equivalent with respect to Reidemeister moves and torus twists.

Similar to knots and links, it is necessary to limit this enumeration by *prime* 2-structures, i.e. such that meet the following conditions:

- structures must be non-trivially doubly-periodical; one-periodic structures which are similar to Figure 18(a) will not be considered;
- similar to knot theory, 2-structures of connected-sum-type (Figure 18(b)) will not be considered;
- structures must not be a multi-layered disjoined combination of several structures such as that shown in Figure 18(c);
- unit cells of 2-structure(s) must be *elementary*: multiple unit cells will not be considered (Figure 18(d)).

The problem of identification of an elementary unit cell is not trivial. It should be noted that the unit cell definition given in conventional textile terms may be different from the topological definition. The definition of the unit cell used by specialists in textiles is technology-dependent and reflects the periodic manner of operation of the textile machinery. In topological terms, an elementary unit cell is defined as a periodic part of the structure that contains a minimum possible number of crossings. Figure 19 shows examples of some repeats in their traditional representation and corresponding elementary unit cells. The problem of the identification of the minimum unit cell requires specific mathematical tools to be developed.

Since doubly-periodic textiles are our prime area of interest, it is possible to limit the number of structures in question even further. For example, it is possible to reject 2-structures which contain closed components (e.g. Figure 18(e)) or those that can only be produced by manual techniques (e.g. Figure 18(f)).

It is obvious that there are no coherent 2-structures with one crossing. Preliminary analysis has shown that there are exactly two non-trivial 2-structures with two crossings; they are presented in Figure 20.

2_1 2_1^2

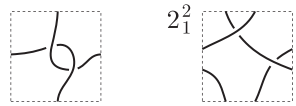

Figure 20 The two-crossing 2-structures: 2_1 – simple linking [23]; 2_1^2 – plain weave.

Discussion and Conclusion

This paper establishes a new principle of description and classification of 2-structures and, in particular, regular textile structures. The proposed methods are based on the application of modern mathematical tools of low-dimensional topology. Textile structures are considered as special ‘knots’ or ‘links’ under the assumption that the constituting threads are smooth spatial curves. The periodic nature of textiles enables a unit cell to be identified for which it is possible to construct a torus diagram. The unit cell can be chosen in an infinite number of ways; for this reason the equivalence of torus diagrams must be considered up to longitudinal and meridional twists of the torus.

The proposed approach to the description of textile structures is completely new and goes hand in hand with the recent developments of low-dimensional topology which made it possible to use topological methods in practical applications [26]. Textile structures have not been the subject of systematic topological studies. An attempt to apply knot theory to textiles requires a number of mathematical problems to be formulated and resolved.

The main problem is enumeration and topological classification of 2-structures and, in particular, textile structures. Textile structures are much more complex than general knots and links because they usually contain a large number of crossings; their analysis requires a more complex space than the usual $\boldsymbol{R}^3$ to be used.

It would be unreasonable to expect that the exhaustive enumeration of textile structures will ever be completed. However, two important problems can be resolved. First of all, it is possible to generate an algorithm which, in principle, can enumerate all possible textile structures and then use it for the classification of basic patterns with a limited number of crossings. Secondly, it should be possible to develop methods for constructing arbitrary complex structures with specified topological properties. This will establish a universal approach to the investigation of ‘structure-properties’ relationship of textiles.

It is necessary to formulate a set of additional rules which define a 2-structure as a textile structure, i.e. one that can be produced using one of the known technological methods. The application of these rules will enable a subset of all different textile structures to be enumerated. This enumeration may help towards finding new textile structures with specific properties.

The enumeration problem cannot be successfully resolved without the construction of invariants for 2-structures.

The fundamental problems outlined in this paper will be considered in more detail in Part II of this series.

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