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A Topological Study of Textile Structures. Part II: Topological Invariants in Application to Textile Structures

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Abstract This paper is the second in the series on topological classification of textile structures. The classification problem can be resolved with the aid of invariants used in knot theory for classification of knots and links. Various numerical and polynomial invariants are considered in application to textile structures. A new Kauffman-type polynomial invariant is constructed for doubly-periodic textile structures. The values of the numerical and polynomial invariants are calculated for some simplest doubly-periodic interlaced structures and for some woven and knitted textiles.

Key words topology of textiles, knot theory, knot invariant, Kauffman polynomial, doubly-periodic interlaced structure

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The main aim of this series of papers is to establish a new technology-independent classification of textile structures based on topological principles. Part I of the series [1] introduced a new approach to the analysis of structural characteristics of textiles based on modern methods of knot theory. It has been shown that textile structures as a special case of doubly-periodic interlaced structures can be considered as specific knots or links which are different from those studied in classical knot theory. A doubly-periodic structure can be represented as a diagram drawn on the surface of the torus, whereas knots usually are represented by diagrams on the plane.

In knot theory [2–4] knots and links are classified using diagrams; the main classification criterion is the number of crossings in the diagram. In this way, the first step in the classification of textiles is enumeration of different torus diagrams starting from the simplest. This is the prime problem of this study. Thus, the intention is to produce a catalog of doubly-periodic structures similar to the catalog of knots and links (see, e.g., [3]).

Algorithms of knot diagram classification usually include two stages – (1) generation, and (2) selection. The main objective of the generation stage is to obtain a set of diagrams that represents all knots with a given number of crossings. As a rule, the set obtained is redundant, because there are many different diagrams of the same knot. Duplicates must be found and eliminated at the selection stage. Classification of doubly-periodic structures follows the same scheme.

For the selection procedure a mathematical tool is needed to determine whether, from the topological point of view, two different diagrams correspond to the same knot (doubly-periodic structure) or different knots; this is not always an easy task (see Figure 1). In knot theory this question is known as the equivalence problem.

The topological equivalence problem can be resolved using ‘invariants’. In simple words, an invariant is some property that holds for all different diagrams of the same knot.

This paper considers some of the knot invariants in application to textile structures and also introduces a specific polynomial invariant that can be used for classification of textile structures.

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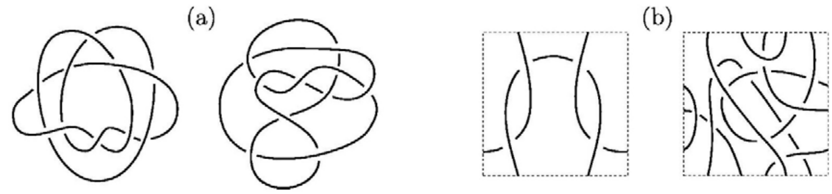


Figure 1 Two different diagrams of a knot (Perko pair) (a), two diagrams of single jersey (b).

Basic Notions and Definitions

Let us briefly outline some definitions [2–4] and assumptions introduced in Part I of this series of papers [1].

Textile structures are considered exclusively from the topological point of view, without taking into account the manufacturing method or the physical properties of threads from which they may be made. For example, in the case of woven structures, no consideration is given to the differences between the warp and the weft.

In the context of this study, the term 'structure' refers to binding patterns of interlacing threads in knitted and woven fabrics without considering any internal structural features of the threads involved.

This study focuses on regular textile structures which have a pattern of interlaced threads infinitely repeating in two transversal directions. In this way, the textile structures are considered as a specific case of *doubly-periodic* structures.

We give the name *2-structure* to any doubly-periodic interlaced structure that is non-trivial, i.e. the structure cannot be divided into separate parts without breaking the constituting threads. 2-structures are called *textile structures* if they can be produced using one of the known methods of textile technology.

Term *unit cell* is used to refer to a *minimal* repeating part of the structure. The unit cell can be chosen in an infinite number of ways, i.e., if an integer lattice is associated with a 2-structure, then an arbitrary parallelogram of unit area such that its sides are integer vectors is a unit cell (see Figure 2(a)). Note that a unit cell is not always equal to what is considered as a *repeat* in textile technology (see [1, Section 3.6], also Figure 14).

A *planar diagram* of a 2-structure is a non-singular projection of 2-structure onto the plane with additional information at each crossing point about which thread is on the top (Figure 2(a)). Along with diagrams, *projections* of 2-structures are also considered that do not distinguish the position of threads in crossings (Figure 2(b)).

The periodic nature of 2-structures is modelled using a *torus diagram*, which is a diagram of a unit cell drawn on the surface of the standard torus (Figure 2(c)). For the simplification of drawings, torus diagrams are presented in an unfolded form as a unit square (Figure 1(b)). A torus diagram consists of several closed smooth curves drawn on the torus surface; each curve is a '*component'* of the diagram.

For torus diagrams important *twisting* operations along longitude and meridian lines are defined (see [1, Figure 14]). Full revolution twists of the torus correspond to the selection of a different unit cell on the planar diagram of 2-structure (Figure 2(a)).

Knot Invariants

In knot theory two knots, K_1 and K_2, are considered as *equivalent* ($K_1 \sim K_2$) if they can be continuously transformed into each other in space without breaks and self-intersections. Such a transformation is called an *isotopy*.

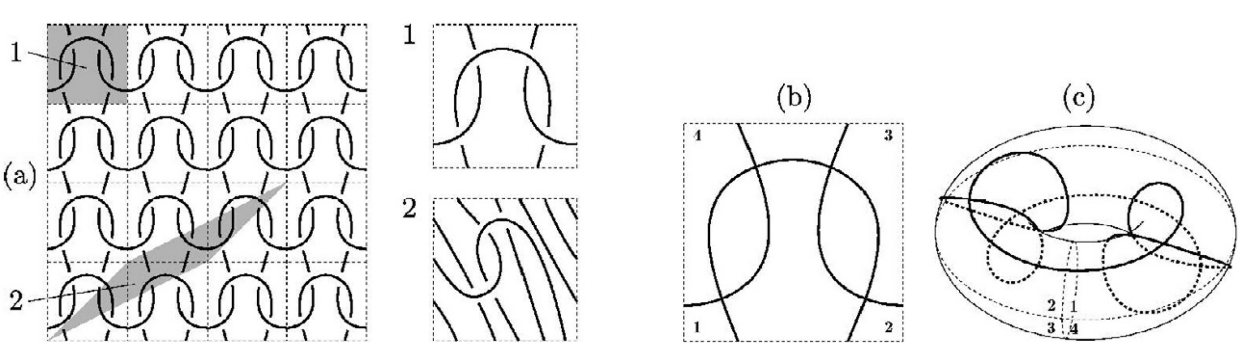


Figure 2 A planar diagram and two different unit cells (a); an unfolded torus diagram –projection in a unit square (b); a torus diagram (c).

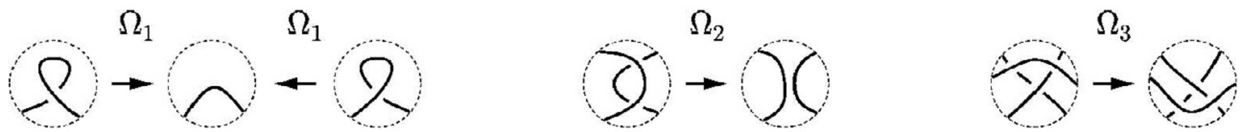


Figure 3 Reidemeister moves.

During an isotopy, the corresponding knot diagram goes through deformations which can be decomposed into a series of elementary transformations known as Reidemeister moves (Figure 3). It is clear that any knot or link can be represented by infinite number of different diagrams.

One of the main problems of knot theory is the *equivalence problem*, i.e. how it can be decided whether two given knots are equivalent or not. The standard way of resolving this problem is to construct invariants.

A *knot invariant* is a function defined on the set of all knots that assumes the same value on knots from the same equivalence class:

$$K_1 \sim K_2 \Rightarrow X(K_1) = X(K_2) \tag{1}$$

The most used invariants are associated with knot diagrams. A function defined on diagrams is a knot invariant if and only if it does not change in Reidemeister moves Ω_1, Ω_2, and Ω_3.

Many different invariants are known. The value set of an invariant can be a numerical set, a set of matrices, a set of polynomials, and so on. Respectively, there are numerical invariants, matrix invariants, or polynomial invariants. The reason to use various invariants simultaneously is that no one of them can discriminate all different knots: the statement opposite to Equation 1

$$X(K_1) = X(K_2) \Rightarrow K_1 \sim K_2$$

can only be true if the invariant $X(\cdot)$ is *complete*.

Recently, a new type of knot invariant has been proposed [5], which was called *finite type invariant* or *Vassiliev invariant*. A Vassiliev invariant is, in fact, an infinite series of numerical invariants where each next invariant is more powerful than the previous one. On the basis of this property it has been suggested that the system of Vassiliev invariants is a complete knot invariant. This proposition has not yet been proved or disproved but it is known that the system of Vassiliev invariants is more powerful than any of the other known invariants.

It has been shown in Part I [1] that textile structures can be represented as diagrams on the torus. An invariant of double-periodic structures may be defined as a function on the set of torus diagrams that does not change under Reidemeister moves (on the torus surface). However, this condition is not sufficient because it has been shown that the same planar diagram can be represented by an infinite number of torus diagrams. For this reason the equivalence of torus diagrams must be considered up to longitudinal and meridional twists of the torus; in other words, any invariant of 2-structures must be independent of torus twists.

The following sections consider some of the invariants that can be used for classification of 2-structures. Figure 4 shows examples of 2-structures (unit cells) that will be used to illustrate the discussion that follows.

Basic Numerical Invariants

In this section the simplest and the most basic invariants are introduced and their application to 2-structures is analysed.

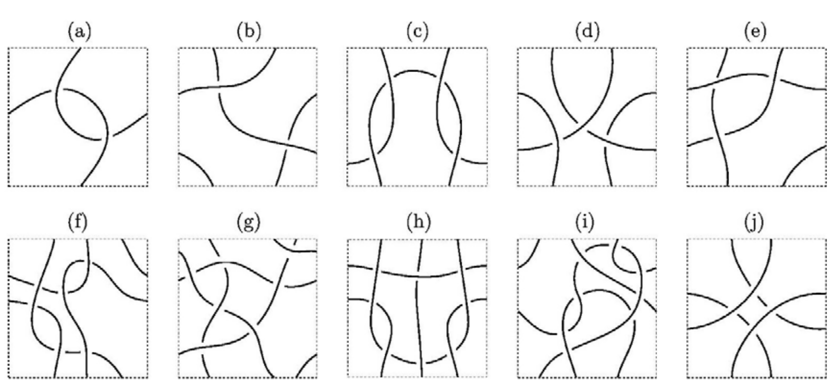


Figure 4 Examples of 2-structures: wire netting (a); plain weave (b); single jersey (c); simple looping (d); triaxial weaving (e); derivative of single jersey (f); 4-axial structure (g); single jersey with inlay threads (h); tricot with opened loops (i); chain-mail (j).

Figure 5 Crossing sign definition for oriented diagrams.

Some of the invariants are defined for *oriented links* [2–4] (2-structures). In this case oriented diagrams are considered (see, for example, Figure 8), and crossing sign $\varepsilon = \pm 1$ can be associated with every crossing according to Figure 5.

Crossing Number

Crossing number is the main classification criterion for knots and links. All existing knot tables enumerate knots and links in order of increasing crossing number. We will classify 2-structures in the same manner. In the case of 2-structures, 'number of crossings' means the number of crossings contained in a torus diagram (in a unit cell).

The number of crossings in a given diagram, D, of a link (2-structure), L, is denoted as $cn(D)$. The number of crossings may change when Reidemeister moves are applied to the diagram (see, e.g., Figure 1(b)). The crossing number of a link (2-structure) L is defined as the minimum number of crossings in any diagram of the link (2-structure) L:

$$cn(L) = \min\{\, cn(D) : D \text{ is diagram of } L\}.$$

Any diagram of the link L which has exactly $cn(L)$ crossings is called *minimal*.

Construction of a minimal diagram is not a simple task; the following straightforward algorithm usually achieves the goal.

1. `While (`Ω`1 and `Ω_2` are applicable) reduce cn(`D`); end;`
2. `Find all possible sequences of `Ω_3` moves;`
3. `Do the next possible sequence of `Ω_3` moves;`
   `If (`Ω_1` and `Ω_2` are applicable) goto 1;`
   `else goto 4; end;`
4. `If (all possible sequences of `Ω_3` moves already tried) exit;`
5. `else goto 3; end.`

The algorithm uses Reidemeister moves Ω_1 and Ω_2 to reduce the number of crossings; Ω_3 move is used to transform the diagram into a form that may make it possible to apply Ω_1 or Ω_2. However, this simple idea does not work in the general case. For example, it is known [2, 3] that there are diagrams for which it is necessary to increase the number of crossings before it can be reduced. Although theoretically this difficulty can be overcome using, in addition to Reidemeister moves, some special diagram transformations (so called flypes [3]), but in practical terms the algorithm based on this approach is very time-consuming even for modern computers.

It can be proven that every diagram in Figure 4 is minimal because neither Reidemeister moves (Ω_1, Ω_2 and Ω_3) nor flypes can reduce the number of crossings. The crossing numbers for 2-structures in Figure 4 are listed in Table 1.

Component Number and Related Invariants

Component number $\mu(L)$ of a link (2-structure) L is another basic invariant that, unlike crossing number, can be easily calculated and may be useful for the classification of textiles.

Table 1 Numerical invariants for structures in Figure 4.

L	*cn*	μ	*lks*	*ax*	*dis*	*spl*	*tu*	*vec*
(a)	2	1	2	1	1	1	1	{(1, 0)}
(b)	2	2	0	2	1	1	1	{(1, 1), (–1, 1)}
(c)	4	1	0	1	2	2	2	{(1, 0)}
(d)	3	1	1	1	1	1	1	{(2, 0)}
(e)	3	3	3	3	1	1	1	{(1, 0), (1, 1), (0, 1)}
(f)	6	1	0	1	2	2	2	{(1, 0)}
(g)	7	4	7	4	1	1	1	{(1, 0), (1, 1), (0, 1), (–1, 1)}
(h)	8	3	2	2	1	1	3	{(1, 0), (0, 1), (0, 1)}
(i)	8	1	0	1	2	2	2	{(1, 0)}
(j)	4	1	0	1	1	2	2	{(1, 0)}

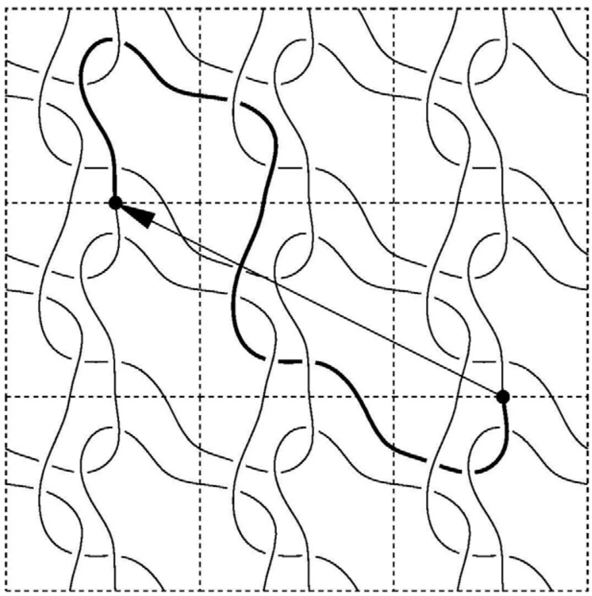


Figure 6 On the definition of periodicity vector of a 2-structure component.

Let us recall that for 2-structures the component number is the number of smooth closed curves which form the torus diagram. The value of $\mu(L)$ of a 2-structure L, obviously, does not depend on torus twists (the unit cell choice); thus, $\mu(L)$ is an invariant of 2-structures. For the 2-structures in Figure 4 the component numbers vary from 1 to 4 (see Table 1).

The number of components should not be confused with the number of threads passing through the unit cell on the planar projection. The latter depends on the unit cell definition and thus is not an invariant.

In the case of 2-structures it is possible to construct specific invariants related to μ, because each component of 2-structure can be associated with an individual characteristic that shows how the component is embedded in the torus. Any component of a 2-structure planar diagram either forms a closed loop (cycle) or infinitely repeats with some period in some direction on the plane (Figure 6). In the latter case it is possible to define the *axis*, along which the component runs, and the *periodicity vector* of the component. For example, in Figure 6 the periodicity vector for one of the threads of the 2-structure Figure 4(f) is shown.

Note that any periodicity vector v has integer coordinates: $v = (m, n), m, n \in \mathbf{Z}$. It is always possible to assume that ordinate $n \geq 0$, and $m \geq 0$ if $n = 0$. This assumption means that components of the 2-structure are considered as ascending, or directed, from left to right.

The number of different axes of a 2-structure L is an important invariant. This invariant will be called the *axis number* and denoted as $ax(L)$. Structures (a), (b), (e), and (g) in Figure 4 give examples of 1-, 2-, 3-, and 4-axis 2-structures respectively. The axis number invariant is related to technological feasibility of 2-structures, for example, most woven fabrics have two axes ($ax = 2$) whereas knitted structures usually have only one axis ($ax = 1$). An example of a knitted structure with inlay threads for which $ax = 2$ is shown in Figure 4(h).

Another invariant associated with technological feasibility of 2-structures is *cycle number*, *cycl*(L), which is the number of cyclic components in the 2-structure L. Note that any cyclic component corresponds to a contractible cycle in the torus diagram (a cycle that does not coil the torus). The chain-mail-type structure (j) in Figure 4 is the only example of a 2-structure with $cycl \neq 0$. Chain-mail-type structures cannot be manufactured using any of the known production methods of textile technology.

A more powerful invariant can be obtained which combines the properties of μ, *ax*, and *cycl* invariants by taking into account the directions of all axes of a 2-structure and the number of components along each axis.

Let us assume that the periodicity vectors of components of 2-structure L are:

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_\mathrm{N}\}, \text{ where } \boldsymbol{v}_i = (m_i, n_i), m_i \in \mathbf{Z}, n_i \in \mathbf{Z}_+ .$$

We will suppose that vectors are listed in counter-clockwise order starting from the left–right direction. For example, the set of vectors in Figure 7(a) is:

$$\{(-5, 4), (-7, 5), (-6, 4), (-6, 3), (-3, 1)\}.$$

The set of vectors $\boldsymbol{V} = \{\boldsymbol{v}_i\}$ by itself is not an invariant because it depends on torus twists; therefore in order to

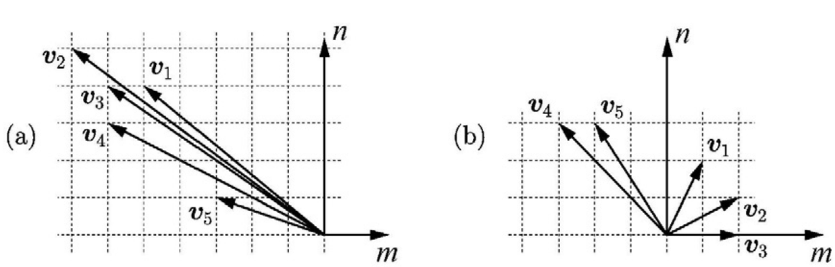


Figure 7 A set of vectors (a) and its canonical form (b).

define an invariant, it is necessary to transform the set V to a canonical form using twists in some way.

Transformation of a vector $\boldsymbol{v}$ by a sequence of torus twists can be represented as a product of the vector $\boldsymbol{v}$ by a unitary matrix $\boldsymbol{U}$ with integer elements ($\boldsymbol{U} \in \mathrm{SL}_2(\boldsymbol{Z})$):

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\boldsymbol{U} \quad \text{or} \quad (\tilde{m}, \tilde{n}) = (m, n)\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$ad - bc = 1, \quad a, b, c, d \in \boldsymbol{Z},$$

where $\mathrm{SL}_2(\boldsymbol{Z})$ is the special linear group of all integer matrices with determinant one.

For example, single positive meridional twist, single negative meridional twist, single positive longitudinal, and single negative longitudinal twists (see [1, Section 3]) are represented, respectively, by matrices

$$\boldsymbol{M}^+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{M}^- = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{L}^+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{L}^- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The canonical form of a set of vectors $\boldsymbol{V} = \{\boldsymbol{v}_i\}$ can be defined in the following way.

Let us associate a quadratic functional Q with the set $\boldsymbol{V} = \{\boldsymbol{v}_i\} = (m_i, n_i)$:

$$Q(V) = \sum_{i}^{N} |\tilde{v}_i|^2 = \sum_{i-1}^{N} \left(m_i^2 + n_i^2\right).$$

A twist sequence defined by a unitary matrix $\boldsymbol{U}$ converts the vectors $\{\boldsymbol{v}_i\}$ to a set $\tilde{\boldsymbol{V}} = \{\tilde{\boldsymbol{v}}_i\}$, $\tilde{\boldsymbol{v}}_i = \boldsymbol{v}_i\boldsymbol{U}$ and the value of the functional changes to

$$Q(\tilde{V}) = \sum_{i}^{N} |\tilde{v}_i|^2 = \sum_{i-1}^{N} (\tilde{m}_i^2 + \tilde{n}_i^2) = \sum_{i=1}^{N} \boldsymbol{v}_i \boldsymbol{U}\boldsymbol{U}^T \boldsymbol{v}_i^T.$$

We can now set the problem as follows: for a given set of vectors $\boldsymbol{V} = \{\boldsymbol{v}_i\}$ find a twist sequence (a unitary matrix $\boldsymbol{U}$) that minimizes the value of the functional Q:

$$Q(\tilde{V}) = \sum_{i}^{N} |\tilde{v}_i|^2 = \sum_{i=1}^{N} \boldsymbol{v}_i \boldsymbol{U}\boldsymbol{U}^T \boldsymbol{v}_i^T \rightarrow \min, \quad \boldsymbol{U} \in \mathrm{SL}_2(\boldsymbol{Z}). \quad (2)$$

It can be shown (given some additional conditions) that the Equation 2 always has a unique non-trivial solution $\boldsymbol{U}_0$.

The *canonical form* of a set of vectors $\boldsymbol{V} = \{\boldsymbol{v}_i\}$ is the set $V_0 = \{\boldsymbol{v}_i\boldsymbol{U}_0\}$ where the matrix $\boldsymbol{U}_0$ is the solution of equation (2). For example, the canonical form of the set in Figure 7(a) is the set

$$\{(0, 2), (2, 1), (1, 2), (-2, 3), (-3, 3)\}$$

shown in Figure 7(b); the values of the functional Q are 222 and 45, respectively.

The canonical form of the set of periodicity vectors of a 2-structure L is an invariant which will be denoted as $vec(L)$. For the 2-structures in Figure 4 the corresponding values of the *vec* invariant are presented in Table 1.

Linking Number

In knot theory one of the important characteristics of a link with several components is the pair-wise linking number.

Let us consider an oriented link $L = K_1 \cup K_2 \cup \ldots$ consisting of components $K_1, K_2, \ldots$ and let D be an arbitrary diagram of the link, and D_i be the part of D corresponding to the component K_i.

Linking number, $lk(K_i, K_j)$, of two components, K_i and K_j, $i \neq j$, is defined as the sum of crossing signs (see Figure 5) over all crossings in which sub-diagrams D_i and D_j cross each other:

$$lk(K_i, K_j) = \sum_{c \in D_i, D_j} \varepsilon_c.$$

For the general links the value of $lk(K_i, K_j)$ is always even whereas for 2-structures that is not always the case.

A (symmetric) $N \times N$ matrix, $\boldsymbol{LK}$, of linking numbers $lk(K_i, K_j)$ can be associated with a link (2-structure) L with N components. Matrix $\boldsymbol{LK}$, if it is considered up to simultaneous permutations of its rows and columns, is an invariant of oriented links.

It is also possible to define *total linking number* for a non-oriented link (2-structure) L as follows:

$$lks(L) = \sum_{i<j} |lk(K_i, K_j)|.$$

The values of the *lks* invariant for the 2-structures in Figure 4 are presented in Table 1.

Note that invariants related to linking number are relatively weak. A good example is that for both the Borromean link and a trivial three-component link matrix $\boldsymbol{LK}$ is a null matrix and thus does not discriminate between these two links (Figure 8(a)). Similar examples can be given for 2-structures (see Figure 8(b)).

Unknotting Number

Switching one of the crossings in a knot diagram, Figure 5, may change the topological type of the knot. The question

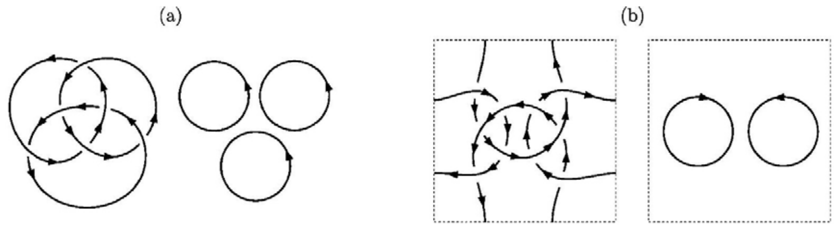


Figure 8 Borromean link (a) and a 2-structure (b) that cannot be discriminated from trivial link using linking numbers

is, how many crossings should be switched to untangle the knot?

The *unknotting number*, $u(K)$, of a knot K is the least number of crossings in a diagram of knot K, which must be switched, for the knot to be transformed into an unknot.

Similar to crossing number, this numerical invariant appears to be simple but it is difficult to calculate. The unknotting number is a weak invariant; the value of u may be the same for very different knots. Thus, it cannot be reliably used for the identification of different knots. However, for 2-structures it is possible to introduce several invariants similar to the unknotting number that can be used to characterize structural stability of textiles.

Let us define *disintegrating number*, $dis(L)$, of 2-structure L as a minimum number of crossings in a diagram of the structure L which must be switched for the structure to lose integrity as a doubly-periodic structure, or to disjoin at least one component from the structure.

Let *splitting number*, $spl(L)$, of 2-structure L be a minimum number of crossings which must be switched in order to disjoin all components of the structure whilst the components themselves may remain knotted.

Finally, the total unknotting number, $tu(L)$, is the minimal number of crossings which must be switched in order to disjoin all components of the structure and to unknot all of them.

All these invariants are very difficult to calculate in the general case. However, for the simplest structures such calculations can be done manually. For example, it is easy to calculate the invariants defined above for the 2-structures in Figure 4 (see Table 1).

Numerical invariants considered in this section do not resolve the classification problem for 2-structures because they do not characterize structures in a unique way, even if they are used together. For example, it can be shown that the values of all the numerical invariants described above derived for all $1/n$ and $n/1$ twills will be equal to those for satins/sateens with $(n + 1) \times (n + 1)$ repeat ($n \geq 4$), i.e.,

Figure 9 Unknotting number of this knot is 1.

$cn = n + 1$, $\mu = 2$, $lks = n - 1$, $ax = 2$, $dis = 1$, $spl = 1$, and $tu = 1$. Some of numeric invariants, such as unknotting number and crossing number, are very difficult to compute and this is another disadvantage.

In the next section we consider a more powerful class of invariants.

Polynomial Invariants

Polynomial invariants are quite simple and yet sufficiently powerful. They play an important role in knot theory and its applications; see, for example, [2–4, 6]. Unlike numerical invariants for which their value is a number, a polynomial invariant is a function that associates a (Laurent) polynomial with each individual knot or link (Figure 10).

The first polynomial invariant for knots was introduced in 1928 by Alexander [2–4]. The Alexander polynomial remained the only invariant of this kind until 1985 when Vaughan Jones introduced his famous invariant, which in some cases is more powerful than the Alexander polynomial. Jones' polynomial discriminates all knots with a crossing number up to nine. Since then, using Jones' 'skein-relations' approach, many more polynomial invariants have been discovered; the most famous are the Conway, Kauffman, and HOMFLY polynomials (see [2, 3]).

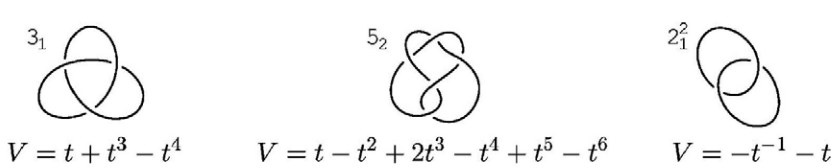


Figure 10 Jones polynomials of some knots and links.

Figure 11 Two types of splitting.

A very simple and pictorial interpretation of the construction procedure for Jones-type polynomials was proposed by Louis Kauffman in terms of *diagram states* [3].

The next section outlines Kauffman's method of constructing the polynomial invariant [2] and proposes a new polynomial invariant for 2-structures based on Kauffman's ideas.

State-sum Model of Diagrams for Links and 2-structures

Let us consider an arbitrary diagram D of a link (2-structure) with n crossings. *Splitting* of a crossing is a local operation (*surgery*) which is applied within a small neighbourhood of the crossing (Figure 11). A splitting results in one of two diagrams which are different from the original diagram only within the neighbourhood of the given crossing.

Let us denote two different splitting operations as A and B. It is essential in the splitting definition that the crossings are considered to be oriented in such a way that the upper strand passes from bottom left to top right.

We shall number the crossings of the diagram in an arbitrary way and apply one of the splittings, A or B, to each crossing. The overall operation can be defined by a binary string σ of length n consisting of letters A and B:

$$\sigma = \underbrace{ABABB\ldots BA}_{n}.$$

This string can be interpreted as an instruction to apply splitting A to crossing No 1, splitting B to crossing No 2 and so on.

It can be said that string σ defines a *state* of the diagram D; this will be denoted as D_σ. Obviously, there are 2^n different states of a diagram D with n crossings.

It can be seen from Figure 12(a) that the diagram of a link in a state σ is a disjoint union of a number of non-knotted and non-intersecting closed curves (a trivial link); each of the components is isotopic to a circle O:

$$D_\sigma = \underbrace{\mathrm{OOO\ldots O}}_{\gamma_\sigma} = \bigcup_{\gamma_\sigma} \mathrm{O} \tag{3}$$

Here D_σ is the diagram D in state σ; γ_σ is the number of components of diagram D_σ.

In the case of 2-structures, any state of a torus diagram corresponds to a set of non-intersecting curves on the torus. But, in contrast to plane diagrams, a trivial torus diagram, together with circles, may also contain a set of non-intersecting closed coils (components) wound around the torus. Such a set will be referred to as a *winding* and denoted as (m, n), where m and n are the number of intersections of the windings with a torus meridian and longitude, respectively. For example, the state of the structure presented in Figure 12(b) contains two circles and a (0, 1)-winding.

The windings have simple properties, as below [7]:

1. No winding can contain components which have different 'slope'.
2. If $g = \gcd(m, n)$ then there are exactly g identical components in the winding.
3. The number of winding components does not change in torus twists.

Similar to equation (3), the state σ of a torus diagram D can be represented as follows:

$$D_\sigma = (m, n) \bigcup \{\underbrace{\mathrm{OOO\ldots O}}_{\gamma_\sigma}\} \tag{4}$$

where γ_σ may be equal to 0 if there are no circles in the diagram D_σ.

If there are no windings in a given state of the diagram, then we can assume that one of the circles is a 'null winding' denoted as (0, 0). In this way, equation (4) will have the same form for all possible states of a torus diagram.

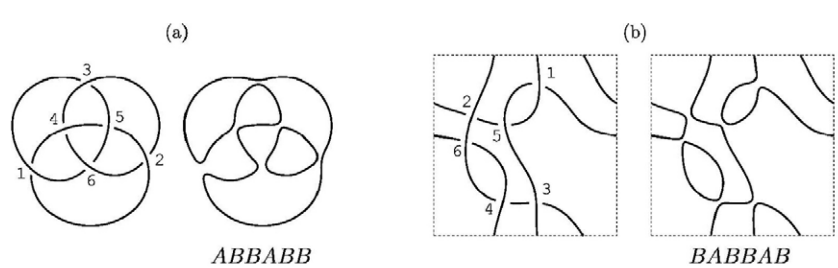


Figure 12 On the definition of the state of a diagram.

Kauffman Bracket

Kauffman's approach is based on the idea of a successive reduction of the number of crossings in the original diagram of a link (2-structure) using splitting operations (Figure 11).

Let us associate a polynomial $\langle D \rangle$, the *Kauffman bracket*, of three variables a, b, and c, to every diagram D according to the relations

(5)

$$\langle D \cup O \rangle = c \langle D \rangle. \tag{6}$$

Equation (5), which is known as the Kauffman *skein-relation*, relates the values of the Kauffman bracket $\langle \cdot \rangle$ on three diagrams which are different only within the neighbourhood of one crossing. It follows from equation (6) that adding an isolated circle O to a diagram D results in multiplying the value of the bracket by a variable c.

Let us assume that the value of the Kauffman bracket for an isolated circle $\langle O \rangle$ is equal to unity:

$$\langle O \rangle = 1 \tag{7}$$

Then, according to equation (6), if diagram D in state σ contains γ_σ circles, then the value of the bracket is:

$$\langle D_\sigma \rangle = c^{\gamma_\sigma - 1}. \tag{8}$$

Skein-relation in equation (5) enables the value of the Kauffman bracket for a diagram with n crossings to be expressed through its values on diagrams with $n - 1$ crossings. Therefore, using equation (5) recursively the value of $\langle D \rangle$ can be expressed through the bracket values on all 2^n states of the original diagram D:

$$\langle D \rangle = \sum_n w_\sigma \langle D_\sigma \rangle, \tag{9}$$

where w_σ is a "weight" of the state σ. The summation is carried out over all 2^n states of D.

It can be seen that the weight w_σ of the state σ depends only on the number of times the bracket $\langle D_\sigma \rangle$ has been multiplied by the variable a or b during the recursion, i.e. on the number of times the letter A or B appears in the string σ. Taking into account equations (8) and (9), the following explicit formula for $\langle D \rangle$ can be obtained:

$$\langle D \rangle = \sum_\sigma a^{\alpha_\sigma} b^{\beta_\sigma} c^{\gamma_\sigma - 1}, \tag{10}$$

where α_σ and β_σ are the number of splits of type A and B, respectively, in the state σ.

It has been shown in the section *State-sum Model of Diagrams for Links and 2-structures* above that in the case of 2-structures a torus diagram D in a state σ may contain a winding (m_σ, n_σ). Thus, it is necessary to define the values of the Kauffman brackets for windings.

Denoting $\langle m_\sigma, n_\sigma \rangle = \langle (m_\sigma, n_\sigma) \rangle$ and applying the same method as has been used above for general links, we obtain an explicit formula for the Kauffman bracket for torus diagrams:

$$\langle D \rangle = \sum_\sigma a^{\alpha_\sigma} b^{\beta_\sigma} c^{\gamma_\sigma - 1} \langle m_\sigma, n_\sigma \rangle, \tag{11}$$

Equation (11) may contain equal multipliers $\langle m_\sigma, n_\sigma \rangle$; collecting similar terms yields

$$\langle D \rangle = \sum_k Q_k(a, b, c) \langle m_k, n_k \rangle\,, \tag{12}$$

where $Q_k(a, b, c)$ are polynomials.

Note that multipliers $\langle m_k, n_k \rangle$ in equation (12) can be considered as additional variables.

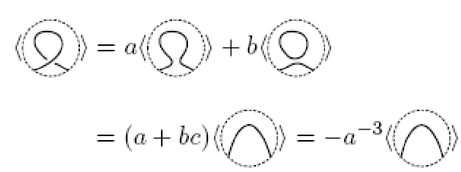
Polynomial Invariant of 2-structures

For the polynomials defined by equations (10) and (11) to be invariants for links and 2-structures respectively it is necessary that they be invariants with respect to Reidemeister moves Ω_1, Ω_2, and Ω_3 and, in the case of 2-structures, also with respect to torus twists.

It can be shown that using freedom in choosing variables a, b, and c, the invariance of the Kauffman bracket with respect to Ω_2 and Ω_3 can be provided by assuming that (see, for example, [2])

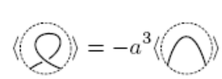
$$b = a^{-1}, \quad c = -(a^2 + a^{-2}). \tag{13}$$

However, the application of Ω_1 to diagram D results in multiplying the polynomial $\langle D \rangle$ by $-a^{\pm 3}$:



(14)

and, in the same way,



(15)

In order to define an isotopy invariant it is necessary to introduce the *self-writhe* number $sw(D)$ of diagram D. Let a link (2-structure) L consist of several components K_i which can be oriented in an arbitrary way and let D_i be the part of diagram D that corresponds to the component K_i. The self-writhe number for the sub-diagram D_i can be defined as the sum of crossing signs over all self-intersections of the sub-diagram:

$$sw(D_i) = \sum_{c \in D_i} \varepsilon_c.$$

For the whole diagram D the self-writhe number is defined by the formula:

$$sw(D) = \sum_i sw(D_i).$$

Note that the self-writhe number $sw(D)$ is not an invariant of the link L (it changes in Ω_1); the value of $sw(D)$ does not depend on the orientation of the diagram components.

Substituting equation (13) into equations (10) and (11), and multiplying the resultant polynomial of variable a by the coefficient $(-a)^{-3sw(D)}$, a *normalized Kauffman polynomial* of the link L [3] can be obtained:

$$X(L) = (-a)^{-3sw(D)} \sum_{\sigma} a^{\alpha_\sigma - \beta_\sigma}(-a^2 - a^{-2})^{\gamma_\sigma - 1}. \qquad (16)$$

In the same way, from equations (11) and (12) a Kauffman-type polynomial for 2-structures can be obtained in the form:

$$X(L) = (-a)^{-3sw(D)} \sum_{\sigma} a^{\alpha_\sigma - \beta_\sigma}(-a^2 - a^{-2})^{\gamma_\sigma} \langle m_\sigma, n_\sigma \rangle, \qquad (17)$$

$$X(L) = \sum_k P_k(a) \langle m_k, n_k \rangle \qquad (18)$$

where $P_k(a) = (-a)^{-3sw(D)} Q_k(a, a^{-1}, -a^2 - a^{-2})$ (see equation (12)) are some polynomials of variable a.

Equations (17) and (18) define a polynomial invariant with respect to Reidemeister moves. Therefore, it is an isotopy invariant for 2-structures. But equations (17) and (18) still depend on the choice of the unit cell because multipliers $\langle m_k, n_k \rangle$, associated with windings, depend on torus twists.

There are various ways to obtain an invariant with respect to torus twists from equations (17) and (18). Paper [7] suggested introducing an additional variable t by assuming that for any winding (m, n) the bracket $\langle m, n \rangle = t^g$, where $g = \gcd(m, n)$ is the number of winding components, which obviously does not depend on torus twists. However, this approach leads to loss of important information about windings as it does not discriminate windings with different slope (e.g. a torus meridian and a longitude are considered to be identical).

The most powerful invariant based on equation (18) can be constructed by considering the set of windings $\{(m_k, n_k)\}$ as a set of integer vectors $\{\boldsymbol{v}_k\}$, similar to periodic vectors in the section *Component Number and Related Invariants*, above (see Figure 7). Then all that is necessary is to transform the set $\{\boldsymbol{v}_k\}$ to canonical form by using torus twists as described under *Component Number and Related Invariants*.

Thus, equation (18), where the set of windings $\{(m_k, n_k)\}$ is in canonical form, defines a Kauffman-type invariant of 2-structures.

Examples

This section gives examples of the calculating procedure of $X(\cdot)$ invariant for some simple structures in Figure 4; "winding multipliers" are highlighted in bold.

The formulae required for the calculations are as follows:


(19)

$$\langle D \cup O \rangle = -(a^2 + a^{-2}) \langle D \rangle, \qquad (20)$$

$$\langle (m_i, n_i) \rangle = (\boldsymbol{\mu}_j, \boldsymbol{v}_j), \qquad (21)$$

$$X(L) = (-a)^{-3sw(D)} \langle D \rangle. \qquad (22)$$

Here D is an arbitrary diagram of the 2-structure L; the set $\{(\mu_j, \nu_j)\}$ is the canonical form of the set of windings $\{(m_i, n_i)\}$.

The following formulae are also useful for simplifying the computation:


(23)

Example 1

Let us calculate the value of polynomial invariant $X(\cdot)$ for the plain weave (2-structure in Figure 4(b)), which has two crossings. First of all, let us number the crossings in the diagram of the plain weave in Figure 4(b) in an arbitrary way and then use Kauffman skein relations in order to calculate the polynomial invariant for the plain weave. Let us apply splittings A and B described under *State-sum Model of Diagrams for Links and 2-structures* above to crossing No 1. This, according to equation (19), enables the value of Kauffman bracket for a plain weave diagram to be expressed through its values on two new diagrams containing one crossing each:

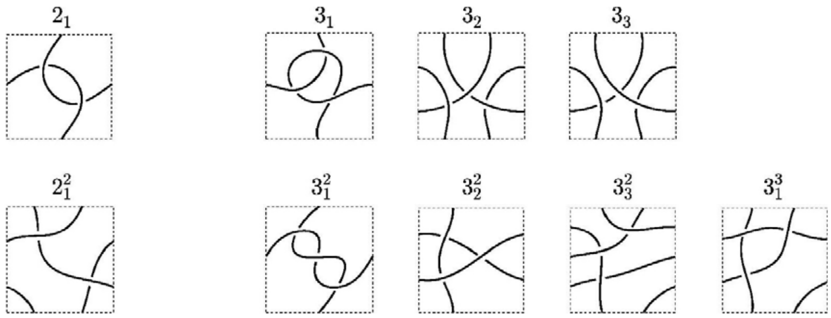
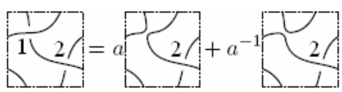
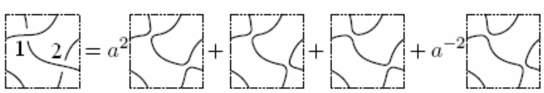


Figure 13 2-structures with 2 and 3 crossings.



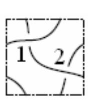
The application of splittings A and B to crossing No 2 in these new diagrams leads to four new diagrams with no crossings and the formula as follows:



(24)

It can be seen that the first and the fourth patterns on the right in equation (24) are equivalent to an isolated circle and thus they can be considered as null windings introduced in the section *State-sum Model of Diagrams for Links and 2-structures*; the winding multiplier for these diagrams is $(\mathbf{0}, \mathbf{0})$. In a similar way, the second pattern is equivalent to two torus meridians whereas the third pattern is equivalent to two torus parallels; their winding multipliers are $(\mathbf{0}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{0})$, respectively.

Substituting this into equation (24) yields:



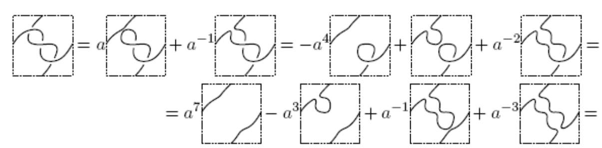
$$= a^2(\mathbf{0},\mathbf{0}) + (\mathbf{0},\mathbf{2}) + (\mathbf{2},\mathbf{0}) + a^{-2}(\mathbf{0},\mathbf{0}).$$

The two components of this structure have no self-intersections, so $sw(D) = 0$; the set of windings $\{(0, 0), (2, 0), (0, 2)\}$ is already in canonical form. Thus, the value of the invariant is

$$X = (a^{-2} + a^2)(\mathbf{0}, \mathbf{0}) + (\mathbf{2}, \mathbf{0}) + (\mathbf{0}, \mathbf{2}).$$

Example 2

The second example considers double wire netting, Figure 13 (2-structure 3_1^2). This example illustrates the case where calculations can be simplified by using equation (23):



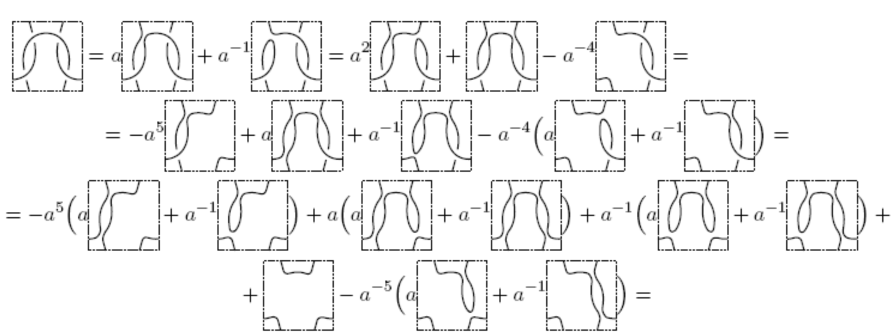
$$= a^7(\mathbf{1}, \mathbf{1}) - a^3(\mathbf{1}, \mathbf{1}) + a^{-1}(\mathbf{1}, \mathbf{1}) + a^{-3}(\mathbf{-1}, \mathbf{1}).$$

The self-writhe number for this structure also is equal to zero. The winding set $\{(1, 1), (-1, 1)\}$ does not require transformation to canonical form. The polynomial is:

$$X = (a^{-1} - a^3 + a^7)(\mathbf{1}, \mathbf{1}) + a^{-3}(\mathbf{-1}, \mathbf{1}).$$

Example 3

Finally, let us calculate polynomial invariant for single jersey fabric (Figure 4(c)).



$$= -a^6(\mathbf{1}, \mathbf{2}) - a^4(\mathbf{1}, \mathbf{0}) + a^2(\mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0}) + a^{-2}(\mathbf{-1}, \mathbf{2}) + (\mathbf{1}, \mathbf{0}) - a^{-4}(\mathbf{1}, \mathbf{0}) - a^{-6}(\mathbf{-1}, \mathbf{2}).$$

Although in this case there are self-intersection points the self-writhe number of the whole diagram is 0.

We again do not need transformation to canonical form. The polynomial of single jersey is:

$$X = -(a^{-4} - 3 + a^{4})(\mathbf{1}, \mathbf{0}) + (a^{2} - a^{6})(\mathbf{1}, \mathbf{2})$$

$$-(a^{-6} - a^{-2})(\mathbf{-1}, \mathbf{2}).$$

Note that the above computations did not require the application of the ‘complicated’ equation (20) due to the simplifications of the diagram achieved by using equation (23).

Topological Invariants of Textile Structures

This section provides tables of invariants considered above for some simple 2-structures and also for some basic woven and knitted textiles.

As was emphasised above, the main purpose of invariants is topological classification of 2-structures.

Let us begin with the full table of 2-structures with crossing number 2 and 3 (Figure 13). It is necessary to emphasize that this table contains all topologically different structures with $cn \leq 3$; it can be proven that there are no other structures of the given complexity.

The first row in Figure 13 shows one-component structures; the second row presents multi-component 2-structures. The denotation of 2-structures in Figure 13 follows a commonly accepted system for knots and links, i.e., knots are denoted as M^n and links as $M_n^{\,k}$, where M is crossing number, n is the ordinal number of knot (or link) within the group with M crossings, and k is the component number.

Table 2 Polynomial invariants for structures in Figure 13.

L	$X(L)$
2_1	$a^{-4}(\mathbf{1}, \mathbf{1}) - (a^{-10} - a^{-6})(\mathbf{-1}, \mathbf{1})$
$2_1^{\,2}$	$(a^{-2} + a^{2})(\mathbf{0}, \mathbf{0}) + (\mathbf{2}, \mathbf{0}) + (\mathbf{0}, \mathbf{2})$
3_1	$-(a^{4} - a^{8})(\mathbf{1}, \mathbf{1}) + (a^{-2} - a^{2})(\mathbf{-1}, \mathbf{1})$
3_2	$(a^{-8} - a^{-4} - 1)(\mathbf{0}, \mathbf{0}) + (a^{-6} - a^{-2}(\mathbf{2}, \mathbf{0}) - a^{-2})(\mathbf{0}, \mathbf{2})$
3_3	$(a^{-12} - 2a^{-8})(\mathbf{0}, \mathbf{0}) - a^{-6}(\mathbf{2}, \mathbf{0}) + (a^{-14} - a^{-10})(\mathbf{0}, \mathbf{2})$
$3_1^{\,2}$	$(a^{-1} - a^{3} + a^{7})(11 + a^{-3})(\mathbf{-1}, \mathbf{1})$
$3_2^{\,2}$	$-a^{2}(\mathbf{2}, \mathbf{1}) - (a^{4} - a^{8})(\mathbf{0}, \mathbf{1}) - a^{2}(\mathbf{-2}, \mathbf{1})$
$3_3^{\,2}$	$a(\mathbf{3}, \mathbf{0}) - (a^{-3} + a + a^{5})(\mathbf{1}, \mathbf{0}) + a^{-1}(\mathbf{1}, \mathbf{2})$
$3_1^{\,3}$	$(a^{-3} + 2a - a^{5})(\mathbf{0}, \mathbf{0}) + a^{-1}(\mathbf{2}, \mathbf{0}) + a^{-1}(\mathbf{2}, \mathbf{2}) + a^{-1}(\mathbf{0}, \mathbf{2})$

It can be seen that most patterns in Figure 13 represent commonly known structures that can be manufactured using one of the methods employed in the textile industry. For example, 2_1 is wire netting, $2_1^{\,2}$ is plain weave, $3_1^{\,3}$ is tri-axial fabric, and $3_3^{\,2}$ is 1/2 twill. Structures 3_1, 3_2, and $3_1^{\,2}$ have been considered by Emery as examples of primary structures (see [8], pages 31, 30, and 62, respectively).

Table 2 gives the values of polynomial invariant for the structures in Figure 13.

It can be seen that all the structures in Figure 13 are discriminated by the polynomial invariant constructed under *Polynomial Invariant of 2-structures* above.

Table 3 contains numerical invariants introduced under *Basic Numerical Invariants* above for 2-structures in Figure 13.

Now let us use invariants for the analysis of some conventional textile fabrics. Figure 14 shows some of the simplest woven structures; for each structure we give a commonly-used representation of the weaving repeat and also select a

Table 3 Numerical invariants for structures in Figure 13.

L	cn	μ	lks	ax	dis	spl	tu	vec
2_1	2	1	2	1	1	1	1	{(1, 0)}
$2_1^{\,2}$	2	2	0	2	1	1	1	{(1, 1), (–1, 1)}
3_1	3	1	3	1	1	1	1	{(1, 0)}
3_2	3	1	1	1	1	1	1	{(2, 0)}
3_3	3	1	1	1	1	1	1	{(2, 0)}
$3_1^{\,2}$	3	2	3	2	1	1	1	{(1, 0), (0, 1)}
$3_2^{\,2}$	3	2	1	2	1	1	1	{(2, 0), (0, 1)}
$3_3^{\,2}$	3	3	1	2	1	1	1	{(2, 1), (–1, 1)}
$3_1^{\,3}$	3	3	3	3	1	1	1	{(1, 0), (1, 1), (0, 1)}

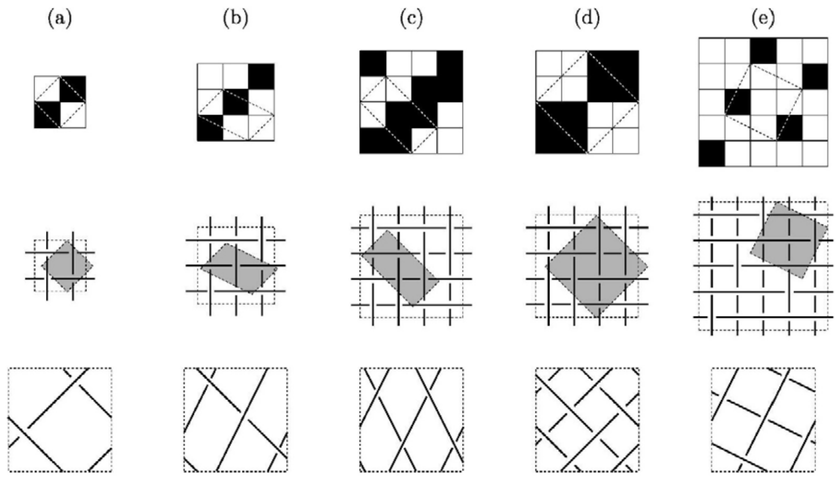


Figure 14 Woven fabrics: weave diagram, weave repeat, unit cell. Plain weave (a), 1/2 twill (b), 2/2 twill (c), hopsack weave 2/2 (d), 5/3 sateen (e).

Table 4 Polynomial invariants for woven fabrics (a) to (e) in Figure 14.

L	$X(L)$
(a)	$(a^{-2} + a^{2})(\mathbf{0}, \mathbf{0}) + (\mathbf{2}, \mathbf{0}) + (\mathbf{0}, \mathbf{2})$
(b)	$a(\mathbf{3}, \mathbf{0}) - (a^{-3} + a + a^{5})(\mathbf{1}, \mathbf{0}) + a^{-1}(\mathbf{1}, \mathbf{2})$
(c)	$(\mathbf{4}, \mathbf{0}) - (a^{-4} + 2 + a^{4})(\mathbf{2}, \mathbf{0}) + (\mathbf{0}, \mathbf{2})$
(d)	$-(a^{-10} + a^{-6} - 2a^{-2} - 2a^{2} + a^{6} + a^{10})(\mathbf{0}, \mathbf{0}) + (\mathbf{4}, \mathbf{0}) - (a^{-8} + 2 + a^{8})(\mathbf{2}, \mathbf{0}) + 2(\mathbf{2}, \mathbf{2}) + (\mathbf{0}, \mathbf{4}) - (a^{-8} + 2 + a^{8})(\mathbf{0}, \mathbf{2}) + 2(\mathbf{-2}, \mathbf{2})$
(e)	$a^{-3}(\mathbf{3}, \mathbf{1}) - (a^{-5} - 2a^{-1} + a^{3})(\mathbf{1}, \mathbf{1}) + a^{3}(\mathbf{-1}, \mathbf{3}) - (a^{-3} - 2a + a^{5})(\mathbf{-1}, \mathbf{1})$

Table 5 Polynomial invariants for knitted fabrics in Figure 15.

L	$X(L)$
(a)	$-(a^{-4} - 3 + a^{4})(\mathbf{1}, \mathbf{0}) + (a^{2} - a^{6})(\mathbf{1}, \mathbf{2}) - (a^{-6} - a^{-2})(\mathbf{-1}, \mathbf{2})$
(b)	$(a^{-12} - a^{-8} - 5a^{-4} + 11 - 5a^{4} - a^{8} + a^{12})(\mathbf{1}, \mathbf{0}) - (a^{-4} - 2 + a^{4})(\mathbf{1}, \mathbf{4}) + (a^{-10} - 4a^{-6} + 3a^{-2} + 3a^{2} - 4a^{6} + a^{10})(\mathbf{1}, \mathbf{2})$ $- (a^{-4} - 2 + a^{4})(\mathbf{-1}, \mathbf{4}) + (a^{-10} - 4a^{-6} + 3a^{-2} + 3a^{2} - 4a^{6} + a^{10})(\mathbf{-1}, \mathbf{2})$
(c)	$(2a^{-10} - 6a^{-6} + 4a^{-2} + 4a^{2} - 6a^{6} + 2a^{10})(\mathbf{0}, \mathbf{0}) + (a^{-12} - 2a^{-8} + a^{-4} + a^{4} - 2a^{8} + a^{12})(\mathbf{2}, \mathbf{0}) - (a^{-4} - 2 + a^{4})(\mathbf{2}, \mathbf{2})$ $+ (a^{-8} - 4a^{-4} + 7 - 4a^{4} + a^{8})(\mathbf{2}, \mathbf{0}) - (a^{-4} - 2 + a^{4})(\mathbf{-2}, \mathbf{2})$
(d)	$-(a^{-4} - 2 + a^{4})(\mathbf{3}, \mathbf{0}) + (a^{-8} - 2a^{-4} + 3 - 2a^{4} + a^{8})(\mathbf{1}, \mathbf{0}) - (a^{-6} - a^{-2} - a^{2} + a^{6})(\mathbf{1}, \mathbf{2}) - (a^{-6} - a^{-2} - a^{2} + a^{6})(\mathbf{-1}, \mathbf{2})$
(e)	$-(a^{-4} - 2 + a^{4})(\mathbf{3}, \mathbf{0}) + (a^{-8} - 2a^{-4} + 3 - 2a^{4} + a^{8})(\mathbf{1}, \mathbf{0}) - (a^{-6} - a^{-2} - a^{2} + a^{6})(\mathbf{1}, \mathbf{2}) - (a^{-6} - a^{-2} - a^{2} + a^{6})(\mathbf{-1}, \mathbf{2})$

unit cell. It is evident that in all cases the traditional representation is not minimal.

Table 4 gives values of polynomial invariants for the woven fabrics represented in Figure 14. It can be noticed that the complexity of the polynomial is related to the crossing number of the 2-structure. Numerical invariants calculated for the structures in Figure 14 are presented in Table 6 below.

Let us consider some knitted structures presented in Figure 15; the corresponding polynomials are tabulated in Table 5.

It is interesting to note that the structures (d) and (e) in Figure 15 have identical polynomials and identical numerical invariants (see Table 6). To the best of our knowledge, this is the only example of different structures with identical values of this polynomial invariant. Thus, even using numerical and polynomial invariants together it cannot be proved that these structures are topologically different. This illustrates the case when more powerful invariants are needed than those considered in this paper.

Finite-type invariants are the most powerful at present. The application of finite-type invariants to 2-structures has

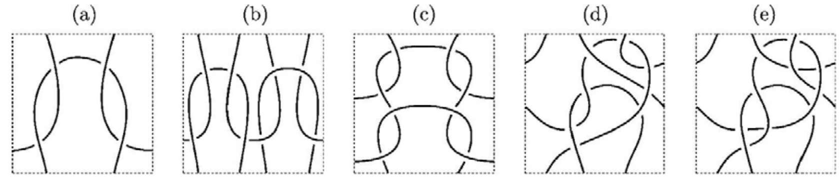


Figure 15 Knitted fabrics: single jersey (a), 1 × 1 rib (b), purl (c), tricot with opened loops (d), tricot with closed loops (e)

Table 6 Numerical invariants for woven and knitted fabrics.

Structure	*cn*	μ	*lks*	*ax*	*dis*	*spl*	*tu*	*vec*
Plain weave	2	2	0	2	1	1	1	(1, 1), (–1, 1)
1/2 twill	3	2	1	2	1	1	1	(2, 1), (–1, 1)
2/2 twill	4	2	0	2	2	2	2	(1, 2), (–1, 2)
2/2 hopsack	8	4	0	2	2	4	4	(1, 1), (1, 1), (–1, 1), (–1, 1)
5/3 sateen	5	2	3	2	1	1	1	(2, 1), (–1, 2)
Single jersey	4	1	0	1	2	2	2	(1, 0)
1 × 1 rib	8	1	0	1	4	4	4	(1, 0)
Purl	8	2	0	1	2	2	2	(1, 0), (1, 0)
Tricot opened	8	1	0	1	2	2	2	(1, 0)
Tricot closed	10	1	0	1	2	2	2	(1, 0)

been suggested in [9], where it has been shown that tricot with opened loops and tricot with closed loops can be discriminated by finite-type invariants.

Conclusion

This paper introduced numerical and polynomial invariants which can be applied to doubly-periodic structures, in particular to textile structures, for their topological classification. It has been shown that the numerical invariants, such as crossing number, linking number, the number of components, and unknotting number, cannot unambiguously distinguish structurally different textiles even if used in combination. It has been noticed that there are no general methods for calculating some numerical invariants despite their apparent simplicity. This in particular concerns crossing number and unknotting number, which are very difficult to compute. However, numerical invariants such as the number of cyclic components and disintegration number may be useful for the analysis of technological feasibility and structural stability of textiles.

A new Kauffman-type polynomial invariant for 2-structures has been suggested which is more powerful than earlier invariants [7]. The construction of the polynomial invariant is based on Kauffman's "state-sum' approach, which is extended to the case of torus diagrams. The independence of the invariant with respect to the choice of the unit cell (torus twists) can be achieved by transforming the set of windings to a canonical form.

Values of the proposed polynomial invariant have been calculated for a set of typical textile structures and have shown its ability to recognize different structures. This polynomial can be used for the classification of topologically different textile structures, which will be based on crossing number as the main criterion. It can be noticed that the conventional classification of weaves is indirectly based on crossing number because it takes into account the number of warp and weft threads in the repeat, see for example [10], although weave repeats considered in this classification are not always minimal unit cells. In this respect it is interesting to note that many textile structures which traditionally have been considered as completely different will fall in the same class of structures because they have the same crossing number. For example, it can be seen from Table 4 that $cn = 4$ for 2/2 twill and for single jersey, $cn = 8$ for 2/2 hopsack, 1 × 1 rib, purl, and tricot with opened loops.

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