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Whitney-Cartan Product Formulae*

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Introduction

We discuss in this paper some ideas which generalize the Whitney product formula for vector bundles and the Cartan product formula for the Steenrod operations.

Let X be a complex and let ξ and η be vector bundles over X. If w_i, $i \geqq 0$, denotes in general the i^{th} Stiefel-Whitney class, the Whitney formula then is:

$$w_t(\xi \oplus \eta) = \sum_{r+s=t} w_r(\xi) \cup w_s(\eta). \tag{A}$$

In Part I we consider the problem: Given a higher order characteristic class (see below), determine the "Whitney formula" satisfied by the class.

Suppose now that u and v are (mod p) cohomology classes of X. If P^i, $i \geqq 0$, denotes the i^{th} mod p Steenrod reduced power the Cartan formula then is:

$$\mathrm{P}^t(u \cup v) = \sum_{r+s=t} \mathrm{P}^r(u) \cup \mathrm{P}^s(v). \tag{B}$$

In Part II we consider the problem: Given a higher order cohomology operation, determine the "Cartan formula" satisfied by the operation.

Before discussing the Whitney and Cartan formulae separately, we make some general remarks which apply to both. We work in the category of spaces with basepoint (denoted always by $*$), so maps and homotopies preserve basepoints. Given spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. By an abuse of notation we will let the same letter stand for a map and its homotopy class. Suppose now that A, B, C are spaces and m a map, as below:

$$A \times B \xrightarrow{m} C.$$

Given a space X and maps $\xi \in [X, A]$, $\eta \in [X, B]$, we set

$$\xi \oplus \eta = m_*(\xi, \eta),$$

in $[X, C]$. Let h^* be a cohomology theory [17], and let $w \in h^* C$. For a map $\zeta \in [X, C]$, we set

$$w(\zeta) = \zeta^* w \in h^* X;$$

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we call w a (primary) characteristic class of C. Suppose that $m^* w \in h^* A \otimes h^* B$; say,

$$m^* w = \sum w_i' \otimes w_i'',$$

where $w_i' \in h^* A$, $w_i'' \in h^* B$. Then,

$$w(\xi \oplus \eta) = w_i' \xi \cup w_i'' \eta,$$

giving a Whitney-Cartan formula for the class w.

For the classical Whitney formula we take A, B, and C to be appropriate classifying spaces for vector bundles and w to be the Whitney class w_t. For the classical Cartan formula A, B, C are Eilenberg-MacLane spaces and w a class representing the operation P^t.

Higher Order Characteristic Classes

Suppose that we have a map $p: E \to C$. A class $\theta \in h^* E$ will be called, in general, a *higher order characteristic class* of C. Given a map $\zeta \in [X, C]$ if there is a map $f \in [X, E]$ such $p_*(f) = \zeta$, we say that θ is defined on ζ and set

$$\theta(\zeta) = \bigcup f^* \theta \subset h^* X,$$

where the union is over all maps f such that $p_* f = \zeta$.

Now let $\xi \in [X, A]$, $\eta \in [X, B]$. In the remainder of the paper we consider the following two problems.

Problem 1. *When is θ defined on $\xi \oplus \eta$?*

Problem 2. *If θ is defined, find a Whitney-Cartan formula for $\theta(\xi \oplus \eta)$.*

The paper is organized as follows. In Part I (§§ 1–4) we study Whitney formulae while Part II (§§ 5–6) is devoted to Cartan formulae. (In a brief section at the end, § 7, we discuss the distinction between these.) In both cases our approach is geometric, via the notion of a Whitney map (§ 2) and a Cartan map (§ 5). We give a general "Whitney formula" in Theorem (3.3), and then apply this in § 4 to a variety of examples involving higher order characteristic classes for sphere-fibrations. In particular we obtain as special cases some recent results of Peterson-Stein [9] and Mosher [7, 8]. Similarly we obtain a general "Cartan formula" in Theorems (5.6) and (6.5), and in § 6 we give an example for a mod p secondary cohomology operation.

Remark. There is some overlap between the material in Part II (Cartan formulae) and a recent paper of Milgram [6], and in working out my ideas for this part I benefitted from seeing a preprint of [6]. However the approach taken here (via the Cartan map, (5.2)) is different from that used in [6]. In a series of recent papers (see *Mathematica Scandinavica*, 16 (1965), 17 (1965)), L. Kristensen has developed a Cartan formulae via cochain operations; this approach differs considerably from that used in [6] and here.

Part I. Whitney Formulae

1. Cohomology Calculations

In this preliminary section we review material needed in the rest of the paper. Our main concern is calculating the cohomology of a principal fibration.

For a space C we denote by PC the function space $(C,*)^{(I,0)}$, where $I=[0,1]$. If B is a space and w a map from B to C, let $p\colon E_w\to B$ denote the principal fibration with w as classifying map. That is, E_w is the subspace of $B\times PC$ consisting of those pairs (b,λ) such that $\lambda(1)=w(b)$; and $p(b,\lambda)=b$. Define an action map

$$m\colon \Omega C\times E_w\to E_w$$

by

$$(\omega,(b,\lambda))\mapsto(b,\omega\vee\lambda),$$

where $\vee$ denotes the usual path composition of the loop ω with the path λ.

Let i and r denote respectively the inclusion and projection, as shown below:

$$((\Omega C,*)\times E)\xleftarrow{i}\Omega C\times E\xrightarrow{r}E.$$

One easily shows [17, 2.2] that there is a unique morphism

$$\mu\colon H^*(E)\to H^*((\Omega C,*)\times E) \tag{1.1}$$

such that

$$i^*\mu=m^*-r^*.$$

(It is easily seen that i^* is injective.) The image of μ is known to be the kernel of a transgression operator [15, Theorem 1], which is easily computed, and hence we gain some information about $H^*(E)$.

In most applications we consider not just a space B but a pair (B,T), and the map w is a map of pairs, $w\colon(B,T)\to(C,*)$. We may think of T embedded in E_w, $s\colon T\subset E_w$, with s given by

$$x\mapsto(x,*),\qquad x\in T,$$

where $*$ denotes the constant path at the basepoint of C. Thus p becomes a map of pairs

$$p\colon(E_w,T)\to(B,T).$$

We define

$$v=m\circ(1\times s)\colon(\Omega C,*)\times T\to(E_w,T). \tag{1.2}$$

In [14, p. 13] we defined a transgression operator

$$\tau\colon H^*((\Omega C,*)\times T)\to H^*(B,T).$$

τ is a relation in the sense of MacLane [3, p. 51] – i.e., τ is defined on a subgroup of $H^*((\Omega C,*)\times T)$ and takes values in a quotient of $H^*(B,T)$ (see [14]). Moreover,

$$\text{Image } v^*=\text{Kernel }\tau. \tag{1.3}$$

Even more, through a certain range of dimensions one has an exact sequence. Suppose that the inclusion map $T \subset B$ is simple [11, p. 440] and that

$$H_i(C)=0, \quad 0<i\leqq b,$$

and that $H_j(B,T)=0$, $0\leqq j<a$. Then the sequence

$$(1.4) \qquad \cdots \xrightarrow{\tau} H^i(B,T) \xrightarrow{p^*} H^i(E,T) \xrightarrow{\nu^*} H^i((\Omega C,*)\times T) \xrightarrow{\tau} \cdots$$

is exact, for $0\leqq i\leqq a+b-1$.

In what follows we will need one other cohomological fact. Suppose that g is a map of pairs,

$$g\colon (X,*)\times(B,A)\to(Y,*).$$

We define a map

$$(1.5) \qquad \hat{g}\colon (PX,*)\times(B,A)\to(PY,*),$$

by

$$(\hat{g}(\lambda,b))(t)=g(\lambda(t),b), \qquad t\in I.$$

Notice that by restriction, $\hat{g}$ gives a map (which we denote by the same symbol)

$$(1.6) \qquad \hat{g}\colon (\Omega X,*)\times(B,A)\to(\Omega Y,*).$$

We will need the following fact. Suppose that u is a class in $H^*(Y,*)$ such that g^*u lies in $H^*(X,*)\otimes H^*(B,A)$ – say $g^*u=\sum x_i\otimes b_i$. Then,

$$(1.7) \qquad \hat{g}^*(\sigma u)=\sum \sigma x_i\otimes b_i,$$

where σ denotes the cohomology operator from the cohomology of a space to that of its loop space. For a proof of (1.7), see [2, § 5].

2. *Whitney Maps*

Suppose now that we have spaces A_1, A_2, and B (changing our notation slightly) and a map m, as shown below:

$$A_1\times B\xrightarrow{m}A_2.$$

Suppose, moreover, that there are principal fibrations $p_i\colon E_i\to A_i$, with classifying maps $w_i\colon A_i\to K_i$, $i=1,2$. Problem 1, given in the Introduction, now becomes the Whitney problem: When does there exist a map

$$l\colon E_1\times B\to E_2$$

such that the following diagram commutes:

$$(2.1) \qquad \begin{array}{ccc} E_1\times B & \xrightarrow{l} & E_2 \\ \downarrow{\scriptstyle p_1\times 1} & & \downarrow{\scriptstyle p_2} \\ A_1\times B & \xrightarrow{m} & A_2. \end{array}$$

And Problem 2 becomes: Given $\theta\in h^*E_2$, determine $l^*\theta\in h^*(E_1\times B)$.

Since p_2 is a principal fibration, the lifting exists if and only if $w_2 \circ m \circ (p_1 \times 1)$ is null-homotopic. However, this answer does not suffice to determine $l^* \theta$ – for this we need a geometric description of l.

We will say that the map m, from $A_1 \times B$ to A_2, is a *Whitney map* if there is a map

$$n\colon (K_1, *) \times B \to (K_2, *)$$

such that the following diagram homotopy-commutes:

$$\begin{array}{ccc} A_1 \times B & \xrightarrow{m} & A_2 \\ \Big\downarrow{\scriptstyle w_1 \times 1} & & \Big\downarrow{\scriptstyle w_2} \\ K_1 \times B & \xrightarrow{n} & K_2 . \end{array} \tag{2.2}$$

Clearly, if m is a Whitney map, there is a map l satisfying (2.1) since $w_1 \circ p_1$ is null-homotopic. But we now can give a specific choice for l, which will in some instances enable us to solve Problem 2. Recall that given a square as in (2.2), one may alter the square, up to homotopy type, to obtain one which is strictly commutative. For example, make the map w_2 into a Hurewicz fibration [11, p. 99]. Thus, without loss of generality, we may assume that (2.2) commutes.

Recall that $E_i \subset A_i \times PK_i$, $i=1,2$. We define

$$l\colon E_1 \times B \to E_2$$

by

$$\big((a,\lambda), b\big) \mapsto \big(m(a,b), \hat{n}(\lambda, b)\big). \tag{2.3}$$

Here $(a,\lambda) \in E_1$, $b \in B$ and

$$\hat{n}\colon PK_1 \times B \to PK_2$$

is the map defined in (1.5). Clearly l does take values in the subspace E_2 of $A_2 \times PK_2$, since

$$w_2\big(m(a,b)\big) = n\big(w_1(a), b\big) = n\big(\lambda(1), b\big) = \hat{n}(\lambda, b)(1).$$

(We use here the fact that $w_1(a) = \lambda(1)$, since $(a,\lambda) \in E_1$.) Also

$$\hat{n}(\lambda, b)(0) = n\big(\lambda(0), b\big) = n(*, b) = *,$$

by assumption on n. Finally,

$$p_2\big(m(a,b), \hat{n}(\lambda, b)\big) = m(a,b) = m \circ (p_1 \times 1)\big(((a,\lambda), b)\big),$$

and so l satisfies (2.1).

We show in the next section that this definition of l is useful for computing $l^* \theta$, when θ is defined using the morphism μ. (See (1.1).) However, if θ is defined using the morphism ν^* (as in (1.2)), we then need a more elaborate approach.

Relative Whitney Maps. Suppose that we have pairs of spaces

$$(A_1, S_1), \quad (A_2, S_2), \quad (B, T),$$

with S_1, S_2 and T non-empty. Assume, moreover, that we have a map m of pairs:

$$m\colon (A_1, S_1) \times (B, T) \to (A_2, S_2).$$

Finally, let $w_i\colon (A_i, S_i) \to (K_i, *)$, be maps of pairs, $i=1,2$. As above, let $p_i\colon E_i \to A_i$ denote the principal fibration with classifying map w_i. Using the map $s \mapsto (s, *)$, $s \in S_i$, we identify S_i with a subspace of E_i, and so we regard p_i as a map of pairs

$$p_i\colon (E_i, S_i) \to (A_i, S_i), \quad i=1,2.$$

In this relative setting the Whitney problem becomes: Find a map l so that the following diagram commutes:

$$\begin{array}{ccc} (E_1, S_1) \times (B, T) & \xrightarrow{l} & (E_2, S_2) \\ \downarrow{\scriptstyle p_1 \times 1} & & \downarrow{\scriptstyle p_2} \\ (A_1, S_1) \times (B, T) & \xrightarrow{m} & (A_2, S_2). \end{array} \tag{2.4}$$

We will say that m is a (relative) Whitney map if there is a map n so that the following diagram commutes:

$$\begin{array}{ccc} (A_1, S_1) \times (B, T) & \xrightarrow{m} & (A_2, S_2) \\ \downarrow{\scriptstyle w_1 \times 1} & & \downarrow{\scriptstyle w_2} \\ (K_1, *) \times (B, T) & \xrightarrow{n} & (K_2, *). \end{array} \tag{2.5}$$

Given n, one then defines l as in (2.2); the fact that l is a map of pairs is easily checked.

We wish now to relate l to the map v, given in (1.2). Define a map

$$k\colon \big((\Omega K_1, *) \times S_1\big) \times (B, T)\big) \to (\Omega K_2, *) \times S_2,$$

by the rule

$$(\omega, s, b) \mapsto \big(\hat{n}(\omega, b), m(s, b)\big),$$

where $\omega \in \Omega K_1$, $s \in S_1$, $b \in B$, and where $\hat{n}$ is defined in (1.6). The fact that k is a map of pairs is easily checked. Let $v_i\colon (\Omega K_i, *) \times S_i \to (E_i, S_i)$, $i=1,2$ be the map defined in (1.2). We shall prove

(2.6) **Theorem.** *Let m be a relative Whitney map and let l be the map defined in* (2.2). *Then the following diagram commutes:*

$$\begin{array}{ccc} \big((\Omega K_1, *) \times S_1\big) \times (B, T) & \xrightarrow{k} & (\Omega K_2, *) \times S_2 \\ \downarrow{\scriptstyle v_1 \times 1} & & \downarrow{\scriptstyle v_2} \\ (E_1, S_1) \times (B, T) & \xrightarrow{l} & (E_2, S_2) \\ \downarrow{\scriptstyle p_1 \times 1} & & \downarrow{\scriptstyle p_2} \\ (A_1, S_1) \times (B, T) & \xrightarrow{m} & (A_2, S_2). \end{array}$$

Proof. We have already shown that the bottom square commutes. To show that the top square commutes, let (ω, s, b) be the point given above. Then,

$$\nu_2(k(\omega, s, b)) = \nu_2(\hat{n}(\omega, b), m(s, b)) = (m(s, b), \hat{n}(\omega, b) \vee *),$$

by (1.2). On the other hand,

$$(\nu_1 \times 1)(\omega, s, b) = ((s, \omega \vee *), b),$$

and so

$$l(\nu_1 \times 1)(\omega, s, b) = l((s, \omega \vee *), b) = (m(s, b), \hat{n}(\omega \vee *, b)).$$

But

$$\hat{n}(\omega \vee *, b) = \hat{n}(\omega, b) \vee n(*, b) = \hat{n}(\omega, b) \vee *,$$

and so $\nu_2 \circ k = l \circ (\nu_1 \times 1)$ as asserted.

In the following section we use Theorem (2.6) to express $l^* \theta$.

(2.7) *Remark.* We have assumed that diagram (2.5) strictly commutes. In practice (note § 4) we will often be given a diagram which simply homotopy-commutes. However, again by making w_2 into a fiber map, one may alter spaces and maps up to homotopy to obtain a commutative diagram.

3. *Whitney Product Formulae*

In Section 1 we gave two different methods for describing the cohomology of a principal fibration – one using μ and one using ν^*. We now relate the map l, defined in (2.2), to these morphisms. We use the notation given in § 2.

As in § 1, we have the (injective) morphism

$$h^*((\Omega K_1, *) \times E_1 \times B) \xrightarrow{i^*} h^*(\Omega K_1 \times E_1 \times B),$$

and a unique morphism

$$\hat{\mu}\colon\ h^*(E_1 \times B) \to h^*((\Omega K_1, *) \times E_1 \times B)$$

such that

$$i^* \hat{\mu} = (m_1 \times 1)^* - r^*,$$

where m_1 is the action map for p_1 and where $r\colon \Omega K_1 \times E_1 \times B \to E_1 \times B$ is the projection.

Define

$$j\colon\ ((\Omega K_1, *) \times E_1 \times B) \to ((\Omega K_2, *) \times E_2)$$

by

$$(\omega, e, b) \to (n_1(\omega, b), l(e, b)).$$

(3.1) **Theorem.** *Let $l\colon E_1 \times B \to E_2$ be the map defined in (2.2). Then, the following diagram commutes:*

$$\begin{array}{ccc} h^*(\Omega K_1 \times E_1 \times B, E_1 \times B) & \xleftarrow{j^*} & h^*(\Omega K_2 \times E_2, E_2) \\ \uparrow{\scriptstyle \hat{\mu}} & & \uparrow{\scriptstyle \mu_2} \\ h^*(E_1 \times B) & \xleftarrow{l^*} & h^* E_2 . \end{array}$$

Proof. It suffices to show that

$$i^* \hat{\mu}\, l^* = i^* j^* \mu_2 .$$

But this follows at once from the commutativity of the following diagram:

$$\begin{array}{ccc} \Omega K_1 \times E_1 \times B & \xrightarrow{j} & \Omega K_2 \times E_2 \\ \Big\downarrow{\scriptstyle m_1 \times 1} & & \Big\downarrow{\scriptstyle m_2} \\ E_1 \times B & \xrightarrow{l} & E_2 . \end{array}$$

We leave it to the reader to check that the diagram commutes and hence to complete the proof of Theorem (3.1).

We now consider Theorem (2.6) in more detail. In practice, how do classes in $H^* E_2$ arise? We describe the present-day view of obstruction theory.

We assume familiarity with the semi-tensor product of an algebra with a Hopf algebra, as described in [5]; we use the notation and ideas given in Section 4 of [17]. In addition we will need the following sign convention. Let $\mathscr{A}$ be any graded algebra. Given $\alpha \in \mathscr{A}$, of homogeneous degree, we set

$$|\alpha| = \begin{cases} \alpha, & \text{if } \deg \alpha \text{ even} \\ -\alpha, & \text{if } \deg \alpha \text{ odd}. \end{cases}$$

Now choose a fixed prime p and let $\mathscr{A}_p$ denote the mod p Steenrod algebra. Take cohomology with coefficients mod p, let $\iota_1, \ldots, \iota_r$ be classes in $H^*(K_2)$, and set

$$a_i = w_2^* \iota_i \in H^*(A_2, S_2),$$

$i = 1, \ldots, r$. Suppose there are elements $\alpha_1, \ldots, \alpha_r$ in $\mathscr{A}_p(A_2)$ such that

$$\sum \alpha_i \cdot a_i = 0 .$$

Then, by the exactness given in (1.3), one sees that there is a class $\theta \in H^*(E_2, S_2)$ such that

$$v_2^* \theta = \sum |\alpha_i| \cdot (\sigma \iota_i \otimes 1),$$

where $\mathscr{A}_p(A_2)$ acts on $H^*\big((\Omega K_2, *) \times S_2\big)$ via the composite map

$$\Omega K_2 \times S_2 \xrightarrow{\text{proj}} S_2 \subset A_2 .$$

Problem 2 now becomes: *Compute $l^* \theta$ where l is given in* (2.2). Using the commutative diagram given in Theorem (2.6), one has

$$(v_1 \times 1)^* l^* \theta = k^* v_2^* \theta = k^* \big(\sum |\alpha_i| \cdot (\sigma \iota_i \otimes 1)\big) = \sum m^* |\alpha_i| \cdot k^* (\sigma \iota_i \otimes 1),$$

in $H^*\big((\Omega K_1, *) \times S_1\big) \otimes H^*(B, T)\big)$. Here $m^* |\alpha_i|$ acts on this cohomology via the composite map

$$\Omega K_1 \times S_1 \times B \xrightarrow{\text{proj}} S_1 \times B \xrightarrow{m} S_2 \subset A_2 .$$

The point to stress here is this: the terms $m^* |\alpha_i|$ and $k^*(\sigma \iota_i \otimes 1)$ in the above expression can be explicitly calculated, using (1.7) for the latter term. Assuming this calculation has been made, one can then choose linearly independent classes in $H^*(B,T)$, $b_1,\dots,b_s$, say, and classes $\hat{\theta}_1,\dots,\hat{\theta}_s$ in $H^*((\Omega K_1,*)\times S_1)$, so that one can write

$$(v_1\times 1)^* l^*\theta=\sum m^*|\alpha_i|\cdot k^*(\sigma\iota_i\otimes 1)=\sum\hat{\theta}_j\otimes b_j. \tag{3.2}$$

Now let $n=$dimension θ and let r be a positive integer (less than n) such that

$$H^i(B,T)=0,\qquad \text{for } 0\leqq i<r.$$

(3.3) **Theorem.** *Assume that the morphism v_1^* is injective through dimension $n-r$. Then for each class $\hat{\theta}_j$ there is a unique class θ_j in $H^*(E_1,S_1)$ such that*

$$v_1^*\theta_j=\hat{\theta}_j.$$

Moreover,

$$l^*\theta=\sum\theta_j\otimes b_j,$$

in $H^*(E_1,S_1)\otimes H^*(B,T)$.

Proof. By (1.3) the following sequence is exact:

$$H^*(E_1,S_1)\xrightarrow{v_1^*}H^*((\Omega K_1,*)\times S_1)\xrightarrow{\tau}H^*(A_1,S_1).$$

Thus,

$$(\tau\otimes 1)(v_1\otimes 1)^*(l^*\theta)=0,$$

which means, by (3.2), that

$$\sum\tau\hat{\theta}_j\otimes b_j=0.$$

But the classes $b_1,\dots,b_s$ were chosen as part of a basis for $H^*(B,T)$. Thus,

$$\tau\hat{\theta}_j=0,\qquad 1\leqq j\leqq s,$$

and so by exactness there are classes $\theta_j\in H^*(E_1,S_1)$ such that $v_1^*\theta_j=\hat{\theta}_j$. Since dimension $b_j\geqq r$, and since v_1^* is injective in dimensions $\leqq n-r$, the classes θ_j are unique. Finally,

$$(v_1\times 1)^* l^*\theta=\sum\hat{\theta}_j\otimes b_j=(v_1\times 1)^*(\sum\theta_j\otimes b_j),$$

and so

$$l^*\theta=\sum\theta_j\otimes b_j,$$

since $(v_1\times 1)^*$ is also injective in dimension n. This completes the proof.

Now let X be a complex and let $\xi\in[X,A_1]$, $\eta\in[X,B]$.

(3.4) **Corollary.** *Suppose that $w_1(\xi)=0$. Then $w_2(\xi\oplus\eta)=0$, and so the class $\theta(\xi\oplus\eta)$ is defined. Moreover,*

$$\sum\theta_j(\xi)\cup b_j(\eta)\in\theta(\xi\oplus\eta).$$

4. Examples of Whitney Formulae

We give a variety of examples illustrating Theorem (3.4). Each is based on a sphere fibration, so we describe the relevant notation.

We denote by B_n, $n \geqq 2$, the classifying map for oriented $(n-1)$-sphere bundles. For each n there is a natural inclusion π_n: $B_{n-1} \subset B_n$, which up to homotopy, represents the universal $(n-1)$-sphere bundle over B_n. We denote by $\chi_n \in H^n(B_n, B_{n-1}; Z)$ the relative Euler class for this fibration. (Equally well, we may think of (B_n, B_{n-1}) as the Thom complex [13] and then χ_n simply denotes the Thom class.) We let $w_i \in H^i(B_n; Z_2)$, $i \geqq 0$, denote the Stiefel-Whitney class.

Set $K_n = K(Z_2, n)$, $K_n^* = K(Z, n)$, $n \geqq 1$. We regard the class χ_n as a map

$$\chi_n\colon (B_n, B_{n-1}) \to (K_n^*, *).$$

The following diagram portrays a (mod 2) Postnikov resolution [4, 14], of π_n through dimension $n+3$.

$$\text{(4.1)}\qquad \begin{array}{ccc}
(E_n^3, B_{n-1}) & \xrightarrow{\gamma_n^3} & (K_{n+3}, *) \\
\downarrow{\scriptstyle p_3} & & \\
(E_n^2, B_{n-1}) & \xrightarrow{(\beta_n^2, \beta_n^3)} & (K_{n+2} \times K_{n+3}, *) \\
\downarrow{\scriptstyle p_2} & & \\
(E_n^1, B_{n-1}) & \xrightarrow{(\alpha_n^1, \alpha_n^3)} & (K_{n+1} \times K_{n+3}, *) \\
\downarrow{\scriptstyle p_1} & & \\
(B_n, B_{n-1}) & \xrightarrow{\chi_n} & (K_n^*, *).
\end{array}$$

The classes $\alpha_n^1, \ldots, \gamma_n^3$ are described using the exact sequence (1.4), as shown in the following table: (We let ι_q denote the q-characteristic class [11] for both K_q and K_q^*, $q \geqq 1$.)

(4.2)

Invariant	Range of v^*	Image by v^*
α_n^1	$H^*((K_{n-1}^*, *) \times B_{n-1})$	$\phi_2 \cdot (\iota_{n-1} \otimes 1)$
α_n^3	$H^*((K_{n-1}^*, *) \times B_{n-1})$	$\phi_4 \cdot (\iota_{n-1} \otimes 1)$
β_n^2	$H^*((K_n \times K_{n+2}, *) \times B_{n-1})$	$\phi_2 \cdot (\iota_n \otimes 1 \otimes 1)$
β_n^3	$H^*((K_n \times K_{n+2}, *) \times B_{n-1})$	$\phi_3 \cdot (\iota_n \otimes 1 \otimes 1) + \mathrm{Sq}^1 \cdot (1 \otimes \iota_{n+2} \otimes 1)$
γ_n^3	$H^*((K_{n+1} \times K_{n+2}, *) \times B_{n-1})$	$\phi_2 \cdot (\iota_{n+1} \otimes 1 \otimes 1) + \mathrm{Sq}^1 \cdot (1 \otimes \iota_{n+2} \otimes 1)$

The classes ϕ_i lie in $\mathscr{A}_2(B_n)$, and are defined by

$$\phi_2 = w_2 \otimes 1 + 1 \otimes \mathrm{Sq}^2, \qquad \phi_3 = w_3 \otimes 1 + 1 \otimes \mathrm{Sq}^2\,\mathrm{Sq}^1,$$
$$\phi_4 = w_4 \otimes 1 + 1 \otimes \mathrm{Sq}^4.$$

For details see [4, 14] and [16].

We shall prove the following result.

(4.3) **Theorem.** *Let ξ and η be oriented sphere bundles over a complex X, with* $\dim \xi = r$, $\dim \eta = s$, $r, s \geqq 2$. *Set $t = r + s$.*

(i) *Suppose that* $\chi_r(\xi)=0$. *Then,*

$$\alpha_t^1(\xi\oplus\eta)\equiv\alpha_r^1(\xi)\cup\chi_s(\eta).$$
$$\alpha_t^3(\xi\oplus\eta)\equiv\alpha_r^3(\xi)\cup\chi_s(\eta)+\alpha_r^1(\xi)\cup\big(\chi_s(\eta)\cdot w_2(\eta)\big).$$

(ii) *Suppose that* $\chi_r(\xi)=0$ *and* $(\alpha_r^1,\alpha_r^3)(\xi)\equiv 0$. *Then*

$$\beta_t^2(\xi\oplus\eta)\equiv\beta_r^2(\xi)\cup\chi_s(\eta),$$
$$\beta_t^3(\xi\oplus\eta)\equiv\beta_r^3(\xi)\cup\chi_s(\eta).$$

(iii) *Suppose that* $\chi_r(\xi)=0$, $(\alpha_r^1,\alpha_r^3)(\xi)\equiv 0$, *and* $(\beta_r^2,\beta_r^3)(\xi)\equiv 0$. *Then*

$$\gamma_t^3(\xi\oplus\eta)\equiv\gamma_r^3(\xi)\cup\chi_s(\eta).$$

The result for the classes α_t^1 and α_t^3 is given by Peterson-Stein in Theorem of [9]. Our point here is to show how the theorem follows in a purely mechanical way from Theorem (3.3).

We do out the details only for the classes α_t^3 and β_t^3. The remaining cases are similar (and slightly easier).

The Whitney sum of bundles gives a map of pairs

$$m\colon (B_r, B_{r-1})\times(B_s, B_{s-1})\to(B_t, B_{t-1}).$$

Since $m^*\chi_t=\chi_r\otimes\chi_s$ by the usual Whitney product formula, it follows that m is a relative Whitney map, in the sense of § 2, if we take

$$n\colon (K_r^*, *)\times(B_s, B_{s-1})\to(K_t^*, *)$$

to be the map given by the cohomology class $\iota_r\otimes\chi_s$. (Note Remark (2.7), to ensure that the resulting diagram commutes.) Thus by Theorem (2.6) there is a map $l_1\colon (E_r^1, B_{s-1})\times(B_s, B_{s-1})\to(E_t^1, B_{t-1})$ so that the following diagram commutes:

$$\begin{array}{ccc}
H^*\big((K_{r-1}^*, *)\times B_{r-1}\big)\otimes H^*(B_s, B_{s-1}) & \xleftarrow{k_1^*} & H^*\big((K_{t-1}^*, *)\times B_{t-1}\big)\\
\big\uparrow{\scriptstyle v_r^{1*}\otimes 1} & & \big\uparrow{\scriptstyle v_t^{1*}}\\
H^*\big((E_r^1, B_{r-1}\big)\otimes H^*(B_s, B_{s-1}) & \xleftarrow{l_1^*} & H^*(E_t^1, B_{t-1}).
\end{array}$$

We proceed to compute $l_1^*\alpha_t^3$. To use Theorem (3.3) we need to calculate the classes

$$m^*\phi_4 \quad\text{and}\quad k_1^*(\iota_{t-1}\oplus 1).$$

By the usual Whitney formula,

$$m^*\phi_4=(w_4\otimes 1\otimes 1+w_2\otimes w_2\otimes 1+1\otimes w_4\otimes 1)+1\otimes 1\otimes \mathrm{Sq}^4,$$

in $\mathscr{A}(B_r\times B_s)$; whereas, by (1.7)

$$k_1^*(\iota_{t-1}\otimes 1)=\iota_{r-1}\otimes 1\otimes\chi_s,$$

in $H^*((K^*_{r-1}, *)\times B_{r-1})\otimes H^*(B_s, B_{s-1})$. Using Table (4.2) and the fact that $\mathrm{Sq}^i \chi_s = \chi_s \cdot w_i$, $i \geqq 1$, we find that

$$k_1^* v_t^{1*} \alpha_t^3 = [\phi_4 \cdot (\iota_{r-1}\otimes 1)]\otimes \chi_s + [\phi_2 \cdot (\iota_{r-1}\otimes 1)]\otimes \chi_s \cdot w_2 .$$

Here the $\cdot$ indicates the action of $\mathscr{A}(B_r)$ on $K^*_{r-1}\times B_{r-1}$ using the composite map

$$K^*_{r-1}\times B_{r-1} \xrightarrow{\text{proj}} B_{r-1} \xrightarrow{\pi r} B_r .$$

But

$$v_r^{1*}\alpha_r^1 = \phi_2 \cdot (\iota_{r-1}\otimes 1), \qquad v_r^{1*}\alpha_r^3 = \phi_4 \cdot (\iota_{r-1}\otimes 1),$$

and so by Theorem (3.3),

$$l_1^* \alpha_t^3 = \alpha_r^1 \otimes \chi_s w_2 + \alpha_r^3 \otimes \chi_s ,$$

from which the result stated in Theorem (4.3) follows at once. (We have used here the facts that $H^i(B_s, B_{s-1})=0$, $0 \leqq i < s$, and that v_r^* is injective through dimension $r+3$. Note [14] and [9].)

We now obtain a Whitney formula for β_t^3. Recall that the fibration p_2: $E_n^2 \to E_n^1$, has as classifying map the classes

$$(E_n^1, B_{n-1}) \xrightarrow{(\alpha_n^1, \alpha_n^3)} (K_{n+1}\times K_{n+3}, *).$$

Take $n=t$. By what we have proved for α_t^3 (and analogously for α_t^1),

$$l_1^* (\alpha_t^1, \alpha_t^3) = (\alpha_r^1 \otimes \chi_s, \alpha_r^3 \otimes \chi_s + \alpha^1 \otimes \chi_s \cdot w_2),$$

in $H^*((E^1, B_{r-1})\times (B_s, B_{s-1}))$. Thus l_1 is a relative Whitney if we define the map n to be given by the pair of maps (a, b),

$$(K_{r+1}\times K_{r+3}, *)\times (B_s, B_{s-1}) \xrightarrow{(a,b)} (K_{t+1}\times K_{t+3}, *)$$

where

(4.4)
$$a^* \iota_{t+1} = \iota_{r+1} \otimes 1 \otimes \chi_s ,$$
$$b^* \iota_{t+3} = 1 \otimes \iota_{r+3} \otimes \chi_s + \iota_{r+1} \otimes 1 \otimes \chi_s \cdot w_2 .$$

Again we use Remark (2.7) to ensure that the resulting diagram commutes. And by Theorem (2.6) there is then a lifting l_2 of l_1,

$$l_2 \colon (E_r^2, B_{r-1})\times (B_s, B_{s-1}) \to (E_t^2, B_{t-1}),$$

so that the following diagram commutes:

$$\begin{array}{ccc} H^*((K_r\times K_{r+2}, *)\times B_{r-1})\otimes H^*(B_s, B_{s-1}) & \xleftarrow{k_2^*} & H^*((K_t\times K_{t+2}, *)\times B_{t-1}) \\ \uparrow {\scriptstyle v_r^{2*}\otimes 1} & & \uparrow {\scriptstyle v_t^{2*}} \\ H^*(E_r^2, B_{r-1})\otimes H^*(B_s, B_{s-1}) & \xleftarrow{l_2^*} & H^*(E_t^2, B_{t-1}). \end{array}$$

To use Theorem (3.3) we see by Table (4.2) that we need to calculate $l_1^* \phi_3$, $k_2^*(\iota_t \otimes 1 \otimes 1)$, and $k_2^*(1 \otimes \iota_{t+2} \otimes 1)$. (At this point we regard ϕ_3 as an element in $\mathscr{A}(E_t^1)$ – namely, $\phi_3 = p_1^* w_3 \otimes 1 + 1 \otimes \mathrm{Sq}^2\,\mathrm{Sq}^1$.) Since l_1 is a lifting of m (see 2.2),

$$l_1^* \phi_3 = p_1^* w_3 \otimes 1 \otimes 1 + 1 \otimes w_3 \otimes 1 + 1 \otimes 1 \otimes \mathrm{Sq}^2\,\mathrm{Sq}^1,$$

in $\mathscr{A}(E_r^1 \times B_s)$; whereas, by (4.4) and (1.7),

$$k_2^*(\iota_t \otimes 1 \otimes 1) = \iota_r \otimes 1 \otimes 1 \otimes \chi_s,$$
$$k_2^*(1 \otimes \iota_{t+2} \otimes 1) = 1 \otimes \iota_{r+2} \otimes 1 \otimes \chi_s + \iota_r \otimes 1 \otimes 1\, \chi_s \cdot w_2.$$

Using the fact that

$$\mathrm{Sq}^i \chi_s = \chi_s \cdot w_i, \quad i \geqq 1, \qquad \mathrm{Sq}^1 w_2 = w_3,$$

we find that

$$k_2^* v_t^{2*} \beta_t^3 = [\phi_3 \cdot (\iota_r \otimes 1 \otimes 1) + \mathrm{Sq}^1 \cdot (1 \otimes \iota_{r+2} \otimes 1)] \otimes \chi_s.$$

Therefore, by Theorem (3.3),

$$l_2^* \beta_t^3 = \beta_r^3 \otimes \chi_s,$$

since $v_r^{2*} \beta_r^3 = \phi_3 \cdot (\iota_r \otimes 1 \otimes 1) + \mathrm{Sq}^1 \cdot (1 \otimes \iota_{r+2} \otimes 1)$. This gives the desired Whitney formula for β_t^3. To obtain the formula for γ_t^3, one uses the results for β_t^2, β_t^3 to show that l_2 is a Whitney map. Thus the lifting l_3 exists and the argument proceeds as before. This completes the proof.

Let ξ and η be bundles over X, as before. In (4.3) we considered the case $\chi_r(\xi) = 0$. We now see what happens if, in addition, $\chi_s(\eta) = 0$. First, by Theorem (4.3), we see that

$$(\alpha_t^1, \alpha_t^3)(\xi \oplus \eta) \equiv 0$$

and so the classes

$$\beta_t^i(\xi \oplus \eta), \qquad i = 2, 3$$

are defined.

(4.5) **Theorem.** *Let ξ and η be bundles over X, as in* (4.3). *Suppose that $\chi_r(\xi) = 0$, $\chi_s(\eta) = 0$. Then,*

$$\beta_t^2(\xi \oplus \eta) \equiv \alpha_r^1(\xi) \cup \alpha_s^1(\eta),$$
$$\beta_t^3(\xi \oplus \eta) \equiv \alpha_r^1(\xi) \cup \mathrm{Sq}^1 \alpha_s^1(\eta)$$
$$\equiv \mathrm{Sq}^1 \alpha_r^1(\xi) \cup \alpha_s^1(\eta).$$

The result for β_t^2 has been given by Mosher [8]. Again the point of our doing the example here is simply to illustrate the functioning of Theorem (3.3) – we do only the case β_t^3.

We start with the map

$$l_1 : (E_r^1, B_{r-1}) \times (B_s, B_{s-1}) \to (E_t^1, B_{t-1})$$

given in the proof of (4.3). Rather than resolve over E_r^1, as in (4.3), we now resolve B_s (since we are assuming that $\chi_s(\eta) = 0$.) In other words, we want a map $\bar{l}_2$, which makes the following diagram commutative:

$$\begin{array}{ccc} (E_r^1, B_{r-1}) \times (E_s^1, B_{s-1}) & \xrightarrow{\bar{l}_2} & (E_t^2, B_{t-1}) \\ \downarrow {\scriptstyle 1\times p_s^1} & & \downarrow {\scriptstyle p_t^2} \\ (E_r^1, B_{r-1}) \times (B_s, B_{s-1}) & \xrightarrow{l_1} & (E_t^1, B_{t-1}). \end{array}$$

By Theorem (4.3), l_1 is a relative Whitney map if we take the map n to be given by the pair of maps (c, d),

$$(E_r^1, B_{r-1}) \times (K_s^*, *) \xrightarrow{(c,d)} (K_{t+1} \times K_{t+3}, *),$$

where

$$\begin{aligned} c^*(\iota_{t+1} \otimes 1) &= \alpha_r^1 \otimes \iota_s, \\ d^*(1 \otimes \iota_{t+3}) &= \alpha_r^3 \otimes \iota_s + \alpha_r^1 \otimes \mathrm{Sq}^2\, \iota_s, \end{aligned} \tag{4.6}$$

since $\mathrm{Sq}^2 \chi_s = \chi_s \cdot w_2$. Thus by Theorem (2.6) the lifting $\bar{l}_2$ exists and moreover the following diagram is commutative:

$$\begin{array}{ccc} H^*(E_r^1, B_{r-1}) \otimes H^*((K_{s-1}^*, *) \times B_{s-1}) & \xleftarrow{\bar{k}_2^*} & H^*((K_t \times K_{t+2}, *) \times B_{t-1}) \\ \uparrow {\scriptstyle 1\otimes v_s^{1*}} & & \uparrow {\scriptstyle v_t^{2*}} \\ H^*(E_r^1, B_{r-1}) \otimes H^*(E_s^1, B_{s-1}) & \xleftarrow{\bar{l}_2^*} & H^*(E_t^2, B_{t-1}). \end{array}$$

In order to use Theorem (3.3), we see by Table (4.2) that we need to calculate

$$l_1^* \phi_3, \quad \bar{k}_2^*(\iota_t \otimes 1 \otimes 1), \quad \text{and} \quad \bar{k}_2^*(1 \otimes \iota_{t+2} \otimes 1).$$

The calculation of $l_1^* \phi_3$ has already been done in the proof (4.3). By (4.6) and (1.7),

$$\begin{aligned} \bar{k}_2^*(\iota_t \otimes 1 \otimes 1) &= \alpha_r^1 \otimes \iota_{s-1} \otimes 1, \\ \bar{k}_2^*(1 \otimes \iota_{t+2} \otimes 1) &= \alpha_r^3 \otimes \iota_{s-1} \otimes 1 + \alpha_r^1 \otimes \mathrm{Sq}^2\, \iota_{s-1} \otimes 1. \end{aligned}$$

A simple calculation now gives

$$\begin{aligned} \bar{k}_2^*\, v_t^{2*}\, \beta_t^3 = {} & \alpha_r^1 \cdot p_r^* w_3 \otimes \iota_{s-1} \otimes 1 + \alpha_r^1 \otimes \iota_{s-1} \otimes w_3 + \mathrm{Sq}^2 \mathrm{Sq}^1 \alpha_r^1 \otimes \iota_{s-1} \otimes 1 \\ & + \mathrm{Sq}^1 \alpha_r^3 \otimes \iota_{s-1} \otimes 1 + \alpha_r^1 \otimes \mathrm{Sq}^3\, \iota_{s-1} \otimes 1. \end{aligned}$$

Using the fact that v_r^{1*} is injective one easily checks that

$$\alpha_r^1 \cdot p_r^* w_3 + \mathrm{Sq}^2 \mathrm{Sq}^1 \alpha_r^1 + \mathrm{Sq}^1 \alpha_r^3 = 0,$$

and so

$$\begin{aligned} \bar{k}_2^*\, v_t^{2*}\, \beta_t^3 &= \alpha_r^1 \otimes \iota_{s-1} \otimes w_3 + \alpha_r^1 \otimes \mathrm{Sq}^3\, \iota_{s-1} \otimes 1 \\ &= \alpha_r^1 \otimes [\mathrm{Sq}^1(\phi_2 \cdot (\iota_{s-1} \otimes 1))] = \alpha_r^1 \otimes v_s^{1*}(\mathrm{Sq}^1 \alpha_s^1). \end{aligned}$$

Thus, by Theorem (3.3),

$$\bar{l}_2^*\, \beta_t^3 = \alpha_r^1 \otimes \mathrm{Sq}^1 \alpha_s^1.$$

Since

$$\alpha_r^1 \otimes \mathrm{Sq}^1 \alpha_s^1 + \mathrm{Sq}^1 \alpha_r^1 \otimes \alpha_s^1 = \mathrm{Sq}^1(\alpha_r^1 \otimes \alpha_s^1) \in \text{Indeterminacy of } \beta_t^3,$$

the proof is complete.

Remarks. (i) One may also prove Theorem (2.4) in [7] by the same method as given above.

(ii) Some applications of Theorem (4.5) will be given in a forthcoming paper on the embedding problem for manifolds.

Part II. Cartan Formulae

5. Cartan Maps

We turn now to a consideration of Cartan formulae for higher order cohomology operations, defering until § 7 a discussion of the distinction made between Cartan formulae and Whitney formulae.

We change notation slightly and assume now that we have three spaces A_1, A_2, A_3 and a map

$$(A_1, *)\times(A_2, *)\xrightarrow{m}(A_3, *).$$

Suppose, moreover, that over each space A_i we have a principal fibration p_i: $E_i \to A_i$, with classifying map w_i: $A_i \to K_i$, $i=1,2,3$. Problem 1, given in the Introduction, now becomes: when does there exist a map

$$l\colon (E_1, *)\times E_2, *)\to(E_3, *)$$

such that the following diagram commutes?

$$\text{(5.1)}\qquad \begin{array}{ccc} E_1\times E_2 & \xrightarrow{l} & E_3 \\ \downarrow{\scriptstyle p_1\times p_2} & & \downarrow{\scriptstyle p_3} \\ A_1\times A_2 & \xrightarrow{m} & A_3. \end{array}$$

And Problem 2 reads: given $\theta\in h^* E_3$, determine $l^*\theta$ in $h^*(E_1\times E_2)$.

Of course the map l, in (5.1), exists if and only if $w_3\circ m\circ(p_1\times p_2)$ is null-homotopic. However, the point is to describe l in such a way as to enable one to solve Problem 2.

In order to state our criterion we assume that K_3 is an H-space (with strict identity); let $g\colon K_3\times K_3\to K_3$ denote the multiplication.

We will say that m is a *Cartan map* (with respect to w_1, w_2, w_3) if there are maps

$$(K_1, *)\times(A_2, *)\xrightarrow{n_1}(K_3, *),$$
$$(A_1, *)\times(K_2, *)\xrightarrow{n_2}(K_3, *),$$

such that the following diagram commutes:

$$\text{(5.2)}\qquad \begin{array}{ccccc} A_1\times A_2 & \xrightarrow{m} & A_3 & \xrightarrow{w_3} & K_3 \\ \downarrow{\scriptstyle \Delta\times\Delta} & & & & \uparrow{\scriptstyle g} \\ (A_1\times A_1)\times(A_2\times A_2) & & & & K_3\times K_3 \\ \downarrow{\scriptstyle 1\times t\times 1} & & & & \uparrow{\scriptstyle n_1\times n_2} \\ (A_1\times A_2)\times(A_1\times A_2) & & \xrightarrow{(w_1\times 1)\times(1\times w_2)} & & (K_1\times A_2)\times(A_1\times K_2). \end{array}$$

Here Δ denotes (generically) the diagonal map and t the transposition map given by $(a_1, a_2) \mapsto (a_2, a_1)$, $a_i \in A_i$. We now can define our desired map l as follows: let $(a_i, \lambda_i) \in E_i$, $i=1,2$; i.e., $\lambda_i \in PK_i$ with $\lambda_i(1) = w_i(a_i)$. We set

$$(5.3) \qquad l((a_1, \lambda_1), (a_2, \lambda_2)) = (m(a_1, a_2), \hat{g}(\hat{n}_1(\lambda_1, a_2), \hat{n}_2(a_1, \lambda_2)).$$

Here $\hat{g}$: $PK_3 \times PK_3 \to PK_3$ is defined by

$$\hat{g}(\lambda, \lambda')(t) = g(\lambda(t), \lambda'(t)), \qquad t \in I,$$

while $\hat{n}_1, \hat{n}_2$ are the maps associated with n_1, n_2 by (1.5). The fact that l is well-defined and satisfies diagram (5.1) is easily checked (using diagram (5.2)); we leave the details to the reader.

Remark. If diagram (5.2) is given as only homotopy-commutative, make w_3 into an Hurewicz fibration, and then m can be altered, up to homotopy, to make the diagram commute.

We turn now to Problem 2. In the context of cohomology operations we usually use the Serre exact sequence to calculate the cohomology of fiber spaces. In general, given a fibration

$$F \xrightarrow{i} E \xrightarrow{\pi} B,$$

we say that an integer n is *stable* (with respect to π) if $n \leqq$ connectivity F + connectivity $B+1$. For then, by Serre [10], given a class $u \in H^n(E)$, $i^* u = 0$ if and only if $u \in \pi^* H^*(B)$. We will use this notion in characterizing the class $l^* \theta$ in $H^*(E_1 \# E_2)$, $\theta \in H^*(E_3)$. (For convenience, we now adopt the notation $X \# Y$ for the space $(X, *) \times (Y, *)$.)

Let j_i: $\Omega K_i \to E_i$, $i=1,2,3$, denote the fiber inclusions.

(5.4) **Proposition.** *Let* $u \in H^*(E_1 \# E_2)$, *where* $\dim u$ *is stable with respect to* p_1 *and* p_2. *Then,* $u \in \text{Image}(p_1 \# p_2)^*$ *if and only if*

$$(j_1 \# 1)^* u = 0, \qquad (1 \# j_2)^* u = 0.$$

Proof. Suppose we have vector spaces V_i, W_i, $i=1,2$, and homomorphisms α_i: $V_i \to W_i$. Set $K_i = \text{kernel } \alpha_i$. By elementary linear algebra one then has: a class u in $V_1 \otimes V_2$ belongs to $K_1 \otimes K_2$ if, and only if,

$$(\alpha_1 \otimes 1)(u) = 0, \qquad (1 \otimes \alpha_2)(u) = 0.$$

We apply this to the proposition by taking $V_i = H^*(E_i)$, $W_i = H^*(\Omega K_i)$, $\alpha_i = j_i^*$, $i=1,2$. By the stability assumption, the Serre exact sequence holds and so $K_i = \text{Image } p_i^*$; thus the proposition follows.

We apply the proposition to solve Problem 2. By (1.6) the composite maps

$$K_1 \# E_2 \xrightarrow{1 \# p_2} K_1 \# A_2 \xrightarrow{n_1} K_3,$$
$$E_1 \# K_2 \xrightarrow{p_1 \# 1} A_1 \# K_2 \xrightarrow{n_2} K_3,$$

induce respective maps (note [6, 2.1.1, 2.1.2])

$$(5.5)\qquad \begin{aligned} k_1 &: \Omega K_1 \# E_2 \to \Omega K_3, \\ k_2 &: E_1 \# \Omega K_2 \to \Omega K_3. \end{aligned}$$

(5.6) **Theorem.** *Let $m: A_1 \# A_2 \to A_3$ be a Cartan map and let $l: E_1 \# E_2 \to E_3$ be the map defined in* (5.3). *Suppose that θ is a class in $H^*(E_3)$, with* $\dim \theta$ *stable with respect to p_1 and p_2. Then $l^* \theta$ is determined, up to image $(p_1 \# p_2)^*$, by*

$$k_1^* j_3^* \theta \quad \textit{and} \quad k_2^* j_3^* \theta.$$

Proof. Consider the following diagram

$$\begin{array}{ccccc} E_1 \# \Omega K_2 & & \xrightarrow{\quad k_2 \quad} & & \Omega K_3 \\ & & & \nearrow^{k_1} & \\ \Big\downarrow{\scriptstyle 1 \# j_2} & & \Omega K_1 \# E_2 & & \Big\downarrow{\scriptstyle j_3} \\ & \swarrow^{j_1 \# 1} & & & \\ E_1 \# E_2 & & \xrightarrow{\quad l \quad} & & E_3. \end{array}$$

We will show that the diagram commutes and so the theorem then follows form Proposition (5.4).

Let $(\omega, (a, \lambda)) \in \Omega K_1 \times E_2$. By (5.5) and (1.6),

$$k_1(\omega, (a, \lambda)) = \hat{n}_1(\omega, a)$$

and so

$$j_3 \circ k_1(\omega, (a, \lambda)) = (*, \hat{n}_1(\omega, a)).$$

On the other hand,

$$\begin{aligned} l \circ (j_1 \# 1)(\omega, (a, \lambda)) &= l((*, \omega), (a, \lambda)) \\ &= (m(*, a), g_1(\hat{n}_1(\omega, a), \hat{n}_2(*, \lambda)) \\ &= (*, g_1(\hat{n}_1(\omega, a), *)) \\ &= (*, \hat{n}_1(\omega, a)). \end{aligned}$$

(Here we use the fact that $*$ is a strict identity in K_3.) Thus

$$j_3 \circ k_1 = l \circ (j_1 \# 1).$$

Commutativity in the other half of the diagram is proved in the same way and so the proof is complete.

Remark. Suppose we are given a map l, as defined in (5.3). One then obtains from diagram (5.1) the following diagram, by using construction (1.7):

$$\begin{array}{ccc} \Omega E_1 \# E_2 & \xrightarrow{\ \hat{l}\ } & \Omega E_3 \\ \Big\downarrow{\scriptstyle \Omega p_1 \times p_2} & & \Big\downarrow{\scriptstyle \Omega p_3} \\ \Omega A_1 \# A_2 & \xrightarrow{\ \hat{m}\ } & \Omega A_3. \end{array}$$

Moreover, Theorem (5.6) remains true for $\hat{m}$, $\hat{l}$, etc. In this way one obtains Cartan formulae for stable higher order operations. See [6, §§ 1–2] for more details.

6. Cartan Product Formulae

We now use Theorem (5.6) to give a rather more explicit form for a Cartan formula. Recall how cohomology classes in E_3 (i.e., higher order operations) arise. Let $\iota_1, \ldots, \iota_r$ be classes in $H^*(K_3)$ (Z or Z_p coefficients, p a prime) and let $\alpha_1, \ldots, \alpha_r \in \mathscr{A}_p$ (the mod p Steenrod algebra). Suppose that

$$(6.1) \qquad \sum \alpha_i (w_3^* \iota_i) = 0,$$

and that the dimension of this relation is stable with respect to p_3. Then, by the Serre exact sequence, there is a class $\phi \in H^*(E_3)$, such that

$$(6.2) \qquad j_3^* \phi = \sum |\alpha_i| (\sigma \iota_i).$$

(Recall that $|\alpha| = (-1)^{\deg \alpha} \alpha$, in $\mathscr{A}_p$.) We wish to express $l^* \phi$ in terms of this data. By Theorem (5.6) we need to calculate $k_a^*(\sigma \iota_i)$, $a = 1, 2$, $i = 1, \ldots, r$; let us consider this problem in isolation. Suppose that ι is any class in $H^*(K_3)$, and suppose that

$$n_1^*(\iota) = \sum c_j \otimes b_j,$$

in $H^*(K_1) \otimes H^*(A_2)$. Then, by (5.5),

$$(6.3) \qquad k_1^*(\sigma \iota) = \sum \sigma c_j \otimes p_2^* b_j,$$

in $H^*(\Omega K_1) \otimes H^*(E_2)$. A similar expression obtains for $k_2^*(\sigma \iota)$, given an expression for $n_2^*(\iota)$. Thus, the point is: if we know $n_1^*(\iota_i)$ and $n_2^*(\iota_i)$, we then can calculate $k_1^* j_3^* \phi$ and $k_2^* j_3^* \phi$, using (6.2) and (6.3).

Notice that

$$k_1^* j_3^*(\phi) \in H^*(\Omega K_1) \otimes p_2^* H^*(A_2),$$
$$k_2^* j_3^*(\phi) \in p_1^* H^*(A_1) \otimes H^*(\Omega K_2).$$

Thus we can choose classes $\{a_i'\}$ in $H^*(A_1)$ and $\{a_j''\}$ in $H^*(A_2)$ such that the classes $\{p_1^* a_i'\}$, $\{p_2^* a_j''\}$ are linearly independent in $p_1^* H^*(A_1)$, respectively, $p_2^* H^*(A_2)$, and such that

$$(6.4) \qquad \begin{aligned} k_1^* j_3^* \phi &= \sum \kappa_j' \otimes p_2^* a_j'', \\ k_2^* j_3^* \phi &= \sum p_1^* a_i' \otimes \kappa_i'', \end{aligned}$$

where $\{\kappa_j'\}$ are classes in $H^*(\Omega K_1)$ and $\{\kappa_i''\}$ classes in $H^*(\Omega K_2)$.

(6.5) **Theorem.** *Let ϕ be a class in $H^*(E_3)$ with $k_a^* j_3^* \phi$, $a = 1, 2$, given in* (6.4). *Suppose that* $\dim \phi$ *is stable with respect to p_1 and p_2. Then there are classes $\{\psi_j'\}$ in $H^*(E_1)$ and $\{\psi_i''\}$ in $H^*(E_2)$ such that*

$$j_1^* \psi_j' = \kappa_j', \qquad j_2^* \psi_i'' = \kappa_i''.$$

Moreover,

$$l^* \phi \equiv \sum \psi'_j \otimes p_2^* a''_j + \sum p_1^* a'_i \otimes \psi''_i,$$

modulo image $(p_1 \# p_2)^*$.

Proof. Recall the classifying map $w_i: A_i \to K_i$, $i=1, 2$. Set

$$w'_i = \Omega w_i: \Omega A_i \to \Omega K_i.$$

Now $j_i \circ w'_i \simeq *$, and so (taking $i=1$),

$$(w'_1 \# 1)^* k_1^* j_3^* \phi = (w'_1 \# 1)^* (j_1 \# 1)^* l^* \phi = 0.$$

Therefore, by (6.4) we find that

$$\sum w'^*_i \kappa'_j \otimes p_2^* a''_j = 0.$$

But the classes $\{p_2^* a''_j\}$ are linearly independent. Hence, $w'^*_i \kappa'_j = 0$, for each j, and so by exactness there is a class ψ'_j with $j_1^* \psi'_j = \kappa'_j$. Similarly, we obtain the classes $\{\kappa''_i\}$. Set

$$\omega = \sum \psi'_j \otimes p_2^* a''_j + \sum p_1^* a'_i \otimes \psi''_i.$$

Since $j_i^* p_i^* = 0$, $i=1, 2$, we find that

$$(j_1 \# 1)^* \omega = (j_1 \# 1)\, l^* \phi,$$
$$(1 \# j_2)^* \omega = (1 \# j_2)^* l^* \phi,$$

and so by (5.4), $l^* \phi \equiv \omega$, up to image $(p_1 \# p_2)^*$. This completes the proof.

Let Φ denote the higher order operation given by ϕ and $\{\Psi'_j\}$, $\{\Psi''_i\}$ the operations corresponding to $\{\psi'_j\}$, $\{\psi''_i\}$. Let X be a complex and let $\xi_i \in [X, A_i]$, $i=1, 2$. We now set

$$\xi_1 \cup \xi_2 = m_*(\xi_1 \# \xi_2).$$

Finally, let u be a class in $H^*(A_1) \otimes H^*(A_2)$ such that

$$l^* \phi - \left(\sum \psi'_j \otimes p_2^* a''_j + \sum p_1^* a'_i \otimes \psi''_i\right) = (p_1 \# p_2)^* u.$$

Say, $u = \sum u'_k \otimes u''_k$.

(6.6) **Corollary.** *Suppose that* $w_i(\xi_i) = 0$, $i=1, 2$. *Then* $\Phi(\xi_1 \cup \xi_2)$ *is defined and*

$$\sum [\Psi'_j(\xi_1) \cup a''_j(\xi_2)] + \sum [a'_i(\xi_1) \cup \Psi''_i(\xi_2)] + \sum u'_k(\xi_1) \cup u''_k(\xi_2) \subset \Phi(\xi_1 \cup \xi_2).$$

Of course the unknown term here is the class u. However, in many applications one finds that $(p_1 \# p_2)^*$ is injective in $\dim \phi$ (i.e., $u=0$). Note also Remark 4.2.6 in [6].

Remark. An advantage to the cochain method of Kristensen is that one obtains explicit information about the class u.

Examples. One can obtain the various examples given by Milgram [6] using Theorem (6.5). These examples all have mod 2 coefficients, so for variety we do out an example using mod p coefficients, $p > 2$.

Let P^i denote the i^{th} mod p Steenrod reduced power, $i \geqq 0$, and let β denote the mod p Bockstein operator. By Adem [1] one has the following relation: for $t \geqq 1$,

(6.8) $$\mathrm{P}^{t+1}\beta + t\beta\,\mathrm{P}^{t+1} - \mathrm{P}^1\beta\,\mathrm{P}^t = 0.$$

Let Φ_{t+1} denote a (stable) secondary operation associated with this relation. Φ_{t+1} has degree $a(t+1)$, where $a = 2(p-1)$. If $1 \leqq t \leqq p-1$, one can show that Φ_{t+1} is unique; there are two choices for Φ_{p+1}, differing by the primary operation $\mathrm{P}^p\,\mathrm{P}^1$.

To state our result we also will need the unique operation – call it Ψ - associated with the relation

$$\mathrm{P}^{p-1}\,\mathrm{P}^1 = 0.$$

(6.9) **Theorem.** *Let u and v be* mod p *cohomology classes for a space X and suppose that*

$$\beta u = \mathrm{P}^1 u = \mathrm{P}^p u = 0,$$

$$\beta v = \mathrm{P}^p v = 0.$$

Then, $\Phi_{p+1}(u \cup v)$ is defined and (setting $\varepsilon = (-1)^{\dim u}$),

$$\Phi_{p+1}(u) \cup v + \varepsilon u \cup \Phi_{p+1}(v) - \varepsilon\,\Psi(u) \cup \beta\,\mathrm{P}^1(v) + \sum_{n=1}^{p-1} \Phi_{n+1}(u) \cup \mathrm{P}^{p-n}(v) \equiv \Phi_{p+1}(u \cup v)$$

modulo the common indeterminacy.

Proof. We work in the universal example. Set

$$L_n = K_{n+1} \times K_{n+ap}, \qquad n \geqq 1,$$

and define

$$w_n\colon K_n \to L_n$$

by

$$w_n^*\,\iota_{n+1} = \beta\,\iota_n,$$

$$w_n^*\,\iota_{n+ap} = \mathrm{P}^p\,\iota_n.$$

Let

$$\Omega L_n \xrightarrow{j_n} E_n \xrightarrow{p_n} K_n,$$

denote the principal fibre space with w_n as classifying map. Notice that taking $t = p$ in relation (6.8), the middle term drops out; we choose a representative ϕ_{p+1} for Φ_{p+1}, to be a class in $H^{n+a(p+1)}(E_n)$ such that

(6.10) $$j_n^*\,\phi_{p+1} = \mathrm{P}^{p+1}\,\iota_n + \mathrm{P}^1\beta\,\iota_{n+ap-1}.$$

(We use the fact that β anti-commutes with the transgression [12].)

Suppose now that $r = \deg u$, and $s = \deg v$; set $t = r + s$. Over K_r we consider the following fibration. Define

$$v_r\colon K_r \to L_r \times K_{r+a}$$

by

$$v_r^* \iota_{r+1} = \beta \iota_r, \quad v_r^* \iota_{r+ap} = \mathrm{P}^p \iota_r, \quad v_r^* \iota_{r+a} = \mathrm{P}^1 \iota_r.$$

Let

$$\Omega(L_r \times K_{r+a}) \xrightarrow{\tilde{j}_r} \tilde{E}_r \xrightarrow{\tilde{p}_r} K_r$$

denote the principal fibration with v_r as classifying map. Let $\tilde{\phi}_{n+1}$, $1 \leqq n \leqq p$ be the unique class in $H^{r+a(n+1)}(\tilde{E}_r)$ such that

(6.11) (a) for $1 \leqq n \leqq p-2$,

$$\tilde{j}_r^* \tilde{\phi}_{n+1} = \mathrm{P}^{n+1} \iota_r - \frac{n}{n+1} \beta \mathrm{P}^n \iota_{r+a-1} + \frac{1}{n} \mathrm{P}^1 \beta \mathrm{P}^{n-1} \iota_{r+a-1}.$$

(b) $$\tilde{j}_r^* \tilde{\phi}_p = \mathrm{P}^p \iota_r + \beta \iota_{r+ap-1} + \frac{1}{p-1} \mathrm{P}^1 \beta \mathrm{P}^{p-2} \iota_{r+a-1},$$

(c) $$\tilde{j}_r^* \tilde{\phi}_{p+1} = \mathrm{P}^{p+1} \iota_r + \mathrm{P}^1 \beta \iota_{r+ap-1}.$$

Here the fractions $n/n+1$, etc., are to be taken in the multiplicative field of mod p residue classes. One can check, using the Adem relations that the classes in (a)-(c) do indeed transgress to relation (6.8) in the base – using the fact that $\mathrm{P}^1 \mathrm{P}^b = (1+b) \mathrm{P}^{1+b}$. Finally, let ψ denote the unique class in $H^{r+ap-1}(E_r)$ such that

(6.11) $$\tilde{j}_r^*(\psi) = \mathrm{P}^{p-1} \iota_{r+a-1}.$$

Now define

$$m\colon K_r \# K_s \to K_t$$

by $m^* \iota_t = \iota_r \otimes \iota_s$. We show in a moment that m is a Cartan map (relative to v_r, w_s, w_t) and hence there is a lifting $l\colon (\tilde{E}_r \# E_s) \to E_t$. The proof of Theorem (6.9) consists then in showing (setting $e_n = p_n^* \iota_n$, $n = r, s, t$):

(6.12) $$l^* \phi_{p+1} = \tilde{\phi}_{p+1} \otimes e_s + \varepsilon e_r \otimes \phi_{p+1} - \varepsilon \psi \otimes \beta \mathrm{P}^1 e_s + \sum_{n=1}^{p-1} \tilde{\phi}_{n+1} \otimes \mathrm{P}^{p-n} e_s.$$

To show that m is a Cartan map we define maps

$$(L_r \times K_{r+a}) \times K_s \xrightarrow{n_1} L_t,$$
$$K_r \times L_s \xrightarrow{n_2} L_t,$$

by

(6.13)
$$n_1^* \iota_{t+1} = \iota_{r+1} \otimes \iota_s,$$
$$n_1^* \iota_{t+ap} = \iota_{r+ap} \otimes \iota_s + \sum_{i=1}^{p-1} \left(\frac{1}{i}\right) \mathrm{P}^{i-1} \iota_{r+a} \otimes \mathrm{P}^{p-i} \iota_s,$$
$$n_2^* \iota_{t+1} = \varepsilon \iota_r \otimes \iota_{s+1},$$
$$n_2^* \iota_{t+ap} = \iota_r \otimes \iota_{s+ap}.$$

(Here $\varepsilon=(-1)^r$.) Since

$$\beta(\iota_r\otimes\iota_s)=\beta\,\iota_r\otimes\iota_s+\varepsilon\,\iota_r\otimes\beta\,\iota_s,$$

$$\mathrm{P}^p(\iota_r\otimes\iota_s)=\sum_{i=0}^{p}\mathrm{P}^i\,\iota_r\otimes\mathrm{P}^{p-i}\,\iota_s,$$

one easily checks that m is a Cartan map, using n_1, n_2 as defined above. Thus the lifting l exists.

Let

$$\Omega(L_r\times K_{r+a})\times E_s\xrightarrow{k_1}\Omega L_t,$$

$$E_r\times\Omega L_s\xrightarrow{k_2}\Omega L_t,$$

be the maps given in (5.5). By (6.13) and (1.7) we have:

$$k_1^*\,\iota_t=\iota_r\otimes e_s,$$

$$k_1^*\,\iota_{t+ap-1}=\iota_{r+ap-1}\otimes e_s+\sum_{i=1}^{p-1}\left(\frac{1}{i}\right)\mathrm{P}^{p-i}\,\iota_{r+a-1}\otimes\mathrm{P}^{p-i}\,e_s,$$

$$k_2^*\,\iota_t=\varepsilon\,e_r\otimes\iota_s,$$

$$k_2^*\,\iota_{t+ap-1}=e_r\otimes\iota_{s+ap-1}.$$

Now by construction,

$$\beta\,e_r=\mathrm{P}^1\,e_r=\mathrm{P}^p\,e_r=0,$$

$$\beta\,e_s=\mathrm{P}^p\,e_s=0.$$

Using this fact we find that

$$\begin{aligned}k_1^*j_t^*\,\phi_{p+1}=&\sum_{i=1}^{p}\mathrm{P}^{i+1}\,\iota_r\otimes\mathrm{P}^{p-i}\,e_s+\mathrm{P}^1\beta\,\iota_{r+ap-1}\otimes e_s+\beta\,\iota_{r+ap-1}\otimes\mathrm{P}^1\,e_s\\&+\sum_{i=1}^{p-1}\left(\frac{1}{i}\right)[\mathrm{P}^1\beta P^{i-1}\,\iota_{r+a-1}\otimes\mathrm{P}^{p-i}\,e_s+\beta\,\mathrm{P}^{i-1}\,\iota_{r+a-1}\otimes\mathrm{P}^1\,\mathrm{P}^{p-i}\,e_s]\\&-\sum_{i=1}^{p-1}\left(\frac{\varepsilon}{i}\right)[\mathrm{P}^1\,\mathrm{P}^{i-1}\,\iota_{r+a-1}\otimes\beta\,\mathrm{P}^{p-i}\,e_s+\mathrm{P}^{i-1}\,\iota_{r+a-1}\otimes\mathrm{P}^1\beta\,\mathrm{P}^{p-i}\,e_s].\end{aligned}$$

Using the fact that

$$\mathrm{P}^1\beta\,\mathrm{P}^{p-i}\,e_s=-i\,\beta\,\mathrm{P}^{p-i+1}\,e_s,$$

the above equation can be simplified to read:

$$\begin{aligned}k_1^*j_t^*\,\phi_{p+1}=&(\mathrm{P}^{p+1}\,\iota_r+\mathrm{P}^1\beta\,\iota_{r+ap-1})\otimes e_s\\&+\left(\mathrm{P}^p\,\iota_r+\beta\,\iota_{r+ap-1}+\frac{1}{p-1}\,\mathrm{P}^1\beta\,\mathrm{P}^{p-2}\,\iota_{r+a-1}\right)\otimes\mathrm{P}^1\,e_s\\&+\sum_{j=1}^{p-2}\left(\mathrm{P}^{j+1}\,\iota_r+\frac{1}{j}\,\mathrm{P}^1\beta\,\mathrm{P}^{j-1}\,\iota_{r+a-1}-\frac{j}{j+1}\,\beta\,\mathrm{P}^j\,\iota_{r+a-1}\right)\otimes\mathrm{P}^{p-j}\,e_s\\&+\mathrm{P}^{p-1}\,\iota_{r+a-1}\otimes\beta\,\mathrm{P}^1\,e_s.\end{aligned}$$

Therefore, by (6.11),

$$k_1^* j_t^* \phi_{p+1} = \sum_{i=1}^{p} j_r^* \tilde{\phi}_{i+1} \otimes \mathrm{P}^{p-i} e_s + j_r^* \psi \otimes \beta \, \mathrm{P}^1 e_s.$$

Similarly, we show that

$$k_2^* j_t^* \phi_{p+1} = \varepsilon e_r \otimes j_s^* \phi_{p+1},$$

and so by Theorem (6.5),

$$l^* \phi_{p+1} = w + (p_r \# p_s)^* u$$

where

$$w = \sum_{i=1}^{p} \tilde{\phi}_{i+1} \otimes \mathrm{P}^{p-i} e_s + \varepsilon e_r \otimes \phi_{p+1} - \varepsilon \psi \otimes \beta \mathrm{P}^1 e_s,$$

and where $u \in H^{t+a(p+1)}(K_r \# K_s)$. Since $\beta e_s = \mathrm{P}^1 e_s = \mathrm{P}^p e_s = 0$, it is easily seen that $u = \lambda(\mathrm{P}^p \mathrm{P}^1 \iota_r \otimes \iota_s)$, where $\lambda \in Z_p$. Set

$$\tilde{\phi}'_{p+1} = \tilde{\phi}_{p+1} + \lambda(\mathrm{P}^p \mathrm{P}^1 e_r).$$

This class also represents Φ_{p+1}, and with this choice Eq. (6.12) is now proved. This completes the proof of the theorem.

7. *A Distinction*

We have discussed separately Whitney formulae and Cartan formulae, although, as observed in the Introduction, there really is but a single question: given a higher order characteristic class θ, evaluate θ on $\xi \oplus \eta$. However, it seems from recent applications that one does need to consider two distinct cases. In both cases one starts with three classifying spaces (A, B, C in the Introduction). One can then either (i) take principal fibrations over only two of these, as in § 2; or (ii), take principal fibrations over all three, as in § 5. In each case one then has the problem of finding the lifting l and then evaluating l^* on a class θ. In practice the results for higher order characteristic classes for bundles have been of type (i), while higher order cohomology operations have fallen under type (ii). And so we have adopted the names Whitney and Cartan to describe in general these respective cases. Notice, finally, that Theorem 4.5 is the proverbial exception that proves the rule. While it is so that here we take fibrations over all three classifying spaces, the point is that we do this in two steps – each of which is of type (i).

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