

THE SUSPENSION OF THE GENERALIZED PONTRJAGIN COHOMOLOGY OPERATIONS

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1. The main theorem. In a previous paper [9] I have defined a sequence of new cohomology operations, called the *generalized Pontrjagin operations*. These operations use as coefficient groups the summands of a certain type of graded ring: namely, *a ring with divided powers* (defined by H. Cartan in [1]), which is termed a Γ -ring in [9]. Let $A = \sum_k A_k$ be a ring with divided powers such that each summand A_k is a cyclic group of infinite or prime power order; we termed such rings *p-cyclic* in [9]. Then, the Pontrjagin operations are functions

$$\mathfrak{P}_t: H^{2n}(X; A_{2k}) \longrightarrow H^{2tn}(X; A_{2tk}) \quad (k, n > 0; t = 0, 1, \dots)$$

where $H^q(Y, B; G)$ denotes the q th (singular) cohomology group of the pair (Y, B) with coefficients in the group G .

Let C be a cohomology operation relative to integers r, s and coefficient groups G, H . That is, C is a natural transformation

$$C: H^r(Y, B; G) \longrightarrow H^s(Y, B; H).$$

With each operation C we associate a second operation, $S(C)$, called the *suspension* of C . $S(C)$ is a natural transformation

$$H^{r-1}(Y, B; G) \longrightarrow H^{s-1}(Y, B; H);$$

its definition is given in § 3.

The purpose of this note is to determine $S(\mathfrak{P}_t)$, where $\mathfrak{P}_t$ is the generalized Pontrjagin operation. In order to state our result concerning $S(\mathfrak{P}_t)$, we need an additional cohomology operation, the Postnikov square (see [3], [10]). This was defined in [9], but only for a restricted class of coefficient groups. In this paper we will define the Postnikov square as a cohomology operation

$$p: H^q(Y, B; A_{2k}) \longrightarrow H^{2q+1}(Y, B; A_{4k}), \quad (q, k > 0)$$

where A_{2k} is an even summand of a p -cyclic ring with divided powers.

We now may state the main result of the paper.

THEOREM I. *For any cohomology operation C , let $S(C)$ denote the suspension of the operation C . Then,*

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- (i) $S(\mathfrak{F}_2) = \mathfrak{p}$
(ii) $S(\mathfrak{F}_t) = 0$, ($t > 2$)

where 0 denotes the zero cohomology operation.

The proof of Theorem I is given in § 5. In § 2 we define the operation $\mathfrak{p}$, while in § 3 we give the definition of the suspension. In § 4 we discuss relative cohomology operations, while in § 6 we give some additional properties of the operation $\mathfrak{p}$. In particular, we show that $S(\mathfrak{p}) = 0$. Finally, the last section gives the theorem, $\delta S(C) = C\delta$, for any operation C .

I would like to thank Professor N. E. Steenrod for the valuable suggestions made to me at the time of revising the paper. In particular the definition of the suspension in § 3 and Theorem 7.1 are due to him.

2. The definition of the Postnikov square. The definition of the Postnikov square, $\mathfrak{p}$, is obtained by first defining a "model operation", p , which uses only a restricted category $\mathcal{C}$ of coefficient groups. The category $\mathcal{C}$ is defined as follows: let $Z_r = Z/rZ$ ($r = 0, 1, \dots$), where $Z = \text{integers} = Z_0$. Denote by $\mathcal{C}$ the category of all groups of the form Z_θ , where θ is zero or a power of a prime. For each group Z_θ in $\mathcal{C}$ we have defined a p -cyclic ring with divided powers,

$$G(Z_\theta) = G_0(Z_\theta) + \dots + G_t(Z_\theta) + \dots \text{ (direct sum) (see [9; 1.17]).}$$

In particular,

$$G_2(Z_\theta) = \begin{cases} Z_\theta, & \text{if } \theta \text{ is zero or odd} \\ Z_{2\theta}, & \text{if } \theta \text{ is a power of 2.} \end{cases}$$

We define a generator for $G_2(Z_\theta)$ by

$$g_2(1_\theta) = \begin{cases} 1_\theta, & \text{if } \theta \text{ is zero or odd} \\ 1_{2\theta}, & \text{if } \theta \text{ is a power of 2} \end{cases}$$

where $1_r = 1 \bmod r$ ($r = 0, 1, \dots$). The group $G_2(Z_\theta)$ will be the coefficient domain for the operation $\mathfrak{p}$. As remarked in [9; § 2], once we have defined the operation $\mathfrak{p}$ for the category of *regular cell complexes*, the definition easily extends to the category of all topological spaces. Hence, in what follows we restrict attention to regular cell complexes, which we will simply term *complexes*.

Let K be a complex and L a subcomplex of K . Let Z_θ be a group in the category $\mathcal{C}$; that is, θ is zero or a power of a prime. We define an operation

$$p: H^q(K, L; Z_\theta) \longrightarrow H^{2q+1}(K, L; G_2(Z_\theta))$$

as follows. Let $u \in H^q(K, L; Z_\theta)$; let β be the homomorphism from Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$. Define

$$(2.1) \quad p(u) = \beta_*(u \cup \delta_* u) .$$

Here, δ_* is the Bockstein coboundary operator associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{\theta} Z \longrightarrow Z_\theta \longrightarrow 0 ,$$

and the cup-product is taken relative to the natural pairing $Z_\theta \otimes Z \approx Z_\theta$.

It is easily seen that this agrees with the usual definition of the operation p (see [3] and [10]). For let $\bar{u} \in C^q(K, L; Z)$ be a cochain representing u ; that is, $\delta\bar{u} = \theta\bar{v}$, for some cochain $\bar{v} \in C^{q+1}(K, L; Z)$. Then, a cocycle representing $\beta_*(u \cup \delta_* u)$ is given by $\bar{u} \cup \delta\bar{u}$, which coincides with the definition given in [10].

In [9; 8.14] we defined a function w which goes from $H^q(K; Z_\theta)$ to $H^{2q+1}(K; Z)$. This function can be extended to the relative case, following the method given in § 4. When this is done it is easily shown that

$$(2.2) \quad p(u) = \beta_* w(u) ,$$

a result we will need later.

The Postnikov square, $\mathfrak{p}$, is defined using the operation p as follows: let $u \in H^q(K, L; A_{2k})$, where A_{2k} is an even summand of a p -cyclic ring with divided powers. By hypothesis, A_{2k} is a cyclic group whose order is infinite or a power of a prime. Thus, there is an integer θ such that A_{2k} is isomorphic to Z_θ , where $Z_\theta \in \mathcal{C}$. Let ν be an isomorphism from A_{2k} to Z_θ . Then, by 3.1 in [9], for each non-negative integer r we have defined a homomorphism ζ_r mapping $G_r(Z_\theta)$ to A_{2rk} , which is an extension of ν^{-1} . We define the operation $\mathfrak{p}$ by

$$(2.3) \quad \mathfrak{p}(u) = \zeta_2^* p \nu_*(u) ;$$

that is, $\mathfrak{p}$ is the composition of the following functions:

$$\begin{aligned} H^q(K, L; A_{2k}) &\xrightarrow{\nu_*} H^q(K, L; Z_\theta) \xrightarrow{p} \\ H^{2q+1}(K, L; G_2(Z_\theta)) &\xrightarrow{\zeta_2^*} H^{2q+1}(K, L; A_{4k}) . \end{aligned}$$

We show the independence of this definition from the particular choice of the isomorphism ν (and hence ζ_2). This is a consequence of the fact that

$$(2.4) \text{ LEMMA.} \quad p\alpha_* = G_2(\alpha)_* p ,$$

where α is a homomorphism from Z_θ to a group Z_τ in $\mathcal{C}$, and $G_2(\alpha)$ is the homomorphism from $G_2(Z_\theta)$ to $G_2(Z_\tau)$ induced by the functor G (see [9; 1.23]).

Using 2.2, the proof of 2.4 is entirely similar to that given for 5.22 in [9] and is omitted here. From 2.4 the proof of the independence of

the definition of $\mathfrak{p}$ follows along exactly the same lines as 3.5 and 3.6 in [9]; we omit the details.

3. Suspension of cohomology operations. The definition of the suspension used here is due to N. E. Steenrod¹. Let I denote the unit interval, $[0, 1]$, and $\dot{I}$ the subspace $\{0\} \cup \{1\}$. The group $H^1(I, \dot{I}; Z)$ is cyclic infinite; let v be a fixed generator. For each space X and coefficient group G define a function ϕ from $H^q(X; G)$ to $H^{q+1}(I \times X, \dot{I} \times X; G)$ by

$$(3.1) \quad \phi(u) = v \times u.$$

We use singular cohomology for X , and the natural pairing $Z \otimes G \approx G$ for the cross-product. In § 7 we prove the following lemma.

(3.2) LEMMA. *The function ϕ is an isomorphism mapping $H^q(X; G)$ onto $H^{q+1}(I \times X, \dot{I} \times X; G)$ ($q > 0$).*

Consider now any cohomology operation C , which is defined on relative cohomology groups; say, C maps $H^r(X, A; G)$ to $H^s(X, A; H)$ for each pair (X, A) . Define an absolute cohomology operation, $S(C)$, which maps $H^{r-1}(Y; G)$ to $H^{s-1}(Y; H)$, for each space Y , by

$$(3.3) \quad S(C)(u) = \phi^{-1} C\phi(u) \quad (u \in H^{r-1}(Y; G)).$$

Using the method described in § 4 we may extend $S(C)$ to an operation defined on relative cohomology groups, an operation which we continue to denote by $S(C)$. We wish to apply this construction to the operation $\mathfrak{A}$; as defined in [9], this is just an absolute operation. Thus, to use Definition 3.3 we must first extend the definition of $\mathfrak{A}_t$ to the relative case.

4. Relative cohomology operations. Let $O(q, r; G, H)$ denote the set of absolute cohomology operations relative to dimensions q, r and coefficient groups G, H ; that is, if $C \in O(q, r; G, H)$, then $C: H^q(X; G) \rightarrow H^r(X, H)$ for each space X . As is well-known the set $O(q, r; G, H)$ is in 1-1 correspondance with the group $H^r(K; H)$, where K is an Eilenberg-MacLane space of type (G, q) . The correspondance is obtained by assigning $C(\iota)$ to ι , where ι is the fundamental class in $H^q(K; G)$. Choose now a base point $e \in K$, and let $\alpha^*: H^*(K, e; A) \approx H^*(K; A)$ be the isomorphism induced by the inclusion $K \subset (K, e)$. For any CW-complex X and subcomplex A , the homotopy classes of maps $(X, A) \rightarrow (K, e)$

¹ This definition has the advantage that it can be used in the case of cohomology with local coefficients.

are in one-to-one correspondance with $H^q(X, A; G)$. Thus we define a relative cohomology operation, C' , associated with an absolute operation, C , as follows:

$$(4.1) \quad C'(u) = f^* \alpha^{*-1} C(\iota) ,$$

where $u \in H^q(X, A; G)$ and f is a map $(X, A) \rightarrow (K, e)$ such that

$$f^* \alpha^{*-1}(\iota) = u .$$

With the operation C' defined, one is then interested in whether the properties of C extend to the operation C' . We now prove a general lemma which essentially asserts that all the properties of C' do carry over to C' .

Let $O(q_1, \dots, q_n, r; G_1, \dots, G_n, H)$ denote the group of absolute cohomology operations, T , in n variables; that is, if $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$), then, $T(u_1, \dots, u_n) \in H^r(X; H)$. The operation T extends to a relative operation, T' , using the method just given for operations of a single variable. Suppose now we are given absolute cohomology operations

$$C \in O(q_1, \dots, q_n, r; G_1, \dots, G_n, H) ,$$

$$E \in O(s_1, \dots, s_p, r; H_1, \dots, H_p, H) ,$$

$$\text{and} \quad D_i \in O(q_1, \dots, q_n, s_i; G_1, \dots, G_n, H_i) \quad (i = 1, 2, \dots, p).$$

Let C', E', D_i , be the corresponding relative operations.

(4.2) PROPOSITION. *Suppose that for each space X and cohomology classes $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$), we have*

$$C(u_1, \dots, u_n) = E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n)) .$$

Then, for each pair (X, A) and classes $u'_i \in H^{q_i}(X, A; G_i)$ ($i = 1, \dots, n$), we have

$$C'(u'_1, \dots, u'_n) = E'(D'_1(u'_1, \dots, u'_n), \dots, D'_p(u'_1, \dots, u'_n)) .$$

We give the proof at the end of this section, first illustrating the theorem by giving several corollaries.

(4.3) COROLLARY 1. *Let $C \in O(q, s; R, S)$, $D_i \in O(q_i, s_i; R, S)$ ($i = 1, 2$), where R, S are rings, $q = q_1 + q_2$, and $s = s_1 + s_2$. Suppose that*

$$C(u_1 \cup u_2) = D_1(u_1) \cup D_2(u_2)$$

for all classes $u_i \in H^{q_i}(X; R)$. Then,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) ,$$

for all classes $u'_i \in (H^{q_i}(X, A; R))$.

Proof. Let $E_R \in O(q_1, q_2, q; R, R, R)$ and $E_S \in O(s_1, s_2, s; S, S, S)$ be the respective cup-products. Let F be the composite operation $C \circ E_R$. Using Proposition 4.2 we see that $F' = C' \circ E'_R$. But since $F(u_1, u_2) = E_S(D_1(u_1), D_2(u_2))$, again using 4.2 we see that

$$F'(u'_1, u'_2) = E'_S(D'_1(u'_1), D'_2(u'_2)) ;$$

that is,

$$C'(u'_1 \cup u'_2) = D'_1(u'_1) \cup D'_2(u'_2) ,$$

as was to be shown.

Let C, D_1, D_2 be the same operations as in Corollary 1. Then,

(4.4) COROLLARY 2. $C'(u'_1 \times u'_2) = D'_1(u'_1) \times D'_2(u'_2)$, where $u_i \in H^{q_i}(X_i, A_i; R)$ ($i = 1, 2$).

Proof. Let $p_1: (X_1 \times X_2, A_1 \times X_2) \rightarrow (X_1, A_1)$, $p_2: (X_1 \times X_2, X_1 \times A_2) \rightarrow (X_2, A_2)$ be projections. Then,

$$u'_1 \times u'_2 = p_1^*(u'_1) \cup p_2^*(u'_2) .$$

Thus,

$$\begin{aligned} C'(u'_1 \times u'_2) &= C'(p_1^*u'_1 \cup p_2^*u'_2) = D'_1(p_1^*u'_1) \cup D'_2(p_2^*u'_2) \\ &= p_1^*(D'_1u'_1) \cup p_2^*(D'_2u'_2) = (D'_1u'_1) \times (D'_2u'_2) . \end{aligned}$$

Here we have used Corollary 1 and the naturality of the cohomology operations involved.

To apply this to the operations $\mathfrak{P}_t$, recall the way in which these operations were defined (see § 3 in [9]). We defined a set of “model operations”, P_t , which used as coefficient groups only the groups of the category $\mathcal{C}$ (see § 2). The operations $\mathfrak{P}_t$ were then defined by composing the operation P_t with coefficient group homomorphisms; that is, precisely the same pattern as followed in Definition 2.3. Thus, the operations $\mathfrak{P}_t$ are defined in the relative case by simply applying the method given in this section to the operations P_t .

Let P'_t be the relative operation obtained from P_t . We note several facts needed later.

(4.5) LEMMA. Let $u_i \in H^{q_i}(X_i, A_i; Z_\theta)$ ($i = 1, 2$), where $Z_\theta \in \mathcal{C}$. Then

$$(1) \quad P'_t(u_1 \times u_2) = P'_t(u_1) \times P'_t(u_2) \quad (t \text{ odd}) .$$

If $t=2$ and θ is a power of 2, then,

$$(2) \quad \begin{aligned} P'_2(u_1 \times u_2) &= P'_2(u_1) \times P'_2(u_2) \\ &+ \nu_*[Sq_1(u_1) \times \mu_* w(u_2) + \mu_* w(u_1) \times Sq_1(u_2)]. \end{aligned}$$

Here, ν is the homomorphism of Z_2 to $G_2(Z_\theta)$ given by $\nu(1_\theta) = \theta g_2(1_\theta)$, and μ is the factor homomorphism $Z_\theta \rightarrow Z_2$. The functions Sq and w are defined respectively in 9.6 and 8.14 of [9].

Proof. The first statement is a consequence of Corollary 4.3 and the fact that the absolute operations P_i satisfy this formula². Equation 4.5(2) was remarked in [9; § 13] for the absolute operations P_i , and the case $\dim u_i$ odd. But it follows from 8.12 in [9] that 4.5(2) holds in general. In fact Theorem 8.11 in [9] can be obtained at once from equation 4.5(2). The extension of the equation to the relative operation P'_i , follows then from application of Proposition 4.2.

Combining Proposition 4.2 and 8.2 of [9] we also obtain

(4.6) LEMMA. *Let t be an integer where $t = p_k \cdots p_1$ (p_i prime). Let $u \in H^{2q}(X, A; Z)$. ($Z \in \mathcal{C}$). Then,*

$$P'_t(u) = P'_{p_k} \circ \cdots \circ P'_{p_1}(u).$$

Since it is in fact the relative operation, P'_i , we will work with, from now on we drop the prime, writing only P_i for both the relative and absolute operation.

Proof of Proposition 4.2. Let $Y = K(G_1, q_1) \times \cdots \times K(G_n, q_n)$, where each $K(G_i, q_i)$ is on Eilenberg-MacLane space of type (G_i, q_i) . Let $\pi_j: Y \rightarrow K(G_j, q_j)$ ($j = 1, \dots, n$), be the projection map and set $\bar{\tau}_j = \pi_j^*(\tau_j)$, where τ_j is the characteristic class in $H^{q_j}(K(G_j, q_j); G_j)$. Let e_j be a base point in $K(G_j, q_j)$ and set $Y' = (K(G_1, q_1), e_1) \times \cdots \times (K(G_n, q_n), e_n)$. Let $\tau'_j, \bar{\tau}'_j$ be the equivalent of τ_j and $\bar{\tau}_j$. Then, Proposition 4.2 follows at once from the following three lemmas (we keep the same notation as used in Proposition 4.2)

$$(4.7) \quad C(u_1, \dots, u_n) = E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n))$$

if and only if

$$C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)).$$

$$(4.8) \quad C'(u'_1, \dots, u'_n) = E'(D'_1(u'_1, \dots, u'_n), \dots, D'_p(u'_1, \dots, u'_n))$$

if and only if

$$C'(\bar{\tau}'_1, \dots, \bar{\tau}'_n) = E'(D'_1(\bar{\tau}'_1, \dots, \bar{\tau}'_n), \dots, D'_p(\bar{\tau}'_1, \dots, \bar{\tau}'_n))$$

² The operations $\mathbb{P}_i$ are easily defined for odd dimensional classes: see [9; § 7].

(4.9) If $C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n))$

then,

$$C'(\bar{\tau}'_1, \dots, \bar{\tau}'_n) = E'(D'_1(\bar{\tau}'_1, \dots, \bar{\tau}'_n), \dots, D'_p(\bar{\tau}'_1, \dots, \bar{\tau}'_n)) .$$

We give only the proof of Lemma 4.7, the others being entirely similar. Assume first we are given classes $u_i \in H^{q_i}(X; G_i)$ ($i = 1, \dots, n$). Let $f_j: X \rightarrow K(G_j; q_j)$ be mappings such that $f_j^*(\iota_j) = u_j$. Set $f = f_1 \times \dots \times f_n: X \rightarrow Y$. Then, by naturality, one has

$$(4.10) \quad (a) \quad C(u_1, \dots, u_n) = f^*C(\bar{\tau}_1, \dots, \bar{\tau}_n) , \\ (b) \quad D_i(u_1, \dots, u_n) = f^*D_i(\bar{\tau}_1, \dots, \bar{\tau}_n) \quad (i = 1, \dots, p).$$

Suppose now that

$$C(\bar{\tau}_1, \dots, \bar{\tau}_n) = E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) .$$

Then, by 4.10,

$$C(u_1, \dots, u_n) = f^*E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) .$$

But E is natural with respect to mappings. Therefore,

$$\begin{aligned} & f^*E(D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) \\ &= E(f^*D_1(\bar{\tau}_1, \dots, \bar{\tau}_n), \dots, f^*D_p(\bar{\tau}_1, \dots, \bar{\tau}_n)) \\ &= E(D_1(u_1, \dots, u_n), \dots, D_p(u_1, \dots, u_n)) , \end{aligned}$$

again by 4.10, which completes the proof of this assertion. The proof in the opposite direction is trivial.

5. The proof of Theorem I. Recall that the operation $\mathfrak{P}_t$ is defined by means of the model operations P_t and coefficient group homomorphisms. But it is clear that the isomorphism ϕ , defined in 3.1, commutes with coefficient group homomorphisms. Thus, it suffices to prove Theorem I with $\mathfrak{P}_t$ replaced by P_t , the operation $\mathfrak{p}$ replaced by p , and the group A_{2k} taken to be a group in the category $\mathcal{C}$, say $A_{2k} = Z_0$.

Assume first that t is an odd prime p . Since ϕ is an isomorphism, the proof of Theorem I (ii) consists simply in showing

$$P_p\phi(u) = 0 , \quad u \in H^r(X; Z_0).$$

But this is immediate; for

$$P_p\phi(u) = P_p(v \times u) = P_p(\bar{v} \times u) = P_p(\bar{v}) \times P_p(u) ,$$

by Lemma 4.5(1). Here, $\bar{v}$ is a generator of $H^1(I, \dot{I}; Z_0)$. However, $P_p(\bar{v}) = 0$, by dimensionality considerations. Thus, $P_p\phi(u) = 0$; and hence, $S(P_p) = 0$.

Now, suppose that t is any integer > 1 which is not a power of 2; say, $t = mp$, where p is an odd prime. Then, by Lemma 4.6

$$P_t \phi(u) = P_m \circ P_p \phi(u) = P_m(0) = 0 .$$

Consequently,

$$S(P_t) = 0 .$$

Thus, we have proved Theorem I(ii) for the case t is not a power of 2. Before concluding the proof of part (ii), we must prove part (i). Let the classes u and v be as above, where u has coefficients in the group Z_θ . If θ is zero or odd, then by Proposition 7.4 in [9], we have

$$P_2(v \times u) = P_2(\bar{v} \times u) = (\bar{v} \times u)^2 = \pm \bar{v}^2 \times u^2 = 0 ,$$

since $\bar{v}^2 = 0$. Thus, in this case $S(P_2) = 0$. Suppose now that θ is a power of 2.

Let η be the factor map $Z \rightarrow Z_\theta$. Then, $v \times u = (\eta_* v) \times u$, where the right hand side uses the pairing $Z_\theta \otimes Z_\theta \approx Z_\theta$. Thus, using Lemma 4.5(2), we have

$$\begin{aligned} P_2(v \times u) &= P_2(\eta_* v \times u) = P_2(\eta_* v) \times P_2(u) \\ &+ \nu_* [Sq_1(\eta_* v) \times \mu_* w(u) + \mu_* w(\eta_* v) \times Sq_1(u)] . \end{aligned}$$

Now, $P_2(\eta_* v) = 0$, $w(\eta_* v) = 0$ by dimensionality considerations. Also, since $\eta_* v$ is a 1-dimensional class, $Sq_1(\eta_* v) = \xi_* v$, where ξ is the natural map $Z \rightarrow Z_2$ (see Steenrod [4; 12.6]). Thus,

$$(5.1) \quad P_2(v \times u) = \nu_* [\xi_* v \times \mu_* w(u)] .$$

Consider the following commutative diagram:

$$\begin{array}{ccc} Z \otimes Z_\theta & \xrightarrow{1 \otimes \beta} & Z \otimes G_2(Z_\theta) \\ \xi \otimes \mu \downarrow & & \approx \downarrow \omega \\ Z_2 \otimes Z_2 & \xrightarrow{\omega'} & Z_2 \xrightarrow{\nu} G_2(Z_\theta) , \end{array}$$

where β is the homomorphism of Z_θ to $G_2(Z_\theta)$ given by $\beta(1_\theta) = \theta g_2(1_\theta)$ (see 2.1). Then, from 5.1,

$$\begin{aligned} P_2(v \times u) &= \nu_* \omega'_*(\xi \otimes \mu)_*[v \otimes w(u)] \\ &= \omega_*(1 \otimes \beta)_*[v \otimes w(u)] \\ &= v \times \beta_* w(u) \\ &= v \times p(u) , \text{ by 2.2 .} \end{aligned}$$

Therefore,

$$P_2\phi(u) = P_2(v \times u) = v \times p(u) = \phi p(u) .$$

That is,

$$S(P_2) = p .$$

This proves part (i) of Theorem I. To complete the proof of the theorem we must show that

$$P_{2^r}\phi(u) = 0 , \quad (r > 1).$$

But by part (i) of Theorem I and Lemma 4.6, we have

$$\begin{aligned} P_{2^r}\phi(u) &= P_{2^{r-1}} P_2\phi(u) = P_{2^{r-1}} \phi p(u) \\ &= P_{2^{r-2}} P_2\phi p(u) = P_{2^{r-2}} \phi p(p(u)) = 0 . \end{aligned}$$

Here, we use property 6.6 of the function p , which is proved independently in the next section. This completes the proof of Theorem I.

6. The properties of the operation $\mathfrak{p}$. We give here the main properties of the Postnikov square, $\mathfrak{p}$.

(6.1) **THEOREM.** *Let X be a space, and let $A = \sum_k A_k$ be a p -cyclic ring with divided powers. Suppose that $u \in H^q(X; A_{2k})$ ($q, k > 0$). Then,³*

$$(6.2) \quad \mathfrak{p}(u) = 0, \text{ if order } A_{2k} \text{ is odd or infinite,}$$

$$(6.3) \quad 2\mathfrak{p}(u) = 0 ,$$

$$(6.4) \quad \mathfrak{p} \text{ is a homomorphism,}$$

$$(6.5) \quad \text{if order } A_{2k} = 2^i \text{ } (i > 1) \text{ and } 2u = 0, \text{ then } \mathfrak{p}(u) = 0,$$

$$(6.6) \quad \mathfrak{p}(\mathfrak{p}(u)) = 0 ,$$

$$(6.7) \quad f^*\mathfrak{p}(u) = \mathfrak{p}f^*(u) ,$$

$$(6.8) \quad \alpha_*\mathfrak{p}(u) = \mathfrak{p}\alpha_*(u) ,$$

where f^* is induced by a map f from a space Y to X , and α_* is induced by a homomorphism α from A to a p -cyclic ring with divided powers A' .

The proof of Theorem 6.1 falls into 2 parts. Suppose first that we have proved 6.2 through 6.7 with the operation $\mathfrak{p}$ replaced by the operation p , and the coefficient group A_{2k} restricted to be a group in the category $\mathcal{C}$. Then, the proof of 6.2–6.7 for the general case of the

³ With the exception of 6.5 and 6.6, these properties are noted by J. H. C. Whitehead in [10].

function $\mathfrak{p}$ follows at once, using definition 2.3; that is, $\mathfrak{p} = \zeta_2^* p \nu_*$. In particular, 6.2–6.5 are simple consequences of the fact that ζ_2^* and ν_* are homomorphisms; 6.6 follows from 6.3 and 6.5, and 6.7 follows from the fact that f^* commutes with all coefficient group homomorphisms. Finally, to prove 6.8 for the operation $\mathfrak{p}$, one uses 2.4 and exactly the same argument as that used to prove I(9) in § 4 of [9]. Thus, we are left with proving 6.2 through 6.7 for the operation p . Let $u \in H^q(K; Z)$, where $Z_\theta \in \mathcal{C}$. Then,

$$(i) \quad p(u) = 0, \quad \text{if } \theta \text{ is zero or odd.}$$

This follows at once from 2.1. For if θ is zero or odd, the homomorphism β is zero.

$$(ii) \quad 2p(u) = 0$$

This again is immediate from 2.1; for it is always the case that $2\beta = 0$.

$$(iii) \quad p \text{ is a homomorphism}$$

In § 5 we showed that the operation p is the suspension of the operation P_2 . But by 7.4 in [6], all operations which are suspensions are homomorphisms.

$$(iv) \quad \text{If } \theta = 2^i \ (i > 1), \text{ and } 2u = 0, \text{ then } p(u) = 0.$$

Since $2u = 0$, we may use Lemma 13.3 of [9]: namely, there are classes $x \in H^{q-1}(K; Z_2)$ and $y \in H^q(K; Z_2)$ such that

$$u = \lambda_* \delta_*(x) + \nu_*(y),$$

where δ_* is the coboundary associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0,$$

λ is the natural factor map $Z \rightarrow Z_2$, and ν maps Z_2 to Z_θ by $\nu(1_2) = (\theta/2)1_\theta$ (recall that $\theta = 2^i$, $i > 1$). Hence, by (iii) above,

$$\begin{aligned} p(u) &= p\lambda_* \delta_*(x) + p\nu_*(y) \\ &= G_2(\lambda)_* p\delta_*(x) + G_2(\nu)_* p(y) \\ &= G_2(\nu)_* p(y), \end{aligned}$$

by 2.4 and (i) above, since $\delta_*(u)$ has integer coefficients. Now,

$$G_2(\nu)_* p(y) = G_2(\nu)_* \beta_* w(u),$$

by 2.2. We show that $p(u) = 0$ by showing that

$$G_2(\nu)\beta = 0.$$

From Definition 2.1 we recall that β maps Z_2 to $G_2(Z_2)$ by $\beta(1_2) = 2g_2(1_2)$. Hence, using 1.21 and 1.24 in [9],

$$\begin{aligned} G_2(\nu)\beta(1_2) &= 2G_2(\nu)g_2(1_2) = 2g_2(\nu 1_2) \\ &= 2g_2((\theta/2)1_\theta) = 2(\theta^2/4)g_2(1) = (\theta^2/2)1_{2\theta} = 0. \end{aligned}$$

For, $\theta^2/2 = 2^{2i}/2 = 2^{2i-1}$; and, $2\theta = 2^{i+1}$. But by hypothesis, $i \geq 2$; thus $2i - 1 \geq i + 1$.

$$(v) \quad p(p(u)) = 0$$

This follows at once from (ii) and (iv) above.

$$(vi) \quad f^*p(u) = pf^*(u).$$

This is simply a special case of Theorem 3.6 of [7]. This, then completes the proof of Theorem 6.1.

We consider one more property of the operation $\mathfrak{p}$: namely, its behaviour with respect to suspension. We continue to denote by $S(C)$ the suspension of a cohomology operation C .

(6.9) PROPOSITION. $S(\mathfrak{p}) = 0$, where 0 denotes the trivial cohomology operation.

Proof. By the same reasoning as given in § 5, it suffices to prove Proposition 6.9 with $\mathfrak{p}$ replaced by the operation p , and the coefficient group A_{2k} taken to be a group in the category $\mathcal{C}$, say $A_{2k} = Z$. Thus, we need simply show that $p\phi(u) = 0$, where $u \in H^q(L; Z_\beta)$. Now by Nakaoka [2] we have⁴:

$$p(v_1 \times v_2) = P_2(v_1) \times p(v_2) + p(v_1) \times P_2(v_2),$$

for classes $v_i \in H^{q_i}(X_i, A_i; Z)$ ($i = 1, 2$).

Thus,

$$p\phi(u) = p(\bar{v} \times u) = P_2(\bar{v}) \times p(u) + p(\bar{v}) \times P_2(u) = 0,$$

since $P_2(\bar{v}) = p(\bar{v}) = 0$ by dimensionality considerations. Here, $\bar{v}$ is the image of v in $H^1(I, \dot{I}; Z_\beta)$. Hence, $S(p) = 0$, as was to be proved.

7. The relation $\delta S(C) = C\delta$. We give here a theorem, whose proof is due to N. E. Steenrod.

(7.1) THEOREM. *Let C be a cohomology operation, and let δ be the relative cohomology coboundary operator. Then,*

⁴ Nakaoka only proves this for the case $\dim v_1, v_2$ even; but the result is true in general, as is easily shown using Definition 2.1.

$$\delta S(C) = C\delta ,$$

where $S(C)$ is the suspension of C .

We sketch the proof; let X be a space and $A \subset X$ a subspace. Let X' denote the mapping cylinder of the inclusion map $A \subset X$. That is, unite $I \times A$ and X by identifying $1 \times A$ with A in X . Let $A' = 0 \times A$. The inclusions

$$(X', A') \longrightarrow (X', I \times A) \longleftarrow (X, A)$$

induce isomorphisms of the cohomology sequence of (X, A) and (X', A') with local coefficients. Thus, we may discuss the behaviour of the coboundary δ in the cohomology sequence of the pair (X', A') .

Consider the following hexagonal diagram (see [8], page 42):

$$(7.2) \quad \begin{array}{ccccc} & & H^q(I \times X) & & \\ & n_1^* \swarrow & \downarrow j^* & \searrow n_0^* & \\ H^q(0 \times X) & & & & H^q(1 \times X) \\ & \nwarrow d_1^* & & \nearrow d_0^* & \\ & & H^q(\dot{I} \times X) & & \\ & k_1^* \uparrow & \downarrow i_1^* & \downarrow i_0^* & k_0^* \uparrow \\ H^q(\dot{I} \times X, 1 \times X) & & & & H^q(\dot{I} \times X, 0 \times X) \\ & \searrow \delta_1 & \downarrow \delta & \swarrow \delta_0 & \\ & & H^{p+}(\dot{I} \times X, \dot{I} \times X) & & \end{array}$$

Here all homomorphisms other than δ , δ_1 , and δ_2 are induced by inclusions. Standard arguments, using exactness and homotopy equivalence, show that the arrows around the peripheries are isomorphisms. We agree to identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, where e is the unit of $H^0(0; Z)$. At the end of this section we will use diagram 7.2 to prove the following lemma:

(7.3) LEMMA. *Let ϕ be the function defined in 3.1. Then,*

$$\phi = \delta_1 k_1^{*-1} ,$$

where k_1^* , δ_1 are the functions defined in diagram 7.2

Notice that this proves Lemma 3.2; for the functions δ_1 , k_1^* are isomorphisms. Now let $g^*: H^{q+1}(X', A' \cup X) \rightarrow H^{q+1}(I \times A, \dot{I} \times A)$ be induced by the inclusion. Using the fact that $\dot{I}$ is a strong deformation retract of a neighborhood of $\dot{I}$ in I (see [8]; Chapter 1, 11.6), together with excision, one shows that g^* is an isomorphism onto.

(7.4) LEMMA. *The following diagram is commutative, where f^* is induced by the inclusion*

$$\begin{array}{ccc}
H^{q+1}(I \times A, \dot{I} \times A) & \xrightarrow{g^{*-1}} & H^{q+1}(X', A' \cup X) \\
\uparrow \phi & & \downarrow f^* \\
H^q(A') & \xrightarrow{\delta} & H^{q+1}(X', A') .
\end{array}$$

Thus $\delta = f^*g^{*-1}\phi$.

This is a consequence of Lemma 7.3 and commutativity relations in a slightly enlarged diagram. We omit the details.

The proof of Theorem 7.1 is an immediate consequence of Lemma 7.4. For let $u \in H^q(A')$. Then, by this lemma,

$$C\delta(u) = Cf^*g^{*-1}\phi(u) .$$

Using the naturality of the operation C , we have

$$Cf^*g^{*-1}\phi(u) = f^*g^{*-1}C\phi(u) .$$

But by Definition 3.1, $C\phi = \phi S(C)$.

Thus,

$$C\delta(u) = f^*g^{*-1}\phi S(C)(u) = \delta S(C)(u) ,$$

again using Lemma 7.4. This completes the proof of Theorem 7.1.

Proof of Lemma 7.3. We apply diagram 7.2 to the case $X = \emptyset$, $q = 0$, and coefficient group = integers. Then, the unit class of $H^0(\dot{I}; Z)$ can be represented as a sum $v_0 + v_1$, where

$$v_0 = i_1^*k_1^{*-1}d_1^*(v_0 + v_1), \quad v_1 = i_0^*k_0^{*-1}d_0^*(v_0 + v_1).$$

Thus,

$\delta(v_0) = -\delta(v_1) = v$ = a generator of $H^1(I, \dot{I}; Z)$. Therefore, by Definition 3.1,

$$\phi(u) = v \times u = (\delta v_0) \times u .$$

But by the axioms for the cross-product, we may write

$$(\delta v_0) \times u = \delta(v_0 \times u) .$$

Furthermore, we have

$$v_0 = i_1^*k_1^{*-1}(e) ,$$

where $e = d_1^*(v_0 + v_1)$ = unit of $H^0(0; Z)$. Thus,

$$\begin{aligned}
\delta(v_0 \times u) &= \delta(i_1^*k_1^{*-1}(e) \times u) \\
&= \delta i_1^*k_1^{*-1}(e \times u) = \delta_1 k_1^{*-1}(e \times u) .
\end{aligned}$$

Here we have used the naturality of the cross-product and the commutativity of diagram 7.2. If we now identify $H^q(X)$ with $H^q(0 \times X)$ by sending $u \rightarrow e \times u$, we then have

$$\phi(u) = \delta(v_0 \times u) = \delta_! h_1^{*-1}(u) ,$$

as was asserted.

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